Chapter

5

Indefinite Integral

Contents

Eudoxus

The origin of the integral calculus goes back to the early period of development of mathematics and it is related to the method of exhaustion developed by the mathematicians of ancient Greece. This method arose in the solution of problems on calculating areas and volumes of solid bodies etc. In this sense, the method of exhaustion can be regarded as an early method of integration.

***The greatest development of method of exhaustion in the early period was obtained in the works of* Eudoxus (440 B.C.) *and* Archimedes (300 B.C.)**

Conclusively, the fundamental concepts and theory and integral calculus and primarily its relationship with differential calculus were developed in the work of P.de Fermat, I. Newton and G.Leibnitz at the end of 17th century A.D.

Indefinite Integral

Introduction

A function $\phi(x)$ is called a primitive or an antiderivative of a function $f(x)$ if $\phi'(x)=f(x)$.

For example, $\frac{x^5}{5}$ is a primitive of x^4 because $\frac{d}{dx}\left(\frac{x^5}{5}\right)=x^4$

Let $\phi(x)$ be a primitive of a function $f(x)$ and let c be any constant.

Then $\frac{d}{dx}[\phi(x)+c]=\phi'(x)=f(x)$ $\quad [\because \ \phi'(x)=f(x)]$

$\Rightarrow$ $\phi(x)+c$ is also a primitive of $f(x)$.

Thus, if a function $f(x)$ possesses a primitive, then it possesses infinitely many primitives which are contained in the expression $\phi(x)+c$, where c is a constant.

For example $\frac{x^5}{5}, \frac{x^5}{5}+2, \frac{x^5}{5}-1$ etc. are primitives of x^4.

Note : ❑ If $F_1(x)$ and $F_2(x)$ are two antiderivatives of a function $f(x)$ on an interval $[a, b]$, then the difference between them is a constant.

5.1 Definition

Let $f(x)$ be a function. Then the collection of all its primitives is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)dx$.

Thus, $\frac{d}{dx}(\phi(x)+c)=f(x) \Rightarrow \int f(x)dx=\phi(x)+c$

where $\phi(x)$ is primitive of $f(x)$ and c is an arbitrary constant known as the constant of integration.

Here $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration.

The process of finding an indefinite integral of a given function is called integration of the function.

It follows from the above discussion that integrating a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}(\phi(x)) = f(x)$.

5.2 Comparison between Differentiation and Integration

(1) Differentiation and integration both are operations on functions and each gives rise to a function.

(2) Each function is not differentiable or integrable.

(3) The derivative of a function, if it exists, is unique. The integral of a function, if it exists, is not unique.

(4) The derivative of a polynomial function decreases its degree by 1, but the integral of a polynomial function increases its degree by 1.

(5) The derivative has a geometrical meaning, namely, the slope of the tangent to a curve at a point on it. The integral has also a geometrical meaning, namely, the area of some region.

(6) The derivative is used in obtaining some physical quantities like velocity, acceleration etc. of a particle. The integral is used in obtaining some physical quantities like centre of mass, momentum etc.

(7) Differentiation and integration are inverse of each other.

5.3 Properties of Integrals

(1) The differentiation of an integral is the integrand itself or the process of differentiation and integration neutralize each other, *i.e.*, $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$.

(2) The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function, *i.e.*, $\int cf(x)dx = c\int f(x)dx$.

(3) Integral of the sum or difference of two functions is equal to the sum or difference of their integrals, *i.e.*, $\int \{f_1(x) \pm f_2(x)\}dx = \int f_1(x)dx \pm \int f_2(x)dx$

In the general form, $\int \{k_1 . f_1(x) \pm k_2 . f_2(x) \pm k_3 . f_3(x) \pm .....\}dx = k_1\int f_1(x)dx \pm k_2\int f_2(x)dx \pm k_3\int f_3(x)dx \pm ....$

5.4 Fundamental Integration Formulae

(1) (i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1 \quad \because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ (ii) $\int dx = x + c$

(iii) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c,$ (iv) $\int (ax+b)^n dx = \frac{1}{a}.\frac{(ax+b)^{n+1}}{n+1} + c$

(2) (i) $\int \frac{1}{x}dx = \log|x| + c$ $\quad \because \frac{d}{dx}(\log|x|) = \frac{1}{x}$ $\quad$ (ii) $\int \frac{1}{ax+b}dx = \frac{1}{a}(\log|ax+b| + c$

(3) $\int e^x dx = e^x + c$ $\quad \because \frac{d}{dx}(e^x) = e^x$

(4) $\int a^x dx = \frac{a^x}{\log_e a} + c$ $\quad \because \frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x$

(5) $\int \sin x\, dx = -\cos x + c$ $\quad \because \frac{d}{dx}(-\cos x) = \sin x$

(6) $\int \cos x\, dx = \sin x + c$ $\quad \because \frac{d}{dx}(\sin x) = \cos x$

(7) $\int \sec^2 x\, dx = \tan x + c$ $\quad \because \frac{d}{dx}(\tan x) = \sec^2 x$

(8) $\int \mathrm{cosec}^2 x\, dx = -\cot x + c$ $\quad \because \frac{d}{dx}(-\cot x) = \mathrm{cosec}^2 x$

(9) $\int \sec x \tan x\, dx = \sec x + c$ $\quad \because \frac{d}{dx}(\sec x) = \sec x \tan x$

(10) $\int \mathrm{cosec}\, x \cot x\, dx = -\mathrm{cosec}\, x + c$ $\quad \because \frac{d}{dx}(-\mathrm{cosec}\, x) = \mathrm{cosec}\, x \cot x$

(11) $\int \tan x\, dx = -\log|\cos x| + c = \log|\sec x| + c$ $\quad \because \frac{d}{dx}(\log \cos x) = -\tan x$

(12) $\int \cot x\, dx = \log|\sin x| + c = -\log|\mathrm{cosec}\, x| + c$ $\quad \because \frac{d}{dx}(\log \sin x) = \cot x$

(13) $\int \sec x\, dx = \log|\sec x + \tan x| + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$ $\quad \because \frac{d}{dx}\log(\sec x + \tan x) = \sec x$

(14) $\int \mathrm{cosec}\, x\, dx = \log|\mathrm{cosec}\, x - \cot x| + c = \log \tan \frac{x}{2} + c$ $\quad \because \frac{d}{dx}(\log|\mathrm{cosec}\, x - \cot x|) = \mathrm{cosec}\, x$

(15) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$ $\quad \because \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(16) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$ $\quad \because \frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}, \frac{d}{dx}(\cos^{-1}\frac{x}{a}) = \frac{-1}{\sqrt{a^2-x^2}}$

(17) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$ $\quad \because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$, \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(18) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c = \frac{-1}{a}\cot^{-1}\frac{x}{a} + c$ $\quad \because \frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2+x^2} \quad \frac{d}{dx}\left(\cot^{-1}\frac{x}{a}\right) = \frac{-a}{a^2+x^2}$

(19) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\text{cosec}^{-1} x + c$ $\quad \because \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

(20) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c = \frac{-1}{a}\text{cosec}^{-1}\frac{x}{a} + c$ $\quad \because \frac{d}{dx}\left(\sec^{-1}\frac{x}{a}\right) = \frac{a}{x\sqrt{x^2-a^2}}$ $\frac{d}{dx}(\text{cosec}^{-1}\frac{x}{a}) = \frac{-a}{x\sqrt{x^2-a^2}}$

Note : ❑ In any of the fundamental integration formulae, if x is replaced by $ax + b$, then the same formulae is applicable but we must divide by coefficient of x or derivative of $(ax + b)$ *i.e.*, a. In general, if $\int f(x)dx = \phi(x) + c$, then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + c$

$\int \sin(ax+b)dx = \frac{-1}{a}\cos(ax+b) + c,$ $\quad \int \sec(ax+b)dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + c$ *etc.*

Some more results:

(i) $\int \frac{1}{x^2-a^2} = \frac{-1}{a}\coth^{-1}\frac{x}{a} + c = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c,$ when $x > a$

(ii) $\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c,$ when $x < a$

(iii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\{|x + \sqrt{x^2-a^2}|\} + c = \cosh^{-1}\left(\frac{x}{a}\right) + c$

(iv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\{|x + \sqrt{x^2+a^2}|\} + c = \sinh^{-1}\left(\frac{x}{a}\right) + c$

(v) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + c$

(vi) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\log\{x + \sqrt{x^2-a^2}\} + c = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\cosh^{-1}\left(\frac{x}{a}\right) + c$

(vii) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\log\{x + \sqrt{x^2+a^2}\} + c = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\sinh^{-1}\left(\frac{x}{a}\right) + c$

Important Tips

☞ *The signum function has an antiderivative on any interval which doesn't contain the point x = 0, and does not possess an anti-derivative on any interval which contains the point.*

☞ *The antiderivative of every odd function is an even function and vice-versa*

Example: 1 $\int \frac{(x+1)^2}{x(x^2+1)}dx$ is equal to **[MP PET 2003]**

(a) $\log_e x$ (b) $\log_e x + 2\tan^{-1} x$ (c) $\log_e \frac{1}{x^2+1}$ (d) $\log_e\{x(x^2+1)\}$

Solution: (b) $\int \frac{(x+1)^2}{x(x^2+1)}dx = \int \frac{x^2+1+2x}{x((x^2+1)}dx = \int \frac{x^2+1}{x(x^2+1)}dx + 2\int \frac{x}{x(x^2+1)}dx = \int \frac{dx}{x} + 2\int \frac{dx}{x^2+1} = \log_e x + 2\tan^{-1} x$

Example: 2 $\int \frac{ax^3 + bx^2 + c}{x^4} dx$ equals to

(a) $a \log x + \frac{b}{x^2} + \frac{c}{3x^3} + k$ (b) $a \log x + \frac{b}{x} - \frac{c}{3x^3} + k$ (c) $a \log x - \frac{b}{x} - \frac{c}{3x^3} + k$ (c) None of these

Solution: (c) $I = \int \frac{ax^3 + bx^2 + c}{x^4} dx = \int \left[\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^4}\right] dx = a \log x - \frac{b}{x} - \frac{c}{3x^3} + k$

Example: 3 $\int \frac{1 + x + \sqrt{x + x^2}}{\sqrt{x} + \sqrt{1 + x}} dx =$ **[EAMCET 2003]**

(a) $\frac{1}{2}\sqrt{1 + x} + c$ (b) $\frac{2}{3}(1 + x)^{3/2} + c$ (c) $\sqrt{1 + x} + c$ (d) $2(1 + x)^{3/2} + c$

Solution: (b) $\int \frac{1 + x + \sqrt{x + x^2}}{\sqrt{x} + \sqrt{1 + x}} dx = \int \frac{\sqrt{1 + x}\left[\sqrt{1 + x} + \sqrt{x}\right]}{(\sqrt{x} + \sqrt{1 + x})} dx = \int \sqrt{1 + x}\, dx = \frac{2}{3}(1 + x)^{3/2} + c$

Example: 4 $\int \left(\sin^4 x - \cos^4 x\right) dx =$ **[Rajasthan PET 2003]**

(a) $-\frac{\cos 2x}{2} + c$ (b) $-\frac{\sin 2x}{2} + c$ (c) $\frac{\sin 2x}{2} + c$ (d) $\frac{\cos 2x}{2} + c$

Solution: (b) $\int (\sin^4 x - \cos^4 x) dx = \int (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) dx$

$= \int (\sin^2 x - \cos^2 x) dx = -\int (\cos^2 x - \sin^2 x) dx = -\int \cos 2x\, dx = \frac{-\sin 2x}{2} + c$

Example: 5 $\int \sqrt{1 + \sin\left(\frac{x}{4}\right)}\, dx =$ **[Karnataka CET 2003]**

(a) $8\left(\sin \frac{x}{8} - \cos \frac{x}{8}\right) + c$ (b) $\left(\sin \frac{x}{8} + \cos \frac{x}{8}\right) + c$ (c) $\frac{1}{8}\left(\sin \frac{x}{8} - \cos \frac{x}{8}\right) + c$ (d) $8\left(\cos \frac{x}{8} - \sin \frac{x}{8}\right) + c$

Solution: (a) $\int \sqrt{1 + \sin\left(\frac{x}{4}\right)} dx = \int \sqrt{\left[\left(\sin^2 \frac{x}{8} + \cos^2 \frac{x}{8}\right) + \left(2 \sin \frac{x}{8} . \cos \frac{x}{8}\right)\right]} dx = \int \sqrt{\left(\sin \frac{x}{8} + \cos \frac{x}{8}\right)^2}$

$= \int \left(\sin \frac{x}{8} + \cos \frac{x}{8}\right) dx = \frac{-\cos x/8}{1/8} + \frac{\sin x/8}{1/8}$

$= 8\left(\sin \frac{x}{8} - \cos \frac{x}{8}\right) + c$

5.5 Integration by Substitution

(1) **When integrand is a function *i.e.,* $\int f[\phi(x)]\phi'(x) dx$:**

Here, we put $\phi(x) = t,$ so that $\phi'(x) dx = dt$ and in that case the integrand is reduced to $\int f(t) dt$.

In this method, the integrand is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

Example: 6 $\int \frac{dx}{2\sqrt{x}(1+x)} =$ **[Rajasthan PET 2002]**

(a) $\frac{1}{2}\tan^{-1}(\sqrt{x})+c$ (b) $\tan^{-1}(\sqrt{x})+c$ (c) $2\tan^{-1}(\sqrt{x})+c$ (d) None of these

Solution: (b) $I=\int \frac{dx}{2\sqrt{x}(1+x)}$, put $\sqrt{x}=t \Rightarrow \frac{1}{2\sqrt{x}}dx=dt$

$\therefore\ I=\int \frac{dt}{1+t^2}=\tan^{-1}t+c=\tan^{-1}\sqrt{x}+c$

(2) **When integrand is the product of two factors such that one is the derivative of the others *i.e.*,** $I=\int f'(x).\ f(x).\ dx$

In this case we put $f(x)=t$ and convert it into a standard integral.

Example: 7 For any natural number m $\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}dx,\quad (x>0)=$ **[IIT 2002]**

(a) $\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{6(m+1)}$ (b) $\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m}{m+1}}}{6(m+1)}$

(c) $-\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{6(m+1)}$ (d) $\frac{-(2x^{3m}+3x^{2m}+6x^m)^{\frac{m}{m+1}}}{6(m+1)}$

Solution: (a) $I=\int (x^{3m-1}+x^{2m-1}+x^{m-1})(2x^{3m}+3x^{2m}+6x^m)^{1/m}dx$

Put $2x^{3m}+3x^{2m}+6x^m=t \Rightarrow 6m(x^{3m-1}+x^{2m-1}+x^{m-1})dx=dt$

$I=\int \frac{1}{6m}t^{1/m}dt=\frac{1}{6m}\frac{t^{\frac{1}{m}+1}}{\frac{1}{m}+1}+c=\frac{1}{6(m+1)}t^{(m+1)/m}+c$

Again put the value of t, then we get option (a).

Example: 8 $\int \frac{x^2\tan^{-1}x^3}{1+x^6}dx$ is equal to

(a) $\tan^{-1}(x^3)+c$ (b) $\frac{1}{6}(\tan^{-1}x^3)^2+c$ (c) $-\frac{1}{2}(\tan^{-1}x^3)^2+c$ (d) $\frac{1}{2}(\tan^{-1}x^3)^2+c$

Solution: (b) Put $\tan^{-1}x^3=t$

$\frac{1}{1+x^6}.3x^2dx=dt \Rightarrow \frac{x^2}{1+x^6}dx=\frac{dt}{3} \Rightarrow \frac{1}{3}\int tdt=\frac{1}{3}.\frac{t^2}{2}=\frac{(\tan^{-1}x^3)^2}{6}+c$

Example: 9 $\int x^{-3}5^{1/x^2}dx=k.5^{1/x^2}+c$, then k is

(a) $-\frac{1}{2\log 5}$ (b) $-2\log 5$ (c) $\frac{2}{\log 5}$ (d) $\frac{-2}{\log 5}$

Solution: (a) Put $x^{-2}=t \Rightarrow -2x^{-3}dx=dt \Rightarrow x^{-3}dx=-\frac{dt}{2}$

$\int x^{-3} 5^{1/x^2} dx = -\frac{1}{2}\int 5^t dt = -\frac{1}{2}\frac{5^t}{\log_e 5} + c \ = \frac{-1}{2\log_e 5}.5^{1/x^2} + c$. On comparing, $k = -\frac{1}{2\log_e 5}$

(3) **Integral of a function of the form $f(ax+b)$:** Here we put $ax + b = t$ and convert it into standard integral. Obviously if $\int f(x)dx = \phi(x)$, then, $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b)+c$

Example: 10 $\int \tan(3x-5)\sec(3x-5)dx =$ **[MP PET 1988]**

(a) $\sec(3x-5)+c$ (b) $\frac{1}{3}\sec(3x-5)+c$ (c) $\tan(3x-5)+c$ (d) None of these

Solution: (b) $\because \int \sec x \tan x\, dx = \sec x + c$

$\therefore \int \sec(3x-5)\tan(3x-5)dx \ = \frac{\sec(3x-5)}{3} + c$

(4) **If integral of a function of the form $\frac{f'(x)}{f(x)}dx = \log[f(x)] + c$**

Example: 11 $\int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}dx$ is equal to **[MNR 1987; Rajasthan PET 1991]**

(a) $\frac{1}{b^2-a^2}\log(a^2\sin^2 x + b^2\cos^2 x)+c$ (b) $\frac{1}{a^2-b^2}\log(a^2\sin^2 x + b^2\cos^2 x)+c$

(c) $\log(a^2\sin^2 x - b^2\cos^2 x)+c$ (d) None of these

Solution: (b) Put $a^2\sin^2 x + b^2\cos^2 x = t \Rightarrow (a^2.2\sin x\cos x - b^2.2\cos x\sin x)dx = dt \Rightarrow \sin 2x(a^2 - b^2)dx = dt$

$\int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}dx = \frac{1}{(a^2-b^2)}\int \frac{dt}{t} = \frac{1}{a^2-b^2}\log t + c = \frac{1}{a^2-b^2}\log(a^2\sin^2 x + b^2\cos^2 x)+c$

Example: 12 $\int \frac{dx}{1+e^x} =$ **[Rajasthan PET 1990, 94; MP PET 1991; Roorkee 1977; SCRA 1999]**

(a) $\log(1+e^x)$ (b) $-\log(1+e^{-x})$ (c) $-\log(1-e^{-x})$ (d) $\log(e^{-x}+e^{-2x})$

Solution: (b) $I = \int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{e^{-x}+1}dx$, Put $e^{-x}+1 = t \Rightarrow -e^{-x}dx = dt \Rightarrow I = -\int \frac{dt}{t} = -\log t \ = -\log(1+e^{-x})$

Example: 13 The value of the integral $\int \frac{x}{1+x\tan x}dx$ is equal to

(a) $\log|x\cos x + \sin x| + c$ (b) $\log|\cos x + x|$ + c (c) $\log|\cos x + x\sin x| + c$ (d) None of these

Solution: (c) $I = \int \frac{x}{1+x\tan x}dx = \int \frac{x\cos x}{\cos x + x\sin x}dx$

Put $(\cos x + x\sin x) = t \Rightarrow (-\sin x + x\cos x + \sin x)dx = dt \Rightarrow x\cos x\, dx = dt$

$I = \int \frac{dt}{t} = \log|t| + c = \log|\cos x + x\sin x| + c$

(5) If integral of a function of the form, $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c \quad [n \neq -1]$

Example: 14 $\int x\sqrt{1+x^2}\,dx =$ **[MP PET 1989]**

(a) $\frac{1+2x^2}{\sqrt{1+x^2}} + c$ (b) $\sqrt{1+x^2} + c$ (c) $3(1+x^2)^{3/2} + c$ (d) $\frac{1}{3}(1+x^2)^{3/2} + c$

Solution: (d) Put $1+x^2 = t \Rightarrow 2xdx = dt$, $\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int t^{1/2}dt = \frac{1}{2}.\frac{t^{3/2}}{3/2} + c = \frac{1}{3}(1+x^2)^{3/2} + c$

Example: 15 $\int \sqrt{2+\sin 3x}\,.\cos 3x\,dx =$ **[IIT 1976]**

(a) $\frac{2}{9}\sqrt{2+\sin 3x} + c$ (b) $\frac{2}{3}(2+\sin 3x)^{2/3} + c$ (c) $\frac{2}{3}(2+\sin 3x)^{3/2} + c$ (d) $\frac{2}{9}(2+\sin 3x)^{3/2} + c$

Solution: (d) Put $(2+\sin 3x) = t \Rightarrow 3\cos 3x\,dx = dt$

$$\int \sqrt{2+\sin 3x}\,.\cos 3x\,dx = \frac{1}{3}\int \sqrt{t}\,dt = \frac{1}{3}.\frac{t^{3/2}}{3/2} + c = \frac{2}{9}.(2+\sin 3x)^{3/2} + c\,.$$

(6) If the integral of a function of the form, $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

Example: 16 $\int \frac{\sqrt{\tan x}}{\sin x\,.\cos x}\,dx$ **[MP PET 2002; Rajasthan PET 1985, 88; Bihar CEE 1974]**

(a) $2\sqrt{\sec x} + c$ (b) $2\sqrt{\tan x} + c$ (c) $\frac{2}{\sqrt{\tan x}} + c$ (d) $\frac{2}{\sqrt{\sec x}} + c$

Solution: (b) $\int \frac{\sqrt{\tan x}}{\sin x\,.\cos x}dx = \int \frac{\tan x}{\sqrt{\tan x}\,.\sin x\cos x}dx = \int \frac{\sin x\,.\sec x}{\sqrt{\tan x}\,\sin x\cos x}dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $t = \tan x \Rightarrow dt = \sec^2 x\,dx$, then it reduces to, $\int \frac{1}{\sqrt{t}}dt = 2t^{1/2} + c = 2\sqrt{\tan x} + c$

Example: 17 $\int \frac{x^5}{\sqrt{1+x^3}}dx$ **[IIT 1975, 85]**

(a) $\frac{2}{9}(1+x^3)^{3/2} + c$ (b) $\frac{2}{9}(1+x^3)^{3/2} + \frac{2}{3}(1+x^3)^{1/2} + c$

(c) $\frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + c$ (d) None of these

Solution: (c) $\int \frac{x^5}{\sqrt{1+x^3}}dx = \int \frac{x^3.x^2}{\sqrt{1+x^3}}dx$. Put $1+x^3 = t \Rightarrow 3x^2dx = dt \Rightarrow x^2dx = \frac{dt}{3}$

$$= \frac{1}{3}\int \frac{(t-1)}{\sqrt{t}}dt = \frac{1}{3}\int\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)dt = \frac{1}{3}\left[\frac{t^{3/2}}{3/2} - 2\sqrt{t}\right] + c = \frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + c$$

(7) Standard substitutions

	Integrand form	Substitution
(i)	$\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$, a^2-x^2	$x = a\sin\theta$, $x = a\cos\theta$

(ii)	$\sqrt{x^2+a^2}$, $\frac{1}{\sqrt{x^2+a^2}}$, x^2+a^2	$x = a\tan\theta$ or $x = a\sinh\theta$
(iii)	$\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$, x^2-a^2	$x = a\sec\theta$ or $x = a\cosh\theta$
(iv)	$\sqrt{\frac{x}{a+x}}$, $\sqrt{\frac{a+x}{x}}$, $\sqrt{x(a+x)}$, $\frac{1}{\sqrt{x(a+x)}}$	$x = a\tan^2\theta$
(v)	$\sqrt{\frac{x}{a-x}}$, $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$
(vi)	$\sqrt{\frac{x}{x-a}}$, $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}$, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(viii)	$\sqrt{\frac{x-\alpha}{\beta-x}}$, $\sqrt{(x-\alpha)(\beta-x)}$, $(\beta>\alpha)$	$x = \alpha\cos^2\theta+\beta\sin^2\theta$

Example: 18 $\int \frac{dx}{(a^2+x^2)^{3/2}}$ is equal to **[Rajasthan PET 2000]**

(a) $\frac{x}{(a^2+x^2)^{1/2}}+c$ (b) $\frac{x}{a^2(a^2+x^2)^{1/2}}+c$ (c) $\frac{1}{a^2(a^2+x^2)^{1/2}}+c$ (d) None of these

Solution: (b) $I = \int \frac{dx}{(a^2+x^2)^{3/2}}$

Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$\therefore I = \int \frac{a\sec^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^{3/2}} = \int \frac{a\sec^2\theta\, d\theta}{a^3(\sec^2\theta)^{3/2}} = \frac{1}{a^2}\int \frac{\sec^2\theta\, d\theta}{\sec^3\theta}$

$\Rightarrow I = \frac{1}{a^2}\int \frac{d\theta}{\sec\theta} = \frac{1}{a^2}\int \cos\theta\, d\theta \Rightarrow I = \frac{1}{a^2}\sin\theta + c \Rightarrow \frac{x}{a^2(x^2+a^2)^{1/2}}+c$

Example: 19 $\int \sqrt{\frac{1-x}{1+x}}dx =$ **[IIT 1971]**

(a) $\sin^{-1}x - \frac{1}{2}\sqrt{1-x^2}+C$ (b) $\sin^{-1}x + \frac{1}{2}\sqrt{1-x^2}+C$ (c) $\sin^{-1}x - \sqrt{1-x^2}+C$ (d) $\sin^{-1}x + \sqrt{1-x^2}+C$

Solution: (d) Put $x = \cos 2\theta$, then $\theta = \frac{1}{2}\cos^{-1}x \Rightarrow dx = -2\sin 2\theta\, d\theta$

$\therefore I = -2\int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}.\sin 2\theta\, d\theta \Rightarrow I = -2\int \sqrt{2\sin^2\theta/2\cos^2\theta}.\sin 2\theta.d\theta$

$\Rightarrow I = -2\int \tan\theta.2\sin\theta\cos\theta\, d\theta \Rightarrow I = -2.2\int \sin^2\theta\, d\theta \Rightarrow I = -2\int (1-\cos 2\theta)d\theta \Rightarrow I = -2\left[\theta - \frac{\sin 2\theta}{2}\right]+C_1$

$\Rightarrow I = -2\theta + \sin 2\theta + C_1 \Rightarrow I = -\cos^{-1}x + \sqrt{1-x^2}+C_1 \Rightarrow I = -\frac{\pi}{2}+\sin^{-1}x+\sqrt{1-x^2}+C_1$

$\Rightarrow I = \sin^{-1}x + \sqrt{1-x^2}+C$ [where $C = C_1 - \frac{\pi}{2}$]

Trick : Rationalization of denominator and put $1-x^2=t^2$.

5.6 Integration by Parts

(1) **When integrand involves more than one type of functions :** We may solve such integrals by a rule which is known as integration by parts. We know that,

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx} \Rightarrow d(uv)=udv+vdu \Rightarrow \int d(uv)=\int udv+\int vdu$$

If u and v are two functions of x, then $\int \underset{I}{u}\,\underset{II}{v}\,dx=u\int v\,dx-\int\{\frac{du}{dx}.\int vdx\}dx$ *i.e.*, the integral of the product of two functions = (First function) × (Integral of second function) – Integral of {(Differentiation of first function) × (Integral of second function)}

Integration with the help of above rule is called integration by parts. Before applying this rule proper choice of first and second function is necessary. Normally we use the following methods :

(i) In the product of two functions, one of the function is not directly integrable (*i.e.*, $\log|x|,\ \sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ ......etc), then we take it as the first function and the remaining function is taken as the second function.

(ii) If there is no other function, then unity is taken as the second function *e.g.* In the integration of $\int \sin^{-1}x\,dx, \int \log x\,dx, 1$ is taken as the second function.

(iii) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable.

Usually, we use the following preference order for the first function. (Inverse, Logarithmic, Algebraic, Trigonometric, exponential). This rule is simply called as "I LATE".

Important Tips

☞ *If* $I_n=\int x^n.e^{ax}dx$, *then* $I_n=\frac{x^ne^{ax}}{a}-\frac{n}{a}I_{n-1}$

☞ *If* $I_n=\int(\log x)dx$, *then* $I_n=x\log x-x$

☞ *If* $I_n=\int\frac{1}{\log x}dx$, *then* $I_n=\log(\log x)+\log x+\frac{(\log x)^2}{2.(2!)}+\frac{(\log x)^3}{3(3!)}+..........\ ....\ ...$

☞ *If* $I_n=\int(\log x)^n dx$; *then* $I_n=x(\log x)^n-n.I_{n-1}$

☞ *Successive integration by parts can be performed when one of the functions is* x^n *(n is positive integer) which will be successively differentiated and the other is either of the following* $\sin ax, \cos ax, e^{ax}, e^{-ax}, (x+a)^m$ *which will be successively integrated.*

☞ ***Chain rule :*** $\int u.v\,dx=uv_1-u'v_2+u''v_3-u'''v_4+..........\ ....+(-1)^{n-1}u^{n-1}v_n+(-1)^n\int u^n.v_n\,dx$ *Where* u^n *stands for* n^{th} *differential coefficient of u and* v_n *stands for* n^{th} *integral of v.*

Example: 20 If $I_n = \int (\log x)^n \, dx$, then $I_n + nI_{n-1} =$ **[Karnataka CET 2003]**

(a) $x(\log x)^n$ (b) $(x \log x)^n$ (c) $(\log x)^{n-1}$ (d) $n(\log x)^n$

Solution: (a) $I_n = \int (\log x)^n \, dx$, $\therefore I_{n-1} = \int (\log x)^{n-1} dx$

Now $I_n = \int (\log x)^n .1 \, dx = (\log x)^n . x - n \int (\log x)^{n-1} . \frac{1}{x} . x \, dx = (\log x)^n . x - n \int (\log x)^{n-1} dx$

$I_n = x(\log x)^n - n I_{n-1} \quad \therefore I_n + n I_{n-1} = x(\log x)^n$

Example: 21 If $\int x \sin x \, dx = -x \cos x + A$, then $A =$ **[MP PET 1992, 2000; Rajasthan PET 1997]**

(a) $\sin x + \text{constant}$ (b) $\cos x + \text{constant}$ (c) Constant (d) None of these

Solution: (a) Since $\int x \sin x \, dx = -x \cos x + A \Rightarrow -x \cos x + \sin x + \text{constant} = -x \cos x + A$

Equating it, we get $A = \sin x + \text{constant}$

Example: 22 $\int x \sec^2 x \, dx =$ **[Rajasthan PET 1996, 2003; MP PET 1987, 97]**

(a) $\tan x + \log \cos x + c$ (b) $\frac{x^2}{2} \sec^2 x + \log \cos x + c$ (c) $x \tan x + \log \sec x + c$ (d)

Solution: (d) $\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \log(\cos x) + c$

Example: 23 $\int [f(x)g''(x) - f''(x)g(x)]dx$ is equal to **[MP PET 2001]**

(a) $\frac{f(x)}{g'(x)}$ (b) $f'(x)g(x) - f(x)g'(x)$ (c) $f(x)g'(x) - f'(x)g(x)$ (d) $f(x)g'(x) + f'(x)g(x)$

Solution: (c) $\int [f(x)g''(x) - f''(x)g(x)]dx = \int f(x)g''(x)dx - \int f''(x)g(x)dx$

$= [f(x)g'(x) - \int f'(x)g'(x)dx] - [g(x)f'(x) - \int g'(x)f'(x)dx] = f(x)g'(x) - f'(x)g(x)$

Example: 24 $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$ is equal to **[IIT 1983; WBJEE 1992]**

(a) $\frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x(1-x)}] - x + c$ (b) $\frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} - \sqrt{x(1-x)}] + x + c$

(c) $\frac{\pi}{2}\left[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x(1-x)}\right] - x + c$ (d) None of these

Solution: (a) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx = \frac{2}{\pi}\left[\int \sin^{-1}\sqrt{x} \, dx - \int \cos^{-1}\sqrt{x} \, dx\right] \left[\because \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}\right]$

Now we solve first and second expressions separately. For first expression, $\int \sin^{-1}\sqrt{x} \, dx$

Put $x = \sin^2\theta \Rightarrow \cos 2\theta = 1 - 2x \Rightarrow dx = \sin 2\theta \, d\theta$

$\int \theta . \sin 2\theta \, d\theta = \frac{-\theta \cos 2\theta}{2} + \frac{1}{2}\int \cos 2\theta \, d\theta = \frac{-\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4} + c_1 = \frac{(2x-1)}{2}\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x(1-x)} + c_1$

For second expression, $\int \cos^{-1}\sqrt{x} \, dx$

Put $x=\cos^2\theta \Rightarrow dx=-2\cos\theta\sin\theta\, d\theta=-\sin 2\theta\, d\theta$

$$\int \cos^{-1}\sqrt{x}\,dx=-\int \theta.\sin 2\theta\, d\theta=\frac{\theta\cos 2\theta}{2}-\frac{\sin 2\theta}{4}+c_2$$

Therefore, $I=\frac{2}{\pi}\left[\frac{(2x-1)}{2}\{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}\}+\sqrt{x(1-x)}\right]+c$

$$I=\frac{2}{\pi}(2x-1)\sin^{-1}\sqrt{x}-\frac{(2x-1)}{2}.\frac{\pi}{2}+\sqrt{x(1-x)}\Big]+c \Rightarrow I=\frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}+\sqrt{x(1-x)}\right]-x+c$$

Example: 25 $\int \frac{x^2 dx}{(x\sin x+\cos x)^2}=$ **[MNR 1989; Rajasthan PET 2000]**

(a) $\frac{\sin x+\cos x}{x\sin x+\cos x}$ (b) $\frac{x\sin x-\cos x}{x\sin x+\cos x}$ (c) $\frac{\sin x-x\cos x}{x\sin x+\cos x}$ (d) None of these

Solution: (c) Differentiation of $x\sin x+\cos x$ is $x\cos x$. Then, $I=\int\frac{x^2dx}{(x\sin x+\cos x)^2}=\int\frac{x\cos x}{(x\sin x+\cos x)^2}.\frac{x}{\cos x}dx$

Integrate by parts, $\left[\because \int\frac{1}{t^2}dt=-\frac{1}{t}\right]$

$$\therefore I=\frac{-1}{(x\sin x+\cos x)}.\left(\frac{x}{\cos x}\right)+\int\frac{1}{(x\sin x+\cos x)}.\frac{\cos x.1-x(-\sin x)}{\cos^2 x}dx \Rightarrow I=\frac{-1}{x\sin x+\cos x}.\frac{x}{\cos x}+\int\sec^2 x\,dx$$

$$I=\frac{-1}{x\sin x+\cos x}.\frac{x}{\cos x}+\frac{\sin x}{\cos x} \Rightarrow I=\frac{-x+x\sin^2 x+\sin x\cos x}{(x\sin x+\cos x)\cos x} \Rightarrow I=\frac{\sin x\cos x-x(1-\sin^2 x)}{(x\sin x+\cos x)\cos x}=\frac{\sin x-x\cos x}{x\sin x+\cos x}$$

(2) **Integral is of the form** $\int e^x\{f(x)+f'(x)\}dx$ **:** If the integral is of the form $\int e^x\{f(x)+f'(x)\}dx$, then by breaking this integral into two integrals integrate one integral by parts and keeping other integral as it is, by doing so, we get

(i) $\int e^x[f(x)+f'(x)]dx=e^x f(x)+c$

(ii) $\int e^{mx}[mf(x)+f'(x)]dx=e^{mx}f(x)+c$

(iii) $\int e^{mx}\left[f(x)+\frac{f'(x)}{m}\right]dx=\frac{e^{mx}f(x)}{m}+c$

Example: 26 $\int e^x(1-\cot x+\cot^2 x)dx$ **[MP PET 2002]**

(a) $e^x\cot x+c$ (b) $e^x \operatorname{cosec} x+c$ (c) $-e^x\cot x+c$ (d) $e^x\operatorname{cosec} x+c$

Solution: (c) $\int e^x(\operatorname{cosec}^2 x-\cot x)dx=\int e^x[-\cot x+\operatorname{cosec}^2 x]=-e^x\cot x+c$

Example: 27 $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$ **[Rajasthan PET 2000]**

(a) $-e^x\tan\frac{x}{2}+c$ (b) $-e^x\cot\frac{x}{2}+c$ (c) $-\frac{1}{2}e^x\tan\frac{x}{2}+c$ (d) $\frac{1}{2}e^x\cot\frac{x}{2}+c$

Solution: (b) $\because \frac{1-\sin x}{1-\cos x}=\frac{1}{2\sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}=\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}-\cot\frac{x}{2}$

$\therefore \int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = \int e^x\left(\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx = -e^x\cot\frac{x}{2}+c$

Example: 28 $\int \frac{e^{2\tan^{-1}x}(1+x)^2}{1+x^2}dx =$ **[WB JEE 1998]**

(a) $xe^{\tan^{-1}x}+c$ (b) $xe^{2\tan^{-1}x}+c$ (c) $2xe^{\tan^{-1}x}+c$ (d) None of these

Solution: (b) Put $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$\therefore I = \int \frac{e^{2\tan^{-1}x}(1+x)^2}{1+x^2}dx = \int e^{2t}(1+\tan t)^2 dt = \int e^{2t}(\sec^2 t + 2\tan t)dt$

$I = \int e^{2t}\sec^2 t\,dt + 2\left[\tan t\frac{e^{2t}}{2} - \int \sec^2 t.\frac{e^{2t}}{2}dt\right] \Rightarrow I = \int e^{2t}\sec^2 t\,dt + e^{2t}\tan t - \int e^{2t}\sec^2 t\,dt + c$

$I = e^{2t}\tan t + c \Rightarrow I = xe^{2\tan^{-1}x}+c$

Example: 29 $\int \frac{(x+3)e^x}{(x+4)^2}dx$ is equal to **[Karnataka CET 2000]**

(a) $\frac{e^x}{x+4}+c$ (b) $\frac{e^x}{x+3}+c$ (c) $\frac{1}{(x+4)^2}+c$ (d) $\frac{e^x}{(x+4)^2}+c$

Solution: (a) $\int \frac{(x+3)e^x}{(x+4)^2}dx = \int \frac{(x+4-1)e^x}{(x+4)^2}dx = \int \frac{(x+4)e^x}{(x+4)^2}dx - \int \frac{e^x}{(x+4)^2}dx = \int \frac{e^x}{(x+4)}dx - \int \frac{e^x}{(x+4)^2}dx$

$= \left[\frac{e^x}{(x+4)} + \int \frac{e^x}{(x+4)^2}dx\right] - \int \frac{e^x}{(x+4)^2}dx + c = \frac{e^x}{x+4}+c$

(3) **Integral is of the form $[x\,f'(x)+f(x)]dx$:** If the integral is of the form $\int [x\,f'(x)+f(x)]dx$ then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is, by doing so, we get, $\int [x\,f'(x)+f(x)]dx = x\,f(x)+c$

Example: 30 If $\frac{d}{dx}f(x) = x\cos x + \sin x$ and $f(0)=2$, then $f(x)=$ **[MP PET 1989]**

(a) $x\sin x$ (b) $x\cos x+\sin x+2$ (c) $x\sin x+2$ (d) $x\cos x+2$

Solution: (c) $\frac{d}{dx}f(x) = x\cos x+\sin x \Rightarrow f(x) = \int(x\cos x+\sin x)dx = x\sin x + c$

Since $f(0)=2 \Rightarrow c=2$, $\therefore f(x) = x\sin x + 2$

Example: 31 $\int \frac{x+\sin x}{1+\cos x}dx =$ **[Roorkee 1980; AMU 1997; UPSEAT 1999]**

(a) $-x\tan x/2+c$ (b) $x\tan x/2+c$ (c) $x\tan x+c$ (d) $\frac{1}{2}x\tan x+c$

Solution: (b) $\int \frac{x+\sin x}{1+\cos x}dx = \frac{1}{2}\int x\sec^2\frac{x}{2}dx + \int \tan\frac{x}{2}dx = \frac{1}{2}\frac{x\tan x/2}{1/2} - \int \tan\frac{x}{2}dx + \int \tan\frac{x}{2}dx = x\tan\frac{x}{2}+c$

Trick : By inspection, $\frac{d}{dx}\left\{x\tan\frac{x}{2}+c\right\} = x\sec^2\frac{x}{2}+\tan\frac{x}{2} = \frac{1}{2}\left[\frac{x}{\cos^2 x/2} + \frac{2\sin x/2}{\cos x/2}\right] = \frac{x+\sin x}{1+\cos x}$

Hence the result.

(4) **Integrals of the form $\int e^{ax}\sin bx dx, \int e^{ax}\cos bx\,dx$:**

Working rule : To evaluate $\int e^{ax}\sin bx\,dx$ or $\int e^{ax}\cos bx\,dx$, proceed as follows

(i) Put the given integral equal to *I*.

(ii) Integrate by parts, taking e^{ax} as the first function.

(iii) Again, integrate by parts taking e^{ax} as the first function. This will involve *I*.

(iv) Transpose and collect terms involving *I* and then obtain the value of *I*.

Let $I=\int e^{ax}\sin bx\,dx$. Then $I=\int \underset{\text{I}}{e^{ax}}.\underset{\text{II}}{\sin bx}\,dx=-e^{ax}.\frac{\cos bx}{b}-\int ae^{ax}.\left(\frac{-\cos bx}{b}\right)dx$

$$=\frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b}\int \underset{\text{I}}{e^{ax}}.\underset{\text{II}}{\cos bx}\,dx \;=\frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b}\left[\frac{e^{ax}.\sin bx}{b}-\int ae^{ax}.\frac{\sin bx}{b}dx\right]$$

$$=\frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b^2}e^{ax}.\sin bx-\frac{a^2}{b^2}\int e^{ax}.\sin bx\,dx \;=\frac{-1}{b}e^{ax}.\cos bx+\frac{a}{b^2}e^{ax}.\sin bx-\frac{a^2}{b^2}I$$

$$I+I.\frac{a^2}{b^2}=\frac{e^{ax}}{b^2}(a\sin bx-b\cos bx)\;\Rightarrow I=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c$$

Thus, $\int e^{ax}\sin bx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c\;=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin(bx-\tan^{-1}\frac{b}{a})+c$

Similarly $\int e^{ax}.\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+c\;=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+c$

Note : ❑ $\int e^{ax}.\sin(bx+c)dx=\frac{e^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]+k=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$

$$\int e^{ax}.\cos(bx+c)dx=\frac{e^{ax}}{a^2+b^2}[a\cos(bx+c)+b\sin(bx+c)]+k=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$$

Important Tips

☞ $\int x e^{ax}\sin(bx+c)dx=\frac{xe^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]-\frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin bx(bx+c)-2ab\cos bx(bx+c)]+k$

☞ $\int x.e^{ax}\cos(bx+c)dx=\frac{x.e^{ax}}{a^2+b^2}[a\cos(bx+c)+b\sin(bx+c)]-\frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos(bx+c)+2ab\sin(bx+c)]+k$

☞ $\int a^x.\sin(bx+c)dx=\frac{a^x}{(\log a)^2+b^2}[(\log a)\sin(bx+c)-b\cos(bx+c)]+k$

☞ $\int a^x.\cos(bx+c)dx=\frac{a^x}{(\log a)^2+b^2}[(\log a)\cos(bx+c)+b\sin(bx+c)]+k$

Example: 32 If $\int e^x\sin x\,dx=\frac{1}{2}e^x.a+c$, then $a=$ **[MP PET 1989,99; Rajasthan PET 1985; IIT 1978]**

(a) $\sin x-\cos x$ (b) $\cos x-\sin x$ (c) $\tan x+c$ (d) None of these

Solution: (a) $\int e^x\sin x\,dx=\frac{e^x}{1^2+1^2}(1.\sin x-1.\cos x)+c=\frac{e^x}{2}(\sin x-\cos x)+c$

Clearly $a=(\sin x-\cos x)$

Example: 33 If $u = \int e^{ax} \cos bx\, dx$ and $v = \int e^{ax} \sin bx\, dx$, then $(a^2 + b^2)(u^2 + v^2) =$

(a) $2e^{ax}$ (b) $(a^2 + b^2)e^{2ax}$ (c) e^{2ax} (d) $(a^2 - b^2)e^{2ax}$

Solution: (c) $u = \int e^{ax} \cos bx\, dx = e^{ax}\dfrac{\sin bx}{b} - \dfrac{a}{b}\int e^{ax}.bx\, dx = \dfrac{e^{ax}\sin bx}{b} - \dfrac{a}{b}v$

$\Rightarrow bu + av = e^{ax}\sin bx$ ......(i)

Similarly, $bv - au = -e^{ax}\cos bx$ .....(ii)

Squaring (i) and (ii) and adding. We get, $(a^2 + b^2)(u^2 + v^2) = e^{2ax}$.

5.7 Evaluation of the Various forms of Integrals by use of Standard Results

(1) Integral of the form $\int \dfrac{dx}{ax^2 + bx + c}$, where $ax^2 + bx + c$ can not be resolved into factors.

(2) Integral of the form $\int \dfrac{px + q}{ax^2 + bx + c}dx$.

(3) Integral of the form $\int \dfrac{dx}{\sqrt{ax^2 + bx + c}}$.

(4) Integral of the form $\int \dfrac{px + q}{\sqrt{ax^2 + bx + c}}dx$.

(5) Integral of the form $\int \dfrac{f(x)}{ax^2 + bx + c}dx$, where $f(x)$ is a polynomial of degree 2 or greater than 2.

(6) Integral of the form (i) $\int \dfrac{x^2 + 1}{x^4 + kx^2 + 1}dx$, (ii) $\int \dfrac{x^2 - 1}{x^4 + kx^2 + 1}dx$, where k is any constant

(7) Integral of the form $\int \sqrt{ax^2 + bx + c}\, dx$ (8) Integral of the form $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$

(9) Integral of the form $\int \dfrac{dx}{P\sqrt{Q}}$

(1) **Integrals of the form $\int \dfrac{dx}{ax^2 + bx + c}$, where $ax^2 + bx + c$ can not be resolved into factors.**

We have, $ax^2 + bx + c = a\left(x^2 + \dfrac{b}{a}.x + \dfrac{c}{a}\right) = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \left(\dfrac{b^2}{4a^2} - \dfrac{c}{a}\right)\right] = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \left(\dfrac{b^2 - 4ac}{4a^2}\right)\right]$

Case (i) : When $b^2 - 4ac > 0$

$$\therefore \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2-a^2}\right]$$

$$= \frac{1}{a}\cdot\frac{1}{2.\frac{\sqrt{b^2-4ac}}{2a}} \log\left|\frac{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}}{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}}\right| + c = \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + c$$

Case (ii) : When $b^2-4ac<0$

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2+a^2}\right]$$

$$= \frac{1}{a}\cdot\frac{1}{\frac{\sqrt{4ac-b^2}}{2a}} \tan^{-1}\left[\frac{x+\frac{b}{2a}}{\frac{\sqrt{4ac-b^2}}{2a}}\right] + c = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right] + c$$

Working rule for evaluating $\int \frac{dx}{ax^2+bx+c}$ **:** To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking 'a' common from ax^2+bx+c.

(ii) Express the terms containing x^2 and x in the form of a perfect square by adding and subtracting the square of half of the coefficient of x.

(iii) Put the linear expression in x equal to t and express the integrals in terms of t.

(iv) The resultant integrand will be either in $\int \frac{dx}{x^2+a^2}$ or $\int \frac{dx}{x^2-a^2}$ or $\int \frac{dx}{a^2-x^2}$ standard form. After using the standard formulae, express the results in terms of x.

Example: 34 $\int \frac{dx}{2x^2+x+1} =$ **[Rajasthan PET 1997]**

(a) $\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$ (b) $\frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$ (c) $\frac{1}{2}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$ (d)

Solution: (d) $I = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{2}} \Rightarrow I = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$

$$I = \frac{1}{2}\cdot\frac{4}{\sqrt{7}}.\tan^{-1}\left[\frac{x+\frac{1}{4}}{(\sqrt{7}/4)}\right]+c \Rightarrow I = \frac{2}{\sqrt{7}}\tan^{-1}\left[\frac{4x+1}{\sqrt{7}}\right]+c$$

Example: 35 $\int \frac{\cos x}{\sin^2 x + 4\sin x + 5}dx$ equals

(a) $\tan^{-1}(\sin x)+c$ (b) $\tan^{-1}(\sin x+2)+c$ (c) $\tan^{-1}(\sin x+1)+c$ (d) None of these

Solution: (b) $I = \int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx = \int \frac{\cos x}{(\sin x + 2)^2 + 1} dx$

Put $\sin x + 2 = t \Rightarrow \cos x\, dx = dt$

$I = \int \frac{dt}{t^2 + 1} = \tan^{-1} t + c = \tan^{-1}(\sin x + 2) + c$.

(2) **Integral of the form** $\int \frac{px + q}{ax^2 + bx + c} dx$ **:** The integration of the function $\frac{px+q}{ax^2 + bx + c}$ is effected by breaking $px + q$ into two parts such that one part is the differential coefficient of the denominator and the other part is a constant.

If M and N are two constants, then we express $px + q$ as $px + q = M \frac{d}{dx}(ax^2 + bx + c) + N = M.(2ax + b) + N = (2aM)x + Mb + N$.

Comparing the coefficients of x and constant terms on both sides, we have, $p = 2aM \Rightarrow M = \frac{p}{2a}$ and $q = Mb + N \Rightarrow N = q - Mb = q - \frac{p}{2a}b$.

Thus, M and N are known. Hence, the given integral is $\int \frac{px+q}{ax^2+bx+c} dx = \int \frac{\frac{p}{2a}(2ax+b) + \left(q - \frac{p}{2a}b\right)}{ax^2 + bx + c} dx$

$= \frac{p}{2a} \int \frac{2ax+b}{ax^2+bx+c} dx + \left(q - \frac{p}{2a}b\right) \int \frac{dx}{ax^2+bx+c} = \frac{p}{2a} \log|ax^2 + bx + c| + \left(q - \frac{p}{2a}b\right) \int \frac{dx}{ax^2+bx+c} + C$

The integral on R.H.S. can be evaluated by the method discussed in previous section.

(i) If $b^2 - 4ac < 0$, then $\int \frac{px+q}{ax^2+bx+c} dx = \frac{p}{2a} \log|ax^2 + bx + c| + \frac{(2aq - bp)}{a\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac - b^2}} + k$

(ii) If $b^2 - 4ac > 0$, then $\int \frac{px+q}{ax^2+bx+c} dx = \frac{p}{2a} \log|ax^2 + bx + c| + \frac{(2aq - bp)}{2a\sqrt{b^2 - 4ac}} \log\left|\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right| + k$

Example: 36 $\int \frac{2\sin 2\theta - \cos\theta}{6 - \cos^2\theta - 4\sin\theta} d\theta =$

(a) $2\log|\sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + c$
(b) $2\log|\sin^2\theta - 4\sin\theta + 5| - 7\tan^{-1}(\sin\theta - 2) + c$

(c) $-2\log|\sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + c$
(d) $-2\log|\sin^2\theta - 4\sin\theta + 5| - 7\tan^{-1}(\sin\theta - 2) + c$

Solution: (a) $I = \int \frac{2(2\sin\theta\cos\theta) - \cos\theta}{6 - (1 - \sin^2\theta) - 4\sin\theta} d\theta \Rightarrow I = \int \frac{(4\sin\theta - 1)\cos\theta}{\sin^2\theta - 4\sin\theta + 5} d\theta$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$, $\therefore I = \int \frac{4t - 1}{t^2 - 4t + 5} dt$

Let $4t - 1 = M \frac{d}{dt}(t^2 - 4t + 5) + N \Rightarrow 4t - 1 = M(2t - 4) + N$

Comparing the coefficient of t and constant terms on both side, then $M = 2,\ N = 7\ \therefore I = \int \frac{2(2t-4)+7}{t^2-4t+5}dt$

$\Rightarrow I = 2\int \frac{2t-4}{t^2-4t+5}dt + \int \frac{7dt}{t^2-4t+5} \Rightarrow I = 2\log|t^2-4t+5| + 7\int \frac{dt}{(t-2)^2+1}$

$\Rightarrow I = 2\log|t^2-4t+5| + 7\tan^{-1}(t-2)+c = 2\log|\sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2)+c$

(3) **Integral of the form** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ **:** To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking $\sqrt{a}$ common from $\sqrt{ax^2+bx+c}$.

Then, $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}$.

(ii) Put $x^2+\frac{b}{a}x+\frac{c}{a}$, by the method of completing the square in the form, $\sqrt{A^2-X^2}$ or $\sqrt{X^2+A^2}$ or $\sqrt{X^2-A^2}$ where, X is a linear function of x and A is a constant.

(iii) After this, use any of the following standard formulae according to the case under consideration

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c \Rightarrow \int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|+c$ and $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+c.$

Note : ❑ If $a<0,\ b^2-4ac>0$, then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)+k.$

❑ If $a>0,\ b^2-4ac<0$, then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\sinh^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right]+k$

❑ If $a>0,\ b^2-4ac>0$ $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\cosh^{-1}\frac{2ax+b}{\sqrt{b^2-4ac}}+k$

Example: 37 $\int \frac{dx}{\sqrt{2-3x-x^2}}$ equals **[EAMCET 1996]**

(a) $\tan^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$ (b) $\sec^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$ (c) $\sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$ (d) $\cos^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$

Solution: (c) $I = \int \frac{dx}{\sqrt{2-3x-x^2}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x+\frac{3}{2}\right)^2}}$

Put $x+3/2 = t \Rightarrow dx = dt$

$$I = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}} = \sin^{-1}\left(\frac{t}{\sqrt{17}/2}\right) + c = \sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c$$

Example: 38 $\int \frac{dx}{\sqrt{x^2 - 4x + 2}} =$

(a) $\log|x - 2 + \sqrt{x^2 + 2 - 4x}| + c$ (b) $\log|x - 2 - \sqrt{x^2 + 2 - 4x}| + c$

(c) $\log|x - 2 + \sqrt{x^2 + 2 + 4x}| + c$ (d) $\log|x - 2 - \sqrt{x^2 + 2 + 4x}| + c$

Solution: (a) $I = \int \frac{dx}{\sqrt{(x-2)^2 - 2}} \Rightarrow I = \int \frac{dt}{\sqrt{t^2 - (\sqrt{2})^2}}$

Put $x - 2 = t \Rightarrow dx = dt \Rightarrow I = \log|t + \sqrt{t^2 - 2}| + c \Rightarrow I = \log|x - 2 + \sqrt{x^2 - 4x + 2}| + c$

(4) **Integral of the form** $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ **:** To evaluate this form of integrals, first we write,

$$px + q = M\frac{d}{dx}(ax^2 + bx + c) + N \Rightarrow px + q = M(2ax + b) + N$$

Where M and N are constants.

By equating the coefficients of x and constant terms on both sides, we get

$p = 2aM \Rightarrow M = \frac{p}{2a}$ and $q = bM + N \Rightarrow N = q - \frac{bp}{2a}$

In this way, the integral breaks up into two parts given by

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \left(q - \frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}} = I_1 + I_2, \text{ (say)}$$

Now, $I_1 = \frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx$

Putting $ax^2 + bx + c \Rightarrow (2ax+b)dx = dt,$ we have,

$$I_1 = \frac{p}{2a}\int t^{-1/2}dt = \frac{p}{2a}\cdot\frac{t^{1/2}}{\frac{1}{2}} + C_1 = \frac{p}{a}\sqrt{ax^2+bx+c} + C_1$$

and I_2 is calculated as in the previous section.

Note : ❑ $\frac{px+q}{\sqrt{ax^2+bx+c}}dx = \frac{p}{a}\sqrt{ax^2+bx+c} + \frac{2aq-bp}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

Example: 39 $\int \frac{5-2x}{\sqrt{6+x-x^2}}dx$ **=**

(a) $2\sqrt{6+x-x^2} - 4\sin^{-1}\left(\frac{2x-1}{5}\right) + c$ (b) $2\sqrt{6+x-x^2} + 4\sin^{-1}\left(\frac{2x-1}{5}\right) + c$

(c) $-2\sqrt{6+x-x^2} - 4\sin^{-1}\left(\frac{2x-1}{5}\right) + c$ (d) $-2\sqrt{6+x-x^2} + 4\sin^{-1}\left(\frac{2x-1}{5}\right) + c$

Solution: (b) $I = \int \frac{5-2x}{\sqrt{6+x-x^2}}\,dx$

Let $5-2x = M\frac{d}{dx}(6+x-x^2)+N \Rightarrow 5-2x = M(1-2x)+N$

Equating the coefficients of x and constant terms on both sides, we get

$-2=-2M \Rightarrow M=1$ and $5=M+N \Rightarrow N=5-1=4 \quad \therefore\ 5-2x=(1-2x)+4$

Hence, $I = \int \frac{(1-2x)+4}{\sqrt{6+x-x^2}}\,dx = \int \frac{1-2x}{\sqrt{6+x-x^2}}\,dx + 4\int \frac{dx}{\sqrt{6+x-x^2}} = I_1 + 4I_2$, (say)

Now, $I_1 = \int \frac{1-2x}{\sqrt{6+x-x^2}}\,dx$

Putting $6+x-x^2 = t \Rightarrow (1-2x)dx = dt$, we have,

$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t}+C_1 = 2\sqrt{6+x-x^2}+C_1$ and $I_2 = \int \frac{dx}{\sqrt{6+x-x^2}} = \int \frac{dx}{\sqrt{6-(x^2-x)}}$

$$I = \int \frac{dx}{\sqrt{6-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}} = \int \frac{dx}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}} = \int \frac{du}{\sqrt{\left(\frac{5}{2}\right)^2-u^2}} \qquad \left(\text{where, } u = x-\frac{1}{2}\right)$$

$$= \sin^{-1}\left(\frac{u}{5/2}\right)+C_2 = \sin^{-1}\left[\frac{2}{5}\left(x-\frac{1}{2}\right)\right]+C_2 = \sin^{-1}\left(\frac{2x-1}{5}\right)+C_2$$

$$\therefore\ I = I_1+4I_2 = 2\sqrt{6+x-x^2}+4\sin^{-1}\left(\frac{2x-1}{5}\right)+C \quad (\text{where, } C = C_1+4C_2)$$

Example: 40 $\int \frac{x+2}{\sqrt{x^2-2x+4}}$ equals

(a) $\sqrt{x^2-2x+4}+3\sinh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right]+c$

(b) $\sqrt{x^2-2x+4}-3\sinh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right]+c$

(c) $\sqrt{x^2-2x+4}+3\cosh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right]+c$

(d) $\sqrt{x^2-2x+4}-3\cosh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right]+c$

Solution: (a) $\int \frac{x+2}{\sqrt{x^2-2x+4}}\,dx = \int \frac{x+2}{\sqrt{(x-1)^2+\left(\sqrt{3}\right)^2}}\,dx = \int \frac{x-1+3}{\sqrt{(x-1)^2+\left(\sqrt{3}\right)^2}}\,dx = \int \frac{x-1}{\sqrt{x^2-2x+4}}\,dx + \int \frac{3\,dx}{\sqrt{(x-1)^2+(\sqrt{3})^2}}$

Put $x^2-2x+4=t^2$ in the first expression $\Rightarrow 2(x-1)dx = 2tdt$

$\Rightarrow (x-1)dx = tdt$

$$\int \frac{x+2}{\sqrt{x^2-2x+4}}\,dx = \int \frac{tdt}{t}+3\int \frac{dx}{\sqrt{(x-1)^2+(\sqrt{3})^2}}$$

$$\int \frac{x+2}{\sqrt{x^2-2x+4}}\,dx = \sqrt{x^2-2x+4}+3\sinh^{-1}\left[\frac{x-1}{\sqrt{3}}\right]+c$$

(5) **Integrals of the form $\int \frac{f(x)}{ax^2+bx+c}\,dx$, where $f(x)$ is a polynomial of degree 2 or greater than 2:**

To evaluate the integrals of the above form, divide the numerator by the denominator. Then, the integrals take the form given by $\frac{f(x)}{ax^2+bx+c} = Q(x) + \frac{R(x)}{ax^2+bx+c}dx$

where, $Q(x)$ is a polynomial and $R(x)$ is a linear polynomial in x.

Then, we have $\int \frac{f(x)}{ax^2+bx+c}dx = \int Q(x)dx + \int \frac{R(x)}{ax^2+bx+c}dx$

The integrals on R.H.S. can be obtained by the methods discussed earlier.

Example: 41 $\frac{(x^3+8)(x-1)}{x^2-2x+4}dx$ equals. **[Rajasthan PET 1989]**

(a) $\left(\frac{x^3}{3}\right)+\left(\frac{x^2}{2}\right)-2x+c$ (b) x^3+x^2-2x+c (c) $\frac{(x^3+x^2-x)}{3}+c$ (d) None of these

Solution: (a) $\int \frac{(x^3+8)(x-1)}{x^2-2x+4}dx = \int \frac{(x+2)(x^2-2x+4)(x-1)}{x^2-2x+4} = \int (x+2)(x-1)dx = \int (x^2+x-2)dx = \frac{x^3}{3}+\frac{x^2}{2}-2x+c$

Example: 42 The value of $\int \frac{2x^3-3x^2+5x+6}{x^2+3x+2}dx$ is

(a) $x^2+3x+4\ln|x^2+3x+2|+12\ln\frac{x+1}{x+2}+c$ (b) $x+3x^2+4\ln|x+1|-12\ln(x+2)+c$

(c) $(x^2+3x)+8\ \ln\frac{|x+1}{(x+2)^2}+c$ (d) None of these

Solution: (c) $\int \frac{2x^3-3x^2+5x+6}{x^2+3x+2}dx$

$= \int \left((2x+3)-\frac{8x}{x^2+3x+2}\right)dx = \int (2x+3)-4\int \frac{(2x+3)dx}{x^2+3x+2}+12\int \frac{dx}{(x+1)(x+2)}$

$=(x^2+3x)-4\ln|x^2+3x+2|+12\int \frac{dx}{x+1}-12\int \frac{dx}{x+2}$

$= (x^2+3x)-4\ln(x+1)-4\ln(x+2)+12\ln\frac{x+1}{x+2}+c$

$= x^2-3x+8\ln(x+1)-16\ln(x+2)+c = (x^2-3x)+8\ln\frac{|x+1|}{(x+2)^2}+c$.

(6) **Integrals of the form** $\int \frac{x^2+1}{x^4+kx^2+1}dx$ **and** $\int \frac{x^2-1}{x^4+kx^2+1}dx$ **:** To evaluate the integral of the form $I = \int \frac{x^2+1}{x^4+kx^2+1}dx$, proceed as follows

(i) Divide the numerator and denominator by x^2 to get $I = \int \frac{1+\frac{1}{x^2}}{x^2+k+\frac{1}{x^2}} dx$.

(ii) Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ and $x^2 + \frac{1}{x^2} - 2 = t^2 \Rightarrow x^2 + \frac{1}{x^2} = t^2 + 2$.

Then, the given integral reduces to the form $I = \int \frac{dt}{t^2+2+k}$, which can be integrand as usual.

(iii) To evaluate $I = \int \frac{x^2-1}{x^4+kx^2+1} dx$, we divide the numerator and denominator by x^2 and get

$$I = \int \frac{1-\frac{1}{x^2}}{x^2+k+\frac{1}{x^2}} dx$$

Then, we put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$ and $x^2 + \frac{1}{x^2} + 2 = t^2 \Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$.

Thus, we have $t = \int \frac{dt}{t^2-2+k}$, which can be evaluated as usual.

Important Tips

☞ ***Algebraic twins:*** $\int \frac{2x^2}{x^4+1} dx = \int \frac{x^2+1}{x^4+1} dx + \int \frac{x^2-1}{x^4+1} dx$

$\int \frac{2}{x^4+1} dx = \int \frac{x^2+1}{x^4+1} dx - \int \frac{x^2-1}{x^4+1} dx$, $\int \frac{2x^2}{x^4+1+kx^2} dx$, $\int \frac{2}{(x^4+1+kx^2)} dx$

We know the result of $I_1 = \int \frac{x^2+1}{x^4+1} dx$ *and* $I_2 = \int \frac{x^2-1}{x^4+1} dx$, *so for* $\int \frac{x^2}{x^4+1} dx$ *and for* $\int \frac{dx}{x^4+1}$, *we can use the result of* $\frac{I_1+I_2}{2}$ *and* $\frac{I_1-I_2}{2}$.

☞ ***Trigonometric twins:*** $\int \sqrt{\tan x}\, dx$, $\int \sqrt{\cot x}\, dx$, $\int \frac{dx}{(\sin^4 x + \cos^4 x)}$, $\int \frac{dx}{\sin^6 x + \cos^6 x}$, $\int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cos} dx$

Example: 43 $\int \frac{x^2-1}{x^4+x^2+1} dx =$

(a) $\frac{1}{2}\log\left[\frac{x^2+x+1}{x^2-x+1}\right]+c$ (b) $\frac{1}{2}\log\left[\frac{x^2-x-1}{x^2+x+1}\right]+c$ (c) $\log\left[\frac{x^2-x+1}{x^2+x+1}\right]+c$ (d) $\frac{1}{2}\log\left[\frac{x^2-x+1}{x^2+x+1}\right]+c$

Solution: (d) $\int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}} dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1} dx = \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|+c = \frac{1}{2}\log\left|\frac{x^2+1-x}{x^2+1+x}\right|+c = \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+c$

Example: 44 $\int \sqrt{\tan x}\, dx =$

(a) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) + \frac{1}{2\sqrt{2}} \log\left|\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right| + c$

(b) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) - \frac{1}{2\sqrt{2}} \log\left|\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right| + c$

(c) $\log\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) + c$

(d) None of these

Solution: (a) Let $I = \int \sqrt{\tan x}\, dx$

Putting $\sqrt{\tan x} = t \Rightarrow \tan x = t^2$, we have $\sec^2 x dx = 2t dt \Rightarrow (1 + \tan^2 x) dx = 2t dt$

i.e., $(1 + t^4) dx = 2t dt \Rightarrow dx = \frac{2t}{1 + t^4} dt$

$\therefore I = \int \sqrt{\tan x} dx = \int t . \frac{2t}{1 + t^4} dt = \int \frac{2t^2}{1 + t^4} dt = \int \frac{(t^2 + 1) + (t^2 - 1)}{t^4 + 1} dt = \int \frac{t^2 + 1}{t^4 + 1} dt + \int \frac{t^2 - 1}{t^4 + 1} dt$

$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt + \int \frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - 2} dt$.

Again putting $t - \frac{1}{t} = u \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = du$ in the first integral and $t + \frac{1}{t} = v \Rightarrow \left(1 - \frac{1}{t^2}\right) dt = dv$ in the second integral, we have,

$I = \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{dv}{v^2 - (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \log\left|\frac{v - \sqrt{2}}{v + \sqrt{2}}\right| + c$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \log\left|\frac{t + \frac{1}{t} - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}}\right| + c = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{t\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \log\left|\frac{t^2 - t\sqrt{2} + 1}{t^2 + t\sqrt{2} + 1}\right| + c$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) + \frac{1}{2\sqrt{2}} \log\left|\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right| + c$

Example: 45 $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$ **[IIT 1989; Roorkee 1984; Karnataka CET 1991; Pb. CET 1997]**

(a) $\sqrt{2} \tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + c$

(b) $\sqrt{2} \tan^{-1}\left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}}\right) + c$

(c) $\tan^{-1}\left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}}\right) + c$

(d) None of these

Solution: (a) $I = \int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt \Rightarrow dx = \frac{2t}{1+t^4} dt$

$$I = \int \left(t + \frac{1}{t}\right) \cdot \frac{2t}{1+t^4} dt = 2\int \frac{\left(1 + \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2} + 2 - 2} dt = 2\int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + \left(\sqrt{2}\right)^2}$$

Put $t - \frac{1}{t} = p \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dp$ $= \int \frac{dp}{p^2 + \left(\sqrt{2}\right)^2} = \frac{2}{\sqrt{2}} \tan^{-1} \frac{p}{\sqrt{2}} + c = \sqrt{2} \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + c$

$$= \sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + c$$

(7) **Integrals of the forms** $\int \sqrt{ax^2 + bx + c}\, dx$ **:** To evaluate this form of integrals, express $ax^2 + bx + c$ in the form $a\left[(x+\alpha)^2 + \beta^2\right]$ by the method of completing the square and apply the standard result discussed in the above section according to the case as may be.

Note : ❑ $\int \sqrt{ax^2 + bx + c}\; dx = \frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2+bx+c}}$

Example: 46 $\int \sqrt{x^2 + 8x + 12}\, dx$ **[CBSE 1999]**

(a) $\frac{1}{2}(x+4)\sqrt{x^2+8x+12} + 2\log|\, x + 4 + \sqrt{x^2+8x+12}\,| + c$

(b) $\frac{1}{2}(x+4)\sqrt{x^2+8x+12} - 2\log|\, x + 4 + \sqrt{x^2+8x+12}\,| + c$

(c) $(x+4)\sqrt{x^2+8x+12} + \log|\, x + 4 + \sqrt{x^2+8x+12}\,| + c$

(d) $(x+4)\sqrt{x^2+8x+12} - \log|\, x + 4 + \sqrt{x^2+8x+12}\,| + c$

Solution: (b) Let $I = \int \sqrt{x^2+8x+12}\, dx = \int \sqrt{(x^2+8x+16) - 4}\, dx$

$= \int \sqrt{(x+4)^2 - 2^2} dx = \int \sqrt{t^2 - 2^2} dx = \int \sqrt{t^2 - 2^2} dt.$ (putting $x + 4 = t \Rightarrow dx = dt$)

$= \frac{1}{2} t\sqrt{t^2 - 2^2} - \frac{1}{2}.2^2 \log|\, t + \sqrt{t^2 - 2^2}\,| + c = \frac{1}{2}(x+4)\sqrt{(x+4)^2 - 4} - 2\log|\, x + 4 + \sqrt{(x+4)^2 - 4}\,| + c$

$= \frac{1}{2}(x+4)\sqrt{x^2+8x+12} - 2\log|\, x + 4 + \sqrt{x^2+8x+12}\,| + c\,.$

Example: 47 $\int \sqrt{2ax - x^2} dx =$

(a) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x-a}{a}\right) + c$

(b) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} - \frac{1}{2}a^2 \sin^{-1}\left(\frac{x-a}{a}\right) + c$

(c) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{1}{2}a^2 \cos^{-1}\left(\frac{x-a}{a}\right) + c$

(d) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} - \frac{1}{2}a^2 \cos^{-1}\left(\frac{x-a}{a}\right) + c$

Solution: (a) $\int \sqrt{2ax - x^2}\, dx = \int \sqrt{a^2 - a^2 + 2ax - x^2}\, dx = \int \sqrt{a^2 - (x^2 - 2ax + a^2)}\, dx = \int \sqrt{a^2 - (x-a)^2}\, dx$

$$= \frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{1}{2}a^2 \sin^{-1}\frac{(x-a)}{a} + c$$

(8) **Integrals of the form** $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ **:** To evaluate this form of integral, proceed as follows:

(i) First express $(px+q)$ as $px+q = M\frac{d}{dx}(ax^2+bx+c)+N \Rightarrow px+q = M(2ax+b)+N$

Where, M and N are constant.

(ii) Compare the coefficients of x and constant terms on both sides, will get

$p = 2aM \Rightarrow M = \frac{p}{2a}$ and $q = Mb+N \Rightarrow N = q-Mb = q-\frac{p}{2a}b.$

(iii) Now, write the given integral as

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \left(q-\frac{p}{2a}b\right)\int \sqrt{ax^2+bx+c}\,dx$$

$$= \frac{p}{2a}I_1 + \left(q-\frac{p}{2a}b\right)I_2, \text{ (say)}.$$

(iv) To evaluate I_1, put $ax^2+bx+c=t$ and to evaluate I_2, follows the method discussed in (7)

Example: 48 $\int (2x+3)\sqrt{x^2+4x+3}\,dx =$

(a) $\log|(x+2)+(\sqrt{x^2+4x+3})|+c$ (b) $\log|(x+2)+(\sqrt{x^2+4x+3})|+c$

(c) $\log|(x-2)+(\sqrt{x^2+4x+3})|+c$ (d) None of these

Solution: (d) Let $2x+3 = M\frac{d}{dx}(x^2+4x+3)+N \Rightarrow 2x+3 = M(2x+4)+N$

Equating the coefficients of x and constant terms on both sides, we get

$2 = 2M \Rightarrow M = 1$ and $3 = 4M+N \Rightarrow N = 3-4\times 1 = -1$

$\therefore\ 2x+3 = (2x+4)-1$

Hence, $I = \int [(2x+4)-1]\sqrt{x^2+4x+3dx} = \int (2x+4)\sqrt{x^2+4x+3}\,dx - \int \sqrt{x^2+4x+3}\,dx = I_1 - I_2, (say)$

Now, $I_1 = \int (2x+4)\sqrt{x^2+4x+3}\,dx$

Putting $x^2+4x+3 = t \Rightarrow (2x+4)dx = dt$, we have

$I_1 = \int t^{1/2}dt = \frac{t^{3/2}}{3/2}+c_1 = \frac{2}{3}(x^2+4x+3)^{3/2}+c_1$ $I_2 = \int \sqrt{x^2+4x+3}\,dx = \int \sqrt{(x+2)^2-1^2}\,dx$

$= \frac{1}{2}(x+2)\sqrt{(x+2)^2-1^2} - \frac{1}{2}.1^2\log\left|x+2+\sqrt{(x+2)^2-1^2}\right|+c_2 = \frac{1}{2}(x+2)\sqrt{x^2+4x+3} - \frac{1}{2}\log\left|x+2+\sqrt{x^2+4x+3}\right|+c_2$

$\therefore\ I = I_1 - I_2 = \frac{2}{3}(x^2+4x+3)^{3/2} - \left[\frac{1}{2}(x+2)\sqrt{x^2+4x+3} - \frac{1}{2}\log\left|x+2+\sqrt{x^2+4x+3}\right|\right]+c$, (where, $c = c_1 - c_2$),

Example: 49 $\int (2x-5)\sqrt{x^2-4x+3}\,dx =$ **[CBSE 1995]**

(a) $\log|x-2+\sqrt{x^2-4x+3}|+c$ (b) $\log|x-2-\sqrt{x^2-4x+3}|+c$

(c) $\log|x+2+\sqrt{x^2-4x+3}|+c$ (d) None of these

Solution: (d) Let $2x-5 = M\frac{d}{dx}(x^2-4x+3)+N \Rightarrow 2x-5 = M(2x-4)+N$

Equating the coefficients of x and constant terms on both sides, we get

$2 = 2M \Rightarrow M = 1$

$-5 = -4M + N \Rightarrow N = 4M - 5 = -1$

Hence, $I = \int \{(2x-4)-1\}\sqrt{x^2-4x+3}\,dx \Rightarrow I = \int (2x-4)\sqrt{x^2-4x+3}\,dx - \int \sqrt{x^2-4x+3}\,dx \quad I = I_1 - I_2$ (say)

$I_1 = \int (2x-4)\sqrt{x^2-4x+3}\,dx = \frac{2}{3}(x^2-4x+3)^{3/2} + c_1 \quad I_2 = \int \sqrt{x^2-4x+3}\,dx = \int \sqrt{x^2-4x+4-4+3}\,dx$

$= \int \sqrt{(x-2)^2 - 1^2} \Rightarrow I_2 = \frac{1}{2}(x-2)\sqrt{x^2-4x+3}\ \frac{-1}{2}.1^2 \log\left\{(x-2)+\sqrt{x^2-4x+3}\right\}$

$= \frac{1}{2}(x-2)\sqrt{x^2-4x+3}\ \frac{-1}{2}\log\left[(x-2)+\sqrt{x^2-4x+3}\right] + c_2$, Therefore $I = I_1 - I_2$

$I = \frac{2}{3}(x^2-4x+3)^{3/2} + \frac{1}{2}(-2)\sqrt{x^2-4x+3} - \frac{1}{2}\log\left[(x-2)+\sqrt{x^2-4x+3}\right] + c$ (Where $c = c_1 + c_2$)

(9) **Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$, (where P and Q and linear or quadratic expressions in x):** To evaluate such types of integrals, we have following substitutions according to the nature of expressions of P and Q in x :

(i) When Q is linear and P is linear or quadratic, we put $Q = t^2$.

(ii) When P is linear and Q is quadratic, we put $P = \frac{1}{t}$.

(iii) When both P and Q are quadratic, we put $x = \frac{1}{t}$.

Example: 50 $\int \frac{dx}{(x-3)\sqrt{x+1}} =$ **[Pb. CET 1991]**

(a) $\frac{1}{2}\log\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}-2}\right)+c$ (b) $\frac{1}{2}\log\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}-2}\right)+c$ (c) $\frac{1}{2}\log\left(\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right)+c$ (d) None of these

Solution: (c) Put $x+1 = t^2 \Rightarrow dx = 2t\,dt$

$\int \frac{dx}{(x-3)\sqrt{x+1}} = \int \frac{2t\,dt}{(t^2-4)t}$ $(\because x+1 = t^2 \Rightarrow x = t^2 - 1 \Rightarrow x - 3 = t^2 - 4)$

$= 2\int \frac{dt}{t^2-2^2} = 2.\frac{1}{2.2}\log\left(\frac{t-2}{t+2}\right)+c = \frac{1}{2}\log\left(\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right)+c$.

5.8 Integrals of the form $\int \frac{dx}{a+b\cos x}$ and $\int \frac{dx}{a+b\sin x}$

To evaluate such form of integrals, proceed as follows:

(1) Put $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

(2) Replace $1+\tan^2\frac{x}{2}$ in the numerator by $\sec^2\frac{x}{2}$.

(3) Put $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt.$

(4) Now, evaluate the integral obtained which will be of the form $\int \frac{dt}{at^2 + bt + c}$ by the method discussed earlier.

(i) $\int \frac{dx}{a + b\cos x}$

Case I: When $a > b$, then $\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) + c$

Case II: When $a < b$, then $\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}}\right| + c$

Case III: When $a = b$, then $\int \frac{dx}{a + b\cos x} = \frac{1}{a}\tan\frac{x}{2} + c$.

(ii) $\int \frac{dx}{a + b\sin x}$

Case I: When $a^2 > b^2$ or $a > 0$ and $a > b$, then $\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}\right] + c$

Case II: When $a^2 < b^2$, then $\int \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{(a\tan\frac{x}{2} + b) - (\sqrt{b^2 - a^2})}{(a\tan\frac{x}{2} + b) + \sqrt{b^2 - a^2}}\right| + c$

Case III: When $a^2 = b^2$

In this case, either $b = a$ or $b = -a$

(a) When $b = a$, then $\int \frac{dx}{a + b\sin x} = \frac{-1}{a}\cot\left(\frac{\pi}{4} + \frac{x}{2}\right) + c = \frac{1}{a}[\tan x - \sec x] + c$

(b) When $b = -a$, then $\int \frac{dx}{a + b\sin x} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$.

Example: 51 $\int \frac{dx}{3 + 4\cos x} =$ **[Rajasthan PET 1997]**

(a) $\frac{1}{\sqrt{7}}\log\left(\frac{\tan(x/2) - \sqrt{7}}{\tan(x/2) + \sqrt{7}}\right) + c$

(b) $\frac{1}{\sqrt{7}}\log\left(\frac{\tan(x/2) + \sqrt{7}}{\tan(x/2) - \sqrt{7}}\right) + c$

(c) $\frac{1}{\sqrt{7}}\log\left(\frac{\sqrt{7} + \tan(x/2)}{\sqrt{7} - \tan(x/2)}\right) + c$

(d) $\frac{1}{\sqrt{7}}\log\left(\frac{\sqrt{7} - \tan(x/2)}{\sqrt{7} + \tan(x/2)}\right) + c$

Solution: (b) If $b > a$ then, $\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}}\log\left[\frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}}\right] + c \Rightarrow I = \frac{1}{\sqrt{7}}\log\left(\frac{\tan\frac{x}{2} + \sqrt{7}}{\tan\frac{x}{2} - \sqrt{7}}\right) + c$

Example: 52 If $\int \frac{dx}{1+\sin x} = \tan\left(\frac{x}{2}+a\right)+b$, then **[Roorkee 1979]**

(a) $a=\frac{\pi}{4},\ b=3$ (b) $a=\frac{-\pi}{4},\ b=3$

(c) $a=\frac{\pi}{4},\ b=$ arbitrary constant (d) $a=\frac{-\pi}{4},\ b=$ arbitrary constant

Solution: (d) If $a=b$, then $\int \frac{dx}{a+b\sin x} = -\frac{1}{a}\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$

$\therefore \int \frac{dx}{1+\sin x} = -\cot\left(\frac{\pi}{4}+\frac{x}{2}\right) = -\tan\left(\frac{\pi}{2}-\frac{\pi}{4}-\frac{x}{2}\right)+c = -\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+c = \tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+c$

Hence $a=\frac{-\pi}{4}$ and $b=$ arbitrary constant.

Example: 53 $\int \frac{dx}{5+4\cos x} =$ **[Roorkee1983; Rajasthan PET 1990, 97]**

(a) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right)+c$ (b) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right)+c$ (c) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$ (d) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$

Solution: (c) If $a>b$, then $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}.\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)+c$

$\therefore \int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$.

5.9 Integrals of the form $\int \frac{dx}{a+b\cos x+c\sin x}, \int \frac{dx}{a\sin x+b\cos x}$

(1) **Integral of the form** $\int \frac{dx}{a+b\cos x+c\sin x}$ **:** To evaluate such integrals, we put $b=r\cos\alpha$ and $c=r\sin\alpha$.

So that, $r^2=b^2+c^2$ and $\alpha=\tan^{-1}\frac{c}{b}$. $\therefore I=\int \frac{dx}{a+r(\cos\alpha\cos x+\sin\alpha\sin x)} = \int \frac{dx}{a+r\cos(x-\alpha)}$

Again, Put $x-\alpha=t \Rightarrow dx=dt$, we have $I=\int \frac{dt}{a+r\cos t}$

Which can be evaluated by the method discussed earlier.

(2) **Integral of the form** $\int \frac{dx}{a\sin x+b\cos x}$ **:** To evaluate this type of integrals we substitute $a=r\cos\theta,\ b=r\sin\theta$ and so $r=\sqrt{a^2+b^2},\ \alpha=\tan^{-1}\frac{b}{a}$

So, $\int \frac{dx}{a\sin x+b\cos x} = \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)} = \frac{1}{r}\int \operatorname{cosec}(x+\alpha)dx$

$$=\frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\alpha}{2}\right)\right| = \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right|+c$$

Note : ❑ The integral of the above form can be evaluated by using $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ and $\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}$.

Important Tips

☞ *If* $a>b,\ a^2>b^2+c^2$, *then* $\int \frac{dx}{a+b\cos x+c\sin x}=\frac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\left[\frac{(a-b)\tan x/2+c}{\sqrt{a^2-b^2-c^2}}\right]+k$

☞ *If* $a>b,\ a^2<b^2+c^2$, *then* $\int \frac{dx}{a+b\cos x+c\sin x}=\frac{1}{\sqrt{b^2+c^2-a^2}}\log\left[\frac{(a-b)\tan x/2+c-\sqrt{b^2+c^2-a^2}}{(a-b)\tan x/2+c+\sqrt{b^2+c^2-a^2}}\right]+k$

☞ *If* $a<b$, $\int \frac{dx}{a+b\cos x+c\sin x}=\frac{-1}{\sqrt{b^2+c^2-a^2}}\log\left[\frac{(b-a)\tan x/2-c-\sqrt{b^2+c^2-a^2}}{(b-a)\tan x/2-c+\sqrt{b^2+c^2-a^2}}\right]+k.$

Example: 54 $\int \frac{dx}{\sin x+\cos x}=$ **[BIT Ranhi 1990; Rajasthan PET 1990, 97, 99; Karnataka CET 1999]**

(a) $\log\tan(\pi/8+x/2)+c$ (b) $\log\tan(\pi/8-x/2)+c$

(c) $\frac{1}{\sqrt{2}}\log\tan(\pi/8+x/2)+c$ (d) None of these

Solution: (c) $\int \frac{dx}{\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}} \Rightarrow \int \frac{\sec^2 x/2}{2\tan x/2+1-\tan^2 x/2}dx$

Put $\tan x/2=t \Rightarrow \frac{1}{2}\sec^2 x/2\ dx=dt$

$\therefore\ I=2\int \frac{dt}{2t+1-t^2}=2\int \frac{dt}{2-(t^2-2t+1)}$

$\Rightarrow\ I=2\int \frac{dt}{(\sqrt{2})^2-(t-1)^2}=\frac{2}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|+c \Rightarrow I=\frac{1}{\sqrt{2}}\log\frac{[(\sqrt{2}-1)+\tan x/2][\sqrt{2}-1]}{[(\sqrt{2}+1)-\tan x/2][\sqrt{2}-1]}+c$

$\Rightarrow\ I=\frac{1}{\sqrt{2}}\log\frac{\tan\pi/8+\tan x/2}{1-(\sqrt{2}-1)\tan x/2}+\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)+c \Rightarrow I=\frac{1}{\sqrt{2}}\log\tan(\pi/8+x/2)+c_1$

where $c_1=\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)+c$

Trick : $I=\frac{1}{\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x}=\frac{1}{\sqrt{2}}\int \frac{dx}{\sin(\pi/4+x)}=\frac{1}{\sqrt{2}}\int \text{cosec}\,(\pi/4+x)dx$

$=\frac{1}{\sqrt{2}}\log\tan(\pi/8+x/2)+c$.

Example: 55 $\int \frac{dx}{1-\cos x-\sin x}=$ **[EAMCET 2002]**

(a) $\log\left|1+\cot\frac{x}{2}\right|+c$ (b) $\log\left|1-\tan\frac{x}{2}\right|+c$ (c) $\log\left|1-\cot\frac{x}{2}\right|+c$ (d) $\log\left|1+\tan\frac{x}{2}\right|+c$

Solution: (c) Given $I=\int \frac{dx}{1-\cos x-\sin x}$

$I=\int \frac{dx}{1-\frac{(1-\tan^2 x/2)}{(1+\tan^2 x/2)}-\frac{2\tan x/2}{1+\tan^2 x/2}} \Rightarrow I=\int \frac{\sec^2 x/2.dx}{1+\tan^2 x/2-1+\tan^2 x/2-2\tan x/2}$

$I=\int \frac{\sec^2 x/2.dx}{2\tan^2 x/2-2\tan x/2} \Rightarrow \int \frac{1/2.\sec^2 x/2.dx}{\tan^2 x/2-\tan x/2}$

Put $\tan x/2 = t$ $\Rightarrow \frac{1}{2}\sec^2 x/2.dx = dt$ therefore $I = \int \frac{dt}{t^2 - t}$ $\Rightarrow I = \int \frac{dt}{t(t-1)}$

$\Rightarrow \int\left[-\frac{1}{t} + \frac{1}{t-1}\right]dt \Rightarrow I = \int \frac{dt}{t-1} - \int \frac{dt}{t}$

$I = \log(t-1) - \log t + c \Rightarrow I = \log\left|\frac{t-1}{t}\right| + c \Rightarrow I = \log\left|\frac{\tan x/2 - 1}{\tan x/2}\right| + c \Rightarrow I = \log|1 - \cot x/2| + c$

Example: 56 The antiderivative of $f(x) = \frac{1}{3 + 5\sin x + 3\cos x}$ whose graph passes through the point $(0, 0)$ is

(a) $\frac{1}{5}\left(\log\left|1 - \frac{5}{3}\tan x/2\right|\right)$ (b) $\frac{1}{5}\left(\log\left|1 + \frac{5}{3}\tan x/2\right|\right)$ (c) $\frac{1}{5}\left(\log\left|1 + \frac{5}{3}\cot x/2\right|\right)$ (d) None of these

Solution: (b) $y = \int \frac{dx}{(3 + 5\sin x + 3\cos x)} = \int \frac{\sec^2 x/2\, dx}{10\tan x/2 + 6} = \frac{1}{5}\log(5\tan x/2 + 3) + c = \frac{1}{5}\log\left|\frac{5}{3}\tan x/2 + 1\right| + c$

Passes through $(0, 0)$

$\therefore c = 0$ then $y = \frac{1}{5}\log\left|1 + \frac{5}{3}\tan x/2\right|$.

5.10 Integrals of the form $\int \frac{dx}{a + b\cos^2 x}, \int \frac{dx}{a + b\sin^2 x}, \int \frac{dx}{a\sin^2 x + b\cos^2 x}, \int \frac{dx}{(a\sin x + b\cos x)^2}, \int \frac{dx}{a + b\sin^2 x + c\cos^2 x}$

To evaluate the above forms of integrals proceed as follows:

(1) Divide both the numerator and denominator by $\cos^2 x$.

(2) Replace $\sec^2 x$ in the denominator, if any by $(1 + \tan^2 x)$.

(3) Put $\tan x = t \Rightarrow \sec^2 x dx = dt$.

(4) Now, evaluate the integral thus obtained, by the method discussed earlier.

Example: 57 $\int \frac{dx}{1 + 3\sin^2 x} =$ **[Roorkee 1989; CBSE PMT 1992; DCE 2001]**

(a) $\frac{1}{3}\tan^{-1}(3\tan^2 x) + c$ (b) $\frac{1}{2}\tan^{-1}(2\tan x) + c$ (c) $\tan^{-1}(\tan x) + c$ (d) None of these

Solution: (b) $\int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x} = \int \frac{\sec^2 x\, dx}{1 + 4\tan^2 x}$

Put $2\tan x = t$

$\sec^2 x\, dx = \frac{dt}{2}$

$\therefore I = \frac{1}{2}\int \frac{dt}{1 + t^2} = \frac{1}{2}\tan^{-1} t + c \Rightarrow I = \frac{1}{2}\tan^{-1}(2\tan x) + c$.

Example: 58 $\int \frac{dx}{4\sin^2 x + 5\cos^2 x}$ **[AISSE 1986; CBSE PMT 1992]**

(a) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + c$ (b) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{\tan x}{\sqrt{5}}\right) + c$ (c) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + c$ (d) None of these

Solution: (c) $\int \frac{\sec^2 x\, dx}{4\tan^2 x + 5}$ $\begin{pmatrix}\text{Put } 2\tan x = t \\ 2\sec^2 x\, dx = dt\end{pmatrix}$

$$\Rightarrow \frac{1}{2}\int \frac{dt}{t^2+(\sqrt{5})^2} = \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right)+c$$

Example: 59 $\int \frac{dx}{(2\sin x+\cos x)^2}$ **[MP PET 1994]**

(a) $\frac{1}{2}\left(\frac{1}{2\tan x+1}\right)$ (b) $\frac{1}{2}\log(2\tan x+1)+c$ (c) $\frac{1}{2+\cot x}+c$ (d) $-\frac{1}{2}\left(\frac{1}{2\tan x-1}\right)+c$

Solution: (c) $\int \frac{dx}{(2\sin x+\cos x)^2} = \int \frac{dx}{\sin^2 x(2+\cot x)^2} = \int \frac{\text{cosec}^2 x\, dx}{(2+\cot x)^2}$

Put $(2+\cot x)=t \Rightarrow -\text{cosec}^2 x\, dx = dt$ $= \int \frac{-dt}{t^2} = \frac{1}{t}+c = \frac{1}{2+\cot x}+c$

Example: 60 $\int \frac{\cos x\, dx}{\cos 3x} =$

(a) $\frac{1}{2\sqrt{3}}\log\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}+c$ (b) $\frac{-1}{2\sqrt{3}}\log\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}+c$

(c) $\frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{3}\tan x)+c$ (d) None of these

Solution: (a) Let $I = \int \frac{\cos x}{\cos 3x}dx = \int \frac{\cos x}{4\cos^3 x - 3\cos x}dx$

Dividing the numerator and denominator by $\cos^2 x$, we have $I = \int \frac{dx}{4\cos^2 x - 3}$

$$I = \int \frac{dt}{1-3t^2} = \frac{1}{3}\int \frac{dt}{\frac{1}{3}-t^2} = \frac{1}{3}\int \frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2} = \frac{1}{3}\cdot\frac{1}{2.\frac{1}{\sqrt{3}}}\log\left|\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right|+c = \frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}t}{1-\sqrt{3}t}\right|+c$$

$$I = \frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+c.$$

Example: 61 $\int \frac{dx}{4\sin^2 x + 4\sin x.\cos x + 5\cos^2 x}$ equals

(a) $\tan^{-1}\left(\tan x+\frac{1}{2}\right)+c$ (b) $\frac{1}{4}\tan^{-1}\left(\tan x+\frac{1}{2}\right)+c$ (c) $4\tan^{-1}\left(\tan x+\frac{1}{2}\right)+c$ (d) None of these

Solution: (b) $I = \int \frac{dx}{4\sin^2 x + 4\sin x.\cos x + 5\cos^2 x} = \int \frac{\sec^2 x\, dx}{4\tan^2 x + 5 + 4\tan x} = \frac{1}{4}\int \frac{\sec^2 x\, dx}{\left(\tan x + \frac{1}{2}\right)^2 + 1}$

Put $\tan x + \frac{1}{2} = t \Rightarrow \sec^2 x\, dx = dt$ $= \frac{1}{4}\int \frac{dt}{t^2+1} = \frac{1}{4}\tan^{-1} t + c \Rightarrow I = \frac{1}{4}\int \tan^{-1}\left(\tan x + \frac{1}{2}\right)+c$.

5.11 Integrals of the form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$ and $\int \frac{a\sin x + b\cos x + q}{c\sin x + d\cos x + r}$

(1) **Integrals of the form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$:** Such rational functions of **sin *x*** and **cos *x*** may be integrated by expressing the numerator of the integrand as follows:

Numerator = *M* (Diff. of denominator)+*N* (Denominator) i.*e.*, $a\sin x + b\cos x = M\frac{d}{dx}(c\sin x + d\cos x) + N(c\sin x + d\cos x)$

The arbitary constants M and N are determined by comparing the coefficients of $\sin x$ and $\cos x$ from two sides of the above identity. Then, the given integral is

$$I = \int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx = \int \frac{M(c\cos x - d\sin x) + N(c\sin x + d\cos x)}{c\sin x + d\cos x}dx = M\int \frac{c\cos x - d\sin x}{c\sin x + d\cos x}dx + N\int 1dx$$

$$= M\log|c\sin x + d\cos x| + Nx + c.$$

(2) **Integrals of the form** $\int \frac{\boldsymbol{a\sin x + b\cos x + q}}{\boldsymbol{c\sin x + d\cos x + r}}\boldsymbol{dx}$ **:** To evaluate this type of integrals, we express the numerator as follows: Numerator $= M(\text{Denominator}) + N(\text{Differentiation of denominator}) + P$

i.e., $(c\sin x + b\cos x + q) = M(c\sin x + d\cos x + r) + N(c\cos x - d\sin x) + P.$

where M, N, P are constants to be determined by comparing the coefficients of $\sin x, \cos x$ and constant term on both sides.

$$\therefore \int \frac{a\sin x + b\cos x + q}{c\sin x + d\cos x + r}dx = \int M\,dx + N\int \frac{\text{Diff.of denominator}}{\text{Denominator}}dx + \int \frac{dx}{c\sin x - d\cos x + r}$$

$$= Mx + N\log|\text{Denominator}| + P\int \frac{dx}{c\sin x + d\cos x + r}.$$

Important Tips

☞ $\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx = \frac{ac+bd}{c^2+d^2}x + \frac{ad-bc}{c^2+d^2}\log|\,c\cos x + d\sin x\,| + c$.

Example: 62 $\int \frac{3\cos x + 3\sin x}{4\sin x + 5\cos x}dx =$ **[EAMCET 1991]**

(a) $\frac{27}{41}x - \frac{3}{41}\log(4\sin x + 5\cos x) + c$

(b) $\frac{27}{41}x + \frac{3}{41}\log(4\sin x + 5\cos x) + c$

(c) $\frac{27}{41}x - \frac{3}{41}\log(4\sin x - 5\cos x) + c$

(d) None of these

Solution: (a) $3\cos x + 3\sin x = M\frac{d}{dx}(4\sin x + 5\cos x) + N(4\sin x + 5\cos x)$

$\Rightarrow$ $3\cos x + 3\sin x = M(4\cos x - 5\sin x) + N(4\sin x + 5\cos x)$

$\Rightarrow$ Comparing the coefficient of $\sin x$ and $\cos x$ on both sides.

$\Rightarrow$ $-5M + 4N = 3$ and $4M + 5N = 3$ $\Rightarrow$ $M = \frac{-3}{41}$, $N = \frac{27}{41}$

$\therefore$ $I = \int \frac{\frac{-3}{41}(4\cos x - 5\sin x) + \frac{27}{41}(4\sin x + 5\cos x)}{4\sin x + 5\cos x}dx \Rightarrow I = \int \frac{27}{41}dx + \left(\frac{-3}{41}\right)\int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}dx$

$\Rightarrow$ $I = \frac{27}{41}x - \frac{3}{41}\log(4\sin x + 5\cos x) + c$.

Example: 63 $\int \frac{dx}{1+\cot x} =$ **[CBSE 1992; PB 1993; HB 1994]**

(a) $\frac{1}{2}x + \frac{1}{2}\log|\,\sin x + \cos x\,| + c$

(b) $\frac{1}{2}x - \frac{1}{2}\log|\,\sin x + \cos x\,| + c$

(c) $\frac{-1}{2}x + \frac{1}{2}\log|\,\sin x + \cos x\,| + c$

(d) None of these

Solution: (b) Here, $I = \int \frac{dx}{1+\cot x} = \int \frac{dx}{1+\frac{\cos x}{\sin x}} = \int \frac{\sin x}{\sin x + \cos x}dx$

Let $\sin x = M\frac{d}{dx}(\sin x + \cos x) + N(\sin x + \cos x)$ $\Rightarrow$ $\sin x = M(\cos x - \sin x) + N(\sin x + \cos x)$

Comparing the coefficients of $\sin x$ and $\cos x$ of both the sides, we have $1 = -M + N$ and $0 = M + N$

Solving these equations, we have $M = \frac{-1}{2}$ and $N = \frac{1}{2}$

$\therefore$ $\sin x = -\frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\sin x + \cos x)$

Hence, $I = \int \frac{\frac{-1}{2}(\cos x - \sin x) + \frac{1}{2}(\sin x + \cos x)}{\sin x + \cos x}dx = -\frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}dx + \frac{1}{2}\int 1\,dx = -\frac{1}{2}\log|\,\sin x + \cos x\,| + \frac{1}{2}x + c.$

5.12 Integration of Rational Functions by using Partial Fractions

(1) **Proper rational functions:** Functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial and $g(x) \neq 0$, are called rational functions of x.

If degree of $f(x)$ is less than degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(2) **Improper rational function :** If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$, is called an improper rational function and every improper rational function can be transformed to a proper rational function by dividing the numerator by the denominator.

For example, $\frac{x^3}{x^2-5x+6}$ is an improper rational function and can be expressed as $(x+5)+\frac{19x-30}{x^2-5x+6}$, which is the sum of a polynomial $(x+5)$ and a proper function $\frac{19x-30}{x^2-5x+6}$.

(3) **Partial fractions:** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of $g(x)$, then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

Depending on the nature of the factors of the denominator, the following cases arise.

Case I: When the denominator consists of non-repeated linear factors: To each linear factor $(x-a)$ occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x-a}$, where A is a constant to be determined.

Case II: When the denominator consists of linear factors, some repeated: To each linear factor $(x-a)$ occurring r times in the denominator of a proper rational function, there corresponds a sum of r partial fractions of the form.

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+..........+\frac{A_r}{(x-a)^r}$$

Where $A's$ are constants to be determined. Of course, A_r is not equal to zero.

Case III: When the denominator consists of quadratic factors: To each irreducible non repeated quadratic factor ax^2+bx+c, there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.

To each irreducible quadratic factor ax^2+bx+c occurring r times in the denominator of a proper rational fraction there corresponds a sum of r partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+................+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Where, A's and B's are constants to be determined.

(4) **General methods of finding the constants**

(i) In the given proper fraction, first of all factorize the denominator.

(ii) Express the given proper fraction into its partial fractions according to rules given above and multiply both the sides by the denominator of the given fraction.

(iii) Equate the coefficients of like powers of x in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable x in the identity obtained after clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of x used as those for which the denominator of the corresponding partial fractions become zero.

Note : ❑ If the given fraction is improper, then before finding partial fractions, the given fraction must be expressed as sum of a polynomial and a proper fraction by division.

(5) **Special cases:** Some times a suitable substitution transform the given function to a rational fraction which can be integrated by breaking it into partial fractions.

Example: 64 $\int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx =$ **[Roorkee 1979]**

(a) $\log[(1+\sin x)(2+\sin x)]+c$ (b) $\log\left[\frac{2+\sin x}{1+\sin x}\right]+c$

(c) $\log\left[\frac{1+\sin x}{2+\sin x}\right]+c$ (d) None of these

Solution: (c) Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\int \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)} = \int \frac{dt}{(1+t)(2+t)} = \int\left[\frac{1}{t+1}-\frac{1}{t+2}\right]dt = \log(t+1)-\log(t+2)+c = \log\left[\frac{t+1}{t+2}\right]+c$$

$$= \log\left[\frac{1+\sin x}{2+\sin x}\right]+c.$$

Example: 65 $\int \frac{3x+1}{(x-2)^2(x+2)} =$ **[Pb. CET 1996]**

(a) $\frac{5}{16}\log\left|\frac{x+2}{x-2}\right|+\frac{7}{4(x-2)}+c$ (b) $\frac{5}{16}\log\left|\frac{x-2}{x+2}\right|+\frac{7}{4(x-2)}+c$

(c) $\frac{16}{5}\log\left|\frac{x-2}{x+2}\right|-\frac{7}{4(x-2)}+c$ (d) None of these

Solution: (b) We have, $\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x+2)}$

$3x+1 = A(x-2)(x+2)+B(x+2)+C(x-2)^2$ ....(i)

Putting $x=2$ and -2 successively in equation (i), we get $B=\frac{7}{4}$, $C=\frac{-5}{16}$

Now, we put $x=0$ and get $A=\frac{5}{16}$

$$\int \frac{3x+1}{(x-2)^2(x+2)}dx = \frac{5}{16}\int \frac{dx}{x-2}+\frac{7}{4}\int \frac{dx}{(x-2)^2}+\frac{-5}{16}\int \frac{dx}{x+2} = \frac{5}{16}\log(x-2)-\frac{7}{4}\frac{1}{(x-2)}-\frac{5}{16}\log(x+2)+c$$

$$= \frac{5}{16}\log\left|\frac{x-2}{x+2}\right|-\frac{7}{4(x-2)}+c$$

Example: 66 $\int \frac{x^2+1}{x(x^2-1)}dx =$ **[MP PET 1999]**

(a) $\log\frac{x^2-1}{x}+c$ (b) $-\log\frac{x^2-1}{x}+c$ (c) $\log\frac{x}{x^2+1}+c$ (d) $-\log\frac{x}{x^2+1}+c$

Solution: (a) $I=\int\left(\frac{2x}{x^2-1}-\frac{1}{x}\right)dx \Rightarrow I=\log(x^2-1)-\log x+c \Rightarrow I=\log\frac{x^2-1}{x}+c$.

Example: 67 If $\int\frac{2x^2+3}{(x^2-1)(x^2+4)}dx=a\log\left(\frac{x-1}{x+1}\right)+b\tan^{-1}\left(\frac{x}{2}\right)+c$ then values of a and b are **[Rajasthan PET 2000]**

(a) 1, −1 (b) −1, 1 (c) 1/2, −1/2 (d) 1/2, 1/2

Solution: (d) Put $x^2=y$

$\therefore\ \frac{2x^2+3}{(x^2-1)(x^2+4)}=\frac{2y+3}{(y-1)(y+4)} \Rightarrow \frac{2y+3}{(y-1)(y+4)}=\frac{A}{(y-1)}+\frac{B}{(y+4)}$

$\therefore\ 2y+3=A(y+4)+B(y-1)$

Comparing the coefficient of y and constant terms

$\Rightarrow A+B=2$ and $4A-B=3$

$\therefore\ A=1$ and $B=1$

$\therefore\ I=\int\frac{1}{y-1}dx+\int\frac{1}{y+4}dx \Rightarrow I=\int\frac{1}{x^2-1}dx+\int\frac{1}{x^2+4}dx \Rightarrow I=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}\frac{x}{2}+c$

$\therefore\ a=\frac{1}{2}$ and $b=\frac{1}{2}$.

Example: 68 $\int\frac{x^4}{(x-1)(x^2+1)}dx=$ **[Roorkee 1986]**

(a) $\frac{x(x+2)}{2}+\frac{\log(x-1)}{2}-\frac{\log(x^2+1)}{4}-\frac{\tan^{-1}x}{2}+c$

(b) $\frac{x(x+2)}{2}+\frac{\log(x-1)}{2}+\frac{\log(x^2+1)}{4}-\frac{\tan^{-1}x}{2}+c$

(c) $\frac{x(x+2)}{2}+\frac{\log(x-1)}{2}+\frac{\log(x^2+1)}{4}+\frac{\tan^{-1}x}{2}+c$

(d) None of these

Solution: (a) $\int\frac{x^4}{(x-1)(x^2+1)}dx=\int\frac{x^4-1}{(x-1)(x^2+1)}dx+\int\frac{1}{(x-1)(x^2+1)}dx=\int\frac{(x+1)(x-1)(x^2+1)}{(x-1)(x^2+1)}dx+\int\frac{dx}{(x-1)(x^2+1)}$

$=\int(x+1)\,dx+\int\frac{dx}{(x-1)(x^2+1)}=\frac{x^2}{2}+x+\left[\frac{1}{2}\log(x-1)-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x\right]+c$

$=\frac{x(x+2)}{2}+\frac{\log(x-1)}{2}-\frac{\log(x^2+1)}{4}-\frac{\tan^{-1}x}{2}+c.$

Example: 69 $\int\frac{\tan\phi+\tan^3\phi}{1+\tan^3\phi}d\phi$

(a) $\frac{1}{3}\log|1+\tan\phi|+\frac{1}{6}\log|\tan^2\phi-\tan\phi+1|+\frac{1}{3}\tan^{-1}\left(\frac{2\tan\phi-1}{\sqrt{3}}\right)+c$

(b) $\frac{-1}{3}\log|1+\tan\phi|+\frac{1}{6}\log|\tan^2\phi-\tan\phi+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\phi-1}{\sqrt{3}}\right)+c$

(c) $\frac{1}{3}\log\left(\frac{1+\tan^2\phi}{\tan^3\phi}\right)+c$

(d) None of these

Solution: (b) Let $I=\int\frac{\tan\phi+\tan^3\phi}{1+\tan^3\phi}d\phi=\int\frac{\tan\phi(1+\tan^2\phi)}{1+\tan^3\phi} \Rightarrow I=\int\frac{\tan\phi\sec^2\phi}{1+\tan^3\phi}d\phi$

Putting $\tan\phi=t\Rightarrow\sec^2\phi\,d\phi=dt$, we have, $I=\int\frac{t}{1+t^3}dt=\int\frac{t\,dt}{(1+t)(1-t+t^2)}$

Let $\frac{t}{(1+t)(1-t+t^2)}=\frac{A}{1+t}+\frac{Bt+C}{1-t+t^2}$

$\Rightarrow t=A(1-t+t^2)+(Bt+C)(1+t)$ ......(i)

Putting $t = -1$ in equation (i), we get, $-1 = A(1+1+1) \Rightarrow A = -\frac{1}{3}$

Comparing the coefficients of t^2 and constant terms on both the sides of equation (i) we get

$0 = A + B \Rightarrow B = -A = \frac{1}{3}$ and $0 = A + C \Rightarrow C = -A = \frac{1}{3}$

$\therefore \frac{t}{(1+t)(1-t+t^2)} = -\frac{1}{3}.\frac{1}{1+r} + \frac{1}{3}.\frac{t+1}{1-t+t^2}$

Hence, $I = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{3}\int \frac{t+1}{t^2-t+1}\,dt = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t+2}{t^2-t+1}dt = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{(2t-1)+3}{t^2-t+1}dt$

$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t-1}{t^2-t+1}dt + \frac{3}{6}.\int \frac{dt}{t^2-t+1} = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t-1}{t^2-t+1}dt + \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}}$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c = -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{3}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + c$$

$$= -\frac{1}{3}\log|1+\tan\phi| + \frac{1}{6}\log|\tan^2\phi - \tan\phi + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\phi - 1}{\sqrt{3}}\right) + c.$$

Example: 70 $\int \frac{dx}{x(x^n+1)} =$ **[Roorkee 1979; Pb. CET 1991; Rajasthan PET 1998; Karnataka CET 1996]**

(a) $\frac{1}{n}\log\frac{x^n}{x^n+1} + c$ (b) $n\log\frac{x^n+1}{x^n} + c$ (c) $\frac{-1}{n}\log\frac{x^n}{x^n+1} + c$ (d) $-n\log\frac{x^n+1}{x^n} + c$

Solution: (a) Let $I = \int \frac{dx}{x(x^n+1)} = \int \frac{x^{n-1}}{x^n(x^n+1)}dx$

Putting $x^n = t \Rightarrow nx^{n-1}dx = dt$, we have

$I = \frac{1}{n}\int \frac{dt}{t(t+1)} = \frac{1}{n}\int \left[\frac{1}{t} - \frac{1}{t+1}\right]dt$, (by resolving into partial fractions)

$= \frac{1}{n}[\log t - \log(t+1)] + c = \frac{1}{n}\log\left|\frac{t}{t+1}\right| + c = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + c.$

5.13 Integration of Trigonometric Functions

(1) **Integral of the form** $\int \sin^m x \cos^n x\, dx$ **:** (i) To evaluate the integrals of the form $I = \int \sin^m x \cos^n x\, dx$, where m and n are rational numbers.

(a) Substitute $\sin x = t$, if n is odd;

(b) Substitute $\cos x = t$, if m is odd;

(c) Substitute $\tan x = t$, if $m + n$ is a negative even integer; and

(d) Substitute $\cot x = t$, if $\frac{1}{2}(n-1)$ is an integer.

(e) If m and n are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitution $\cos x = t$ or $\tan x = t$ is found suitable.

(ii) Integrals of the form $\int R(\sin x, \cos x)dx,$ where R is a rational function of $\sin x$ and $\cos x,$ are transformed into integrals of a rational function by the substitution $\tan\frac{x}{2} = t,$ where $-\pi < x < \pi.$ This is the so called universal substitution. Sometimes it is more convenient to make the substitution $\cot\frac{x}{2} = t$ for $0 < x < 2\pi.$

The above substitution enables us to integrate any function of the form $R(\sin x, \cos x).$ However, in practice, it sometimes leads to extremely complex rational function. In some cases, the integral can be simplified by:

(a) Substituting $\sin x = t,$ if the integral is of the form $\int R(\sin x)\cos x\, dx.$

(b) Substituting $\cos x = t,$ if the integral is of the form $\int R(\cos x)\sin x\, dx.$

(c) Substituting tan *x* = *t, i.e.,* $dx = \frac{dt}{1+t^2},$ if the integral is dependent only on $\tan x.$

(d) Substituting $\cos x = t$, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

(e) Substituting $\sin x = t$, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

(f) Substituting $\tan x = t$, if $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$

Important Tips

☞ *To evaluate integrals of the form* $\int \sin mx \cos nx\, dx, \int \sin mx\, .\sin nx\, dx, \int \cos mx\, .\cos nx\, dx$ *and* $\int \cos mx\, .\sin nx\, dx,$ *we use the following trigonometrical identities.*

$\sin mx\, .\cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \Rightarrow \cos mx\, .\sin nx = \frac{1}{2}[\sin(m+n)x - \sin(m-n)x]$

$\sin mx\, .\sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \Rightarrow \cos mx\, .\cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$

Example: 71 $\int \sin^3 x \cos^2 x\, dx$ **[Pb. CET 1992; EAMCET 1996]**

(a) $\frac{\cos^2 x}{5} - \frac{\cos^3 x}{3} + c$ (b) $\frac{\cos^5}{5} + \frac{\cos^3}{3} + c$ (c) $\frac{\sin^5}{5} - \frac{\sin^3}{3} + c$ (d) $\frac{\sin^5}{5} + \frac{\sin^3}{3} + c$

Solution: (a) $I = \int \sin x(1-\cos^2 x)\cos^2 x\, dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow I = -\int (t^2 - t^4)\,dt = \frac{t^5}{5} - \frac{t^3}{3} + c = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c.$

Example: 72 $\int \frac{d\theta}{\sin\theta . \cos^3\theta} =$

(a) $\log\tan\theta + \tan^2\theta + c$ (b) $\log\tan\theta - \frac{1}{2}\tan^2\theta + c$ (c) $\log\tan\theta + \frac{1}{2}\tan^2\theta + c$ (d) None of these

Solution: (c) $\int \frac{d\theta}{\sin\theta\cos^3\theta} = \int \frac{\sec^2\theta\, d\theta}{\sin\theta\cos\theta} = \int \frac{\sec^2\theta(1+\tan^2\theta)d\theta}{\tan\theta\, d\theta}$

Put $t = \tan\theta$ $\Rightarrow dt = \sec^2\theta\, d\theta$ then it reduces to $\Rightarrow \int \frac{1+t^2}{t}\, dt = \int\left(\frac{1}{t}+t\right)dt$

$= \log t + \frac{t^2}{2} + c = \log\tan\theta + \frac{1}{2}\tan^2\theta + c.$

Example: 73 $\int \frac{\sin^3 2x}{\cos^5 2x}\, dx =$ **[Karnataka CET 1999]**

(a) $\tan^4 x + c$ (b) $\tan 4x + c$ (c) $\tan^4 2x + x + c$ (d) $\frac{1}{8}\tan^4 2x + c$

Solution: (d) Given, $I = \int \frac{\sin^3 2x}{\cos^5 2x}\, dx$. The given equation may be written as $\int \frac{\sin^3 2x}{\cos^3 2x}.\frac{1}{\cos^2 2x}\, dx = \int \tan^3 2x . \sec^2 2x\, dx.$

Put $\tan 2x = t$ and $2\sec^2 2x\, dx = dt$

$I = \frac{1}{2}\int t^3\, dt = \frac{t^4}{8} + c = \frac{\tan^4 2x}{8} + c.$

Example: 74 If $I_n = \int \frac{\sin nx}{\sin x}\, dx$, where $n > 1$, then $I_n - I_{n-2} =$

(a) $\frac{2}{(n-1)}\cos(n-1)x$ (b) $\frac{2}{n-1}\sin(n-1)x$ (c) $\frac{2}{n}\cos n\, x$ (d) $\frac{2}{n}\sin n\, x$

Solution: (b) $I_n = \int \frac{\sin nx}{\sin x}\, dx$

$I_{n-2} = \int \frac{\sin(n-2)x}{\sin x}\, dx \Rightarrow I_n - I_{n-2} = \int \frac{\sin nx - \sin(n-2)x}{\sin x}\, dx = \int \frac{2\cos(n-1)x . \sin x}{\sin x}\, dx$

$I_n - I_{n-2} = \frac{2\sin(n-1)x}{(n-1)}$

(2) Reduction formulae for special cases

(i) $\int \sin^n x\, dx = \frac{-\cos x . \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

(ii) $\int \cos^n x\, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx$

(iii) $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$

(iv) $\int \cot^n x\, dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\, dx$

(v) $\int \sec^n x\, dx = \frac{1}{(n-1)}\left[\sec^{n-2} x . \tan x + (n-2)\int \sec^{n-2} x\, dx\right]$

(vi) $\int \operatorname{cosec}^n x\, dx = \frac{1}{(n-1)}\left[-\operatorname{cosec}^{n-2} x . \cot x + (n-2)\int \operatorname{cosec}^{n-2} x\, dx\right]$

(vii) $\int \sin^p x \cos^q x\, dx = -\frac{\sin^{q+1} x . \cos^{p-1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x . \cos^q x\, dx$

(viii) $\int \sin^p x \cos^q x\, dx = \frac{\sin^{p+1} x . \cos^{q-1} x}{p+q} + \frac{p-1}{p+q}\int \sin^p x . \cos^{q-2} x\, dx$

(ix) $\int \frac{dx}{(x^2+k)^n} = \frac{x}{k(2n-2)(x^2+k)^{n-1}} + \frac{(2n-3)}{k(2n-2)}\int \frac{dx}{(x^2+k)^{n-1}}$

Important Tips

☞ *Reduction formulae for* $I_{(n,m)}=\int\frac{\sin^n x}{\cos^m x}dx$ *is* $I_{(n,m)}=\frac{1}{m-1}.\frac{\sin^{n-1}x}{\cos^{m-1}x}-\frac{(n-1)}{(m-1)}.I_{(n-2,m-2)}$

Example: 75 $\int\tan^5\theta\,d\theta=$ **[EAMCET 1993]**

(a) $\frac{\tan^4\theta}{4}-\frac{\tan^2\theta}{2}+\log\sec\theta+c$ (b) $\frac{\tan^4\theta}{4}-\frac{\tan^2\theta}{2}-\log\sec\theta+c$

(c) $\frac{\tan^4\theta}{4}-\frac{\tan^2\theta}{2}+\log|\sec\theta|+c$ (d) None of these

Solution: (c) $\int\tan^5\theta\,d\theta=I_5=\frac{\tan^4\theta}{4}-I_3\;=\frac{\tan^4\theta}{4}-\frac{\tan^2\theta}{2}+I_1\;=\frac{\tan^4\theta}{4}-\frac{\tan^2\theta}{2}+\log|\sec\theta|+c.$

Example: 76 $\int\text{cosec}^4x\,dx$ **[Rajasthan PET 2002]**

(a) $\cot x+\frac{\cot^3x}{3}+c$ (b) $\tan x+\frac{\tan^3x}{3}+c$ (c) $\cot x-\frac{\cot^3x}{3}+c$ (d) $-\tan x-\frac{\tan^3x}{3}+c$

Solution: (c) $\int\text{cosec}^4x\,dx=\int\text{cosec}^2x(1+\cot^2x)dx\;=\int\text{cosec}^2x\,dx+\int\cot^2x\,\text{cosec}^2x\,dx\;=-\cot x+\frac{(\cot x)^3}{3}+c.$

Example: 77 Integrate $\int\sin^8x\,dx$

(a) $\frac{1}{2^7}\left[\frac{\sin 8x}{8}-8\frac{\sin 6x}{6}+28\frac{\sin 4x}{4}-56\frac{\sin 2x}{2}+35x+c\right]$

(b) $\frac{1}{2^7}\left[-\frac{\sin 8x}{8}+8\frac{\sin 6x}{6}+28\frac{\sin 4x}{4}+56\frac{\sin 2x}{2}-35x+c\right]$

(c) $\frac{\sin 8x}{8}-8\frac{\sin 6x}{6}+28\frac{\sin 4x}{4}-56\frac{\sin 2x}{2}+35x+c$

(d) None of these

Solution: (a) Let $\cos x+i\sin x=y$; then

$2\cos x=y+\frac{1}{y},2\cos nx=y^n+\frac{1}{y^n}\Rightarrow 2i\sin x=y-\frac{1}{y},2i\sin nx=y^n-\frac{1}{y^n}$ (Remember as the standard results)

Thus $2^8i^8\sin^8x=\left(y-\frac{1}{y}\right)^8=\left(y^8+\frac{1}{y^8}\right)-8\left(y^6+\frac{1}{y^6}\right)+28\left(y^4+\frac{1}{y^4}\right)-56\left(y^2+\frac{1}{y^2}\right)+70$

$=2\cos 8x-16\cos 6x+56\cos 4x-112\cos 2x+70$

Thus $\sin^8x=\frac{1}{2^7}(\cos 8x-8\cos 6x+28\cos 4x-56\cos 3x+35),$

and $\int\sin^8x\,dx=\frac{1}{2^7}\left[\frac{\sin 8x}{8}-8\frac{\sin 6x}{6}+28\frac{\sin 4x}{4}-56\frac{\sin 2x}{2}+35x\right]+c$

5.14 Integration of Hyperbolic Functions

(1) $\int\sinh x\,dx=\cos\text{h}x+c$

(2) $\int\cos\text{h}x\,dx=\sin\text{h}x+c$

(3) $\int\sec\text{h}^2x\,dx=\tan\text{h}\,x+c$

(4) $\int\text{cosec}\,\text{h}^2x\,dx=-\cot\text{h}x+c$

(5) $\int\sec\text{h}x\tan\text{h}x\,dx=-\sec\text{h}x+c$

(6) $\int\text{cosec}\,\text{h}x\cot\text{h}\,dx=-\text{cosec}\,\text{h}x+c$

Example: 78 $\int \cos^{-1}\left(\frac{1}{x}\right)dx =$ **[Rajasthan PET 2002]**

(a) $x\sec^{-1}x + \cos h^{-1}x + c$ (b) $x\sec^{-1}x - \cos h^{-1}x + c$

(c) $\cos h^{-1}x - x\sec^{-1}x + c$ (d) None of these

Solution: (b) $\int \cos^{-1}\left(\frac{1}{x}\right)dx = \int \sec^{-1}x\,dx \;= x\sec^{-1}x - \int \frac{1}{x\sqrt{x^2-1}}x\,dx \;= x\sec^{-1}x - \cos h^{-1}x + c.$

Example: 79 $\int \frac{\cos x - \sin x}{\sqrt{\sin 2x}}dx$ equals **[Rajasthan PET 1996]**

(a) $\cos h^{-1}(\sin x + \cos x) + c$ (b) $\sin h^{-1}(\sin x + \cos x) + c$

(c) $-\cos h^{-1}(\sin x + \cos x) + c$ (d) $-\sin h^{-1}(\sin x + \cos x) + c$

Solution: (a) $I = \int \frac{\cos x - \sin x}{\sqrt{\sin 2x}}dx \;= \int \frac{\cos x - \sin x}{\sqrt{1+\sin 2x - 1}}dx \;= \int \frac{\cos x - \sin x}{\sqrt{(\cos x + \sin x)^2 - 1}}dx$

Put $\cos x + \sin x = t \Rightarrow (\cos x - \sin x)dx = dt \Rightarrow \int \frac{dt}{\sqrt{t^2-1}} = \cos h^{-1}t + c = \cos h^{-1}(\sin x + \cos x) + c.$

Example: 80 $\int \sin\text{h}^{-1}x\,dx$ equals **[Rajasthan PET 1996]**

(a) $x\cos\text{h}^{-1}x - \sqrt{x^2+1} + c$ (b) $x\cos\text{h}^{-1}x + \sqrt{x^2+1} + c$

(c) $x\sin\text{h}^{-1}x + \sqrt{x^2+1} + c$ (d) $x\sin\text{h}^{-1}x - \sqrt{x^2+1} + c$

Solution: (d) $\int \underset{\text{I}}{\sin\text{h}^{-1}x}.\underset{\text{II}}{1}\,dx \;= x\sin\text{h}^{-1}x - \int \frac{1}{\sqrt{1+x^2}}.x\,dx \;= x\sin\text{h}^{-1}x - \sqrt{1+x^2} + c.$

Example: 81 $\int \sqrt{\frac{e^x+a}{e^x-a}}\,dx =$

(a) $\cos\text{h}^{-1}(e^x/a) + \sec^{-1}(e^x/a) + c$ (b) $\left(\frac{1}{a}\right)\left[\cos\text{h}^{-1}(e^x/a) + \sec^{-1}(e^x/a)\right] + c$

(c) $\left(\frac{1}{a}\right)\left[\cos\text{h}^{-1}(e^x/a) + \sec^{-1}(e^x/a)\right] + c$ (d) None of these

Solution: (a) $\int \sqrt{\frac{e^x+a}{e^x-a}}\,dx = \int \frac{e^x+a}{\sqrt{e^{2x}-a^2}}dx \;= \int \frac{e^x}{\sqrt{e^{2x}-a^2}}dx + \int \frac{a}{\sqrt{e^{2x}-a^2}}dx \;= \int \frac{e^x/a}{\sqrt{(e^x/a)^2-1^2}}dx + \int \frac{1}{\sqrt{(e^x/a)^2-1^2}}dx$

Put $\frac{e^x}{a} = t \Rightarrow \frac{e^x}{a}dx = dt \;= \int \frac{dt}{\sqrt{t^2-1}} + \int \frac{dt}{t\sqrt{t^2-1}} = \cos\text{h}^{-1}(e^x/a) + \sec^{-1}(e^x/a) + c.$

Example: 82 $\int \frac{dx}{(e^x+e^{-x})^2} =$

(a) $\tan\text{h}x + c$ (b) $\frac{1}{2}\tan\text{h}x + c$ (c) $\frac{1}{4}\tan\text{h}x + c$ (d) None of these

Solution: (c) $\int \frac{dx}{(e^x+e^{-x})^2} = \frac{1}{4}\int \frac{4.dx}{(e^x+e^{-x})^2} = \frac{1}{4}\int \sec\text{h}^2x\,dx = \frac{1}{4}\tan\text{h}x + c.$

Example: 83 $\int \frac{\tan x}{\sqrt{1-\tan^2 x}}dx =$

(a) $\cos\text{h}^{-1}(\sqrt{2}\cos x) + c$ (b) $-\cos\text{h}^{-1}(\sqrt{2}\cos x) + c$ (c) $\frac{1}{\sqrt{2}}\cos^{-1}(\sqrt{2}\cos x) + c$ (d) $\frac{-1}{\sqrt{2}}\cos\text{h}^{-1}(\sqrt{2}\cos x) + c$

Solution: (d) $\int \frac{\tan x\,dx}{\sqrt{1-\tan^2 x}} = \int \frac{\sin x\,dx}{\sqrt{\cos^2 x - \sin^2 x}} = \int \frac{\sin x\,dx}{\sqrt{2\cos^2 x - 1}}$

Put $\sqrt{2}\cos x = t \Rightarrow -\sqrt{2}\sin x\,dx = dt$

$I = \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2-1}} = \frac{-1}{\sqrt{2}}\cosh^{-1}(\sqrt{2}\cos x + c)$.

Example: 84 $\int \sin x \sinh x\,dx =$

(a) $\frac{1}{2}(e^x - e^{-x}) + c$ (b) $\frac{1}{2}[\sin x \cosh x - \cos x \sinh x] + c$

(c) $\frac{1}{2}[\sin x \cosh x + \cos x \cosh x] + c$ (d) None of these

Solution: (b) $I = \int \sin x \sinh x\,dx = -\sinh x \cos x + \int \cosh x \cos x\,dx = \sinh x \cos x + [\cosh x \sin x - \int \sinh x \sin x\,dx]$

$I = \frac{1}{2}[\sin x \cos x - \cos x \sinh x] + c.$

5.15 Integral of the type $f[x, (ax+b)^{m_1/n_1}, (ax+b)^{m_2/n_2} ...]$ where f is a rational function and m_1, n_1, m_2, n_2 are Integers

To evaluate such type of integral, we transform it into an integral of rational function by putting $(ax+b) = t^s$, where s is the least common multiple (*L.C.M.*) of the numbers n_1, n_2.

Integrals of the form $\int x^m (a+bx^n)^p\,dx$

Case I : If $p \in N$ (Natural number). We expand the integral with the help of binomial theorem and integrate.

Example: 85 Evaluate $\int x^{1/3}(2+x^{1/2})dx$

(a) $3x^{4/3} + \frac{7}{3}x^{7/3} + c$ (b) $x^{4/3} + \frac{5}{3}x^{5/3} + c$ (c) $3x^{4/3} + \frac{3}{5}x^{5/3} + c$ (d) $3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + c$

Solution: (d) $I = \int x^{1/3}(2+x^{1/2})^2\,dx$

Since P is natural number,

$I = \int x^{1/3}(4 + x + 4x^{1/2})dx = \int (4x^{1/3} + x^{4/3} + 4x^{5/6})dx = \frac{4x^{4/3}}{4/3} + \frac{x^{7/3}}{7/3} + \frac{4.x^{11/6}}{11/6} + c = 3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + c.$

Case II: If $p \in I^-$ (*i.e.* negative integer). Write $x = t^k$, where k is the *L.C.M.* of denominator of m and n.

Example: 86 Evaluate $\int x^{-2/3}(1+x^{2/3})^{-1}\,dx$

(a) $3\tan^{-1}(x^{1/3}) + c$ (b) $3\tan^{-1}x + c$ (c) $3\tan^{-1}(x^{2/3}) + c$ (d) None of these

Solution: (a) If we substitute $x = t^3$ (as we know $P \in$ negative integer)

$\therefore$ *Let* $x = t^k$, where k is *L.C.M.* of denominator m and n.

$\therefore x = t^3 \Rightarrow dx = 3t^2 dt$ or $I = \int \frac{3t^2 dt}{t^2(1+t^2)} = 3\int \frac{dt}{t^2+1} = 3\tan^{-1}(t) + c \Rightarrow I = 3\tan^{-1}(x^{1/3}) + c$.

Case III: If $\frac{m+1}{n}$ is an integer and $p \in$ fraction put $(a+bx^n)=t^k$, where k is the denominator of the fraction p.

Example: 87 Evaluate $\int x^{-2/3}(1+x^{1/3})^{1/2}dx$

(a) $2(1+x^{1/3})^{2/3}+c$ (b) $2(1+x^{1/3})^{3/2}+c$ (c) $2(1+x^{2/3})^{3/2}+c$ (d) $2(1+x^{2/3})^{2/3}+c$

Solution: (b) If we substitute $1=x^{1/3}=t^2$ then, $\frac{1}{3x^{2/3}}dx=dt$

$\therefore I=\int\frac{t.6t\,dt}{1}=6\int t^2\,dt=2t^3+c$ or $I=2(1+x^{1/3})^{3/2}+c.$

Case IV: If $\left(\frac{m+1}{n}+P\right)$ is an integer and $P\in$fraction. We put $(a+bx^n)=t^k.x^n$, where k is the denominator of the fraction P.

Example: 88 Evaluate $\int x^{-11}(1+x^4)^{-1/2}dx$

(a) $\frac{1}{2}(t^5+t^3+t)+c$ (b) $-\frac{1}{2}\left[\frac{t^5}{5}-\frac{2t^3}{3}+t\right]+c$ (c) $\frac{1}{2}\left[\frac{t^4}{4}+\frac{2t^3}{3}+t\right]+c$ (d) None of these

where $t=\sqrt{1+\frac{1}{x^4}}$

Solution: (b) Here $\left(\frac{m+1}{n}+P\right)=\left[\frac{-11+1}{4}+\frac{1}{2}\right]=-3$

If we substitute then

$1+\frac{1}{x^4}=t^2$ and $\frac{-4}{x^5}dx=2t\,dt\Rightarrow I=\int\frac{dx}{x^{11}(1+x^4)^{1/2}}=\int\frac{dx}{x^{11}.x^2(1+1/x^4)^{1/2}}$

$I=\int\frac{dx}{x^{13}(1+1/x^4)^{1/2}}=\frac{1}{4}\int\frac{2t\,dt}{x^8t}=-\frac{1}{2}\int(t^2-1)^2\,dt=\frac{-1}{2}\int(t^4-2t^2+1)dt=\frac{-1}{2}\left[\frac{t^5}{5}-\frac{2t^3}{3}+t\right]+c,$

where $t=\sqrt{1+\frac{1}{x^4}}$

5.16 Integrals using Euler's substitution

Integrals of the form $\int f(x),\sqrt{(ax^2+bx+c)}\,dx$ are calculated with the aid of one of the there Euler substitution:

(1) $\sqrt{ax^2+bx+c}=t\pm x\sqrt{a}$, if $a>0$.

(2) $\sqrt{ax^2+bx+c}=t\,x\pm\sqrt{c}$, if $c>0$.

(3) $\sqrt{ax^2+bc+c}=(x-\alpha)t$, if $ax^2+bx+c=a(x-\alpha)(x-\beta)$, *i.e.*, if x is real root of (ax^2+bx+c).

Note : ❑ The Euler substitution often lead to rather some calculations, therefore they should be applied only when it is difficult to find another method for calculating the given integral.

Example: 89 Evaluate $\int\frac{dx}{x+\sqrt{x^2-x+1}}$

(a) $2\log_e|t|-\frac{1}{2}\log_e|t-1|-\frac{3}{2}\log_e|t+1|+\frac{3}{(t+1)}+c$ (b) $2\log_e|t|+\frac{1}{2}\log_e|t-1|+\frac{3}{2}\log_e|t+1|+\frac{3}{(t+1)}+c$

(c) $2\log_e|t|-\frac{1}{2}\log_e|t-1|+\frac{3}{2}\log_e|t+1|+\frac{3}{(t+1)}+c$ (d) None of these

$\left(\text{Where } t=\frac{\sqrt{x^2-x+1}+1}{x}\right)$

Solution: (a) Since here $c=1,$ we can apply the second Euler substitution.

$\sqrt{x^2-x+1}=tx-1$

Hence $(2t-1)x=(t^2-1)x^2$; $x=\frac{2t-1}{t^2-1}$

$\therefore dx=\frac{2(t^2-t+1)dt}{(t^2-1)^2}$ and $x+\sqrt{x^2-x+1}=\frac{1}{t-1}$

$\therefore I=\int\frac{dx}{x+\sqrt{x^2-x+1}}=\int\frac{-2t^2+2t-2}{t(t-1)(t+1)^2}dt$

Using partial fractions, we have,

$\frac{-2t^2+2t-2}{t(t-1)(t+1)^2}=\frac{A}{t}+\frac{B}{t-1}+\frac{C}{(t+1)}+\frac{D}{(t+1)^2}$ or $(-2t+2t-2)=A(t-1)(t+1)^2+B(t+1)^2+C(t-1)(t+1)t+Dt.$

We get $A=2, B=1/2, C=-3/2, D=-3.$

Hence $I=2\int\frac{dt}{t}-\frac{1}{2}\int\frac{dt}{t-1}-\frac{3}{2}\int\frac{dt}{(t+1)}-3\int\frac{dt}{(t+1)^2}=2\log_e|t|-\frac{1}{2}\log_e|t-1|-\frac{3}{2}\log_e|t+1|+\frac{3}{(t+1)}+c$

$\left(\text{Where } t=\frac{\sqrt{x^2-x+1}+1}{x}\right)$

Example: 90 Evaluate $\int\frac{x\,dx}{\sqrt{(7x-10-x^2)^3}}$

(a) $\frac{-2}{9}\left(\frac{-5}{t}+2t\right)+c$ (b) $\frac{2}{9}\left(\frac{-5}{t}-2t\right)+c$ (c) $\frac{-1}{9}\left(\frac{5}{t}+2t\right)+c$ (d) None of these

where $t=\frac{\sqrt{7x-10-x^2}}{x-2}$

Solution: (a) In this case $a<0$ and $c<0.$ Therefore neither (I) nor (II) Euler substitution is applicable. But the quadratic $7x-10-x^2$ has real roots $\alpha=2, \beta=5.$

$\therefore$ We use the (III) *i.e.,* $\sqrt{7x-10-x^2}=\sqrt{(x-2)(5-x)}=(x-2)t$

where $(5-x)=(x-2)t^2$ or $5+2t^2=x(1+t^2)$

$\therefore\ x=\frac{5+2t^2}{1+t^2}\Rightarrow(x-2)t=\left(\frac{5+2t^2}{1+t^2}-2\right)t=\frac{3t}{1+t^2}$, $\therefore dx=\frac{-6t}{(1+t^2)^2}dt$

Hence, $I=\int\frac{x\,dx}{(\sqrt{7x-10-x^2})^3}=\int\frac{\left(\frac{5+2t^2}{1+t^2}\right).\frac{-6t}{(1+t^2)^2}dt}{\left(\frac{3t}{1+t^2}\right)^3}=\frac{-6}{27}\int\frac{5+2t^2}{t^2}dt=\frac{-2}{9}\left(\frac{5}{t^2}+2\right)dt=\frac{-2}{9}\left[\frac{-5}{t}+2t\right]+c$

$\therefore\int\frac{x\,dx}{(\sqrt{7x-10-x^2})^3}=\frac{-2}{9}\left(\frac{-5}{t}+2t\right)+c,$ where $t=\frac{\sqrt{7x-10-x^2}}{x-2}$

5.17 Some Integrals which can not be Found

Any function continuous on interval (*a, b*) has an antiderivative in that interval. In other words, there exists a function $F(x)$ such that $F'(x) = f(x)$.

However not every antiderivative $F(x)$, even when it exists is expressible in closed form in terms of elementary functions such as polynomials, trigonometric, logarithmic, exponential etc. function. Then we say that such antiderivatives or integrals "can not be found." Some typical examples are:

(i) $\int \frac{dx}{\log x}$ (ii) $\int e^{x^2} dx$ (iii) $\int \frac{x^2}{1+x^5} dx$

(iv) $\int \sqrt[3]{1+x^2}\, dx$ (v) $\int \sqrt{1+x^3}\, dx$ (vi) $\sqrt{1-k^2 \sin^2 x}\, dx$

(vii) $\int e^{-x^2} dx$ (viii) $\int \frac{\sin x}{x} dx$ (ix) $\int \frac{\cos x}{x} dx$

(x) $\int \sqrt{\sin x}\, dx$ (xi) $\int \sin(x^2)\, dx$ (xii) $\int \cos(x^2)\, dx$ (xiii) $\int x \tan x\, dx$

etc.

Assignment

Properties of Integration, fundamental Integration formulae

Basic Level

1. $\int \sec x \, dx =$ **[MP PET 1988,95; Rajasthan PET 1996]**

(a) $\log \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + c$ (b) $\log(\sec x - \tan x) + c$ (c) $\log\left(\frac{\pi}{4} + x\right) + c$ (d) $\log(\sec x + \tan c) + c$

2. $\int 5 \sin x \, dx =$ **[MP PET 1988]**

(a) $5 \cos x + c$ (b) $-5 \cos x + c$ (c) $5 \sin x + c$ (d) $-5 \sin x + c$

3. $\int (\sec x + \tan x)^2 dx =$ **[MP PET 1987, 92]**

(a) $2(\sec x + \tan x) - x + c$ (b) $\frac{1}{3}(\sec x + \tan x)^3 + c$ (c) $\sec x(\sec x + \tan x) + c$ (d) $2(\sec x + \tan x) + c$

4. $\int \text{cosec}^2 x \, dx$ is equal to **[MP PET 1999]**

(a) $\cot x + C$ (b) $-\cot x + C$ (c) $\tan^2 x + C$ (d) $-\cot^2 x + C$

5. $\int \sec x \tan x \, dx =$ **[Rajasthan PET 2003]**

(a) $\sec x + \tan x + C$ (b) $\sec x + C$ (c) $\tan x + C$ (d) $-\sec x + C$

6. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx =$ **[MP PET 1990]**

(a) $\sin x + c$ (b) $\cos x + c$ (c) $x + c$ (d) $x^2 + c$

7. $\int (3\text{cosec}^2 x + 2 \sin 3x) dx =$ **[AI CBSE 1981]**

(a) $3 \cot x + \frac{2}{3} \cos 3x + c$ (b) $-\left(3 \cot x + \frac{2}{3} \cos 3x\right) + c$ (c) $3 \cot x - \frac{2}{3} \cos 3x + c$ (d) None of these

8. $\int \frac{1 + \cos^2 x}{\sin^2 x} dx =$ **[MP PET 1993; Ranchi BIT 1982]**

(a) $-\cot x - 2x + c$ (b) $-2 \cot x - 2x + c$ (c) $-2 \cot x - x + c$ (d) $-2 \cot x + x + c$

9. The value of $\int \cot x \, dx$ is **[Rajasthan PET 1995]**

(a) $\log \cos x + c$ (b) $\log \tan x + c$ (c) $\log \sin x + c$ (d) $\log \sec x + c$

10. The value of $\int \frac{1}{(x-5)^2} dx$ is

(a) $\frac{1}{x-5} + c$ (b) $-\frac{1}{x-5} + c$ (c) $\frac{2}{(x-5)^3} + c$ (d) $-2(x-5)^3 + c$

11. $\int \frac{x^2}{x^2+4}dx$ equals to **[Rajasthan PET 2001]**

(a) $x-2\tan^{-1}(x/2)+c$ (b) $x+2\tan^{-1}(x/2)+c$ (c) $x-4\tan^{-1}(x/2)+c$ (d) $x+4\tan^{-1}(x/2)+c$

12. $\int x^2 \sec x^3 dx =$ **[MNR 1986; Roorkee 1975]**

(a) $\log(\sec x^3+\tan x^3)$ (b) $3(\sec x^3+\tan x^3)$ (c) $\frac{1}{3}\log(\sec x^3+\tan x^3)$ (d) None of these

13. $\int \frac{\cos 2x-1}{\cos 2x+1}dx =$ **[MP PET 2000]**

(a) $\tan x - x + c$ (b) $x+\tan x + c$ (c) $x-\tan x+c$ (d) $-x-\cot x+c$

14. $\int \sin^{-1}(\cos x)dx =$

(a) $\frac{\pi x}{2}$ (b) $\frac{\pi x^2}{2}$ (c) $\frac{\pi x - x^2}{2}$ (d) $\frac{\pi x + x^2}{2}$

15. $\int(\sin^{-1}x+\cos^{-1}x)dx =$ **[MP PET 1990]**

(a) $\frac{1}{2}\pi x + c$ (b) $x(\sin^{-1}x-\cos^{-1}x)+c$ (c) $x(\cos^{-1}x+\sin^{-1}x)+c$ (d) $\frac{\pi}{2}+x+c$

16. $\int x^{51}(\tan^{-1}x+\cot^{-1}x)dx =$ **[MP PET 1991]**

(a) $\frac{x^{52}}{52}(\tan^{-1}x+\cot^{-1}x)+c$ (b) $\frac{x^{52}}{52}(\tan^{-1}x-\cot^{-1}x)+c$

(c) $\frac{\pi x^{52}}{104}+\frac{\pi}{2}+c$ (d) $\frac{\pi x^{52}}{52}+\frac{\pi}{2}+c$

17. The value of $\int \frac{1}{x^4}dx$ is **[Rajasthan PET 1995]**

(a) $\frac{1}{-3x^3}+c$ (b) $\frac{1}{3x^3}+c$ (c) $\frac{1}{-4x^3}+c$ (d) $-\frac{1}{3x^3}+c$

18. $\int a^x dx =$ **[Rajasthan PET 2003]**

(a) $\frac{a^x}{\log a}+C$ (b) $a^x \log a + C$ (c) $\log a + c$ (d) $a^x + C$

19. $\int a^x da =$ **[MP PET 1994, 96]**

(a) $\frac{a^x}{\log_e a}+C$ (b) $a^x \log_e a + C$ (c) $\frac{a^x}{x+1}+C$ (d) None of these

20. $\int 13^x dx =$ **[Kerala (Engg.) 2002]**

(a) $\frac{13^x}{\log 13}+C$ (b) $13^{x+1}+C$ (c) $14x+C$ (d) $14^{x+1}+C$

21. $\int e^{x\log a}.e^x dx$ is equal to

(a) $(ae)^x$ (b) $\frac{(ae)^x}{\log(ae)}$ (c) $\frac{e^x}{1+\log a}$ (d) None of these

22. $\int a^{3x+3}dx =$ **[Roorkee 1977]**

(a) $\frac{a^{3x+3}}{\log a}+c$ (b) $\frac{a^{3x+3}}{3\log a}+c$ (c) $a^{3x+3}\log a+c$ (d) $3a^{3x+3}\log a+c$

23. $\int e^{\log(\sin x)}dx =$ **[MP PET 1995]**

(a) $\sin x+c$ (b) $-\cos x+c$ (c) $e^{\log\cos x}+c$ (d) None of these

24. The value of $\int e^{m\log x}dx$ is

(a) $\frac{x^{m+1}}{m+1}+k$ (b) $\frac{e^{m\log x}}{m}+k$ (c) $\frac{e^m}{\log x}+k$ (d) $\frac{e^m}{x}+k$

25. $\int \frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}dx =$ **[MNR 1985]**

(a) $e.3^{-3x}+c$ (b) $e^3\log x+c$ (c) $\frac{x^3}{3}+c$ (d) None of these

26. If $f'(x)=\frac{1}{x}+x$ and $f(1)=\frac{5}{2}$, then $f(x)=$

(a) $\log x+\frac{x^2}{2}+2$ (b) $\log x+\frac{x^2}{2}+1$ (c) $\log x-\frac{x^2}{2}+2$ (d) $\log x-\frac{x^2}{2}+1$

27. $\int \sqrt{1+\sin x}\,dx =$ **[MP PET 1995]**

(a) $\frac{1}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)+c$ (b) $\frac{1}{2}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+c$ (c) $2\sqrt{1+\sin x}+c$ (d) $-2\sqrt{1-\sin x}+c$

28. $\int \sqrt{1+\sin\frac{x}{2}}\,dx =$ **[IIT 1980; MP PET 1989]**

(a) $\frac{1}{4}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)+c$ (b) $4\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)+c$ (c) $4\left(\sin\frac{x}{4}-\cos\frac{x}{4}\right)+c$ (d) $4\left(\sin\frac{x}{4}+\cos\frac{x}{4}\right)+c$

29. $\int \sqrt{1-\sin 2x}\,dx =$.......... ......, $x\in(0,\pi/4)$ **[MP PET 1987]**

(a) $-\sin x+\cos x$ (b) $\sin x-\cos x$ (c) $\tan x+\sec x$ (d) $\sin x+\cos x$

30. $\int \frac{dx}{1-\sin x} =$ **[MP PET 1991]**

(a) $x+\cos x+c$ (b) $1+\sin x+c$ (c) $\sec x-\tan x+c$ (d) $\sec x+\tan x+c$

31. $\int \frac{\cos x-1}{\cos x+1}dx =$ **[MP PET 1989, 92]**

(a) $2\tan\frac{x}{2}-x+c$ (b) $\frac{1}{2}\tan\frac{x}{2}-x+c$ (c) $x-\frac{1}{2}\tan\frac{x}{2}+c$ (d) $x-2\tan\frac{x}{2}+c$

32. $\int \sqrt{1+\cos x}\,dx$ equals **[Rajasthan PET 1996]**

(a) $2\sqrt{2}\sin\frac{x}{2}+c$ (b) $-2\sqrt{2}\sin\frac{x}{2}+c$ (c) $-2\sqrt{2}\cos\frac{x}{2}+c$ (d) $2\sqrt{2}\cos\frac{x}{2}+c$

33. $\int \frac{dx}{\sqrt{x}+\sqrt{x-2}} =$ **[MP PET 1990]**

(a) $\frac{1}{3}\left[x^{3/2}-(x-2)^{3/2}\right]+c$ (b) $\frac{2}{3}\left[x^{3/2}-(x-2)^{3/2}\right]+c$ (c) $\frac{1}{3}\left[(x-2)^{3/2}-x^{3/2}\right]+c$ (d) $\frac{2}{3}\left[(x-2)^{3/2}-x^{3/2}\right]+c$

34. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} =$ **[AISSE 1989]**

(a) $\frac{2}{3(b-a)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+c$

(b) $\frac{2}{3(a-b)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+c$

(c) $\frac{2}{3(a-b)}\left[(x+a)^{3/2}+(x+b)^{3/2}\right]+c$

(d) None of these

35. $\int \frac{3x^3-2\sqrt{x}}{x}\,dx =$ **[Roorkee 1976]**

(a) $x^3-\sqrt{x}+c$ (b) $x^3+\sqrt{x}+c$ (c) $x^3-2\sqrt{x}+c$ (d) $x^3-4\sqrt{x}+c$

36. $\int \frac{5(x^6+1)}{x^2+1}dx =$

(a) $5(x^7+x)\tan^{-1}x+c$

(b) $x^5-\frac{5}{3}x^3+5x+c$

(c) $3x^4-5x^2+15x+c$

(d) $5\tan^{-1}(x^2-1)+\log(x^2+1)+c$

37. $\int \frac{dx}{\tan x+\cot x} =$ **[MP PET 1991]**

(a) $\frac{\cos 2x}{4}+c$ (b) $\frac{\sin 2x}{4}+c$ (c) $-\frac{\sin 2x}{4}+c$ (d) $-\frac{\cos 2x}{4}+c$

38. $\int\left(2\sin x+\frac{1}{x}\right)dx$ is equal to **[MP PET 1999]**

(a) $-2\cos x+\log x+C$ (b) $2\cos x+\log x+C$ (c) $-2\sin x-\frac{1}{x^2}+C$ (d) $-2\cos x+\frac{1}{x^2}+C$

39. $\int \sin 2x\cos 3x\,dx =$ **[Roorkee 1976]**

(a) $\frac{1}{2}\left(\cos x+\frac{1}{5}\cos 5x\right)+c$ (b) $\frac{1}{2}\left(\cos x-\frac{1}{5}\cos 5x\right)+c$ (c) $\cos x+\frac{1}{5}\cos 5x+c$ (d) $\cos x-\frac{1}{5}\cos 5x+c$

40. If $\int(\sin 2x-\cos 2x)dx=\frac{1}{\sqrt{2}}\sin(2x-a)+b$, then **[Roorkee 1978; MP PET 2001]**

(a) $a=\frac{\pi}{4}, b=0$ (b) $a=-\frac{\pi}{4}, b=0$ (c) $a=\frac{5\pi}{4}, b=$ any constant (d) $a=-\frac{5\pi}{4}, b=$ any constant

41. If $\int(\sin 2x+\cos 2x)\,dx=\frac{1}{\sqrt{2}}\sin(2x-c)+a$, then the value of a and c is **[Roorkee 1978]**

(a) $c=\frac{\pi}{4}$ and $a=k$ (an arbitrary constant)

(b) $c=-\frac{\pi}{4}$ and $a=\frac{\pi}{2}$

(c) $c=\frac{\pi}{2}$ and a is an arbitrary constant

(d) None of these

42. If $\int \sin 5x\cos 3x\,dx=-\frac{\cos 8x}{16}+A$, then $A=$ **[MP PET 1992]**

(a) $\frac{\sin 2x}{16}+\text{constant}$ (b) $-\frac{\cos 2x}{4}+\text{constant}$ (c) Constant (d) None of these

43. If $\int \sqrt{2}\sqrt{1+\sin x}\,dx=-4\cos(ax+b)+C$ then the value of (a,b) is **[UPSEAT 2002]**

(a) $\frac{1}{2},\frac{\pi}{4}$ (b) $1,\frac{\pi}{2}$ (c) 1,1 (d) None of these

44. $\int \frac{\sin x + \text{cosec } x}{\tan x} dx =$

(a) $\sin x - \text{cosec } x + c$ (b) $\text{cosec } x - \sin x + c$ (c) $\log \tan x + c$ (d) $\log \cot x + c$

45. $\int \frac{1}{\sqrt{1+\sin x}} dx =$ **[Rajasthan PET 1996]**

(a) $2\sqrt{2} \log \tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$ (b) $\frac{1}{\sqrt{2}} \log \tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$ (c) $\sqrt{2} \log \tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$ (d) $\frac{1}{2\sqrt{2}} \log \tan\left(\frac{\pi}{8} + \frac{x}{4}\right) + c$

46. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx =$ **[MP PET 1996]**

(a) $\tan x + \cot x + c$ (b) $\tan x + \text{cosec } x + c$ (c) $-\tan x + \cot x + c$ (d) $\tan x + \sec x + c$

47. $\int \frac{1 - \tan x}{1 + \tan x} dx =$ **[MP PET 1994]**

(a) $\log \sec\left(\frac{\pi}{4} - x\right) + c$ (b) $\log \cos\left(\frac{\pi}{4} + x\right) + c$ (c) $\log \sin\left(\frac{\pi}{4} + x\right) + c$ (d) None of these

48. $\int \frac{\cos 2x}{\cos x} dx$ is equal to **[Rajasthan PET 1991]**

(a) $2\sin x + \log(\sec x - \tan x) + c$ (b) $2\sin x - \log(\sec x - \tan x) + c$

(c) $2\sin x + \log(\sec x + \tan x) + c$ (d) $2\sin x - \log(\sec x + \tan x) + c$

49. $\int \frac{(1+x)^3}{\sqrt{x}} dx$ equals **[Rajasthan PET 1990]**

(a) $\frac{2}{7}x^{7/2} + \frac{6}{5}x^{5/2} + 2x^{3/2} + 2x^{1/2} + c$ (b) $\frac{2}{5}x^{7/2} + 2x^{5/2} + 6x^{3/2} + 2x^{1/2} + c$

(c) $\frac{2}{7}x^{7/2} - \frac{6}{5}x^{5/2} + 2x^{3/2} - 2x^{1/2} + c$ (d) None of these

50. $\int \frac{dx}{9x^2 - 4} =$

(a) $\frac{1}{12}\log\left|\frac{3x+2}{3x-2}\right| + c$ (b) $\frac{1}{6}\log\left|\frac{3x+2}{3x-2}\right| + c$ (c) $\frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right| + c$ (d) $\frac{1}{6}\log\left|\frac{3x-2}{3x+2}\right| + c$

51. $\int \frac{dx}{a^2 - x^2}$ is equal to **[DCE 2002]**

(a) $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ (b) $\frac{1}{2a}\sin\left(\frac{a-x}{a+x}\right)$ (c) $\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$ (d) $\frac{1}{2a}\log\left(\frac{a-x}{a+x}\right)$

52. $\int \sqrt{x^2 + a^2}\, dx$ is equal to **[Rajasthan PET 2001]**

(a) $\frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log\{x + \sqrt{x^2+a^2}\} + C$ (b) $\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\{x + \sqrt{x^2+a^2}\} + C$

(c) $\frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log\{x - \sqrt{x^2+a^2}\} + C$ (d) $\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\{x - \sqrt{x^2+a^2}\} + C$

53. $\int \frac{dx}{4x^2 + 9} =$ **[MP PET 1991; Roorkee 1977; MNR 1974]**

(a) $\frac{1}{2}\tan^{-1}\left(\frac{2x}{3}\right) + c$ (b) $\frac{3}{2}\tan^{-1}\left(\frac{2x}{3}\right) + c$ (c) $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c$ (d) $\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c$

54. $\int (x-a)(x-b)(x-c)\ldots\ldots\ldots.(x-z)dx$ is equal to

(a) Constant (b) $5c + 5d + x$ (c) 0 (d) None of these

55. If $f'(x)=\sqrt{x}$ and $f(1)=2$, then $f(x)=$

(a) $\sqrt{x}+2$ (b) $x\sqrt{x}+2$ (c) $\frac{3}{2}(x\sqrt{x}+2)$ (d) $\frac{2}{3}(x\sqrt{x}+2)$

56. If $f(x)=\int x^{m-1}dx$ then $f^{(m+1)}x=0$, where

(a) m is a negatve integer (b) $m=0$ (c) m is not an integer (d)

57. A primitive of $|x|$, when $x<0$ is **[SCRA 1999]**

(a) $\frac{1}{2}x^2+c$ (b) $\frac{-1}{2}x^2+c$ (c) $x+c$ (d) $-x+c$

58. $\int \frac{\text{cosec }\theta-\cot\theta}{\text{cosec }\theta+\cot\theta}d\theta=$

(a) $2\text{cosec}\,\theta-2\cot\theta-\theta+c$ (b) $2\,\text{cosec}\,\theta-2\cot\theta+\theta+c$ (c) $2\,\text{cosec}\,\theta+2\cot\theta-\theta+c$ (d) None of these

59. $\int(e^{a\log x}+e^{x\log a})dx=$

(a) $x^{a+1}+\frac{a^x}{\log a}+c$ (b) $\frac{x^{a+1}}{a+1}+a^x\log a+c$ (c) $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$ (d) None of these

60. $\int\frac{dx}{4\cos^3 2x-3\cos 2x}=$

(a) $\frac{1}{3}\log[\sec 6x+\tan 6x]+c$ (b) $\frac{1}{6}\log[\sec 6x+\tan 6x]+c$ (c) $\log[\sec 6x+\tan 6x]+c$ (d) None of these

Advance Level

61. $\int\frac{dx}{\sin^2 x\cos^2 x}=$ **[Roorkee1976; Rajasthan PET 1991]**

(a) $\tan x+\cot x+c$ (b) $\cot x-\tan x+c$ (c) $\tan x-\cot x+c$ (d) None of these

62. $\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=$ **[MP PET 1994]**

(a) $2[\sin x+x\cos\alpha]+c$ (b) $2[\sin x+\sin\alpha]+c$ (c) $2[-\sin x+x\cos\alpha]+c$ (d) $-2[\sin x+\sin\alpha]+c$

63. $\int\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=$ **[IIT 1986]**

(a) $\sin 2x+c$ (b) $-\frac{1}{2}\sin 2x+c$ (c) $\frac{1}{2}\sin 2x+c$ (d) $-\sin 2x+c$

64. $\{1+2\tan x(\tan x+\sec x)\}^{1/2}$ **[Roorkee 1987]**

(a) $\log(\sec x+\tan x)+c$ (b) $\log(\sec x+\tan x)^{1/2}+c$ (c) $\log\sec x(\sec x+\tan x)+c$ (d) None of these

65. If $\int\frac{\cos 4x+1}{\cot x-\tan x}dx=A\cos 4x+B$, then

(a) $A=\frac{-1}{2}$ (b) $A=\frac{-1}{8}$ (c) $A=\frac{-1}{4}$ (d) None of these

66. If $x=f''(t)\cos t+f'(t)\sin t, y=-f''(t)\sin t+f'(t)\cos t$, then $\int\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{1/2}dt$ is equal to **[SCRA 1999]**

(a) $f'(t)+f''(t)+c$ (b) $f''(t)+f'''(t)+c$ (c) $f(t)+f''(t)+c$ (d) $f'(t)-f''(t)+c$

67. If $f(0) = f'(0) = 0$ and $f''(x) = \tan^2 x$ then $f(x)$ is

(a) $\log \sec x - \frac{1}{2}x^2$ (b) $\log \cos x + \frac{1}{2}x^2$ (c) $\log \sec x + \frac{1}{2}x^2$ (d) None of these

68. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to

(a) $\frac{-1}{\sin x + \cos x} + C$ (b) $\log(\sin x + \cos x) + C$ (c) $\log(\sin x - \cos x) + C$ (d) $\log(\sin x + \cos x)^2 + C$

69. If $\int \frac{\cos^4 x}{\sin^2 x} dx = A \cot x + b \sin 2x + C\frac{x}{2} + D$, then

(a) $A = -2$ (b) $B = -1/4$ (c) $C = -3$ (d) None of these

70. $\int \frac{\tan x}{\sec x + \tan x} dx =$

(a) $\sec x + \tan x - x + c$ (b) $\sec x - \tan x + x + c$ (c) $\sec x + \tan x + x + c$ (d) $-\sec x - \tan x + x + c$

Integration by substitution

Basic Level

71. A primitive of $\frac{x}{x^2+1}$ is **[SCRA 1996]**

(a) $\log_e(x^2+1)$ (b) $x \tan^{-1} x$ (c) $\frac{\log_e(x^2+1)}{2}$ (d) $\frac{1}{2} x \tan^{-1} x$

72. The value of $\int \frac{x^3}{\sqrt{1+x^4}} dx$ is **[SCRA 1996]**

(a) $(1+x^4)^{1/2} + c$ (b) $-(1+x^4)^{1/2} + c$ (c) $\frac{1}{2}(1+x^4)^{1/2} + c$ (d) $-\frac{1}{2}(1+x^4)^{1/2} + c$

73. $\int \sqrt{\frac{1+x}{1-x}} dx =$ **[Rajasthan PET 2002]**

(a) $-\sin^{-1} x - \sqrt{1-x^2} + c$ (b) $\sin^{-1} x + \sqrt{1-x^2} + c$ (c) $\sin^{-1} x - \sqrt{1-x^2} + c$ (d) $-\sin^{-1} x - \sqrt{x^2-1} + c$

74. $\int x^x(1+\log x)dx$ is equal to

(a) x^x (b) x^{2x} (c) $x^x \log x$ (d) $\frac{1}{2}(1+\log x)^2$

75. $\int \frac{(x+1)(x+\log x)^2}{x} dx =$ **[AI CBSE 1986]**

(a) $\frac{1}{3}(x+\log x) + c$ (b) $\frac{1}{3}(x+\log x)^2 + c$ (c) $\frac{1}{3}(x+\log x)^3 + c$ (d) None of these

76. $\int \frac{x^2+1}{x(x^2-1)} dx$ is equal to **[MP PET 1999]**

(a) $\log \frac{x^2-1}{x} + C$ (b) $-\log \frac{x^2-1}{x} + C$ (c) $\log \frac{x}{x^2+1} + C$ (d) $-\log \frac{x}{x^2+1} + C$

77. $\int \frac{1}{x^2(x^4+1)^{3/4}}dx =$ **[IIT 1984; Rajasthan PET 2000; UPSEAT 2001]**

(a) $\frac{(x^4+1)^{1/4}}{x}+c$ (b) $-\frac{(x^4+1)^{1/4}}{x}+c$ (c) $\frac{3}{4}\frac{(x^4+1)^{3/4}}{x}+c$ (d) $\frac{\frac{4}{3}(x^4+1)^{3/4}}{x}+c$

78. $\int x\cos x^2\,dx$ is equal to **[MP PET 1999]**

(a) $-\frac{1}{2}\sin^2 x+C$ (b) $\frac{1}{2}\sin^2 x+C$ (c) $-\frac{1}{2}\sin x^2+C$ (d) $\frac{1}{2}\sin x^2+C$

79. $\int e^{-x}\,\text{cosec}^2(2e^{-x}+5)dx =$ **[AISSE 1988]**

(a) $\frac{1}{2}\cot(2e^{-x}+5)+c$ (b) $-\frac{1}{2}\cot(2e^{-x}+5)+c$ (c) $2\cot(2e^{-x}+5)+c$ (d) $-2\cot(2e^{-x}+5)+c$

80. $\int \frac{1+\tan x}{x+\log\sec x}dx =$ **[AI CBSE 1986]**

(a) $\log(x+\log\sec x)+c$ (b) $-\log(x+\log\sec x)+c$ (c) $\log(x-\log\sec x)+c$ (d) None of these

81. $\int \frac{x^3}{\sqrt{x^2+2}}dx =$

(a) $\frac{1}{3}(x^2+2)^{3/2}+2(x^2+2)^{1/2}+c$ (b) $\frac{1}{3}(x^2+2)^{3/2}-2(x^2+2)^{1/2}+c$

(c) $\frac{1}{3}(x^2+2)^{3/2}+(x^2+2)^{1/2}+c$ (d) $\frac{1}{3}(x^2+2)^{3/2}-(x^2+2)^{1/2}+c$

82. $\int x^3\sqrt{3+5x^4}\,dx =$ **[DSSE 1982]**

(a) $(3+5x^4)^{3/2}+c$ (b) $\frac{1}{5}(3+5x^4)^{3/2}+c$ (c) $\frac{1}{30}(3+5x^4)^{3/2}+c$ (d) None of these

83. $\int \sin^2 x\cos x\,dx$ is equal to **[SCRA 1996]**

(a) $\frac{\cos^2 x}{2}$ (b) $\frac{\sin^2 x}{3}$ (c) $\frac{\sin^3 x}{3}$ (d) $-\frac{\cos^2 x}{2}$

84. $\int \frac{(1+\log x)^2}{x}dx =$ **[Roorkee1977]**

(a) $(1+\log x)^3+c$ (b) $3(1+\log x)^3+c$ (c) $\frac{1}{3}(1+\log x)^3+c$ (d) None of these

85. $\int \frac{\sin x\,dx}{(a+b\cos x)^2}- =$

(a) $\frac{1}{b}(a+b\cos x)+c$ (b) $\frac{1}{b(a+b\cos x)}+c$ (c) $\frac{1}{b}\log(a+b\cos x)+c$ (d) None of these

86. $\int \cos^2(ax+b)\sin(ax+b)dx =$ **[DSSE 1979]**

(a) $-\frac{\cos^3(ax+b)}{3a}+c$ (b) $\frac{\cos^3(ax+b)}{3a}+c$ (c) $\frac{\sin^3(ax+b)}{3a}+c$ (d) $-\frac{\sin^3(ax+b)}{3a}+c$

87. $\int \frac{1+\tan^2 x}{1-\tan^2 x}dx$ equals to **[Rajasthan PET 2001]**

(a) $\log\left(\frac{1-\tan x}{1+\tan x}\right)+C$ (b) $\log\left(\frac{1+\tan x}{1-\tan x}\right)+C$ (c) $\frac{1}{2}\log\left(\frac{1-\tan x}{1+\tan x}\right)+C$ (d) $\frac{1}{2}\log\left(\frac{1+\tan x}{1-\tan x}\right)+C$

88. $\int \frac{a^{\sqrt{x}}}{\sqrt{x}} dx =$ **[Roorkee 1990; MP PET 2001]**

(a) $2a^{\sqrt{x}} \log_e a + c$ (b) $2a^{\sqrt{x}} \log_a e + c$ (c) $2a^{\sqrt{x}} \log_{10} a + c$ (d) $2a^{\sqrt{x}} \log_a 10 + c$

89. $\int \frac{t}{e^{3t^2}} dt =$ **[MP PET 1997]**

(a) $\frac{1}{6} e^{3t^2} + c$ (b) $-\frac{1}{6} e^{3t^2} + c$ (c) $\frac{1}{6} e^{-3t^2} + c$ (d) $-\frac{1}{6} e^{-3t^2} + c$

90. $\int xe^{x^2} dx =$ **[SCRA 1996; Rajasthan PET 2003]**

(a) $-\frac{e^{x^2}}{2} + C$ (b) $\frac{e^{x^2}}{2} + C$ (c) $\frac{e^x}{2} + C$ (d) $-\frac{e^x}{2} + C$

91. $\int e^{\sqrt{x}} dx$ is equal to **[MP PET 1988]**

(a) $e^{\sqrt{x}} + A$ (b) $\frac{1}{2} e^{\sqrt{x}} + A$ (c) $2(\sqrt{x} - 1)e^{\sqrt{x}} + A$ (d) $2(\sqrt{x} + 1)e^{\sqrt{x}} + A$

(A is an arbitrary constant)

92. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx =$ **[DCE 1999]**

(a) $e^{\sqrt{x}}$ (b) $\frac{e^{\sqrt{x}}}{2}$ (c) $2e^{\sqrt{x}}$ (d) $\sqrt{x} e^{\sqrt{x}}$

93. $\int \frac{a\,dx}{b + ce^x} =$ **[MP PET 1988; BIT Ranchi 1978]**

(a) $\frac{a}{b} \log\left[\frac{e^x}{b + ce^x}\right] + k$ (b) $\frac{a}{b} \log\left[\frac{b + ce^x}{e^x}\right] + k$ (c) $\frac{b}{a} \log\left[\frac{e^x}{b + ce^x}\right] + k$ (d) $\frac{b}{a} \log\left[\frac{b + ce^x}{e^x}\right] + k$

94. $\int \frac{1}{(e^x + e^{-x})^2} dx =$

(a) $-\frac{1}{2(e^{2x} + 1)} + c$ (b) $\frac{1}{2(e^{2x} + 1)} + c$ (c) $-\frac{1}{e^{2x} + 1} + c$ (d) None of these

95. $\int \frac{e^{2x} - 1}{e^{2x} + 1} dx =$ **[MP PET 1987]**

(a) $\frac{e^{2x} - 1}{e^{2x} + 1} + c$ (b) $\log(e^{2x} + 1) - x + c$ (c) $\log(e^{2x} + 1) + c$ (d) None of these

96. $\int \frac{e^{2x} + 1}{e^{2x} - 1} dx$ equals **[Rajasthan PET 1996]**

(a) $\log(e^x - e^{-x}) + c$ (b) $\log(e^x + e^{-x}) + c$ (c) $\log(e^{-x} - e^x) + c$ (d) $\log(1 - e^{-x}) + c$

97. $\int \frac{dx}{e^x + e^{-x}} =$ **[Bihar CEE 1997; MNR 1974]**

(a) $\tan^{-1}(e^{-x})$ (b) $\tan^{-1}(e^x)$ (c) $\log(e^x - e^{-x})$ (d) $\log(e^x + e^x)$

98. What is the value of the integral $I = \int \frac{dx}{(1 + e^x)(1 + e^{-x})}$ **[DCE 1999]**

(a) $\frac{-1}{1 + e^x}$ (b) $\frac{e^x}{1 + e^x}$ (c) $\frac{1}{1 + e^x}$ (d) None of these

99. $\int e^{3\log x}(x^4+1)^{-1}dx =$ **[MP PET 2001]**

(a) $\log(x^4+1)+c$ (b) $\frac{1}{4}\log(x^4+1)+c$ (c) $-\log(x^4+1)+c$ (d) None of these

100. $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx =$ **[MP PET 1987]**

(a) $\log(1+x^2)-c$ (b) $\log e^{\tan^{-1}x}+c$ (c) $e^{\tan^{-1}x}+c$ (d) $\tan^{-1}e^{\tan^{-1}x}+c$

101. $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$ equals to **[Rajasthan PET 2001]**

(a) $e^{\tan^{-1}x}$ (b) $\frac{1}{m}e^{\tan^{-1}x}$ (c) $\frac{1}{m}e^{m\tan^{-1}x}$ (d) None of these

102. $\int e^{\cos^2 x}\sin 2x\,dx =$ **[AI CBSE 1995]**

(a) $e^{\cos^2 x}+c$ (b) $-e^{\cos^2 x}+c$ (c) $\frac{1}{2}e^{\cos^2 x}+c$ (d) None of these

103. $\int e^x\sin(e^x)dx =$ **[MP PET 1995]**

(a) $-\cos e^x+c$ (b) $\cos e^x+c$ (c) $-\text{cosec}\, e^x+c$ (d) None of these

104. $\int \frac{dx}{e^x-1} =$ **[MP PET 1989]**

(a) $\ln(1-e^{-x})+c$ (b) $-\ln(1-e^{-x})+c$ (c) $\ln(e^x-1)+c$ (d) None of these

105. The value of $\int \left(1+\frac{1}{x^2}\right)e^{\left(x-\frac{1}{x}\right)}dx$ equals **[Kurukshetra CEE 1998]**

(a) $e^{x-\frac{1}{x}}+c$ (b) $e^{x+\frac{1}{x}}+c$ (c) $e^{x^2-\frac{1}{x}}+c$ (d) $e^{x^2+\frac{1}{x^2}}+c$

106. $\int \frac{x^2}{\sqrt{1-x^3}}dx$ equals **[Rajasthan PET 1987]**

(a) $\frac{2}{3}\sqrt{1-x^3}+c$ (b) $\frac{-2}{3}\sqrt{1-x^3}+c$ (c) $\frac{1}{3}\sqrt{1-x^3}+c$ (d) $\frac{-1}{3}\sqrt{1-x^3}+c$

107. $\int \frac{x}{1+x^4}dx =$ **[IIT 1978; UPSEAT 2002]**

(a) $\frac{1}{2}\cot^{-1}x^2+c$ (b) $\frac{1}{2}\tan^{-1}x^2+c$ (c) $\cot^{-1}x^2+c$ (d) $\tan^{-1}x^2+c$

108. $\int \frac{3x^2}{x^6+1}dx =$ **[MNR 19981; MP PET 1988; Rajasthan PET 1995]**

(a) $\log(x^6+1)+c$ (b) $\tan^{-1}(x^3)+c$ (c) $3\tan^{-1}(x^3)+c$ (d) $3\tan^{-1}\left(\frac{x^3}{3}\right)+c$

109. $\int \frac{1}{\sqrt{1-e^{2x}}}dx =$ **[MP PET 1993, 2002; Rajasthan PET 1999]**

(a) $x-\log[1+\sqrt{1-e^{2x}}]+c$ (b) $x+\log[1+\sqrt{1-e^{2x}}]+c$ (c) $\log[1+\sqrt{1-e^{2x}}]-x+c$ (d) None of these

110. $\int \frac{\sec^2 x\,dx}{\sqrt{\tan^2 x+4}} =$

(a) $\log\left[\tan x+\sqrt{\tan^2 x+4}\right]+c$

(b) $\frac{1}{2}\log\left[\tan x+\sqrt{\tan^2 x+4}\right]+c$

(c) $\log\left[\frac{1}{2}\tan x+\frac{1}{2}\sqrt{\tan^2 x+4}\right]+c$

(d) None of these

111. $\int \cos x\sqrt{4-\sin^2 x}\;dx =$

(a) $\frac{1}{2}\sin x\sqrt{4-\sin^2 x}-2\sin^{-1}\left(\frac{1}{2}\sin x\right)+c$

(b) $\frac{1}{2}\sin x\sqrt{4-\sin^2 x}+2\sin^{-1}\left(\frac{1}{2}\sin x\right)+c$

(c) $\frac{1}{2}\sin x\sqrt{4-\sin^2 x}+\sin^{-1}\left(\frac{1}{2}\sin x\right)+c$

(d) None of these

112. $\int \frac{3x^2}{\sqrt{9-16x^6}}dx =$

(a) $\frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right)+c$ (b) $\frac{1}{3}\sin^{-1}\left(\frac{4x^3}{3}\right)+c$ (c) $\frac{1}{4}\sin^{-1}x^3+c$ (d) $\frac{1}{3}\sin^{-1}x^3+c$

113. $\int \frac{x}{\sqrt{4-x^4}}dx =$ **[Roorkee 1976]**

(a) $\cos^{-1}\frac{x^2}{2}$ (b) $\frac{1}{2}\cos^{-1}\frac{x^2}{2}$ (c) $\sin^{-1}\frac{x^2}{2}$ (d) $\frac{1}{2}\sin^{-1}\frac{x^2}{2}$

114. $\frac{a^x}{\sqrt{1-a^{2x}}}dx =$ **[MNR 1983, 87]**

(a) $\frac{1}{\log a}\sin^{-1}a^x+c$ (b) $\sin^{-1}a^x+c$ (c) $\frac{1}{\log a}\cos^{-1}a^x+c$ (d) $\cos^{-1}a^x+c$

115. $\int \frac{\sin x\,dx}{3+4\cos^2 x} =$ **[Karnataka CET 2000]**

(a) $\log(3+4\cos^2 x)+c$ (b) $\frac{-1}{2\sqrt{3}}\tan^{-1}\left(\frac{\cos x}{\sqrt{3}}\right)+c$ (c) $\frac{-1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+c$ (d) $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+c$

116. $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}dx =$ **[Roorkee 1977]**

(a) $\frac{1}{b^2}\log(a^2+b^2\sin^2 x)+c$

(b) $\frac{1}{b}\log(a^2+b^2\sin^2 x)+c$

(c) $\log(a^2+b^2\sin^2 x)+c$

(d) $b^2\log(a^2+b^2\sin^2 x)+c$

117. $\int \frac{\sin x\cos x}{a\cos^2 x+b\sin^2 x}dx =$ **[AI CBSE 1988, 89]**

(a) $\frac{1}{2(b-a)}\log(a\cos^2 x+b\sin^2 x)+c$

(b) $\frac{1}{b-a}\log(a\cos^2 x+b\sin^2 x)+c$

(c) $\frac{1}{2}\log(a\cos^2 x+b\sin^2 x)+c$

(d) None of these

118. $\int \frac{1}{x\sqrt{1+\log x}}dx$ **[Roorkee 1997]**

(a) $\frac{2}{3}(1+\log x)^{3/2}+c$ (b) $(1+\log x)^{3/2}+c$ (c) $2\sqrt{1+\log x}+c$ (d) $\sqrt{1+\log x}+c$

119. $\int \frac{dx}{x + x\log x} =$ **[MP PET 1993; Roorkee1977]**

(a) $\log(1+\log x)$ (b) $\log\log(1+\log x)$ (c) $\log x + \log(\log x)$ (d) None of these

120. $\int \frac{\sin 2x}{1+\sin^2 x}dx =$ **[Roorkee 1976]**

(a) $\log \sin 2x + c$ (b) $\log(1+\sin^2 x)+c$ (c) $\frac{1}{2}\log(1+\sin^2 x)+c$ (d) $\tan^{-1}(\sin x)+c$

121. $\int \frac{\sec^2 x}{1+\tan x}dx =$ **[MP PET 1987]**

(a) $\log(\cos x+\sin x)+c$ (b) $\log(\sec^2 x)$ (c) $\log(1+\tan x)$ (d) $-\frac{1}{(1+\tan x)^2}$

122. $\int \frac{cosec^2 x}{1+\cot x}dx =$ **[MNR 1973]**

(a) $\log(1+\cot x)+c$ (b) $-\log(1+\cot x)+c$ (c) $\frac{1}{2(1+\cot x)^2}+c$ (d) None of these

123. $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx =$ **[MP PET 1989]**

(a) $-\frac{1}{2}\cos\sqrt{x}+c$ (b) $-2\cos\sqrt{x}+c$ (c) $\frac{1}{2}\cos\sqrt{x}+c$ (d) $2\cos\sqrt{x}+c$

124. $\int \frac{x+1}{\sqrt{1+x^2}}dx =$ **[MP PET 1991]**

(a) $\sqrt{1+x^2}+\tan^{-1}x+c$ (b) $\sqrt{1+x^2}-\log\{x+\sqrt{1+x^2}\}+c$

(c) $\sqrt{1+x^2}+\log\{x+\sqrt{1+x^2}\}+c$ (d) $\sqrt{1+x^2}+\log(\sec x+\tan x)+c$

125. $\int \frac{3^x}{\sqrt{9^x-1}}dx =$ **[EAMCET 202]**

(a) $\frac{1}{\log 3}\log|3^x+\sqrt{9^x-1}|+c$ (b) $\frac{1}{\log 3}\log|9^x+\sqrt{9^x-1}|+c$

(c) $\frac{1}{\log 9}\log|3^x+\sqrt{9^x-1}|+c$ (d) $\frac{1}{\log 9}\log|3^x-\sqrt{9^x-1}|+c$

126. To find the value of $\int \frac{1+\log x}{x}dx$, the proper substitution is **[MP PET 1988]**

(a) $\log x = t$ (b) $1+\log x = t$ (c) $\frac{1}{x}=t$ (d) None of these

127. $\int \frac{10x^9+10^x\log_e 10}{10^x+x^{10}}dx =$ **[MNR 1979]**

(a) $-\frac{1}{2}\frac{1}{(10^x+x^{10})^2}$ (b) $\log(10^x+x^{10})+c$ (c) $\frac{1}{2}\frac{1}{(10^x+x^{10})^2}+c$ (d) None of these

128. $\int \frac{\sin x}{\sin(x-\alpha)}dx =$ **[Rajasthan PET 1999; Kerala (Engg.) 2002]**

(a) $x\cos\alpha-\sin\alpha\log\sin(x-\alpha)+c$ (b) $x\cos\alpha+\sin\alpha\log\sin(x-\alpha)+c$

(c) $x\sin\alpha-\sin\alpha\log\sin(x-\alpha)+c$ (d) None of these

129. $\int \frac{2x\tan^{-1}x^2}{1+x^4}dx =$ **[Roorkee 1982]**

(a) $[\tan^{-1}x^2]^2+c$ (b) $\frac{1}{2}[\tan^{-1}x^2]^2+c$ (c) $2[\tan^{-1}x^2]^2+c$ (d) None of these

130. $\int \tan^{-1}\frac{2x}{1-x^2}dx =$ **[MP PET 1991]**

(a) $x\tan^{-1}x+c$ (b) $x\tan^{-1}x-\log(1+x^2)+c$ (c) $2x\tan^{-1}x+\log(1+x^2)+c$ (d) $2x\tan^{-1}x-\log(1+x^2)+c$

131. $\int \sin^{-1}(3x-4x^3)dx =$ **[AISSE 1986; DSSE 1984]**

(a) $x\sin^{-1}x+\sqrt{1-x^2}+c$ (b) $x\sin^{-1}x-\sqrt{1-x^2}+c$ (c) $2[x\sin^{-1}x+\sqrt{1-x^2}]+c$ (d) $3[x\sin^{-1}x+\sqrt{1-x^2}]+c$

132. The value of $\int \frac{2\,dx}{\sqrt{1-4x^2}}$ is **[Karnataka CET 2001]**

(a) $\tan^{-1}(2x)+c$ (b) $\cot^{-1}(2x)+c$ (c) $\cos^{-1}(2x)+c$ (d) $\sin^{-1}(2x)+c$

133. $\int \frac{\cot x}{\log \sin x}dx =$ **[MNR 1974]**

(a) $\log(\log \sin x)+c$ (b) $\log(\log \text{cosec}\, x)+c$ (c) $2\log(\log \sin x)+c$ (d) None of these

134. If $\int f(x)dx = f(x)$, then $\int [f(x)]^2 dx$ is **[DCE 2002]**

(a) $\frac{1}{2}[f(x)]^2$ (b) $[f(x)]^3$ (c) $\frac{[f(x)]^3}{3}$ (d) $[f(x)]^2$

135. Integral of $f(x)=\sqrt{1+x^2}$ with respect to x^2 is

(a) $\frac{2}{3}\frac{(1+x^2)^{3/2}}{x}+k$ (b) $\frac{2}{3}(1+x^2)^{3/2}+k$ (c) $\frac{2}{3}x(1+x^2)^{3/2}+k$ (d) None of these

136. $\int \frac{d(x^2+1)}{\sqrt{x^2+2}}$ is equal to

(a) $2\sqrt{x^2+2}+k$ (b) $\sqrt{x^2+2}+k$ (c) $\frac{1}{(x^2+2)^{3/2}}+k$ (d) None of these

137. $\int x\sec x^2 dx$ is equal to

(a) $\frac{1}{2}\log(\sec x^2+\tan x^2)+k$ (b) $\frac{x^2}{2}\log(\sec x^2+\tan x^2)+k$

(c) $2\log(\sec x^2+\tan x^2)+k$ (d) None of these

138. $\int f'(ax+b)\{f(ax+b)\}^n dx$ is equal to

(a) $\frac{1}{n+1}\{f(ax+b)\}^{n+1}+c, \forall n$ except $n=-1$ (b) $\frac{1}{n+1}\{f(ax+b)\}^{n+1}+c, \forall n$

(c) $\frac{1}{a(n+1)}\{f(ax+b)\}^{n+1}+c, \forall n$ except $n=-1$ (d) $\frac{1}{a(n+1)}\{f(ax+b)\}^{n+1}+c, \forall n$

139. $\int \frac{\sin x-\cos x}{\sqrt{1-\sin 2x}}e^{\sin x}\cos x\,dx$ is equal to

(a) $e^{\sin x}+c$ (b) $e^{\sin x-\cos x}+c$ (c) $e^{\sin x+\cos x}+c$ (d) $e^{\cos x-\sin x}+c$

140. $\int 5^{5^{5^x}}.5^{5^x}.5^x\,dx$ is equal to

(a) $\frac{5^{5^x}}{(\log 5)^3}+c$ (b) $5^{5^{5^x}}(\log 5)^3+c$ (c) $\frac{5^{5^{5^x}}}{(\log 5)^3}+c$ (d) None of these

141. If $\int \frac{2x}{\sqrt{1-4^x}} dx = k \sin^{-1}(2^x) + c$, then k is equal to

(a) $\log 2$ (b) $\frac{1}{2}\log 2$ (c) $\frac{1}{2}$ (d) $\frac{1}{\log 2}$

142. $\int \frac{1}{\sqrt{\sin^3 x \cos x}} dx$ is equal to

(a) $\frac{-2}{\sqrt{\tan x}} + c$ (b) $2\sqrt{\tan x} + c$ (c) $\frac{2}{\sqrt{\tan x}} + c$ (d) $-2\sqrt{\tan x} + c$

143. $\int \frac{\sec x \, dx}{\sqrt{\cos 2x}} =$

(a) $\sin^{-1}(\tan x)$ (b) $\tan x$ (c) $\cos^{-1}(\tan x)$ (d) $\frac{\sin x}{\sqrt{\cos x}}$

144. $\int \frac{x \, dx}{1 - x \cot x} =$

(a) $\log(\cos x - x \sin x) + c$ (b) $\log(x \sin x - \cos x) + c$ (c) $\log(\sin x - x \cos x) + c$ (d) None of these

145. To evaluate $\int \frac{\sec^2 x}{(1+\tan x)(2+\tan x)} dx$, the most suitable substitution is

(a) $1 + \tan x = t$ (b) $2 + \tan x = t$ (c) $\tan x = t$ (d) None of these

146. For which of the following functions, the substitution $x^2 = t$ is applicable

(a) $\int x^6 \tan^{-1} x^3 dx$ (b) $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$ (c) $\int x^3 \cos x^2 dx$ (d) None of these

147. $\int x\sqrt{\frac{1-x^2}{1+x^2}} dx =$

(a) $\frac{1}{2}[\sin^{-1} x^2 + \sqrt{1-x^4}] + c$ (b) $\frac{1}{2}[\sin^{-1} x^2 + \sqrt{1-x^2}] + c$

(c) $\sin^{-1} x^2 + \sqrt{1-x^4} + c$ (d) $\sin^{-1} x^2 + \sqrt{1-x^2} + c$

148. $\int \frac{1}{\cos^2 x(1-\tan x)^2} dx =$

(a) $\frac{1}{\tan x - 1} + c$ (b) $\frac{1}{1-\tan x} + c$ (c) $-\frac{1}{3}\frac{1}{(1-\tan x)^3} + c$ (d) None of these

149. $\int \sec^p x \tan x \, dx =$

(a) $\frac{\sec^{p+1} x}{p+1} + c$ (b) $\frac{\sec^p x}{p} + c$ (c) $\int \frac{\tan^{p+1} x}{p+1} + c$ (d) $\frac{\tan^p x}{p} + c$

Advance Level

150. Consider the following statements: **[SCRA 1996]**

Assertion (A): $\frac{1}{x^2 + a^2}$ can be integrated by a substitution $x = a \tan \theta$.

Reason (R): Because all integrands are integrated by the methodof substitution only.

Of these statements

(a) Both A and R are true and R is the correct explanation of A (b) Both A and R are true but R is not the correct explanation of A

(c) A is true but R is false (d) A is false but R is true

151. $\int \sqrt{\frac{a-x}{x}}dx =$

(a) $a\left[in^{-1}\sqrt{\frac{x}{a}}+\sqrt{\frac{x}{a}}.\sqrt{\frac{a-x}{a}}\right]+C$

(b) $a\left[\sin^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}\sqrt{\frac{a-x}{a}}\right]+C$

(c) $-a\left[\sin^{-1}\sqrt{\frac{x}{a}}+\sqrt{\frac{x}{a}}\sqrt{\frac{a-x}{a}}\right]+C$

(d) None of these

152. $\int \frac{\sin x}{\sin x-\cos x}dx =$ **[Roorkee 1988]**

(a) $\frac{1}{2}\log(\sin x-\cos x)+x+c$

(b) $\frac{1}{2}[\log(\sin x-\cos x)+x]+c$

(c) $\frac{1}{2}\log(\cos x-\sin x)+x+c$

(d) $\frac{1}{2}[\log(\cos x-\sin x)+x]+c$

153. $\int \frac{1+x^2}{\sqrt{1-x^2}}dx =$ **[IIT 1977]**

(a) $\frac{3}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+c$

(b) $\frac{3}{2}\sin^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+c$

(c) $\frac{3}{2}\cos^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+c$

(d) $\frac{3}{2}\cos^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+c$

154. If $I=\int \sec^4 x\,\text{cosec}^2 x\,dx = K\tan^3 x+L\tan x+M\cot x+\text{constant}$, then

(a) $K=\frac{1}{3}, L=1, M=2$ (b) $K=\frac{1}{3}, L=2, M=-1$ (c) $K=-1, L=0, M=1$ (d) None of these

155. $\int \frac{\log(x+1)-\log x}{x(x+1)}dx =$

(a) $-\log\left(\frac{x+1}{x}\right)+c$

(b) $-\log\left[\log\left(\frac{x+1}{x}\right)\right]+c$

(c) $-\left(\frac{1}{2}\right)\left[\log\left(\frac{x+1}{x}\right)^2\right]+c$

(d) $c-\frac{1}{2}\left[(\log(x+1))^2-(\log x)^2\right]$

156. $\int \frac{dx}{\sqrt{1+\sin x}} =$

(a) $\sqrt{2}\log\tan(x/4+\pi/8)$

(b) $\sqrt{2}\log[\text{cosec}\,(x/2+\pi/4)-\cot(x/2+\pi/4)]$

(c) $\sqrt{2}\log[sec\,(x/2-\pi/4)+\tan(x/2-\pi/4)]$

(d) All (a), (b) and (c)

157. $\int \frac{x+1}{x(1+xe^x)^2}dx = \log|1-f(x)|+f(x)+c$, then $f(x)=$

(a) $\frac{1}{x+e^x}$ (b) $\frac{1}{1+xe^x}$ (c) $\frac{1}{(1+xe^x)^2}$ (d) None of these

158. If $l^r(x)$ means $\log\log\log......\log x$, the log being repeated r times, then $\int \frac{1}{xl(x)l^2(x)l^3(x).....l^r(x)}dx =$

(a) $l^{r+1}(x)+c$ (b) $\frac{l^{r+1}(x)}{r+1}+c$ (c) $l^r(x)+c$ (d) None of these

159. $\int x\sqrt{\left[\frac{2\sin(x^2-1)-\sin 2(x^2-1)}{2\sin(x^2-1)+\sin 2(x^2-1)}\right]}dx =$, (where $x^2-1\neq n\pi$)

(a) $\log\frac{1}{2}\sec(x^2-1)$ (b) $\log\sec\left(\frac{x^2-1}{2}\right)$ (c) $\frac{1}{2}\log\sec(x^2-1)$ (d) None of these

160. $\int\cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}dx$ is equal to

(a) $\frac{1}{8}(x^2-1)+k$ (b) $\frac{1}{2}x^2+k$ (c) $\frac{1}{2}x+k$ (d) None of these

161. Let the equation of a curve passing through the point (0,1) be given by $y=\int x^2.e^{x^3}dx$. If the equation of the curve is written in the form $x=f(y)$ then $f(y)$ is

(a) $\sqrt{\log_e(3y-2)}$ (b) $\sqrt[3]{\log_e(3y-2)}$ (c) $\sqrt[3]{\log_e(2-3y)}$ (d) None of these

162. $\int\sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}dx$ is equal to

(a) $\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+c$ (b) $\frac{3}{2}\sin^{-1}(\cos^{3/2}x)+c$ (c) $\frac{2}{3}\cos^{-1}(\cos^{3/2}x)+c$ (d) None of these

163. The value of $\int\frac{ax^2-b}{x\sqrt{c^2x^2-(ax^2+b)^2}}dx$ is

(a) $\sin^{-1}\left[\frac{ax+\frac{b}{x}}{c}\right]+k$ (b) $\sin^{-1}\left[\frac{ax^2+\frac{b}{x^2}}{c}\right]+k$ (c) $\cos^{-1}\left(\frac{ax+\frac{b}{x}}{c}\right)+k$ (d) $\cos^{-1}\left(\frac{ax^2+\frac{b}{x^2}}{c}\right)+k$

164. $\int\sqrt{\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)}dx =$ **[IIT 1985]**

(a) $\cos^{-1}\sqrt{x}+\sqrt{1-x}.(\sqrt{x}-2)+c$ (b) $\cos^{-1}\sqrt{x}-\sqrt{1-x}.(\sqrt{x}-2)+c$

(c) $\cos^{-1}\sqrt{x}+\sqrt{1-x}.(\sqrt{x-2})+c$ (d) None of these

165. $\int\frac{\sin 2x}{\sin^4 x+\cos^4 x}dx =$ **[Rajasthan PET 1995]**

(a) $\cot^{-1}(\tan^2 x)+c$ (b) $\tan^{-1}(\tan^2 x)+c$ (c) $\cot^{-1}(\cot^2 x)+c$ (d) $\tan^{-1}(\cot^2 x)+c$

166. $\int\tan^3 2x\sec 2x\,dx =$ **[IIT 1977]**

(a) $\frac{1}{6}\sec^3 2x-\frac{1}{2}\sec 2x+c$ (b) $\frac{1}{6}\sec^3 2x+\frac{1}{2}\sec 2x+c$ (c) $\frac{1}{9}\sec^2 2x-\frac{1}{3}\sec 2x+c$ (d) None of these

167. The value of $\int\frac{\sqrt{(x^2-a^2)}}{x}dx$ will be **[UPSEAT 1999]**

(a) $\sqrt{(x^2-a^2)}-\tan^{-1}\left[\frac{\sqrt{(x^2-a^2)}}{a}\right]$ (b) $\sqrt{(x^2-a^2)}+\tan^{-1}\left[\frac{\sqrt{(x^2-a^2)}}{a}\right]$

(c) $\sqrt{(x^2-a^2)}+a^2\tan^{-1}\left[\sqrt{x^2-a^2}\right]$ (d) $\tan^{-1}\frac{x}{a}+c$

Integration by Parts

Basic Level

168. $\int (1-x^2)\log x\,dx =$ **[DSSE 1982]**

(a) $\left(x-\frac{x^3}{3}\right)\log x - \left(x-\frac{x^3}{9}\right)+c$ (b) $\left(x-\frac{x^3}{3}\right)\log x + \left(x-\frac{x^3}{9}\right)+c$

(c) $\left(x+\frac{x^3}{3}\right)\log x + \left(x+\frac{x^3}{9}\right)+c$ (d) None of these

169. $\int \frac{1}{x^2}\log(x^2+a^2)dx$ **[MNR 1980]**

(a) $\frac{1}{x}\log(x^2+a^2)+\frac{2}{a}\tan^{-1}\frac{x}{a}+c$ (b) $-\frac{1}{x}\log(x^2+a^2)+\frac{2}{a}\tan^{-1}\frac{x}{a}+c$

(c) $-\frac{1}{x}\log(x^2+a^2)-\frac{2}{a}\tan^{-1}\frac{x}{a}+c$ (d) None of these

170. $\int \frac{\log x}{x^3}dx =$ **[Roorkee 1986]**

(a) $\frac{1}{4x^2}(2\log x-1)+C$ (b) $-\frac{1}{4x^2}(2\log x+1)+C$ (c) $\frac{1}{4x^2}(2\log x+1)+C$ (d) $\frac{1}{4x^2}(1-2\log x)+C$

171. $\int x^3\log x\,dx =$ **[Karnataka CET 2002]**

(a) $\frac{x^4\log x}{4}+C$ (b) $\frac{1}{16}[4x^4\log x - x^4]+C$ (c) $\frac{1}{8}[4x^4\log x-4x^2]+C$ (d) $\frac{1}{16}[4x^4\log x+x^4]+C$

172. $\int x\sin^{-1}x\,dx =$ **[MP PET 1991]**

(a) $\left(\frac{x^2}{2}-\frac{1}{4}\right)\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+c$ (b) $\left(\frac{x^2}{2}+\frac{1}{4}\right)\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+c$

(c) $\left(\frac{x^2}{2}-\frac{1}{4}\right)\sin^{-1}x-\frac{x}{4}\sqrt{1-x^2}+c$ (d) $\left(\frac{x^2}{2}+\frac{1}{4}\right)\sin^{-1}x-\frac{x}{4}\sqrt{1-x^2}+c$

173. $\int \cos(\log_e x)dx$ is equal to **[MP PET 2003]**

(a) $\frac{1}{2}x\{\cos(\log_e x)+\sin(\log_e x)\}$ (b) $x\{\cos(\log_e x)+\sin(\log_e x)\}$

(c) $\frac{1}{2}x\{\cos(\log_e x)-\sin(\log_2 x)\}$ (d) $x\{\cos(\log_e x)-\sin(\log_e x)\}$

174. If $\int xe^{2x}dx$ is equal to $e^{2x}f(x)+c$ where c is constant of integration, then $f(x)$ is **[UPSEAT 2001]**

(a) $(3x-1)/4$ (b) $(2x+1)/2$ (c) $(2x-1)/4$ (d) $(x-4)/6$

175. $\int e^{2x}\left(\frac{\sin 4x-2}{1-\cos 4x}\right)dx =$ **[Mathematics Olympiad 1986]**

(a) $\frac{1}{2}e^{2x}\cot 2x+c$ (b) $-\frac{1}{2}e^{2x}\cot 2x+c$ (c) $-2e^{2x}\cot 2x+c$ (d) $2e^{2x}\cot 2x+c$

176. $\int x\cos x\ dx =$ **[MP PET 1988]**

(a) $x\sin x+\cos x$ (b) $x\sin x-\cos x$ (c) $x\cos x+\sin x$ (d) $x\cos x-\sin x$

177. $\int x\cos^2 x\,dx =$ **[IIT 1972]**

(a) $\frac{x^4}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+c$ (b) $\frac{x^4}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+c$

(c) $\frac{x^4}{4}-\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+c$ (d) $\frac{x^4}{4}+\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+c$

178. If $\frac{d}{dx}f(x)=x\cos x+\sin x$ and $f(0)=2$, then $f(x)=$ **[MP PET 1989]**

(a) $x\sin x$ (b) $x\cos x+\sin x+2$ (c) $x\sin x+2$ (d) $x\cos x+2$

179. $\int e^{x/2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx =$ **[Roorkee 1982]**

(a) $e^{x/2}\cos\frac{x}{2}+c$ (b) $\sqrt{2}\,e^{x/2}\cos\frac{x}{2}+c$ (c) $e^{x/2}\sin\frac{x}{2}+c$ (d) $\sqrt{2}\,e^{x/2}\sin\frac{x}{2}+c$

180. $\int x\sin^2 x\,dx =$ **[Ranchi BIT 1977; IIT 1972]**

(a) $\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x+c$ (b) $\frac{x^2}{4}-\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x+c$

(c) $\frac{x^2}{4}+\frac{x}{4}\sin 2x-\frac{1}{8}\cos 2x+c$ (d) $\frac{x^2}{4}-\frac{x}{4}\sin 2x-\frac{1}{8}\cos 2x+c$

181. $\int\frac{x-\sin x}{1-\cos x}dx =$ **[AISSE 1989]**

(a) $x\cot\frac{x}{2}+c$ (b) $-x\cot\frac{x}{2}+c$ (c) $\cot\frac{x}{2}+c$ (d) None of these

182. $\int x^2\sin 2x\,dx =$ **[IIT 1974]**

(a) $\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$ (b) $-\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$

(c) $\frac{1}{2}x^2\cos 2x-\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$ (d) None of these

183. $\int\log x\,dx =$ **[MNR 1979; Ranchi BIT 1992; SCRA 1996]**

(a) $x+x\log x+c$ (b) $x\log x-x+c$ (c) $x^2\log x+c$ (d) $\frac{1}{x}\log x+x+c$

184. $\int\log_{10}x\ dx =$ **[Roorkee 1973]**

(a) $x\log_{10}x+c$ (b) $x(\log_{10}x+\log_{10}e)+c$ (c) $\log_{10}x+c$ (d) $x(\log_{10}x-\log_{10}e)+c$

185. $\int\frac{\log x}{(1+\log x)^2}dx =$

(a) $\frac{1}{1+\log x}+c$ (b) $\frac{x}{(1+\log x)^2}+c$ (c) $\frac{x}{1+\log x}+c$ (d) $\frac{1}{(1+\log x)^2}+c$

186. $\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx =$

(a) $\frac{1}{\log x}+c$ (b) $\frac{x}{\log x}+c$ (c) $\frac{x}{(\log x)^2}$ (d) None of these

187. $\int e^{2x} \sin 3x \, dx =$ **[Pb. CET 1994]**

(a) $\frac{e^{2x}}{13}(2\sin 3x + 3\cos 3x) + c$
(b) $\frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + c$
(c) $\frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + c$
(d) $\frac{e^{2x}}{13}(2\cos 3x - 3\sin 3x) + c$

188. If $I = \int e^x \sin 2x \, dx$, then for what value of K, $KI = e^x(\sin 2x - 2\cos 2x) + \text{const.}$ **[MP PET 1992]**

(a) 1 (b) 3 (c) 5 (d) 7

189. If $\int g(x) dx = g(x)$, then $\int g(x)\{f(x) + f'(x)\} dx$ is equal to

(a) $g(x)f(x) - g(x)f'(x) + c$
(b) $g(x)f'(x) + c$
(c) $g(x)f(x) + c$
(d) $g(x)f^2(x) + c$

190. The primitive of the function $x|\cos x|$ when $\frac{\pi}{2} < x < \pi$ is given by

(a) $\cos x + x\sin x$ (b) $-\cos x - x\sin x$ (c) $x\sin x - \cos x$ (d) None of these

191. $\int \log(x+1) dx =$ **[Roorkee 1974]**

(a) $(x+1)\log(x+1) - x + c$ (b) $(x+1)\log(x+1) + x + c$ (c) $(x-1)\log(x+1) - x + c$ (d) None of these

192. $\int \frac{1}{\log_x e} dx =$ **[MP PET 1994]**

(a) $\log\log_x e + c$ (b) $\frac{1}{(\log_x e)^2} + c$ (c) $x\log\left(\frac{x}{e}\right) + c$ (d) None of these

193. $\int (\log x)^2 dx =$ **[IIT 1971, 77]**

(a) $x(\log x)^2 - 2x\log x - 2x + c$
(b) $x(\log x)^2 - 2x\log x - x + c$
(c) $x(\log x)^2 - 2x\log x + 2x + c$
(d) $x(\log x)^2 - 2x\log x + x + c$

194. $\int x \log x dx =$ **[MP PET 1987]**

(a) $\frac{x^2}{2}\log x - \frac{x^2}{2} + c$ (b) $\frac{x^2}{2}\log x - \frac{x^2}{4} + c$ (c) $\frac{x^2}{2}\log x + \frac{x^2}{2} + c$ (d) None of these

195. If $\int \ln(x^2 + x) dx = x \ln(x^2 + x) + A$, then $A =$ **[MP PET 1992]**

(a) $2x + \ln(x+1) + \text{const.}$ (b) $2x - \ln(x+1) + \text{const.}$ (c) Constant (d) None of these

196. The value of $\int \frac{\log x}{(x+1)^2} dx$ is **[UPSEAT 1999]**

(a) $\frac{-\log x}{x+1} + \log x - \log(x+1)$
(b) $\frac{\log x}{x+1} + \log x - \log(x+1)$
(c) $\frac{\log x}{x+1} - \log x - \log(x+1)$
(d) $\frac{-\log x}{x+1} - \log x - \log(x+1)$

197. $\int \tan^{-1} x dx =$ **[Roorkee 1977]**

(a) $x\tan^{-1} x + \frac{1}{2}\log(1+x^2)$ (b) $x\tan^{-1} x - \frac{1}{2}\log(1+x^2$ (c) $(x-1)\tan^{-1} x$ (d) $x\tan^{-1} x - \log(1+x^2)$

198. $\int x \tan^{-1} x dx =$ **[Roorkee 1979]**

(a) $\frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + c$ (b) $\frac{1}{2}(x^2-1)\tan^{-1} x - \frac{1}{2}x + c$

(c) $\frac{1}{2}(x^2+1)\tan^{-1} x + \frac{1}{2}x + c$ (d) $\frac{1}{2}(x^2+1)\tan^{-1} x - x + c$

199. $\int e^x(1+\tan x)\sec x dx =$

(a) $e^x \cot x$ (b) $e^x \tan x$ (c) $e^x \sec x$ (d) $e^x \cos x$

200. $\int e^x\left[\sin^{-1}\frac{x}{a} + \frac{1}{\sqrt{a^2-x^2}}\right]dx =$

(a) $\frac{1}{a}e^x \sin^{-1}\frac{x}{a} + c$ (b) $ae^x \sin^{-1}\frac{x}{a} + c$ (c) $e^x \sin^{-1}\frac{x}{a} + c$ (d) $\frac{e^x}{\sqrt{a^2-x^2}} + c$

201. $\int e^x \sin x(\sin x + 2\cos x)dx =$ **[MP PET 1988]**

(a) $e^x \sin^2 x + c$ (b) $e^x \sin x + c$ (c) $e^x \sin 2x$ (d) None of these

202. $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx =$ **[AISSE 1983; MP PET 1994, 96; MNR 1990]**

(a) $-\frac{e^x}{x^2} + c$ (b) $\frac{e^x}{x^2} + c$ (c) $\frac{e^x}{x} + c$ (d) $-\frac{e^x}{x} + c$

203. $\int e^x[f(x) + f'(x)]dx$ is equal to **[DCE 2002]**

(a) $e^x f(x)$ (b) e^x (c) $e^x f'(x)$ (d) None of these

204. $\int\left(\frac{x+2}{x+4}\right)e^x dx$ is equal to **[AMU 2000]**

(a) $e^x\left(\frac{x}{x+4}\right) + C$ (b) $e^x\left(\frac{x+2}{x+4}\right) + C$ (c) $e^x\left(\frac{x-2}{x+4}\right) + C$ (d) $\left(\frac{2xe^x}{x+4}\right) + C$

205. $\int \frac{x-1}{(x+1)^3}e^x dx =$ **[IIT 1983; MP PET 1990]**

(a) $\frac{-e^x}{(x+1)^2} + c$ (b) $\frac{e^x}{(x+1)^2} + c$ (c) $\frac{e^x}{(x+1)^3} + c$ (d) $\frac{-e^x}{(x+1)^3} + c$

206. $\int \frac{xe^x}{(1+x)^2}dx =$ **[MP PET 1997; UPSEAT 2001; Rajasthan PET 2002]**

(a) $\frac{e^{-x}}{1+x} + C$ (b) $-\frac{e^{-x}}{1+x} + c$ (c) $\frac{e^x}{1+x} + c$ (d) $-\frac{e^x}{1+x} + c$

207. $\int e^x \frac{(x^2+1)}{(x+1)^2}dx =$

(a) $\left(\frac{x-1}{x+1}\right)e^x + c$ (b) $e^x\left(\frac{x+1}{x-1}\right) + c$ (c) $e^x(x+1)(x-1) + c$ (d) None of these

208. The value of $\int e^{2x}(2\sin 3x + 3\cos 3x)dx$ is **[MP PET 2003]**

(a) $e^{2x}\sin 3x$ (b) $e^{2x}\cos 3x$ (c) e^{2x} (d) $e^{2x}(2\sin 3x)$

209. $\int e^x.\left(\frac{1+\sin x}{1+\cos x}\right)dx$ is equal to **[Rajasthan PET 1997; Karnataka CET 2003]**

(a) $e^x.\tan\left(\frac{x}{2}\right)+C$ (b) $e^x.\cot\left(\frac{x}{2}\right)+C$ (c) $e^x.\tan x+C$ (d) $e^x.\cot x+C$

210. $\int e^x.(1+\tan x+\tan^2 x)dx =$ **[Karnataka CET 1999]**

(a) $e^x \sin x+C$ (b) $e^x \cos x+C$ (c) $e^x \tan x+C$ (d) $e^x \sec x+C$

211. $\int (1+x-x^{-1})e^{x+x^{-1}}dx =$ **[EAMCET 2003]**

(a) $(x+1)e^{x+x^{-1}}+C$ (b) $(x-1)e^{x+x^{-1}}+C$ (c) $-xe^{x+x^{-1}}+C$ (d) $xe^{x+x^{-1}}+C$

212. $\int e^{-x}(1-\tan x)\sec x dx$ is equal to

(a) $e^{-x}\sec x+C$ (b) $e^{-x}\tan x+C$ (c) $-e^{-x}\tan x+C$ (d) None of these

213. Let $\int e^x\{f(x)-f'(x)\}dx=\phi(x)$ Then $\int e^x f(x)dx$ is

(a) $\phi(x)+e^x f(x)$ (b) $\phi(x)-e^x f(x)$ (c) $\frac{1}{2}\{\phi(x)+e^x f(x)\}$ (d) $\frac{1}{2}\{\phi(x)+e^x f'(x)\}$

214. If $\int f(x)dx=g(x)$, then $\int f^{-1}(x)dx$ is equal to **[MP PET 2003]**

(a) $g^{-1}(x)$ (b) $xf^{-1}(x)-g(f^{-1}(x))$ (c) $xf^{-1}(x)-g^{-1}(x)$ (d) $f^{-1}(x)$

215. $\int \sin\sqrt{x}dx =$ **[Roorkee 1977]**

(a) $2[\sin\sqrt{x}-\cos\sqrt{x}]+c$ (b) $2[\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x}]+c$ (c) $2[\sin\sqrt{x}+\cos\sqrt{x}]+c$ (d) $2[\sin\sqrt{x}+\sqrt{x}\cos\sqrt{x}]+c$

216. $\int \cos\sqrt{x}dx =$ **[BIT Ranchi 1990; IIT 1997; Rajasthan PET 1999]**

(a) $2[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}]+c$ (b) $2[\sqrt{x}\sin\sqrt{x}-\cos\sqrt{x}]+c$ (c) $2[\cos\sqrt{x}-\sqrt{x}\sin\sqrt{x}]+c$ (d) $-2[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}]+c$

217. $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx =$ **[MNR 1978; EAMCET 1982; IIT 1984]**

(a) $x-\sqrt{1-x^2}\sin^{-1}x+c$ (b) $x+\sqrt{1-x^2}\sin^{-1}x+c$ (c) $\sqrt{1-x^2}\sin^{-1}x-x+c$ (d) None of these

218. $\int \frac{x\tan^{-1}x\,dx}{(1+x^2)^{3/2}}$

(a) $\frac{x+\tan^{-1}x}{\sqrt{1+x^2}}+c$ (b) $\frac{x-\tan^{-1}x}{\sqrt{1+x^2}}+c$ (c) $\frac{\tan^{-1}x-x}{\sqrt{1+x^2}}+c$ (d) None of these

Advance Level

219. $\int[\sin(\log x)+\cos(\log x)]dx =$ **[MP PET 1991]**

(a) $x\cos(\log x)+c$ (b) $\sin(\log x)+c$ (c) $\cos(\log x)+c$ (d) $x\sin(\log x)+c$

220. $\int \cos 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta =$ **[IIT 1994]**

(a) $(\cos\theta-\sin\theta)^2 \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$ (b) $(\cos\theta+\sin\theta)^2 \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$

(c) $\frac{(\cos\theta-\sin\theta)^2}{2}\log\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$ (d) $\frac{1}{2}\sin 2\theta \log\tan\left(\frac{\pi}{4}+\theta\right)-\frac{1}{2}\log\sec 2\theta$

221. $\int \left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx =$

(a) $x\log(\log x)+\frac{x}{\log x}+c$ (b) $x\log(\log x)-\frac{x}{\log x}+c$ (c) $x\log(\log x)+\frac{\log x}{x}+c$ (d) $x\log(\log x)-\frac{\log x}{x}+c$

222. If $f(x)=\begin{vmatrix} 0 & x^2-\sin x & \cos x-2 \\ \sin x - x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$, then $\int f(x)dx$ is equal to

(a) $\frac{x^3}{3}-x^2\sin x+\sin 2x$ (b) $\frac{x^3}{3}-x^2\sin x-\cos 2x$ (c) $\frac{x^3}{3}-x^2\cos x+\cos 2x$ (d) Constant

223. If integral of $\frac{2\sin x-\sin 2x}{x^3}$ $(x\neq 0)$ is $g(x)$, then $\lim_{x\to 0} g'(x)$ will be equal to

(a) 1 (b) -1 (c) 0 (d) None of these

224. The value of $\int e^{\sec x}.\sec^3 x(\sin^2 x+\cos x+\sin x+\sin x\cos x)dx$ is

(a) $e^{\sec x}(\sec^2 x+\sec x.\tan x)+c$ (b) $e^{\sec x}+c$

(c) $e^{\sec x}(\sec x+\tan x)+c$ (d) None of these

225. $\int e^{\tan^{-1}x}\left(1+\frac{x}{1+x^2}\right)dx$ is equal to

(a) $x.e^{\tan^{-1}x}$ (b) $\frac{1}{2}xe^{\tan^{-1}x}$ (c) $e^{\tan^{-1}x}$ (d) $\frac{1}{2}e^{\tan^{-1}x}$

226. $\int \sin 2x.\log\cos x dx$ is equal to

(a) $\cos^2 x\left(\frac{1}{2}+\log\cos x\right)+k$ (b) $\cos^2 x.\log x+k$

(c) $\cos^2 x\left(\frac{1}{2}-\log\cos x\right)+k$ (d) None of these

227. If $\int x\log(1+x^2)dx=\phi(x).\log(1+x^2)+\psi(x)+c$ then

(a) $\phi(x)=\frac{1+x^2}{2}$ (b) $\psi(x)=\frac{1+x^2}{2}$ (c) $\psi(x)=-\frac{1+x^2}{2}$ (d) $\phi(x)=-\frac{1+x^2}{2}$

228. If $\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}dx=\sqrt{1+x^2}f(x)+A\log(x+\sqrt{x^2+1})+c$, then

(a) $f(x)=\tan^{-1}x, A=-1$ (b) $f(x)=\tan^{-1}x, A=1$ (c) $f(x)=2\tan^{-1}x, A=-1$ (d) $f(x)=2\tan^{-1}x, A=1$

229. $\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$ is equal to

(a) $\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}\left[\log\left(1+\frac{1}{x^2}\right)+\frac{2}{3}\right]+c$

(b) $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+c$

(c) $\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)+\frac{2}{3}\right]+c$

(d) None of these

230. If $\int x\log(1+\frac{1}{x})dx = f(x).\log(x+1)+g(x).x^2+Ax+c$, then

(a) $f(x)=\frac{1}{2}x^2$ (b) $g(x)=\log x$ (c) $A=1$ (d) None of these

231. If $\int \frac{xe^x}{\sqrt{1+e^x}}dx = f(x)\sqrt{1+e^x}-2\log g(x)+c$, then

(a) $f(x)=x-1$ (b) $g(x)=\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}$ (c) $g(x)=\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}$ (d) $f(x)=2(x-2)$

232. $\int \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx=$ **[AISSE 1983, 87]**

(a) $\frac{x}{\sqrt{(1-x^2)}}\sin^{-1}x+\frac{1}{2}\log(1-x^2)+c$

(b) $\frac{x}{\sqrt{(1-x^2)}}\sin^{-1}x-\frac{1}{2}\log(1-x^2)+c$

(c) $\frac{1}{\sqrt{(1-x^2)}}\sin^{-1}x-\frac{1}{2}\log(1-x^2)+c$

(d) $\frac{1}{\sqrt{(1-x^2)}}\sin^{-1}x+\frac{1}{2}\log(1-x^2)+c$

233. If $\int f(x)dx = F(x)$, then $\int x^3 f(x^2)dx$ is equal to

(a) $\frac{1}{2}\left[x^2F(x^2)-\int F(x^2)\,d(x^2)\right]$

(b) $\frac{1}{2}\left[x^2F(x^2)-\int F(x^2)dx\right]$

(c) $\frac{1}{2}\left[x^2F(x)-\frac{1}{2}\int F(x^2)\,dx\right]$

(d) None of these

Evaluation of the Various forms of Integrals by use of Standard Results

Basic Level

234. $\int \frac{dx}{x^2+4x+13}$ is equal to **[Kerala CET 2002]**

(a) $\log(x^2+4x+13)+c$ (b) $\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$ (c) $\log(2x+4)+c$ (d) $\frac{2x+4}{(x^2+4x+13)^2}+c$

235. $\int \frac{dx}{x^2+8x+20}=$ **[Pb. CET 1996]**

(a) $\tan^{-1}\left(\frac{x+4}{2}\right)+c$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+c$ (c) $-\tan^{-1}\left(\frac{x+4}{2}\right)+c$ (d) $-\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+c$

236. $\int \frac{dx}{1+x-x^2} =$

(a) $\frac{1}{\sqrt{5}} \log\left[\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right]+c$

(b) $\frac{1}{\sqrt{5}} \log\left[\frac{\sqrt{5}-1-2x}{\sqrt{5}+1+2x}\right]+c$

(c) $-\frac{1}{\sqrt{5}} \log\left[\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right]+c$

(d) $-\frac{1}{\sqrt{5}} \log\left[\frac{\sqrt{5}-1-2x}{\sqrt{5}+1+2x}\right]+c$

237. The value of $\int \frac{dx}{3-2x-x^2}$ will be **[UPSEAT 1999]**

(a) $\frac{1}{4}\log\left(\frac{3+x}{1-x}\right)$ (b) $\frac{1}{3}\log\left(\frac{3+x}{1-x}\right)$ (c) $\frac{1}{2}\log\left(\frac{3+x}{1-x}\right)$ (d) $\log\left(\frac{1-x}{3+x}\right)$

238. If $\int \frac{2x+3}{x^2-5x+6}\,dx = 9\ln(x-3)-7\ln(x-2)+A$, then $A=$

(a) $5\ln(x-2)+$ constant (b) $-4\ln(x-3)+$ constant (c) Constant (d) None of these

239. $\int \frac{x\,dx}{x^2+4x+5}$ **[Rajasthan PET 2002]**

(a) $\frac{1}{2}\log(x^2+4x+5)+2\tan^{-1}x+c$

(b) $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2)+c$

(c) $\frac{1}{2}\log(x^2+4x+5)+\tan^{-1}(x+2)+c$

(d) $\frac{1}{2}\log(x^2+4x+5)-2\tan^{-1}(x+2)+c$

240. $\int \frac{2x-3}{x^2+3x-18}dx =$

(a) $\log|x^2+3x-18|-\frac{2}{3}\log\left|\frac{x-3}{x+6}\right|+c$

(b) $\log|x^2+3x-18|+\frac{2}{3}\log\left|\frac{x-3}{x+6}\right|+c$

(c) $-\log|x^2+3x-18|-\frac{2}{3}\log\left|\frac{x-3}{x+6}\right|+c$

(d) $-\log|x^2+3x-18|+\frac{2}{3}\log\left|\frac{x-3}{x+6}\right|+c$

241. $\int \sqrt{x^2-8x+7}\,dx =$

(a) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log[x-4+\sqrt{x^2-8x+7}]+c$

(b) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log[x-4+\sqrt{x^2-8x+7}]+c$

(c) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log[x-4+\sqrt{x^2-8x+7}]+c$

(d) None of these

242. $\int \frac{dx}{\sqrt{2x-x^2}} =$ **[MP PET 1991; Karnataka CET 2002]**

(a) $\cos^{-1}(x-1)+c$ (b) $\sin^{-1}(x-1)+c$ (c) $\cos^{-1}(1+x)+c$ (d) $\sin^{-1}(1-x)+c$

243. $\int \frac{x^4+x^2+1}{x^2-x+1}dx =$

(a) $\frac{1}{3}x^3+\frac{1}{2}x^2+x+c$ (b) $\frac{1}{3}x^3-\frac{1}{2}x^2+x+c$ (c) $\frac{1}{3}x^3+\frac{1}{2}x^2-x+c$ (d) None of these

244. $\int \frac{dx}{x[(\log x)^2 + 4\log x - 1]} =$

(a) $\frac{1}{2\sqrt{5}} \log\left[\frac{\log x + 2 - \sqrt{5}}{\log x + 2 + \sqrt{5}}\right] + c$ (b) $\frac{1}{\sqrt{5}} \log\left[\frac{\log x + 2 - \sqrt{5}}{\log x + 2 + \sqrt{5}}\right] + c$

(c) $\frac{1}{2\sqrt{5}} \log\left[\frac{\log x + 2 + \sqrt{5}}{\log x + 2 - \sqrt{5}}\right] + c$ (d) $\frac{1}{\sqrt{5}} \log\left[\frac{\log x + 2 + \sqrt{5}}{\log x + 2 - \sqrt{5}}\right] + c$

245. $\int \frac{dx}{7 + 5\cos x} =$ **[EAMCET 2002]**

(a) $\frac{1}{\sqrt{6}} \tan^{-1}\left(\frac{1}{\sqrt{6}} \tan \frac{x}{2}\right) + c$ (b) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x}{2}\right) + c$

(c) $\frac{1}{4} \tan^{-1}\left(\tan \frac{x}{2}\right) + c$ (d) $\frac{1}{7} \tan^{-1}\left(\tan \frac{x}{2}\right) + c$

246. $\int \frac{dx}{\sin x + \sqrt{3}\cos x} =$

(a) $\log \tan\left(\frac{x}{2} + \frac{\pi}{2}\right) + c$ (b) $\frac{1}{2} \log \tan\left(\frac{x}{2} + \frac{\pi}{6}\right) + c$ (c) $\log \cot\left(\frac{x}{2} + \frac{\pi}{6}\right) + c$ (d) $\frac{1}{2} \log \cot\left(\frac{x}{2} + \frac{\pi}{6}\right) + c$

247. $\int \frac{dx}{1 - \sin x + \cos x} =$ **[Pb. CET 1992]**

(a) $\log\left|1 - \tan \frac{x}{2}\right| + c$ (b) $-\log\left|1 - \tan \frac{x}{2}\right| + c$ (c) $\log\left|1 + \tan \frac{x}{2}\right| + c$ (d) None of these

248. $\int \frac{dx}{\sin x - \cos x + \sqrt{2}}$ equals **[MP PET 2002]**

(a) $-\frac{1}{\sqrt{2}} \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$ (b) $\frac{1}{\sqrt{2}} \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$ (c) $\frac{1}{\sqrt{2}} \cot\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$ (d) $-\frac{1}{\sqrt{2}} \cot\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$

249. $\int \frac{dx}{1 + 2\sin x + \cos x} =$ **[Rajasthan PET 1991]**

(a) $\log[1 + 2\tan(x/2)] + c$ (c) $\log[1 - 2\tan(x/2)] + c$ (c) $\frac{1}{2} \log[1 + 2\tan(x/2)] + c$ (d) None of these

250. $\int \frac{c^2 \sin 2x}{a^2 + b^2 \sin^2 x} dx =$

(a) $\frac{c^2}{b^2} \log(a^2 + b^2 \sin^2 x) + k$ (b) $\frac{c^2}{a^2} \log(a^2 + b^2 \sin^2 x) + k$ (c) $\frac{b^2}{c^2} \log(a^2 + b^2 \sin^2 x) + k$ (d) None of these

251. $\int \frac{1}{1 + \sin^2 x} dx =$

(a) $\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + k$ (b) $\sqrt{2} \tan^{-1}(\sqrt{2} \tan x) + k$ (c) $-\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + k$ (d) $-\sqrt{2} \tan^{-1}(\sqrt{2} \tan x) + k$

252. $\int \frac{1}{1 + \cos^2 x} dx =$

(a) $\frac{1}{\sqrt{2}} \tan^{-1}(\tan x) + c$ (b) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{2} \tan x\right) + c$ (c) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}} \tan x\right) + c$ (d) None of these

253. $\int \frac{dx}{(a\sin x + b\cos x)^2} =$

(a) $\frac{1}{a(a\tan x + b)} + c$ (b) $\frac{-1}{a(a\tan x + b)} + c$ (c) $\frac{1}{a\tan x + b} + c$ (d) $\frac{-1}{a\tan x + b} + c$

254. $\int \frac{2\sin x + 3\cos x}{4\sin x + 5\cos x} dx$

(a) $\frac{2}{41}\log|5\cos x + 4\sin x| + \frac{23}{41}x + c$ (b) $\frac{1}{2}\log|\sin x - \cos x| + \frac{1}{2}x + c$

(c) $\frac{2}{13}\log|2\sin x + 3\cos x| + \frac{3}{13}x + c$ (d) $\frac{1}{2}\log|\sin x + \cos x| + \frac{1}{2}x + c$

255. $\int \frac{dx}{1 - \tan x} =$ **[Pb. CET 1991, 93]**

(a) $\frac{1}{2}x - \frac{1}{2}\log|\cos x - \sin x| + c$ (b) $\frac{1}{2}x + \frac{1}{2}\log|\cos x + \sin x| + c$

(c) $\frac{1}{2}x + \frac{1}{2}\log|\cos x - \sin x| + c$ (d) None of these

256. $\int \frac{dx}{\cos x - \sin x}$ is equal to **[AIEEE 2004]**

(a) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + c$ (b) $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}\right)\right| + c$ (c) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} - \frac{3\pi}{8}\right)\right| + c$ (d) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + c$

257. $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx =$

(a) $\frac{12}{13}x - \frac{5}{13}\log(3\cos x + 2\sin x)$ (b) $\frac{12}{13}x + \frac{5}{13}\log(3\cos x + 2\sin x$

(c) $\frac{13}{12}x + \frac{5}{13}\log(3\cos x + 2\sin x$ (d) None of these

258. $\int \frac{6x + 7}{\sqrt{(x-5)(x-4)}} dx =$

(a) $6\sqrt{x^2 - 9x + 20} + 34\log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + c$ (b) $6\sqrt{x^2 + 9x + 20} + 34\log\left|x - \frac{9}{2} + \sqrt{x^2 + 9x + 20}\right| + c$

(c) $6\sqrt{x^2 - 9x + 20} - 34\log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + c$ (d) $6\sqrt{x^2 + 9x + 20} - 34\log\left|x - \frac{9}{2} + \sqrt{x^2 + 9x + 20}\right| + c$

Advance Level

259. The integral $\int \frac{2x - 3}{(x^2 + x + 1)^2} dx$ is equal to

(a) $-\frac{8x + 7}{3(x^2 + x + 1)} - \frac{16}{3\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + c$ (b) $-\frac{1}{x^2 + x + 1} - \frac{4}{3}\tan^{-1}(4x + 3) + c$

(c) $\frac{1}{2(x^2 + x + 1)} - \frac{(2x + 1)^2}{(x^2 + x + 1)^2} + c$ (d) $\frac{1}{4(x^2 + x + 1)} + \frac{2}{3}\tan^{-1}(2x + 1) + c$

260. The value of the integral $\int \frac{1 + x^2}{1 + x^4} dx$ is equal to

(a) $\tan^{-1} x^2 + c$ (b) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)$ (c) $\frac{1}{2\sqrt{2}}\log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+c$ (d) None of these

261. $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$ is equal to

(a) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)$ (b) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)$ (c) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{x+1}}\right)$ (d) None of these

262. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} =$ **[MNR 1985]**

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right]+c$ (b) $\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{x\sqrt{2}}{\sqrt{1-x^2}}\right]+c$ (c) $\sqrt{2}\tan^{-1}\left[\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right]+c$ (d) $-\sqrt{2}\tan^{-1}\left[\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right]+c$

Integration of Rational functions by using Partial fractions

Basic Level

263. Correct evaluation of $\int \frac{x}{(x-2)(x-1)}dx$ is **[MP PET 1993]**

(a) $\log_e \frac{(x-2)^2}{(x-1)}+p$ (b) $\log_e \frac{(x-1)}{(x-2)}+p$ (c) $\frac{x-1}{x-2}+p$ (d) $2\log_e \frac{(x-2)}{(x-1)}+p$

(where p is an aribitrary constant)

264. $\int \frac{dx}{(x+1)(x+2)} =$ **[MP PET 1987]**

(a) $\log\frac{(x+2)}{(x+1)}+c$ (b) $\log(x+1)+\log(x+2)+c$ (c) $\log\frac{(x+1)}{(x+2)}+c$ (d) None of these

265. $\int \frac{dx}{1-x^2} =$ **[MP PET 1987, 92, 2000]**

(a) $\tan^{-1} x + c$ (b) $\sin^{-1} x + c$ (c) $\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|+c$ (d) $\frac{1}{2}\log\left|\frac{1-x}{1+x}\right|+c$

266. $\int \frac{x-1}{(x-3)(x-2)}dx =$ **[Roorkee 1978]**

(a) $\log(x-3)-\log(x-2)+c$ (b) $\log(x-3)^2-\log(x-2)+c$ (c) $\log(x-3)+\log(x-2)+c$ (d) $\log(x-3)^2+\log(x-2)+c$

267. $\int \frac{1}{x-x^3}dx =$ **[MP PET 1996]**

(a) $\frac{1}{2}\log\frac{(1-x^2)}{x^2}+c$ (b) $\log\frac{(1-x)}{x(1+x)}+c$ (c) $\log x(1-x^2)+c$ (d) $\frac{1}{2}\log\frac{x^2}{(1-x^2)}+c$

268. If $\int \frac{1}{(\sin x+4)(\sin x-1)}dx = A\frac{1}{\tan\frac{x}{2}-1}+B\tan^{-1}f(x)+c$, then

(a) $A=\frac{1}{5},\ B=\frac{-2}{5\sqrt{15}},\ f(x)=\frac{4\tan x+3}{\sqrt{15}}$ (b) $A=-\frac{1}{5},\ B=\frac{1}{\sqrt{15}},\ f(x)=\frac{4\tan\left(\frac{x}{2}\right)+1}{\sqrt{15}}$

(c) $A=\frac{2}{5},\ B=\frac{-2}{5},\ f(x)=\frac{4\tan x+1}{5}$ (d) $A=\frac{2}{5},\ B=\frac{-2}{5\sqrt{15}},\ f(x)=\frac{4\tan\frac{x}{2}+1}{\sqrt{15}}$

269. $\int\frac{dx}{(\sin x-2\cos x)(2\sin x+\cos x)}=$

(a) $\frac{1}{5}\log\left|\frac{\tan x-2}{1+2\tan x}\right|+c$ (b) $-\frac{1}{5}\log\left|\frac{\tan x-2}{1+2\tan x}\right|+c$ (c) $\log\left|\frac{\tan x-2}{1+2\tan x}\right|+c$ (d) $-\log\left|\frac{\tan x-2}{1+2\tan x}\right|+c$

270. $\int\frac{dx}{(x-1)^2(x-2)}=$ **[CBSE PMT 1994]**

(a) $\log\left|\frac{x-2}{x-1}\right|+\frac{1}{x-1}+c$ (b) $\log\left|\frac{x-1}{x-2}\right|+\frac{1}{x-1}+c$ (c) $\log\left|\frac{x-2}{x-1}\right|-\frac{1}{x-1}+c$ (d) $\log\left|\frac{x-1}{x-2}\right|-\frac{1}{x-1}+c$

271. $\int\frac{x-1}{(x+1)^2}dx=$

(a) $\log(x+1)+\frac{2}{x+1}+c$ (b) $\log(x+1)-\frac{2}{x+1}+c$ (c) $\frac{2}{x+1}-\log(x+1)+c$ (d) None of these

272. $\int\frac{2x}{(2x+1)^2}dx=$ **[DSSE 1985]**

(a) $\frac{1}{2}\log(2x+1)+\frac{1}{2(2x+1)}+c$ (b) $\frac{1}{2}\log(2x+1)-\frac{1}{2(2x+1)}+c$

(c) $2\log(2x+1)+\frac{1}{2(2x+1)}+c$ (d) $2\log(2x+1)-\frac{1}{2(2x+1)}+c$

273. $\int\frac{dx}{(x-x^2)}=$ **[Roorkee 1982]**

(a) $\log x-\log(1-x)+c$ (b) $\log(1-x)^2+c$ (c) $-\log x+\log(1-x)+c$ (d) $\log(x-x^2)+c$

274. Value of $\int\frac{x^2}{x^2-a^2}dx=$ **[MNR 1997]**

(a) $x-\frac{a}{2}\log\left(\frac{x-a}{x+a}\right)+c$ (b) $x+\frac{a}{2}\log\left(\frac{x-a}{x+a}\right)+c$ (c) $x-\frac{a}{2}\log\left(\frac{x+a}{x-a}\right)+c$ (d) $x+\frac{a}{2}\log\left(\frac{x+a}{x-a}\right)+c$

275. $\int\frac{1}{(x-1)(x^2+1)}dx=$ **[Roorkee 1984]**

(a) $\frac{1}{2}\log(x-1)-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+c$ (b) $\frac{1}{2}\log(x-1)+\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+c$

(c) $\frac{1}{2}\log(x-1)-\frac{1}{2}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+c$ (d) None of these

276. If $\int\frac{2x+3}{(x-1)(x^2+1)}dx=\log_e\left\{(x-1)^{\frac{5}{2}}(x^2+1)^a\right\}-\frac{1}{2}\tan^{-1}x+A$, Where A is any arbitrary constant, then the value of 'a' is

[MP PET 1998]

(a) $\frac{5}{4}$ (b) $-\frac{5}{3}$ (c) $-\frac{5}{6}$ (d) $-\frac{5}{4}$

277. $\int\frac{dx}{(x^2+1)(x^2+4)}=$ **[MP PET 1995]**

(a) $\frac{1}{3}\tan^{-1}x-\frac{1}{3}\tan^{-1}\frac{x}{2}+c$ (b) $\frac{1}{3}\tan^{-1}x+\frac{1}{3}\tan^{-1}\frac{x}{2}+c$

(c) $\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+c$ (d) $\tan^{-1}x-2\tan^{-1}\frac{x}{2}+c$

278. $\int\frac{x^2}{(x^2+2)(x^2+3)}dx=$ **[AISSE 1990]**

(a) $-\sqrt{2}\tan^{-1}x+\sqrt{3}\tan^{-1}x+c$ (b) $-\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}+c$

(c) $\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}+c$ (d) None of these

279. $\int\frac{1}{(x^2+a^2)(x^2+b^2)}dx=$

(a) $\frac{1}{(a^2-b^2)}\left[\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]+c$ (b) $\frac{1}{(b^2-a^2)}\left[\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]+c$

(c) $\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$ (d) $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+c$

280. $\int\frac{x\,dx}{(x^2-a^2)(x^2-b^2)}=$ **[Roorkee 1976]**

(a) $\frac{1}{(a^2-b^2)}\log\left(\frac{x^2-a^2}{x^2-b^2}\right)+c$ (b) $\frac{1}{a^2-b^2}\log\left(\frac{x^2-b^2}{x^2-a^2}\right)+c$

(c) $\frac{1}{2(a^2-b^2)}\log\left(\frac{x^2-a^2}{x^2-b^2}\right)+c$ (d) $\frac{1}{2(a^2-b^2)}\log\left(\frac{x^2-b^2}{x^2-a^2}\right)+c$

281. If $\int\frac{(2x^2+1)dx}{(x^2-4)(x^2-1)}=\log\left[\left(\frac{x+1}{x-1}\right)^a\left(\frac{x-2}{x+2}\right)\right]+c$, then the values of a and b are respectively **[Roorkee 2000]**

(a) $\frac{1}{2},\frac{3}{4}$ (b) $-1,\frac{3}{2}$ (c) $1,\frac{3}{2}$ (d) $\frac{-1}{2},\frac{3}{4}$

282. If $\int\frac{2x^2+3}{(x^2-1)(x^2+4)}dx=a\log\left(\frac{x-1}{x+2}\right)+b\tan^{-1}\left(\frac{x}{2}\right)+c$, then values of a and b are **[Rajasthan PET 2000]**

(a) $(1,-1)$ (b) $(-1,1)$ (c) $\left(\frac{1}{2},-\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},\frac{1}{2}\right)$

283. For $x>1$, $\int\frac{1}{x(x^4-1)}dx=$ **[Rajasthan PET 1997, 89]**

(a) $\log\frac{x^4-1}{x^4}+k$ (b) $\frac{1}{4}\log\frac{x^4-1}{x^4}+k$ (c) $\log\frac{x^4-1}{x}+k$ (d) $\frac{1}{4}\log\frac{x^4-1}{x}+k$

284. $\int\frac{dx}{e^x+1-2e^x}=$

(a) $\log(e^x-1)-\log(e^x+2)+c$ (b) $\frac{1}{2}\log(e^x-1)-\frac{1}{3}\log(e^x+2)+c$

(c) $\frac{1}{3}\log(e^x-1)-\frac{1}{3}\log(e^x+2)+c$ (d) $\frac{1}{3}\log(e^x-1)+\frac{1}{3}\log(e^x+2)+c$

285. $\int\frac{x^2}{x^2+6x-3}dx=$ **[AICBSE 1999]**

(a) $x+3\log|x^2+6x-3|+\frac{21}{4\sqrt{3}}\log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right|+c$
(b) $x-3\log|x^2+6x-3|+\frac{21}{4\sqrt{3}}\log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right|+c$
(c) $x-3\log|x^2+6x-3|-\frac{21}{4\sqrt{3}}\log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right|+c$
(d) None of these

286. $\int \frac{e^x}{(1+e^x)(2+e^x)}dx =$

(a) $\log[(1+e^x)(2+e^x)]+c$ (b) $\log\left[\frac{1+e^x}{2+e^x}\right]+c$ (c) $\log[(1+e^x)\sqrt{2+e^x}]+c$ (d) None of these

Advance Level

287. $\int \frac{x^3-x-2}{(1-x^2)}dx =$ **[AICBSE 1985]**

(a) $\log\left(\frac{x+1}{x-1}\right)-\frac{x^2}{2}+c$ (b) $\log\left(\frac{x-1}{x+1}\right)+\frac{x^2}{2}+c$ (c) $\log\left(\frac{x+1}{x-1}\right)+\frac{x^2}{2}+c$ (d) $\log\left(\frac{x-1}{x+1}\right)-\frac{x^2}{2}+c$

288. $\int \frac{x^2+x-1}{x^2+x-6}dx =$ **[AISSE 1988]**

(a) $x+\log(x+3)+\log(x-2)+c$
(b) $x-\log(x+3)+\log(x-2)+c$
(c) $x-\log(x+3)-\log(x-2)+c$
(d) None of these

289. $\int \frac{dx}{1+x+x^2+x^3} =$ **[MP PET 1991]**

(a) $\log\sqrt{1+x}-\frac{1}{2}\log\sqrt{1+x^2}+\frac{1}{2}\tan^{-1}x+c$
(b) $\log\sqrt{1+x}-\log\sqrt{1+x^2}+\tan^{-1}x+c$
(c) $\log\sqrt{1+x^2}-\log\sqrt{1+x}+\frac{1}{2}\tan^{-1}x+c$
(d) $\log\sqrt{1+x}+\tan^{-1}x+\log\sqrt{1+x^2}+c$

290. $\int \frac{x^3-1}{x^3+x}dx =$ **[Roorkee 1988, MP PET 2001]**

(a) $x-\log x+\frac{1}{2}\log(x^2+1)+\tan^{-1}x+c$
(b) $x-\log x+\log\sqrt{x^2+1}-\tan^{-1}x+c$
(c) $x+\log x+\log\sqrt{x^2+1}+\tan^{-1}x+c$
(d) None of these

291. $\int \frac{(1+x)^3}{(1-x)^3}dx =$

(a) $x+6\log|1-x|+\frac{12}{1-x}-\frac{4}{(1-x)^2}+c$
(b) $-x+6\log|1-x|+\frac{12}{1-x}-\frac{4}{(1-x)^2}+c$
(c) $-x-6\log|1-x|-\frac{12}{1-x}-\frac{4}{(1-x)^2}+c$
(d) None of these

292. $\int \frac{dx}{x(x^5+1)} =$ **[CBSE 1997]**

(a) $\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+c$ (b) $5\log\left|\frac{x^5}{x^5+1}\right|+c$ (c) $-\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+c$ (d) $-5\log\left|\frac{x^5}{x^5+1}\right|+c$

293. If $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx = Ax+B\log(9e^{2x}-4)+C$ then A, B and C are **[IIT 1990]**

(a) $A=\frac{3}{2},\quad B=\frac{36}{35},\quad C=\frac{3}{2}\log 3+$ constant (b) $A=\frac{3}{2},\quad B=\frac{35}{36},\quad C=\frac{3}{2}\log 3+$ constant

(c) $A=-\frac{3}{2},\quad B=-\frac{35}{36},\quad C=-\frac{3}{2}\log 3+$ constant (d) None of these

294. $\int \frac{x}{x^4-1}dx=$

(a) $\frac{1}{4}\log\left[\frac{x^2-1}{x^2+1}\right]+c$ (b) $\frac{1}{4}\log\left[\frac{x^2+1}{x^2-1}\right]+c$ (c) $\frac{1}{2}\log\left[\frac{x^2-1}{x^2+1}\right]+c$ (d) $\frac{1}{2}\log\left[\frac{x^2+1}{x^2-1}\right]+c$

295. $\int \frac{dx}{\sin x+\sin 2x}=$ **[IIT 1984; J & KCET 1995]**

(a) $\frac{1}{6}\log(1-\cos x)+\frac{1}{2}\log(1+\cos x)-\frac{2}{3}\log(1+2\cos x)+c$ (b) $6\log(1-\cos x)+2\log(1+\cos x)-\frac{2}{3}\log(1+2\cos x)+c$

(c) $6\log(1-\cos x)+\frac{1}{2}\log(1+\cos x)+\frac{2}{3}\log(1+2\cos x)+c$ (d) None of these

296. The value of $\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx$ is

(a) $\sin x-6\tan^{-1}(\sin x)+c$ (b) $\sin x-2(\sin x)^{-1}+c$

(c) $\sin x-2(\sin x)^{-1}-6\tan^{-1}(\sin x)+c$ (d) $\sin x-2(\sin x)^{-1}+5\tan^{-1}(\sin x)+c$

Reduction formulae for some Special cases, Integration of form $\int \sin^m x\cos^n x\,dx,\ \int \sin mx\cos nx\,dx,\ \int \sin mx\sin nx\,dx,\ \int \cos mx\cos nx\,dx$

Basic Level

297. $\int \tan^4 x\,dx=$

(a) $\tan^3 x-\tan x+x+c$ (b) $\frac{1}{3}\tan^3 x-\tan x+x+c$ (c) $\frac{1}{3}\tan^3 x+\tan x+x+c$ (d) $\frac{1}{3}\tan^3 x+\tan x+2x+c$

298. The value of $\int \sec^3 x\,dx$ will be **[UPSEAT 1999]**

(a) $\frac{1}{2}[\sec x\tan x+\log(\sec x+\tan x)]$ (b) $\frac{1}{3}[\sec x\tan x+\log(\sec x+\tan x)]$

(c) $\frac{1}{4}[\sec x\tan x+\log(\sec x+\tan x)]$ (d) $\frac{1}{8}[\sec x\tan x+\log(\sec x+\tan x)]$

299. $\int \sec^{2/3}x\,.\,\text{cosec}^{4/3}x\,dx=$

(a) $-3(\tan x)^{1/3}+c$ (b) $-3(\tan x)^{-1/3}+c$ (c) $3(\tan x)^{-1/3}+c$ (d) $(\tan x)^{-1/3}+c$

300. $\int \sin^4 x\cos^3 x\,dx=$ **[CBSE 1985]**

(a) $\frac{1}{5}\sin^5 x+\frac{1}{7}\sin^7 x+c$ (b) $\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+c$ (c) $-\frac{1}{5}\sin^5 x+\frac{1}{7}\sin^7 x+c$ (d) None of these

301. $\int \sin^3 x\,dx$ is equal to **[SCRA 1996]**

(a) $\sin^2 x+1$ (b) $\sin x^2+x^2+1$ (c) $\frac{\cos^3 x}{3}-\cos x$ (d) $\frac{1}{4}\sin^4 x-\frac{3}{4}\sin^2 x$

302. Which value of constant not integration makes the value of integral of $\sin 3x.\cos 5x$ equal to zero at $x=0$ **[Roorkee 1971]**

(a) 0 (b) –3/16 (c) –5/6 (d) 1/8

303. $\int \sin 2x.\sin 3x\,dx$ equals **[Rajasthan PET 1989]**

(a) $\frac{(\sin x - \sin 5x)}{2} + c$ (b) $\frac{(\sin x - \sin 5x)}{10} + c$ (c) $\frac{(5\sin x - \sin 5x)}{10} + c$ (d) None of these.

304. $\int \cos 2x . \sin 4x dx$ equals

(a) $\frac{\cos 2x}{2} + \frac{\cos 6x}{6} + c$ (b) $-\left(\frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right) + c$ (c) $-\left(\frac{\cos 2x}{4} + \frac{\cos 6x}{12}\right) + c$ (d) None of these.

305. $\int \frac{\sin 5x}{\cos 7x . \cos 2x} dx$ is equal to

(a) $\log|\sec 7x| + c$ (b) $\log|\sec 7x . \sec 2x| + c$ (c) $\log|\sec 7x + \sec 2x| + c$ (d) None of these

306. $\int \cos^5 x \, dx =$

(a) $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$ (b) $\sin x + \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$

(c) $\sin x - \frac{2}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$ (d) None of these

Advance Level

307. If $f(x) = \int \cot^4 x dx + \frac{1}{3}\cot^3 x - \cot x$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ then $f(x) =$

(a) $\frac{\pi}{2} - x$ (b) $x - \pi$ (c) $\pi - x$ (d) x

Integration of Hyperbolic Functions

Basic Level

308. $\int \frac{dx}{1 + \cosh x} =$

(a) $\cot h\left(\frac{x}{2}\right) + c$ (b) $\tan h\left(\frac{x}{2}\right) + c$ (c) $\frac{1}{2}\tan h\left(\frac{x}{2}\right) + c$ (d) None of these

309. $\int \frac{dx}{x\sqrt{(\log x)^2 - 3}}$ is equal to

(a) $\sin \text{h}^{-1}\left(\log \frac{x}{\sqrt{3}}\right) + c$ (b) $\cos \text{h}^{-1}\left(\log \frac{x}{\sqrt{3}}\right) + c$ (c) $\cos \text{h}^{-1}\left(\log \frac{x}{\sqrt{2}}\right) + c$ (d) None of these

310. $\int \frac{(e^x + e^{-x})^2}{(e^x - e^{-x})^2} dx$ is equal to

(a) $2\log(e^x - e^{-x}) + c$ (b) $2\log(e^x + e^{-x}) + c$ (c) $x + \cot \text{h} x + c$ (d) $x - \cot \text{h} x + c$

311. $\int \frac{1}{x}\sqrt{\frac{x-1}{x+1}} dx$ equals

(a) $\cos \text{h}^{-1} x + \sec^{-1} x + c$ (b) $\sin \text{h}^{-1} x - \sec^{-1} x + c$ (c) $\cos \text{h}^{-1} x - \sec^{-1} x + c$ (d) $\sin h^{-1} x + \sec^{-1} x + c$

312. $\int \sqrt{\sec x - 1}\,dx$ is equal to

(a) $2\sin h^{-1}\left\{\sqrt{2}\cos\left(\frac{x}{2}\right)\right\}+c$ (b) $-2\sin h^{-1}\left\{\sqrt{2}\cos\left(\frac{x}{2}\right)\right\}+c$ (c) $-2\cos h^{-1}\left\{\sqrt{2}\cos\left(\frac{x}{2}\right)\right\}+c$ (d) None of these

313. $\int e^x(\sin hx + \cos hx)\,dx$ is equal to **[Karnataka CET 1993]**

(a) $e^x \sec hx + c$ (b) $e^x \cos hx + c$ (c) $\sin h\,2x + c$ (d) $\cos h\,2x + c$

Integration of Surds like Expression

Basic Level

314. $\int \frac{x^{5/2}}{\sqrt{1+x^7}}\,dx$ is

(a) $\frac{2}{7}\log(x^{7/2}+\sqrt{x^7+1})+c$ (b) $\frac{1}{2}\log\frac{x^7+1}{x^7-1}+c$ (c) $2\sqrt{1+x^7}+c$ (d) None of these

315. $\int \frac{dx}{x^{1/5}(1+x^{4/5})^{1/2}}$ is

(a) $\sqrt{1+x^{4/5}}+k$ (b) $\frac{5}{2}\sqrt{1+x^{4/5}}+k$ (c) $x^{4/5}(1+x^{4/5})^{1/2}+k$ (d) None of these

316. $\int x^{-\frac{2}{3}}(1+x^{\frac{1}{2}})^{-\frac{5}{3}}\,dx$ is equal to

(a) $3(1+x^{-1/2})^{-1/3}+c$ (b) $3(1+x^{-1/2})^{-2/3}+c$ (c) $3(1+x^{1/2})^{-2/3}+c$ (d) None of these

317. The value of $\int \frac{(x-x^3)^{1/3}}{x^4}dx$ is

(a) $\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{\frac{4}{3}}+c$ (b) $-\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{\frac{4}{3}}+c$ (c) $\frac{1}{8}\left(1-\frac{1}{x^2}\right)^{\frac{4}{3}}+1$ (d) None of these

318. $\int \frac{(x^4-x)^{1/4}}{x^5}dx$ is equal to

(a) $\frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+c$ (b) $\frac{4}{5}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+c$ (c) $\frac{4}{15}\left(1+\frac{1}{x^3}\right)^{\frac{5}{4}}+c$ (d) None of these

319. If $\int \frac{1}{x\sqrt{1-x^3}}\,dx = a\log\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|+b$, then a is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{1}{3}$ (d) $-\frac{2}{3}$

Advance Level

320. $\int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}dx$ is equal to

(a) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}}+c$ (b) $\frac{4}{3}\left(\frac{x+2}{x-1}\right)^{\frac{1}{4}}+c$ (c) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}}+c$ (d) $\frac{1}{3}\left(\frac{x+2}{x-1}\right)^{\frac{1}{4}}+c$

321. $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}dx$ is equal to

(a) $\frac{1}{2}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right)$ (b) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1+x^2}}\right)$ (c) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right)$ (d) None of these

322. Let $f(x)=\int \frac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0)=0$, then $f(1)$ is

(a) $\log(1+\sqrt{2})$ (b) $\log(1+\sqrt{2})-\frac{\pi}{4}$ (c) $\log(1+\sqrt{2})+\frac{\pi}{4}$ (d) None of these

323. Let $\int \frac{x^{1/2}}{\sqrt{1-x^3}}dx=\frac{2}{3}gof(x)+c$, then

(a) $f(x)=\sqrt{x}$ (b) $f(x)=x^{3/2}$ (c) $f(x)=x^{2/3}$ (d) $g(x)=\sin^{-1}x$

(e) Both b and d

324. $\int \frac{dx}{\sqrt{x+x\sqrt{x}}}$ is equal to

(a) $\log\sqrt{x+x\sqrt{x}}+c$ (b) $\sqrt{1+\sqrt{x}}+c$ (c) $4\sqrt{1+\sqrt{x}}+c$ (d) None of these

325. $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx=$

(a) $\sqrt{x^2+\frac{1}{x^2}+1}$ (b) $\frac{\sqrt{x^4+x^2+1}}{x}$ (c) $\sqrt{\frac{x^4+x^2+1}{x}}$ (d) Both a and b

326. $\int \frac{dx}{(x+\alpha)^{8/7}(x-\beta)^{6/7}}=$

(a) $\frac{6}{\alpha+\beta}\left(\frac{x-\beta}{x+\alpha}\right)^{\frac{1}{6}}$ (b) $\frac{6}{\alpha+\beta}\left(\frac{x+\alpha}{x-\beta}\right)^{\frac{1}{6}}$ (c) $\frac{7}{\alpha+\beta}\left(\frac{x+\alpha}{x-\beta}\right)^{\frac{1}{7}}$ (d) $\frac{7}{\alpha+\beta}\left(\frac{x-\beta}{x+\alpha}\right)^{\frac{1}{7}}$

327. $\int \frac{xdx}{(x^2+1)^{4/5}(x^2+2)^{6/5}}=$

(a) $\frac{2}{5}\left(\frac{x^2+2}{x^2+1}\right)^{\frac{1}{5}}$ (b) $\frac{5}{2}\left(\frac{x^2+2}{x^2+1}\right)^{-\frac{1}{5}}$ (c) $\frac{5}{2}\left(\frac{x^2+1}{x^2+2}\right)^{\frac{1}{5}}$ (d) Both (b) and (c)

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Answer Sheet

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
d	b	a	b	b	c	b	c	c	b	a	c	c	c	a	a	a	a	d	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
b	b	b	a	c	a	d	c	d	d	d	a	a	b	d	b	d	a	b	d
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
a	b	a	a	c	a	c	d	a	c	c	b	c	a	d	d	b	a	c	b
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
c	a	b	c	b	c	a	b	b,c	b	c	c	c	a	c	a	b	d	a	a
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
b	c	c	c	b	a	d	b	d	b	c	c	a	a	b	a	b	a	b	c
101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
c	b	a	a	a	b	b	b	a	a	b	a	d	a	c	a	a	c	a	b
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140
c	b	b	c	a	b	b	b	b	d	d	d	a	a	b	a	a	c	a	c
141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160
d	b	a	c	c	c	a	b	b	c	b	a	a	b	c	d	b	a	b	b
161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
b	c	a	a	b	a	a	a	b	b	b	a	a	c	a	a	b	c	d	d
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200
b	b	b	d	c	b	b	c	c	b	a	c	c	b	d	a	b	a	c	c
201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220
a	c	a	a	b	c	a	a	a	c	d	d	c	b	b	a	a	d	d	b
221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
d	d	c	a	c	a	a	b	d	b,d	a	a	b	b	b	a	c	d	a	c
241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260
b	a	a	a	a	b	b	d	c	a	a	c	b	a	a	a	a	a	a	b
261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280
b	b	a	c	c	b	d	d	a	a	a	a	a	b	a	d	c	b	a	c
281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299	300
a	d	b	c	b	b	d	b	a	b	c	a	d	a	a	c	b	a	b	b
301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320
c	b	c	c	d	a	d	b	b	d	c	c	b	a	b	b	b	a	a	a
321	322	323	324	325	326	327													
c	b	e	c	d	d	d													