MISCELLANEOUS EXERCISE ON CHAPTER 7.

Q No 1 $\int \frac{1}{x-x^3}dx = \int \frac{1}{x(1-x^2)}dx = \int \frac{x}{x^2(1-x^2)}dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore I = \int \frac{dt/2}{t(1-t)} = \frac{1}{2}\int\left(\frac{1}{t} + \frac{1}{1-t}\right)dt$$

$$= \frac{1}{2}\left(\log|t| + \frac{\log|1-t|}{-1}\right) + C = \frac{1}{2}\log\left|\frac{t}{1-t}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

Q No 2 $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}dx.$

$$= \frac{1}{a-b}\int\left((x+a)^{1/2} - (x+b)^{1/2}\right)dx$$

$$= \frac{1}{a-b}\left(\frac{(x+a)^{3/2}}{3/2} - \frac{(x+b)^{3/2}}{3/2}\right) + C$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$$

Q No 3. $\int \frac{1}{x\sqrt{ax-x^2}}dx =$

Sol. Put $x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2}dt$

$$\therefore I = \int \frac{1}{\frac{a}{t}\sqrt{a\left(\frac{a}{t}\right)-\frac{a^2}{t^2}}}\left(-\frac{a}{t^2}\right)dt = \int \frac{-a}{a\cdot a\sqrt{t-1}}dt$$

$$= -\frac{1}{a}\int (t-1)^{-1/2}dt = -\frac{1}{a}\frac{(t-1)^{1/2}}{1/2} + C = -\frac{2}{a}\sqrt{t-1} + C$$

$$= -\frac{2}{a}\sqrt{\frac{a}{x}-1} + C = -\frac{2}{a}\sqrt{\frac{a-x}{x}} + C \quad \left[\because x = a/t \Rightarrow t = a/x\right]$$

Q No 4 $\int \frac{1}{x^2(x^4+1)^{3/4}}dx = \int \frac{1}{x^2\left(x^4\left(1+\frac{1}{x^4}\right)\right)^{3/4}}dx.$

$$= \int \frac{1}{x^2\cdot x^3(1+x^{-4})^{3/4}}dt$$

$$= \int x^{-5}(1-x^{-4})^{-3/4}\,dx = -\frac{1}{4}\int (1+x^{-4})^{-3/4}(-4x^{-5})\,dx$$

$$= -\frac{1}{4}\cdot\frac{(1+x^{-4})^{-3/4+1}}{-3/4+1} + C \qquad \left((f(x))^n f'(x) \right) \text{ type}$$

$$= -(1+x^{-4})^{1/4} + C$$

Q No 5 $\int \frac{1}{x^{1/2}+x^{1/3}}\,dx$. Put $x = t^6 \Rightarrow dx = 6t^5 dt$

$$I = \int \frac{1}{(t^6)^{1/2}+(t^6)^{1/3}}\cdot 6t^5 dt = 6\int \frac{t^5}{t^3+t^2}\,dt = 6\int \frac{t^3}{t+1}\,dt$$

$$= 6\int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt \qquad [\text{By Long Division}]$$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1| \right] + C$$

$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - \log|x^{1/6}+1| + C \qquad [\because x = t^6 \Rightarrow t = x^{1/6}]$$

Q No 6 $\int \frac{5x}{(x+4)(x^2+9)}\,dx$.

Sol. Let $\frac{5x}{(x+4)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$

$\Rightarrow \quad 5x = A(x^2+9) + (Bx+C)(x+1) \quad -(1)$

$\Rightarrow \quad 5x = A(x^2+9) + Bx^2 + Bx + Cx + C$

$\Rightarrow \quad 5x = (A+B)x^2 + (B+C)x + (9A+C) \quad -(11)$

When $x+1 = 0$ ie $x = -1$.

From (1) $\quad 5(-1) = A(1+9) \Rightarrow A = -1/2$

from (11) $\quad A+B = 0 \Rightarrow B = -A = 1/2$

and $\quad 9A + C = 0 \Rightarrow C = -9A = 9/2$.

$$\therefore \int \frac{5x}{(x+4)(x^2+9)}\,dx = \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x - \frac{9}{2}}{x^2+9} \right) dx.$$

$$= -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \left(\frac{x+9}{x^2+9} \right) dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{2}\cdot\frac{1}{2}\int \frac{2x}{x^2+9}\,dx + \frac{9}{2}\int \frac{1}{x^2+9}\,dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}\times\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c.$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\left(\frac{x}{3}\right) + c$$

QNo 7 $\int \frac{\sin x}{\sin(x-\alpha)}dx = \int \frac{\sin(x+\alpha-\alpha)}{\sin(x-\alpha)}dx = \int \frac{\sin(x-\alpha)\cos\alpha + \cos(x-\alpha)\sin\alpha}{\sin(x-\alpha)}dx$

$$= \int(\cos\alpha + \sin\alpha\cot(x-\alpha))dx = \cos\alpha\int 1\cdot dx + \sin\alpha\int\cot(x-\alpha)dx.$$

$$= \cos\alpha\cdot x + \sin\alpha\log|\sin(x-\alpha)| + c$$

QNo 8 $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int \frac{e^{\log x^5} - e^{\log x^4}}{e^{\log x^3} - e^{\log x^2}}dx = \int \frac{x^5 - x^4}{x^3 - x^2}dx.$

$$= \int \frac{x^4(x-1)}{x^2(x-1)}dx = \int x^2 dx = \frac{x^3}{3} + c.$$

QNo 9. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx.$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{4-t^2}} = \sin^{-1}\left(\frac{t}{2}\right) + c = \sin^{-1}\left(\frac{\sin x}{2}\right) + c$$

QNo 10 $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}dx.$

Sol.

$$\sin^8 x - \cos^8 x = (\sin^4 x)^2 - (\cos^4 x)^2 = (\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)$$

$$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)$$

$$= -(\cos^2 x - \sin^2 x)\times 1\times(\sin^4 x + \cos^4 x)$$

$$= -\cos 2x(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x - 2\sin^2 x\cos^2 x)$$

$$= -\cos 2x((\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x)$$

$$= -\cos 2x(1 - 2\sin^2 x\cos^2 x)$$

$$\therefore I = \int \frac{-\cos 2x(1-2\sin^2 x\cos^2 x)}{(1-2\sin^2 x\cos^2 x)}dx = -\int\cos 2x\, dx.$$

$$= -\frac{\sin 2x}{2} + c$$

QNo 11. $\int \frac{1}{\cos(x+a)\cos(x+b)}dx = \frac{1}{\sin(a-b)}\int \frac{\sin(x+a) - \sin(x+b)}{\cos(x+a)\cos(x+b)}dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}dx$$

$$= \frac{1}{\sin(a-b)} \int \left(\tan(x+a) - \tan(x+b) \right) dx$$

$$= \frac{1}{\sin(a-b)} \left(-\log|\cos(x+a)| + \log|\cos(x+b)| \right) + C$$

$$= \frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C$$

Q No. 12 $\int \frac{x^3}{\sqrt{1-x^8}} dx$

Put $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\therefore I = \int \frac{dt/4}{\sqrt{1-t^2}} = \frac{1}{4} \int \frac{1}{\sqrt{1-t^2}} dt = \frac{1}{4} \sin^{-1}(t) + C$$

$$= \frac{1}{4} \sin^{-1}(x^4) + C$$

Q No 13. $\int \frac{e^x}{(1+e^x)(2+e^x)} dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\therefore I = \int \frac{dt}{(1+t)(2+t)} = \int \left(\frac{1}{(1+t)(2+(-1))} + \frac{1}{(1+(-2))(2+t)} \right) dt$$

$$= \int \left(\frac{1}{1+t} - \frac{1}{2+t} \right) dt = \log|1+t| - \log|2+t| + C$$

$$= \log \left| \frac{1+t}{2+t} \right| + C = \log \left| \frac{1+e^x}{2+e^x} \right| + C$$

Q No 14. $\int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3} \int \frac{(x^2+4)-(x^2+1)}{(x^2+1)(x^2+4)} dx$

$$= \frac{1}{3} \int \left(\frac{1}{x^2+1} \right) dx - \frac{1}{3} \int \frac{1}{x^2+4} dx = \frac{1}{3} \left[\tan^{-1} x + \tan^{-1} \frac{x}{2} \right] + C$$

Q No. 15 $\int \cos^3 x . e^{\log \sin x} dx = \int \cos^3 x . \sin x \, dx$

$$= -\int (\cos x)^3 (-\sin x) \, dx = -\frac{\cos^4 x}{4} + C$$

Q No 16 $\int e^{3\log x} (x^4+1)^{-1} dx = \int e^{\log x^3} (x^4+1)^{-1} dx$

$$= \int x^3 (x^4+1)^{-1} dx = \frac{1}{4} \int \frac{4x^3}{x^4+1} dx$$

$$= \frac{1}{4} \log|x^4+1| + C \quad \left(\frac{f'(x)}{f(x)} \text{ type} \right)$$

Q No. 17 $\int [f(ax+b)]^n f'(ax+b)\,dx$

Sol. Substitute $f(ax+b) = t \Rightarrow f'(ax+b).a\,dx = dt$

$$\therefore I = \int (t)^n \frac{dt}{a} = \frac{1}{a}\cdot\frac{t^{n+1}}{n+1} + c = \frac{1}{a}\frac{(f(ax+b))^{n+1}}{n+1} + c \quad n \neq -1$$

If $n = -1$, then $I = \frac{1}{a}\int \frac{1}{t}\,dt = \frac{1}{a}\log|t| = \frac{1}{a}|f(ax+b)| + c$

Q No 18 $\int \frac{dx}{\sin^3 x \sin(x+\alpha)} = \int \sqrt{\frac{\sin x}{\sin^4 x \sin x+\alpha}}\,dx$

$$= \int \sqrt{\frac{\sin x}{\sin(x+\alpha)}}\cdot\frac{1}{\sin^2 x}\,dx$$

Now Put $\sqrt{\frac{\sin(x+\alpha)}{\sin(x)}} = t \Rightarrow \frac{\sin(x+\alpha)}{\sin(x)} = t^2$

$$\Rightarrow \frac{\sin x \cos(x+\alpha) - \cos x \sin(x+\alpha)}{\sin^2 x} = 2t\,dt$$

$$\Rightarrow \frac{\sin[x-(x+\alpha)]}{\sin^2 x}\,dx = 2t\,dt$$

$$\Rightarrow \frac{\sin(-\alpha)}{\sin^2 x}\,dx = 2t\,dt$$

$$\Rightarrow -\frac{\sin\alpha}{\sin^2 x}\,dx = 2t\,dt.$$

$$\therefore I = \int \frac{1}{t}\frac{2t}{-\sin\alpha}\,dt = -\frac{2}{\sin\alpha}\times\int 1\,dt$$

$$= \left(-\frac{2}{\sin\alpha}\right)(t) + c = \frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + c$$

Q No 19 $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$; $x \in [0,1]$

Sol. We know that $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \pi/2$

$$\Rightarrow \cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

$$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\pi/2}\,dx$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \pi/2\right)dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int 1\cdot dx \quad \frac{4}{\pi}I_1 - x + C$$

Let $I_1 = \int \sin^{-1}(\sqrt{x})\,dx$

Put $\sin^{-1}\sqrt{x} = \theta \Rightarrow \sqrt{x} = \sin\theta \Rightarrow (\sqrt{x})^2 = \sin^2\theta \Rightarrow x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\cdot d\theta = \sin 2\theta\, d\theta$

$\therefore I_1 = \int \theta\cdot\sin 2\theta\, d\theta$

Integrating by parts taking θ as first function.

$$I_1 = \theta\cdot\left(\frac{-\cos 2\theta}{2}\right) - \int\left(-\frac{\cos 2\theta}{2}\right)d\theta$$

$$= -\frac{\theta}{2}\cdot\cos 2\theta + \frac{1}{2}\int\cos 2\theta\, d\theta$$

$$= -\frac{1}{2}\theta\cdot\cos 2\theta + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) = -\frac{1}{2}\theta\left[1 - 2\sin^2\theta\right] + \frac{1}{4}\left[2\sin\theta\cos\theta\right]$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}\left(1 - 2x\right) + \frac{1}{4}\left(2\sqrt{x}\cdot\sqrt{1-\sqrt{x}^2}\right) -$$

$$= -\frac{1}{2}\sin^{-1}(\sqrt{x})(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x} + C$$

$$\therefore I = I_1 - x = \frac{4}{\pi}\left(\frac{1}{2}\sin^{-1}\sqrt{x}(2x-1) + \frac{1}{2}\sqrt{x}\sqrt{1-x}\right) + C$$

$$= \frac{2}{\pi}\sin^{-1}\sqrt{x}\,(2x-1) + \sqrt{x-x^2} + C$$

Q No. 20 $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Sol. Put $\sqrt{x} = \cos t \Rightarrow x = \cos^2 t \Rightarrow dx = -2\sin t\cos t\, dt$

$$\therefore I = \int\sqrt{\frac{1-\cos t}{1+\cos t}}\cdot(-2\sin t\cos t)\,dt$$

$$= -2\int\sqrt{\frac{2\sin^2 t/2}{2\cos^2 t/2}}\left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)\cos t\,dt$$

$$= -4\int\sin^2\frac{t}{2}\cos t\,dt = -4\int\frac{1-\cos t}{2}\cdot\cos t\,dt$$

$$= -2\int\left(\cos t - \cos^2 t\right)dt = -2\int\left(\cos t - \frac{1+\cos 2t}{2}\right)dt$$

$$= -2\sin t + t + \frac{\sin 2t}{2} + C$$

$$= -2\sqrt{1-\cos^2 t} + t + \cos t\sqrt{1-\cos^2 t} + C = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

Q No 21 $\int \frac{2+\sin 2x}{1+\cos 2x} e^x \, dx = \int \left(\frac{2(1+2\sin x \cos x)}{2\cos^2 x}\right) e^x \, dx$

$= \int \left(\frac{1}{\cos^2 x} \tan x\right) e^x \, dx = \int (\tan + \sec^2 x) e^x \, dx$

$= \int e^x \left(\tan x + \frac{d}{dx}(\tan x)\right) dx = e^x \tan x + C$

Q No. 22 $\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx = \int$

Now $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$

$\Rightarrow \quad x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2 \quad \cdots (i)$

$\Rightarrow \quad x^2+x+1 = A(x^2+3x+2) + Bx + 2B + C(x^2+1+2x)$

$\Rightarrow \quad x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C) \quad \cdots (ii)$

From (i) when $x+1=0 \Rightarrow x=-1$

$(-1)^2+1-1 = B(-1+2) \Rightarrow B = 1$

Now from (ii) $A+C=1$

$3A+B+2C=1 \Rightarrow 3A+2C=0$

$\therefore \quad 3A+2C=0$

$3A+3C=3$

$-C=-3 \Rightarrow C=3$

$\therefore \quad A+C=1 \Rightarrow A=-2.$

$\therefore \quad I = \int \left(\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\right) dx$

$= -2\log|x+1| + 3\log|x+2| + \frac{(x+1)^{-2+1}}{-2+1} + C$

$= \log\left[\frac{|x+2|^3}{|x+1|^2}\right] - \frac{1}{x+1} + C$

Q No 23. $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}} \, dx$

Sol. Put $x = \cos\theta \Rightarrow dx = -\sin\theta \, d\theta$

$\therefore \quad I = \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} (-\sin\theta \, d\theta) = \int \tan^{-1}\sqrt{\left(\frac{2\sin^2\theta/2}{2\cos^2\theta/2}\right)} d\theta \, (-\sin\theta)$

$$= \int \tan^{-1}\left(\tan\frac{\theta}{2}\right)(-\sin\theta)\,d\theta = -\int \frac{\theta}{2}\sin\theta\,d\theta$$

$$= \frac{1}{2}\int \theta(-\sin\theta)\,d\theta = \frac{1}{2}\left[\theta\cdot\cos\theta - \int 1\cdot(\cos\theta)\,d\theta\right]$$

$$= \frac{1}{2}\left[\theta\cdot\cos\theta\right] - \frac{1}{2}\sin\theta + c = \frac{1}{2}(\cos^{-1}x)x - \frac{1}{2}\sin(\cos^{-1}x) + c.$$

$$= \frac{1}{2}x\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + c.$$

Q No. 24. $\displaystyle\int \frac{\sqrt{x^2+1}\left[\log(x^2+1) - 2\log x\right]}{x^4}\,dx.$

Sol..

$$I = \frac{\sqrt{x^2+1}}{x^4}\int\left(\log(x^2+1) - \log x^2\right)dx = \int \frac{\sqrt{x^2+1}}{x}\left(\log\left(\frac{x^2+1}{x^2}\right)\right)\frac{1}{x^3}\,dx$$

$$= \int \sqrt{\frac{x^2+1}{x^2}}\left(\log\left(\frac{x^2+1}{x^2}\right)\right)\frac{1}{x^3}\,dx.$$

Put $\dfrac{x^2+1}{x^2} = t \Rightarrow 1 + \dfrac{1}{x^2} = t \Rightarrow -\dfrac{2}{x^3}\,dx = dt.$

$$I = \int \sqrt{t}\,\log t\left(-\frac{1}{2}dt\right) = -\frac{1}{2}\int(\log t)\,t^{1/2}\,dt$$

$$= -\frac{1}{2}\left[(\log t)\frac{t^{3/2}}{3/2} - \int \frac{1}{t}\times\frac{t^{3/2}}{3/2}\,dt\right] = -\frac{1}{3}\left[t^{3/2}\log t - \int t^{1/2}\,dt\right]$$

$$= -\frac{1}{3}\left[t^{3/2}\log t - \frac{t^{3/2}}{3/2}\right] + c = -\frac{1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right] + c$$

$$= -\frac{1}{3}\frac{(x^2+1)^{3/2}}{x^3}\left[\log\left(\frac{x^2+1}{x^2}\right) - \frac{2}{3}\right] + c$$

Q No 25. $\displaystyle\int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx = \int_{\pi/2}^{\pi} e^x\left(\frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\left(\frac{x}{2}\right)}\right)dx.$

$$= \int_{\pi/2}^{\pi} e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx = \int_{\pi/2}^{\pi} e^x\left(-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)dx$$

$$= \left[e^x\left(-\cot\frac{x}{2}\right)\right]_{\pi/2}^{\pi} = \left(-e^{\pi}\cot\frac{\pi}{2}\right) - \left(-e^{\pi/2}\cot\left(\frac{\pi}{4}\right)\right)$$

$$= -e^{\pi}\times 0 + e^{\pi/2}\times 1 = e^{\pi/2}.$$

Q.No. 26 $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$

Sol. Put $\sin^2 x = t \Rightarrow 2\sin t \cos t\, d\theta = dt$, When $x=0,\ t=0$; when $x = \pi/4$, $t = \frac{1}{2}$

$$I = \int_0^{1/2} \frac{dt/2}{(1-t)^2 + t^2} = \frac{1}{2}\int_0^{1/2} \frac{dt}{2t^2 - 2t + 1} = \frac{1}{2} \times \frac{1}{2}\int_0^{1/2} \frac{dt}{t^2 - t + \frac{1}{2}} = \frac{1}{4}\int_0^{1/2} \frac{dt}{t^2 - t + \frac{1}{4} + \frac{1}{4}}$$

$$= \frac{1}{4}\int_0^{1/2} \frac{dt}{(t-\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{1}{4} \times \frac{1}{1/2}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{1/2}\right)\right]_0^{1/2} = \frac{1}{2}\left\{\tan^{-1} 0 - \tan^{-1}(-1)\right\}$$

$$= \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}$$

Q.No 27 $\displaystyle I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x / \cos^4 x}{\frac{\cos^2 x}{\cos^4 x} + 4\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^2 x}}\, dx \quad \left(\begin{array}{l}\text{Dividing Numerator and} \\ \text{denominator by } \cos^4 x\end{array}\right)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\, dx}{\sec^2 x + 4\tan^2 x \sec^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

When $x = 0$, $t = 0$ and when $x \to \frac{\pi}{2}$, $t \to \infty$

$$I = \int_0^{\infty} \frac{dt}{(1+t^2) + 4t^2(1+t^2)} = \int_0^{\infty} \frac{dt}{(1+t^2)(1+4t^2)} = \int_0^{\infty}\left(\frac{-1/3}{1+t^2} + \frac{4/3}{1+4t^2}\right) dt \quad \left[\begin{array}{l}\text{Partial} \\ \text{fractions}\end{array}\right]$$

$$= -\frac{1}{3}\left[\tan^{-1} t\right]_0^{\infty} + \frac{4}{3} \times \frac{1}{4}\int_0^{\infty} \frac{dt}{\frac{1}{4} + t^2}$$

$$= -\frac{1}{3}\left(\frac{\pi}{2} - 0\right) + \frac{1}{3} \times \frac{1}{1/2}\left[\tan^{-1}\left(\frac{t}{1/2}\right)\right]_0^{\infty} = -\frac{\pi}{6} + \frac{2}{3}\left(\lim_{t\to\infty} \tan^{-1}(2t) - 0\right)$$

$$= -\frac{\pi}{6} + \frac{2}{3}\left(\frac{\pi}{2} - 0\right) = -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}.$$

Q.No 28 $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$ $\quad$ $\displaystyle I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x)\, dx = dt$

When $x = \pi/6$; $t = \sin \pi/6 - \cos \pi/6 = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$

When $x = \frac{\pi}{3}$; $t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$.

Also $\sin x + \cos x = t$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x\cos x = t^2$

$\Rightarrow 1 + \sin 2x = t^2 \Rightarrow \sin 2x = t^2 - 1$

$$\therefore I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t\right]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) + C$$

QNo.29. $$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}} = \int_0^1 \frac{\sqrt{1+x} + \sqrt{x}}{(\sqrt{1+x}-\sqrt{x})(\sqrt{1+x}+\sqrt{x})} = \int_0^1 \left(\frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}\right) dx$$

$$= \int_0^1 \left((1+x)^{1/2} + x^{1/2}\right) dx = \left[\frac{(1+x)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2}\right]_0^1$$

$$= \frac{2}{3}\left[(1+x)^{3/2} + (x)^{3/2}\right]_0^1 = \frac{2}{3}\left[2^{3/2} + 1^{3/2} - (1+0)\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

QNo.30. $$\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx.$$

Sol. Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

and $(\sin x - \cos x)^2 = t^2 \Rightarrow \sin 2x = 1 - t^2$ (As in above question)

when $x = 0$, $t = -1$ when $x = \pi/4$ $t = 0$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx = \int_{-1}^{0} \frac{dt}{9 + 16(1-t^2)} = \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{16} \times \frac{1}{2 \times \frac{5}{4}}\left[\log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0} = \frac{1}{40}\left(\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{40}\left(\log 1 - \log\frac{1}{9}\right) = \frac{1}{40}\left[\log 1 - \log 1 + \log 9\right] = \frac{1}{40}\log 9.$$

QNo.31. $$\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx. = \int_0^{\pi/2} 2\sin x\cos x \tan^{-1}(\sin x)\, dx$$

Sol. Put $\sin x = t \Rightarrow \cos x\, dx = dt$

when $x = 0$, $t = \sin 0 = 0$

when $x = \pi/2$, $t = \sin \pi/2 = 1$

$$\therefore I = \int_0^1 2.t.\tan^{-1}(t)\, dt = 2\int_0^1 t.\tan^{-1} t\, dt.$$

$$= 2\left[\tan^{-1}(t).\frac{t^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1+t^2}.\frac{t^2}{2}\, dt \qquad \left[\because \tan^{-1} t \text{ as first function.}\right.$$

$$= 2\left(\frac{\tan^{-1}(1)}{2} - 0\right) - \int_0^1 \frac{t^2 dt}{1+t^2} = 2\left(\frac{\pi}{8}\right) - \int_0^1 \frac{(1+t^2)-1}{1+t^2} dt$$

$$= \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt = \frac{\pi}{4} - \left[t - \tan^{-1}t\right]_0^1 = \frac{\pi}{4} - \left[(1-\tan^{-1}1)-(0-0)\right].$$

$$= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

Q.No 32 $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ — (1)

Sol. Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx = \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-(\sec x + \tan x)} dx$$

$$= \int_0^{\pi} \frac{\pi \tan x - x \tan x}{\sec x + \tan x} dx \quad \text{---(2)}$$

Adding (1) and (2)

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx = 2\pi \int_0^{\pi/2} \frac{\tan x}{\sec x + \tan x} dx \quad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ whenever } f(2a-x) = f(x)\right]$$

$$\Rightarrow I = \pi \int_0^{\pi/2} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx = \pi \int_0^{\pi/2} (\sec x \tan x - \tan^2 x)\,dx.$$

$$\therefore I = \pi \int_0^{\pi/2} (\sec x \tan x - \sec^2 x + 1)\,dx = \left[\sec x - \tan x + x\right]_0^{\pi/2}$$

$$= \pi \left\{ \lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) + \frac{\pi}{2} - \sec 0 \right\}$$

$$= \pi \lim_{x \to \pi/2^-} \left(\frac{1 - \sin x}{\cos x}\right) + \frac{\pi^2}{2} - 1 \times \pi = \pi \lim_{x \to \pi/2^-} \frac{1 - \sin^2 x}{\cos x(1 + \sin x)} + \frac{\pi^2}{2} - \pi$$

$$= \pi \times 0 + \frac{\pi^2}{2} - \pi = \frac{\pi^2}{2} - \pi.$$

Q.No 33 $\int_1^4 (|x-1| + |x-2| + |x-3|)\,dx$

Sol. $I = \int_1^2 (|x-1| + |x-2| + |x-3|)\,dx + \int_2^3 (|x-1| + |x-2| + |x-3|)\,dx$

$$+ \int_3^4 (|x-1| + |x-2| + |x-3|)\,dx$$

$$= \int_1^2 [(x-1) - (x-2) - (x-3)]\,dx + \int_2^3 [(x-1) + (x-2) - (x-3)]\,dx + \int_3^4 (x-1+x-2+x-3)\,dx$$

$$= \int_1^2 (-x+4)\,dx + \int_2^3 x\,dx + \int_3^4 (3x-6)\,dx = \left[-\frac{x^2}{2}+4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2}-6x\right]_3^4$$

$$= \left(\frac{(-2)^2}{2}+8\right) - \left(-\frac{1}{2}+4\right) + \frac{1}{2}\left(3^2-2^2\right) + \left(\frac{3}{2}\times 4^2-6\times 4\right) - \left(\frac{3}{2}\times 3^2-6\times 3\right)$$

$$= 6-\frac{7}{2}+\frac{5}{2}+(24-24)-(-9/2) = \frac{12-7+5}{2}+0+\frac{9}{2} = \frac{19}{2}.$$

Q.No 34 Prove $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3}+\log\frac{2}{3}$

Proof: Let $\frac{1}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$

$\Rightarrow \quad 1 = A(x)(x+1)+B(x+1)+Cx^2$ —(i)

$\Rightarrow \quad 1 = (A+C)x^2+(A+B)x+B.$ —(ii)

When $x+1=0$ i.e $x=-1$

$1 = C(-1)^2 \Rightarrow C=1$

From (ii) $A+C=0 \Rightarrow A=-1$

$A+B=0 \Rightarrow B=1$

$$\therefore \int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx = \left[-\log|x|-\frac{1}{x}+\log|x+1|\right]_1^3$$

$$= \left[\log\left|\frac{x+1}{x}\right|-\frac{1}{x}\right]_1^3 = \left(\log\frac{4}{3}-\frac{1}{3}\right)-\left[\left(\log\frac{2}{1}\right)-1\right]$$

$$= \left(\log\frac{4}{3}-\log 2\right)+\left(1-\frac{1}{3}\right) = \log\left(\frac{4}{3}\times\frac{1}{2}\right)+\frac{2}{3} = \log\frac{2}{3}+\frac{2}{3}$$

Q.No 35 Prove that $\int_0^1 x e^x\,dx = 1$

Proof: $\int_0^1 x e^x\,dx = \left[x\int_0^1 e^x\,dx - \int_0^1 1\cdot e^x\,dx\right]$ [Integrating by parts taking x as first fn.]

$$= \left[x e^x\right]_0^1 - \left[e^x\right]_0^1 = \left[(x-1)e^x\right]_0^1$$

$$= (1-1)e-(0-1)e^0 = 0+1 = +1$$

Q.No 36 Prove $\int_{-1}^1 x^{17}\cos^4 x\,dx = 0$

Sol. Let $f(x) = x^{17}\cos^4 x$

then $f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

$\therefore f(x)$ is odd fn.

$$\therefore \int_{-1}^1 x^{17}\cos^4 x\,dx = 0$$

Q No. 37 Prove that $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}$.

Proof: $I = \int_0^{\frac{\pi}{2}} \sin^3 x\,dx = \int_0^{\pi/2} \frac{3\sin x - \sin 3x}{4}\,dx = \frac{3}{4}\left(-\cos x\right)_0^{\frac{\pi}{2}} - \frac{1}{4}\left(\frac{-\cos 3x}{3}\right)_0^{\pi/2}$

$= -\frac{3}{4}(0-1) + \frac{1}{12}(0-1) = \frac{3}{4} - \frac{1}{12} = \frac{9-1}{12} = \frac{8}{12} = \frac{2}{3}$

Q No 38. Prove that $\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx = 1 - \log 2$.

Proof.

Proof $I = \int_0^{\frac{\pi}{4}} 2\tan^2 x \tan x\,dx = 2\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\tan x\,dx$

$= 2\left[\int_0^{\pi/4} \tan x \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan x\,dx\right] = 2\int_0^{\frac{\pi}{4}} (\tan x)^1 \sec^2 x\,dx - 2\left[-\log|\cos x|\right]_0^{\pi/4}$

$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$

$= \tan^2(\pi/4) - 0 + 2\log\left(\frac{1}{\sqrt{2}}\right) - \log 1 = 1^2 + 2\log 2^{-\frac{1}{2}} - 0$

$= 1 - 2\times\frac{1}{2}\log 2 = 1 - \log 2$

Q No 39. Prove that $\int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2} - 1$

Proof:

$I = \int_0^1 1\times \sin^{-1} x\,dx = \left[\sin^{-1} x \cdot x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

$= \frac{\pi}{2} - 0 + \frac{1}{2}\int_0^1 (1-x^2)^{-1/2}(-2x)\,dx = \frac{\pi}{2} + \frac{1}{2}\left[\frac{(1-x^2)^{1/2}}{1/2}\right]_0^1$

$= \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$

Q No 40. Evaluate $\int_0^1 e^{2-3x}\,dx$ as limit of sum.

Proof.

Sol. $\int_0^1 e^{2-3x}\,dx = \int_a^b f(x)\,dx$; $a = 0$, $b = 1$, $nh = b - a = 1 - 0 = 1$

$f(a) = f(0) = e^2$

$f(a+h) = f(h) = e^{2-3h}$

$f(a+2h) = f(2h) = e^{2-6h}$

- - -

$f(a+(n-1)h) = e^{2-3(n-1)h}$

$\therefore \int_0^1 e^{2-3x}\,dx = \lim_{h\to 0} h\left[e^2 + e^{2-3h} + e^{2-6h} + \cdots e^{2-3(n-1)h}\right]$

$$= \lim_{h\to 0} he^2\left[1+e^{-3h}+e^{-6h}+\dots \text{ up to } n \text{ terms}\right]$$

$$= \lim_{h\to 0} he^2\left[\frac{1\times(1-(e^{-3h})^n)}{1-e^{-3h}}\right] \quad ; \quad nh=1$$

$$= \lim_{h\to 0}\frac{e^2(1-e^{-3(1)})}{\frac{1-e^{-3h}}{-3h}(-3)} = \frac{e^2(1-e^{-3})}{3\lim_{h\to 0}\frac{e^{-3h}-1}{-3h}} = \frac{1}{3}(e^2-e^{-1}) \quad \left[\because \lim_{h\to 0}\frac{e^{-3h}-1}{-3h}=1\right.$$

QNo 41 $\int\frac{dx}{e^x+e^{-x}} = \int\frac{dx}{e^x+\frac{1}{e^x}} = \int\frac{e^x\,dx}{(e^x)^2+1}$

Put $e^x=t \Rightarrow e^x dx = dt$

$\therefore I = \int\frac{dt}{t^2+1} = \tan^{-1}(t)+C = \tan^{-1}(e^x)+C$

$\therefore$ Correct option is (A)

QNo 42. $\int\frac{\cos 2x}{(\sin x+\cos x)^2}dx = \int\frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2}dx$

$$= \int\frac{(\cos x-\sin x)(\cos x+\sin x)}{(\sin x+\cos x)(\sin x+\cos x)}dx = \int\frac{\cos x-\sin x}{\sin x+\cos x}dx$$

$$= \log|\sin x+\cos x|+C$$

QNo 43 If $f(a+b-x)=f(x)$, then $\int_a^b x f(x)\,dx$.

Sol. Let $I = \int_a^b x f(x)\,dx \quad -(1)$

then by a property of definite integrals

$$I = \int_a^b (a+b-x)f(a+b-x)\,dx = \int_a^b (a+b-x)f(x)\,dx \quad -(2)$$

Adding (1) and (2) we get

$$2I = \int_a^b (a+b)f(x)\,dx \Rightarrow I = \frac{a+b}{2}\int_a^b f(x)\,dx$$

Correct option is (D).

Q No 44. The Value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

Sol. Let $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx.$

$$= \int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x\cdot(1-x)}\right)dx.$$

$$= \int_0^1 \left(\tan^{-1}x - \tan^{-1}(1-x)\right)dx \quad -(1)$$

Also $I = \int_0^1 \left(\tan^{-1}(1-x) - \tan(1-(1-x))\right)dx \quad -(2)$

Adding (1) and (2) We get.

$$2I = 0$$

$$\Rightarrow I = 0$$

$\therefore$ Option B is Correct option.

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