CHAPTER 18 Definite Integrals and Applications of Integrals

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. $f(x)=\begin{vmatrix} \sec x & \cos x & \sec^2 x+\cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.

Then $\int_0^{\pi/2} f(x)\,dx = ...$ *(1987 - 2 Marks)*

2. The integral $\int_0^{1.5}[x^2]dx$, *(1988 - 2 Marks)*

Where [ ] denotes the greatest integer function, equals ....

3. The value of $\int_{-2}^{2}|1-x^2|\,dx$ is....... *(1989 - 2 Marks)*

4. The value of $\int_{\pi/4}^{3\pi/4}\frac{\phi}{1+\sin\phi}d\phi$ is .... *(1993 - 2 Marks)*

5. The value of $\int_2^3\frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is .... *(1994 - 2 Marks)*

6. If for nonzero x, $af(x)+bf\left(\frac{1}{x}\right)=\frac{1}{x}-5$ where $a \neq b$, then

$\int_1^2 f(x)\,dx = ...$ *(1996 - 2 Marks)*

7. For $n>0$, $\int_0^{2\pi}\frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx = ...$ *(1996 - 1 Mark)*

8. The value of $\int_1^{e^{37}}\frac{\pi\sin(\pi \ln x)}{x}dx$ is .... *(1997 - 2 Marks)*

9. Let $\frac{d}{dx}F(x)=\frac{e^{\sin x}}{x}, x>0$. If $\int_1^4\frac{2e^{\sin x^2}}{x}dx = F(k)-F(1)$ then one of the possible values of k is ....

(1997 - 2 Marks)

B True / False

1. The value of the integral $\int_0^{2a}\left[\frac{f(x)}{\{f(x)+f(2a-x)\}}\right]dx$ is equal to a. *(1988 - 1 Mark)*

C MCQs with One Correct Answer

1. The value of the definite integral $\int_0^1(1+e^{-x^2})\,dx$ is

(a) -1 (b) 2 *(1981 - 2 Marks)*
(c) $1+e^{-1}$ (d) none of these

2. Let a, b, c be non-zero real numbers such that

$$\int_0^1(1+\cos^8 x)(ax^2+bx+c)\,dx=\int_0^2(1+\cos^8 x)(ax^2+bx+c)\,dx.$$

Then the quadratic equation $ax^2+bx+c=0$ has *(1981 - 2 Marks)*

(a) no root in (0, 2) (b) at least one root in (0, 2)
(c) a double root in (0, 2) (d) two imaginary roots

3. The area bounded by the curves $y=f(x)$, the x-axis and the ordinates $x=1$ and $x=b$ is $(b-1)\sin(3b+4)$. Then $f(x)$ is *(1982 - 2 Marks)*

(a) $(x-1)\cos(3x+4)$
(b) $\sin(3x+4)$
(c) $\sin(3x+4)+3(x-1)\cos(3x+4)$
(d) none of these

4. The value of the integral $\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ is

(a) $\pi/4$ (b) $\pi/2$ *(1983 - 1 Mark)*
(c) π (d) none of these

5. For any integer n the integral —

$\int_0^{\pi}e^{\cos^2 x}\cos^3(2n+1)x\,dx$ has the value *(1985 - 2 Marks)*

(a) π (b) 1
(c) 0 (d) none of these

6. Let $f: R \to R$ and $g: R \to R$ be continous functions. Then the value of the integral $\int_{-\pi/2}^{\pi/2} [f(x)+f(-x)][g(x)-g(-x)]dx$ is *(1990 - 2 Marks)*

(a) π (b) 1 (c) -1 (d) 0

7. The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is *(1993 - 1 Marks)*

(a) 0 (b) 1 (c) $\pi/2$ (d) $\pi/4$

8. If $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)dx = \frac{2A}{\pi}$, then constants A and B are *(1995S)*

(a) $\frac{\pi}{2}$ and $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ and $\frac{3}{\pi}$

(c) 0 and $\frac{-4}{\pi}$ (d) $\frac{4}{\pi}$ and 0

9. The value of $\int_{\pi}^{2\pi} [2\,Sinx]\,dx$ where [.] represents the greatest integer function is *(1995S)*

(a) $\frac{-5\pi}{3}$ (b) $-\pi$ (c) $\frac{5\pi}{3}$ (d) -2π

10. If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ equals *(1997 - 2 Marks)*

(a) $g(x)+g(\pi)$ (b) $g(x)-g(\pi)$

(c) $g(x)\,g(\pi)$ (d) $\frac{g(x)}{g(\pi)}$

11. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to *(1999 - 2 Marks)*

(a) 2 (b) –2 (c) 1/2 (d) –1/2

12. If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx$ is *(1999 - 2 Marks)*

(a) $-\pi$ (b) 0 (c) $-\pi/2$ (d) $\pi/2$

13. Let $g(x) = \int_0^x f(t)dt$, where f is such that

$\frac{1}{2} \le f(t) \le 1$, for $t \in [0,1]$ and $0 \le f(t) \le \frac{1}{2}$, for $t \in [1,2]$.

Then $g(2)$ satisfies the inequality *(2000S)*

(a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ (b) $0 \le g(2) < 2$

(c) $\frac{3}{2} < g(2) \le \frac{5}{2}$ (d) $2 < g(2) < 4$

14. If $f(x) = \begin{cases} e^{\cos x}\sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise,} \end{cases}$ then $\int_{-2}^{3} f(x)dx =$ *(2000S)*

(a) 0 (b) 1 (c) 2 (d) 3

15. The value of the integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is: *(2000S)*

(a) 3/2 (b) 5/2 (c) 3 (d) 5

16. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$, $a > 0$, is *(2001S)*

(a) π (b) $a\pi$ (c) $\pi/2$ (d) 2π

17. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is *(2002S)*

(a) 1 (b) 2 (c) $2\sqrt{2}$ (d) 4

18. Let $f(x) = \int_1^x \sqrt{2-t^2}\,dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are *(2002S)*

(a) ± 1 (b) $\pm\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2}$ (d) 0 and 1

19. Let $T > 0$ be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x+T) = f(x)$.

If $I = \int_0^T f(x)dx$ then the value of $\int_3^{3+3T} f(2x)dx$ is *(2002S)*

(a) $3/2I$ (b) $2I$ (c) $3I$ (d) $6I$

20. The integral $\int_{-1/2}^{1/2} \left([x] + \ell n\left(\frac{1+x}{1-x}\right)\right) dx$ equal to *(2002S)*

(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $2\ell n\left(\frac{1}{2}\right)$

21. If $l(m,n) = \int_0^1 t^m (1+t)^n\,dt$, then the expression for $l(m, n)$ in terms of $l(m+1, n-1)$ is *(2003S)*

(a) $\frac{2^n}{m+1} - \frac{n}{m+1} l(m+1, n-1)$

(b) $\frac{n}{m+1} l(m+1, n-1)$

(c) $\frac{2^n}{m+1} + \frac{n}{m+1} l(m+1, n-1)$

(d) $\frac{m}{n+1} l(m+1, n-1)$

22. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in *(2003S)*

(a) $(-2, 2)$ (b) no value of x
(c) $(0, \infty)$ (d) $(-\infty, 0)$

23. The area bounded by the curves $y=\sqrt{x}$, $2y+3=x$ and x-axis in the 1st quadrant is **(2003S)**
(a) 9 (b) 27/4 (c) 36 (d) 18

24. If $f(x)$ is differentiable and $\int_0^{t^2} xf(x)dx=\frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals **(2004S)**
(a) 2/5 (b) –5/2 (c) 1 (d) 5/2

25. The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$ is **(2004S)**
(a) $\frac{\pi}{2}+1$ (b) $\frac{\pi}{2}-1$ (c) -1 (d) 1

26. The area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 sq. unit, then the value of a is **(2004S)**
(a) $1/\sqrt{3}$ (b) 1/2 (c) 1 (d) 1/3

27. $\int_{-2}^{0}\{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}\,dx$ is equal to **(2005S)**
(a) –4 (b) 0 (c) 4 (d) 6

28. The area bounded by the parabolas $y = (x + 1)^2$ and $y = (x – 1)^2$ and the line $y = 1/4$ is **(2005S)**
(a) 4 sq. units (b) 1/6 sq. units
(c) 4/3 sq. units (d) 1/3 sq. units

29. The area of the region between the curves $y=\sqrt{\frac{1+\sin x}{\cos x}}$ and $y=\sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x=\frac{\pi}{4}$ is **(2008)**
(a) $\int_0^{\sqrt{2}-1}\frac{t}{(1+t^2)\sqrt{1-t^2}}dt$ (b) $\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$
(c) $\int_0^{\sqrt{2}+1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$ (d) $\int_0^{\sqrt{2}+1}\frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

30. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)\,dt$, $0\le x\le 1$, and $f(0) = 0$, then **(2009)**
(a) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$
(b) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$
(c) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$
(d) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

31. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt$ is **(2010)**
(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$

32. Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x}f(x)=2+\int_0^x \sqrt{t^4+1}\,dt$, for all $x\in(-1,1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to **(2010)**
(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{e}$

33. The value of $\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}}\frac{x\sin x^2}{\sin x^2+\sin(\ell n6-x^2)}dx$ is **(2011)**
(a) $\frac{1}{4}\ell n\frac{3}{2}$ (b) $\frac{1}{2}\ell n\frac{3}{2}$ (c) $\ell n\frac{3}{2}$ (d) $\frac{1}{6}\ell n\frac{3}{2}$

34. Let the straight line $x = b$ divide the area enclosed by $y=(1-x)^2$, $y=0$, and $x=0$ into two parts R_1 $(0 \le x \le b)$ and R_2 $(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals **(2011)**
(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

35. Let $f : [– 1, 2] \to [0, \infty)$ be a continuous function such that $f(x)=f(1-x)$ for all $x \in [-1, 2]$
Let $R_1=\int_{-1}^{2} xf(x)dx$, and R_2 be the area of the region bounded by $y=f(x)$, $x=-1$, $x=2$, and the x-axis.
Then **(2011)**
(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$
(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

36. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is **(2012)**
(a) 0 (b) $\frac{\pi^2}{2}-4$ (c) $\frac{\pi^2}{2}+4$ (d) $\frac{\pi^2}{2}$

37. The area enclosed by the curves $y = \sin x + \cos x$ and $y=|\cos x-\sin x|$ over the interval $\left[0,\frac{\pi}{2}\right]$ is **(JEE Adv. 2013)**
(a) $4(\sqrt{2}-1)$ (b) $2\sqrt{2}(\sqrt{2}-1)$
(c) $2(\sqrt{2}+1)$ (d) $2\sqrt{2}(\sqrt{2}+1)$

38. Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real number) be a positive, non-constant and differentiable function such that $f'(x)<2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval *(JEE Adv. 2013)*

(a) $(2e-1, 2e)$ (b) $(e-1, 2e-1)$

(c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

39. The following integral $\int_{\pi/4}^{\pi/2}(2\text{cosec } x)^{17}dx$ is equal to *(JEE Adv. 2014)*

(a) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u+e^{-u}\right)^{16}du$

(b) $\int_0^{\log(1+\sqrt{2})} \left(e^u+e^{-u}\right)^{17}du$

(c) $\int_0^{\log(1+\sqrt{2})} \left(e^u-e^{-u}\right)^{17}du$

(d) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u-e^{-u}\right)^{16}du$

40. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{x^2\cos x}{1+e^x}dx$ is equal to *(JEE Adv. 2016)*

(a) $\frac{\pi^2}{4}-2$ (b) $\frac{\pi^2}{4}+2$

(c) $\pi^2-e^{\frac{\pi}{2}}$ (d) $\pi^2+e^{\frac{\pi}{2}}$

41. Area of the region $\left\{(x,y)\in\mathbb{R}^2 : y\geq\sqrt{|x+3|}, 5y\leq x+9\leq 15\right\}$ is equal to *(JEE Adv. 2016)*

(a) $\frac{1}{6}$ (b) $\frac{4}{3}$

(c) $\frac{3}{2}$ (d) $\frac{5}{3}$

D MCQs with One or More than One Correct

1. If $\int_0^x f(t)\,dt = x+\int_x^1 t\,f(t)\,dt$, then the value of $f(1)$ is *(1998 - 2 Marks)*

(a) 1/2 (b) 0 (c) 1 (d) –1/2

2. Let $f(x) = x-[x]$, for every real number x, where $[x]$ is the integral part of x. Then $\int_{-1}^{1} f(x)\,dx$ is *(1998 - 2 Marks)*

(a) 1 (b) 2 (c) 0 (d) 1/2

3. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals 9/2? *(1999 - 3 Marks)*

(a) –4 (b) –2 (c) 2 (d) 4

4. Let $f(x)$ be a non-constant twice differentiable function definied on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right)=0$. Then, *(2008)*

(a) $f''(x)$ vanishes at least twice on [0, 1]

(b) $f'\left(\frac{1}{2}\right)=0$

(c) $\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx = 0$

(d) $\int_0^{1/2} f(t)\,e^{\sin \pi t}dt = \int_{1/2}^{1} f(1-t)\,e^{\sin \pi t}dt$

5. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is *(2009)*

(a) $e-1$ (b) $\int_1^e \ln(e+1-y)\,dy$

(c) $e-\int_0^1 e^x dx$ (d) $\int_1^e \ln y\,dy$

6. If $I_n = \int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx$ $n = 0, 1, 2, ...,$ then *(2009)*

(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(c) $\sum_{m=1}^{10} I_{2m} = 0$ (d) $I_n = I_{n+1}$

7. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are) *(2010)*

(a) $\frac{22}{7}-\pi$ (b) $\frac{2}{105}$

(c) 0 (d) $\frac{71}{15}-\frac{3\pi}{2}$

8. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\, dt$. Then which of the following statement(s) is (are) true? ***(2010)***

(a) $f''(x)$ exists for all $x \in (0,\infty)$

(b) $f'(x)$ exists for all $x \in (0,\infty)$ and f' is continuous on $(0,\infty)$, but not differentiable on $(0,\infty)$

(c) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha,\infty)$

(d) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0,\infty)$

9. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0, x = 0$ and $x = 1$; then ***(2012)***

(a) $S \ge \frac{1}{e}$ (b) $S \ge 1 - \frac{1}{e}$

(c) $S \le \frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$ (d) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$

10. The option(s) with the values of a and L that satisfy the following equation is(are) ***(JEE Adv. 2015)***

$$\frac{\int_0^{4\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt}{\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt} = L?$$

(a) $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (b) $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(c) $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (d) $a = 4, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

11. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are) ***(JEE Adv. 2015)***

(a) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ (b) $\int_0^{\pi/4} f(x)dx = 0$

(c) $\int_0^{\pi/4} xf(x)dx = \frac{1}{6}$ (d) $\int_0^{\pi/4} f(x)dx = 1$

12. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\frac{1}{2}\right) = 0$.

If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are ***(JEE Adv. 2015)***

(a) $m = 13, M = 24$ (b) $m = \frac{1}{4}, M = \frac{1}{2}$

(c) $m = -11, M = 0$ (d) $m = 1, M = 12$

13. Let $f(x) = \lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)\ldots\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)\ldots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all $x > 0$. Then ***(JEE Adv. 2016)***

(a) $f\left(\frac{1}{2}\right) \ge f(1)$ (b) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$

(c) $f'(2) \le 0$ (d) $\frac{f'(3)}{f(3)} \ge \frac{f'(2)}{f(2)}$

E Subjective Problems

1. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$. ***(1981 - 4 Marks)***

2. Show that : $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{6n}\right) = \log 6$ ***(1981 - 2 Marks)***

3. Show that $\int_0^{\pi} xf(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$. ***(1982 - 2 Marks)***

4. Find the value of $\int_{-1}^{3/2} |x \sin \pi x|\, dx$ ***(1982 - 3 Marks)***

5. For any real t, $x = \frac{e^t + e^{-t}}{2}$, $y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by this hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1. ***(1982 - 3 Marks)***

6. Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x}dx$ ***(1983 - 3 Marks)***

7. Find the area bounded by the x-axis, part of the curve $y = \left(1+\frac{8}{x^2}\right)$ and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, find a. ***(1983 - 3 Marks)***

8. Evaluate the following $\int_0^{\frac{1}{2}} \frac{x\sin^{-1} x}{\sqrt{1-x^2}}dx$ ***(1984 - 2 Marks)***

9. Find the area of the region bounded by the x-axis and the curves defined by ***(1984 - 4 Marks)***

$y = \tan x,\ -\frac{\pi}{3} \le x \le \frac{\pi}{3};\ y = \cot x,\ \frac{\pi}{6} \le x \le \frac{3\pi}{2}$

10. Given a function $f(x)$ such that *(1984 - 4 Marks)*
(i) it is integrable over every interval on the real line and
(ii) $f(t+x)=f(x)$, for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a .

11. Evaluate the following : $\int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ *(1985 - 2½ Marks)*

12. Sketch the region bounded by the curves $y=\sqrt{5-x^2}$ and $y=|x-1|$ and find its area. *(1985 - 5 Marks)*

13. Evaluate: $\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}$, $0<\alpha<\pi$ *(1986 - 2½ Marks)*

14. Find the area bounded by the curves, $x^2+y^2=25$, $4y=|4-x^2|$ and $x=0$ above the x-axis. *(1987 - 6 Marks)*

15. Find the area of the region bounded by the curve C : $y=\tan x$, tangent drawn to C at $x=\frac{\pi}{4}$ and the x-axis. *(1988 - 5 Marks)*

16. Evaluate $\int_0^1 \log[\sqrt{1-x}+\sqrt{1+x}]dx$ *(1988 - 5 Marks)*

17. If f and g are continuous function on $[0, a]$ satisfying $f(x)=f(a-x)$ and $g(x)+g(a-x)=2$, then show that $\int_0^a f(x)g(x)dx=\int_0^a f(x)dx$ *(1989 - 4 Marks)*

18. Show that $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx=\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$ *(1990 - 4 Marks)*

19. Prove that for any positive integer k,

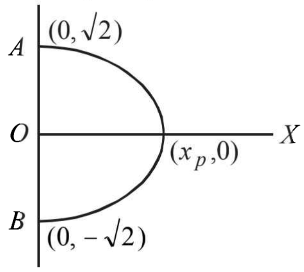
$$\frac{\sin 2kx}{\sin x}=2[\cos x+\cos 3x+.............+\cos(2k-1)x]$$

Hence prove that $\int_0^{\pi/2}\sin 2kx\cot x\,dx=\frac{\pi}{2}$ *(1990 - 4 Marks)*

20. Compute the area of the region bounded by the curves $y=ex\ln x$ and $y=\frac{\ln x}{ex}$ where $\ln e=1$. *(1990 - 4 Marks)*

21. Sketch the curves and identify the region bounded by $x=\frac{1}{2}$, $x=2$, $y=\ln x$ and $y=2^x$. Find the area of this region. *(1991 - 4 Marks)*

22. If 'f' is a continous function with $\int_0^x f(t)\,dt\to\infty$ as $|x|\to\infty$, then show that every line $y=mx$



intersects the curve $y^2+\int_0^x f(t)\,dt=2$! *(1991 - 4 Marks)*

23. Evaluate $\int_0^{\pi}\frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$ *(1991 - 4 Marks)*

24. Sketch the region bounded by the curves $y=x^2$ and $y=\frac{2}{1+x^2}$. Find the area. *(1992 - 4 Marks)*

25. Determine a positive integer $n\le 5$, such that $\int_0^1 e^x(x-1)^n dx=16-6e$ *(1992 - 4 Marks)*

26. Evalute $\int_2^3\frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx.$ *(1993 - 5 Marks)*

27. Show that $\int_0^{n\pi+v}|\sin x|\,dx=2n+1-\cos v$ where n is a positive integer and $0\le v<\pi$. *(1994 - 4 Marks)*

28. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y=4x-x^2$ and $y=x^2-x$? *(1994 - 5 Marks)*

29. Let $I_m=\int_0^{\pi}\frac{1-\cos mx}{1-\cos x}dx$. Use mathematical induction to prove that $I_m=m\pi$, $m=0,1,2,.......$ *(1995 - 5 Marks)*

30. Evaluate the definite integral :

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$$ *(1995 - 5 Marks)*

31. Consider a square with vertices at (1, 1), (– 1, 1), (– 1, – 1) and (1, – 1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area. *(1995 - 5 Marks)*

32. Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \frac{\pi}{4}$. Prove that for $n > 2$,

$$A_n + A_{n-2} = \frac{1}{n-1} \text{ and deduce } \frac{1}{2n+2} < A_n < \frac{1}{2n-2}.$$

(1996 - 3 Marks)

33. Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$.

(1997 - 5 Marks)

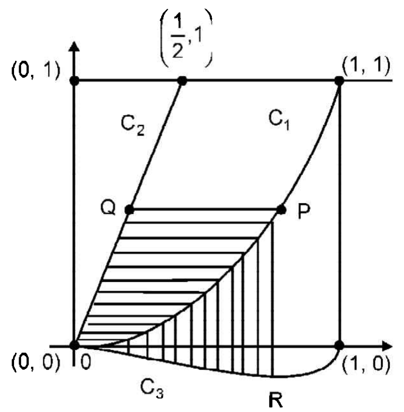
34. Let $f(x) = \text{Maximum}\ \{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$. ***(1997 - 5 Marks)***

35. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2\int_0^1 \tan^{-1} x\,dx$.

Hence or otherwise, evaluate the integral

$\int_0^1 \tan^{-1}(1-x+x^2)\,dx$. ***(1998 - 8 Marks)***

36. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure.) If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$. ***(1998 - 8 Marks)***



37. Integrate $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$. ***(1999 - 5 Marks)***

38. Let $f(x)$ be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$ ***(1999 - 10 Marks)***

Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

39. For $x > 0$, let $f(x) = \int_e^x \frac{\ln t}{1+t}\,dt$. Find the function

$f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

Here, $\ln t = \log_e t$. ***(2000 - 5 Marks)***

40. Let $b \ne 0$ and for $j = 0, 1, 2, \ldots, n$, let S_j be the area of the region bounded by the y–axis and the curve $xe^{ay} = \sin by$, $\frac{j\pi}{b} \le y \le \frac{(j+1)\pi}{b}$. Show that $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$. ***(2001 - 5 Marks)***

41. Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. ***(2002 - 5 Marks)***

42. If f is an even function then prove that ***(2003 - 2 Marks)***

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx.$$

43. If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\,d\theta$, then find $\frac{dy}{dx}$ at $x = \pi$

(2004 - 2 Marks)

44. Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)}\,dx$

(2004 - 4 Marks)

45. Evaluate $\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$

(2005 - 2 Marks)

46. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$. ***(2005 - 4 Marks)***

47. $f(x)$ is a differentiable function and $g(x)$ is a double differentiable function such that $|f(x)| \le 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c).g''(c) < 0$. ***(2005 - 6 Marks)***

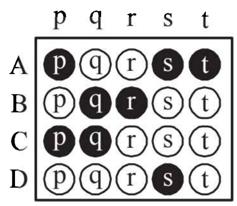
48. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, $f(x)$ is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of $y = f(x)$ with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by $f(x)$ and chord AB. ***(2005 - 6 Marks)***

49. The value of $5050\frac{\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ is. ***(2006 - 6M)***

F Match the Following

DIRECTIONS (Q. 1 and 2) : *Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :*

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



1. Match the following : *(2006 - 6M)*

Column I	*Column II*
(A) $\int_0^{\pi/2} (\sin x)^{\cos x}\left(\cos x \cot x - \log(\sin x)^{\sin x}\right)dx$	(p) 1
(B) Area bounded by $-4y^2 = x$ and $x - 1 = -5y^2$	(q) 0
(C) Cosine of the angle of intersection of curves $y = 3^{x-1}\log x$ and $y = x^x - 1$ is	(r) $6 \ln 2$
(D) Let $\frac{dy}{dx} = \frac{6}{x+y}$ where $y(0) = 0$ then value of y when $x + y = 6$ is	(s) $\frac{4}{3}$

2. Match the integrals in **Column I** with the values in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the *ORS*. *(2007 - 6 marks)*

Column I	*Column II*
(A) $\int_{-1}^{1} \frac{dx}{1+x^2}$	(p) $\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	(q) $2\log\left(\frac{2}{3}\right)$
(C) $\int_2^3 \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

DIRECTIONS (Q. 3) : *Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

3.

List - I	**List - II**
P. The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)\,dx = 1$, is	**1.** 8
Q. The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	**2.** 2
R. $\int_{-2}^{2} \frac{3x^2}{\left(1+e^x\right)}dx$ equals	**3.** 4

S. $\dfrac{\left(\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\displaystyle\int_{0}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ **4.** 0

(JEE Adv. 2014)

	P	Q	R	S
(a)	3	2	4	1
(b)	2	3	4	1
(c)	3	2	1	4
(d)	2	3	1	4

G Comprehension Based Questions

PASSAGE - 1

Let the definite integral be defined by the formula $\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b))$. For more accurate result for $c \in (a, b)$, we can use $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = F(c)$ so that for $c = \frac{a+b}{2}$, we get $\int_a^b f(x)dx = \frac{b-a}{4}(f(a)+f(b)+2f(c))$.

1. $\int_0^{\pi/2} \sin x\, dx =$ *(2006 - 5M, –2)*

(a) $\frac{\pi}{8}(1+\sqrt{2})$ (b) $\frac{\pi}{4}(1+\sqrt{2})$

(c) $\frac{\pi}{8\sqrt{2}}$ (d) $\frac{\pi}{4\sqrt{2}}$

2. If $\lim\limits_{x \to a} \dfrac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$, then $f(x)$ is of maximum degree *(2006 - 5M, –2)*

(a) 4 (b) 3 (c) 2 (d) 1

3. If $f''(x) < 0\ \forall x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to *(2006 - 5M, –2)*

(a) $\frac{f(b)-f(a)}{b-a}$ (b) $\frac{2(f(b)-f(a))}{b-a}$

(c) $\frac{2f(b)-f(a)}{2b-a}$ (d) 0

PASSAGE - 2

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

4. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$ *(2008)*

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$ (c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

5. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is *(2008)*

(a) $\int_a^b \frac{x}{3((f(x))^2-1)}dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3((f(x))^2-1)}dx + bf(b) - af(a)$

(c) $\int_a^b \frac{x}{3((f(x))^2-1)}dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3((f(x))^2-1)}dx - bf(b) + af(a)$

6. $\int_{-1}^{1} g'(x)\, dx =$ *(2008)*

(a) $2g(-1)$ (b) 0

(c) $-2g(1)$ (d) $2g(1)$

PASSAGE - 3

Consider the function f: $(-\infty, \infty) \to (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2.$$

7. Which of the following is true? *(2008)*
   (a) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$
   (b) $(2-a)^2 f''(1) - (2+a)^2 f''(-1) = 0$
   (c) $f'(1) f'(-1) = (2-a)^2$
   (d) $f'(1) f'(-1) = -(2+a)^2$

8. Which of the following is true? *(2008)*
   (a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$
   (b) $f(x)$ is increasing on $(-1, 1)$ and has a local minimum at $x = 1$
   (c) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$
   (d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$

9. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2} dt$. Which of the following is true? *(2008)*
   (a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
   (b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
   (c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
   (d) $g'(x)$ does not change sign on $(-\infty, \infty)$

PASSAGE - 4

Consider the polynomial *(2010)*
$f(x) = 1 + 2x + 3x^2 + 4x^3$.
Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

10. The real numbers lies in the interval
    (a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$
    (c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

11. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval
    (a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$
    (c) $(9, 10)$ (d) $\left(0, \frac{21}{64}\right)$

12. The function $f'(x)$ is
    (a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$
    (b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$
    (c) increasing in $(-t, t)$
    (d) decreasing in $(-t, t)$

PASSAGE - 5

Given that for each $a \in (0, 1)$, $\lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1} dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. *(JEE Adv. 2014)*

13. The value of $g\left(\frac{1}{2}\right)$ is
    (a) π (b) 2π
    (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

14. The value of $g'\left(\frac{1}{2}\right)$ is
    (a) $\frac{\pi}{2}$ (b) π
    (c) $-\frac{\pi}{2}$ (d) 0

PASSAGE - 6

Let F : $\mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F(x) < 0$ for all $x \in \left(\frac{1}{2}, 3\right)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$. *(JEE Adv. 2015)*

15. The correct statement(s) is(are)
    (a) $f'(1) < 0$
    (b) $f(2) < 0$
    (c) $f'(x) \neq 0$ for any $x \in (1, 3)$
    (d) $f'(x) = 0$ for some $x \in (1, 3)$

16. If $\int_1^3 x^2 F'(x)dx = -12$ and $\int_1^3 x^3 F''(x)dx = 40$, then the correct expression(s) is (are)
    (a) $9f'(3) + f'(1) - 32 = 0$ (b) $\int_1^3 f(x)dx = 12$
    (c) $9f'(3) - f'(1) + 32 = 0$ (d) $\int_1^3 f(x)dx = -12$

I Integer Value Correct Type

1. Let $f: \mathbf{R} \to \mathbf{R}$ be a continuous function which satisfies

$$f(x) = \int_0^x f(t)\,dt.$$

Then the value of $f(\ln 5)$ is ***(2009)***

2. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd,} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ is ***(2010)***

3. The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is

(JEE Adv. 2014)

4. Let $f: R \to R$ be a function defined by $f(x) = \begin{cases}[x], & x \le 2 \\ 0, & x > 2\end{cases}$

where $[x]$ is the greatest integer less than or equal to x, if

$I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}dx$, then the value of $(4I - 1)$ is

(JEE Adv. 2015)

5. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t(dt)$ for all $x \in R$ and $f:\left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a)+2$ is the area of the region bounded by $x=0, y=0, y=f(x)$ and $x=a$, then $f(0)$ is ***(JEE Adv. 2015)***

6. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is ***(JEE Adv. 2015)***

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|\,dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is ***(JEE Adv. 2015)***

8. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$ is ***(JEE Adv. 2016)***

GP_3480

Section-B JEE Main / AIEEE

1. $\int_0^{10\pi} |\sin x|\,dx$ is [2002]

(a) 20 (b) 8 (c) 10 (d) 18

2. $I_n = \int_0^{\pi/4} \tan^n x\,dx$ then $\lim_{n\to\infty} n[I_n + I_{n+2}]$ equals [2002]

(a) $\frac{1}{2}$ (b) 1 (c) ∞ (d) zero

3. $\int_0^2 [x^2]dx$ is [2002]

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$
(c) $\sqrt{2}-1$ (d) $-\sqrt{2}-\sqrt{3}+5$

4. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$ is [2002]

(a) $\frac{\pi^2}{4}$ (b) π^2 (c) zero (d) $\frac{\pi}{2}$

5. If $y = f(x)$ makes +ve intercept of 2 and 0 unit on x and y axes and encloses an area of 3/4 square unit with the axes then $\int_0^2 xf'(x)dx$ is [2002]

(a) 3/2 (b) 1 (c) 5/4 (d) –3/4

6. The area bounded by the curves $y = \ln x$, $y = \ln|x|$, $y = |\ln x|$ and $y = |\ln|x||$ is [2002]

(a) 4sq. units (b) 6 sq. units
(c) 10 sq. units (d) none of these

7. The area of the region bounded by the curves $y = |x-1|$ *and* $y = 3-|x|$ is [2003]

(a) 6 sq. units (b) 2 sq. units
(c) 3 sq. units (d) 4 sq. units.

8. If $f(a+b-x) = f(x)$ then $\int_a^b xf(x)dx$ is equal to [2003]

(a) $\frac{a+b}{2}\int_a^b f(a+b+x)dx$ (b) $\frac{a+b}{2}\int_a^b f(b-x)dx$

(c) $\frac{a+b}{2}\int_a^b f(x)dx$ (d) $\frac{b-a}{2}\int_a^b f(x)dx$.

9. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0)=1$ and g(x) be a function that satisfies f(x) + g(x) = x^2. Then the value of the integral $\int_0^1 f(x)g(x)dx$, is [2003]

(a) $e+\frac{e^2}{2}+\frac{5}{2}$ (b) $e-\frac{e^2}{2}-\frac{5}{2}$

(c) $e+\frac{e^2}{2}-\frac{3}{2}$ (d) $e-\frac{e^2}{2}-\frac{3}{2}$.

10. The value of the integral $I = \int_0^1 x(1-x)^n dx$ is [2003]

(a) $\frac{1}{n+1}+\frac{1}{n+2}$ (b) $\frac{1}{n+1}$

(c) $\frac{1}{n+2}$ (d) $\frac{1}{n+1}-\frac{1}{n+2}$.

11. $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ is [2004]

(a) $e+1$ (b) $e-1$ (c) $1-e$ (d) e

12. The value of $\int_{-2}^{3} |1-x^2|dx$ is [2004]

(a) $\frac{1}{3}$ (b) $\frac{14}{3}$ (c) $\frac{7}{3}$ (d) $\frac{28}{3}$

13. The value of $I = \int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}dx$ is [2004]

(a) 3 (b) 1 (c) 2 (d) 0

14. If $\int_0^{\pi} xf(\sin x)dx = A\int_0^{\pi/2} f(\sin x)dx$, then A is [2004]

(a) 2π (b) π (c) $\frac{\pi}{4}$ (d) 0

15. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx$ and $I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}dx$, then the value of $\frac{I_2}{I_1}$ is [2004]

(a) 1 (b) –3 (c) –1 (d) 2

16. The area of the region bounded by the curves $y = |x-2|, x = 1, x = 3$ and the x-axis is [2004]

(a) 4 (b) 2 (c) 3 (d) 1

17. If $I_1 = \int_0^1 2^{x^2} dx$, $I_2 = \int_0^1 2^{x^3} dx$, $I_3 = \int_1^2 2^{x^2} dx$ and $I_4 = \int_1^2 2^{x^3} dx$ then **[2005]**

(a) $I_2 > I_1$ (b) $I_1 > I_2$ (c) $I_3 = I_4$ (d) $I_3 > I_4$

18. The area enclosed between the curve $y = \log_e(x+e)$ and the coordinate axes is **[2005]**

(a) 1 (b) 2 (c) 3 (d) 4

19. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4, y = 4$ and the coordinate axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom; then $S_1 : S_2 : S_3$ is **[2005]**

(a) 1 : 2 : 1 (b) 1 : 2 : 3 (c) 2 : 1 : 2 (d) 1 : 1 : 1

20. Let $f(x)$ be a non – negative continuous function such that the area bounded by the curve $y = f(x)$, x - axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is **[2005]**

$\left(\beta \sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta\right)$. Then $f\left(\frac{\pi}{2}\right)$ is

(a) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$ (b) $\left(\frac{\pi}{4} - \sqrt{2} + 1\right)$

(c) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$ (d) $\left(1 - \frac{\pi}{4} + \sqrt{2}\right)$

21. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x} dx$, a > 0, is **[2005]**

(a) $a\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{a}$ (d) 2π

22. The value of integral, $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x} + \sqrt{x}} dx$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 2 (d) 1

23. $\int_0^{\pi} x f(\sin x) dx$ is equal to **[2006]**

(a) $\pi \int_0^{\pi} f(\cos x) dx$ (b) $\pi \int_0^{\pi} f(\sin x) dx$

(c) $\frac{\pi}{2} \int_0^{\pi/2} f(\sin x) dx$ (d) $\pi \int_0^{\pi/2} f(\cos x) dx$

24. $\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x+\pi)^3 + \cos^2(x+3\pi)] dx$ is equal to **[2006]**

(a) $\frac{\pi^4}{32}$ (b) $\frac{\pi^4}{32} + \frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4} - 1$

25. The value of $\int_1^a [x] f'(x) dx$, $a > 1$ where $[x]$ denotes the greatest integer not exceeding x is **[2006]**

(a) $af(a) - \{f(1) + f(2) + \ldots\ldots\ldots\ldots f([a])\}$

(b) $[a]f(a) - \{f(1) + f(2) + \ldots\ldots\ldots\ldots f([a])\}$

(c) $[a]f([a]) - \{f(1) + f(2) + \ldots\ldots\ldots\ldots f(a)\}$

(d) $af([a]) - \{f(1) + f(2) + \ldots\ldots\ldots\ldots f(a)\}$

26. Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x) = \int_l^x \frac{\log t}{1+t} dt$, Then $F(e)$ equals **[2007]**

(a) 1 (b) 2 (c) 1/2 (d) 0

27. The solution for x of the equation $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2 - 1}} = \frac{\pi}{2}$ is **[2007]**

(a) $\frac{\sqrt{3}}{2}$ (b) $2\sqrt{2}$ (c) 2 (d) None

28. The area enclosed between the curves $y^2 = x$ and $y = |x|$ is **[2007]**

(a) 1/6 (b) 1/3 (c) 2/3 (d) 1

29. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$. Then which one of the following is true?

(a) $I > \frac{2}{3}$ and $J > 2$ (b) $I < \frac{2}{3}$ and $J < 2$

(c) $I < \frac{2}{3}$ and $J > 2$ (d) $I > \frac{2}{3}$ and $J < 2$

30. The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to **[2008]**

(a) $\frac{5}{3}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

31. The area of the region bounded by the parabola $(y-2)^2 = x - 1$, the tangent of the parabola at the point (2, 3) and the x-axis is: **[2009]**

(a) 6 (b) 9 (c) 12 (d) 3

32. $\int_0^{\pi}[\cot x]dx$, where [.] denotes the greatest integer function, is equal to : **[2009]**

(a) 1 (b) –1 (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

33. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$ is **[2010]**

(a) $4\sqrt{2}+2$ (b) $4\sqrt{2}-1$ (c) $4\sqrt{2}+1$ (d) $4\sqrt{2}-2$

34. Let $p(x)$ be a function defined on **R** such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then $\int_0^1 p(x)\,dx$ equals **[2010]**

(a) 21 (b) 41 (c) 42 (d) $\sqrt{41}$

35. The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is **[2011]**

(a) $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{2}\log 2$

(c) $\log 2$ (d) $\pi \log 2$

36. The area of the region enclosed by the curves **[2011]**

$y = x,\ x = e,\ y = \frac{1}{x}$ and the positive x-axis is

(a) 1 square unit (b) $\frac{3}{2}$ square units

(c) $\frac{5}{2}$ square units (d) $\frac{1}{2}$ square unit

37. The area between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is : **[2012]**

(a) $20\sqrt{2}$ (b) $\frac{10\sqrt{2}}{3}$ (c) $\frac{20\sqrt{2}}{3}$ (d) $10\sqrt{2}$

38. If $g(x) = \int_0^x \cos 4t\,dt$, then $g(x+\pi)$ equals **[2012]**

(a) $\frac{g(x)}{g(\pi)}$ (b) $g(x)+g(\pi)$

(c) $g(x)-g(\pi)$ (d) $g(x).g(\pi)$

39. Statement-1 : The value of the integral

$\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$ **[JEE M 2013]**

Statement-2 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$

(a) Statement-1 is true; Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true; Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true; Statement-2 is false.
(d) Statement-1 is false; Statement-2 is true.

40. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis, and lying in the first quadrant is : **[JEE M 2013]**

(a) 9 (b) 36 (c) 18 (d) $\frac{27}{4}$

41. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals: **[JEE M 2014]**

(a) $4\sqrt{3}-4$ (b) $4\sqrt{3}-4-\frac{\pi}{3}$

(c) $\pi-4$ (d) $\frac{2\pi}{3}-4-4\sqrt{3}$

42. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is: **[JEE M 2014]**

(a) $\frac{\pi}{2}-\frac{2}{3}$ (b) $\frac{\pi}{2}+\frac{2}{3}$ (c) $\frac{\pi}{2}+\frac{4}{3}$ (d) $\frac{\pi}{2}-\frac{4}{3}$

43. The area (in sq. units) of the region described by $\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is **[JEE M 2015]**

(a) $\frac{15}{64}$ (b) $\frac{9}{32}$ (c) $\frac{7}{32}$ (d) $\frac{5}{64}$

44. The integral

$\int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)}dx$ is equal to : **[JEE M 2015]**

(a) 1 (b) 6 (c) 2 (d) 4

45. The area (in sq. units) of the region $\{(x, y) : y^2 \ge 2x \text{ and } x^2 + y^2 \le 4x, x \ge 0, y \ge 0\}$ is : **[JEE M 2016]**

(a) $\pi-\frac{4\sqrt{2}}{3}$ (b) $\frac{\pi}{2}-\frac{2\sqrt{2}}{3}$

(c) $\pi-\frac{4}{3}$ (d) $\pi-\frac{8}{3}$

18 Definite Integrals and Applications of Integrals

Section-A : JEE Advanced/ IIT-JEE

A	**1.**	$-\left(\frac{15\pi+32}{60}\right)$	**2.**	$2-\sqrt{2}$	**3.**	4	**4.**	$\pi(\sqrt{2}-1)$	**5.**	$\frac{1}{2}$		
	6.	$\frac{1}{a^2-b^2}\left[a(\log 2-5)+\frac{7b}{2}\right]$	**7.**	π^2	**8.**	2	**9.**	16				
B	**1.**	T										
C	**1.**	(d)	**2.**	(b)	**3.**	(c)	**4.**	(a)	**5.**	(c)	**6.**	(d)
	7.	(d)	**8.**	(d)	**9.**	(a)	**10.**	(a)	**11.**	(a)	**12.**	(c)
	13.	(b)	**14.**	(c)	**15.**	(b)	**16.**	(c)	**17.**	(b)	**18.**	(a)
	19.	(c)	**20.**	(a)	**21.**	(a)	**22.**	(d)	**23.**	(d)	**24.**	(a)
	25.	(b)	**26.**	(a)	**27.**	(c)	**28.**	(d)	**29.**	(b)	**30.**	(c)
	31.	(b)	**32.**	(b)	**33.**	(a)	**34.**	(b)	**35.**	(c)	**36.**	(b)
	37.	(b)	**38.**	(d)	**39.**	(a)	**40.**	(a)	**41.**	(c)		
D	**1.**	(a)	**2.**	(a)	**3.**	(b, d)	**4.**	(a, b, c, d)	**5.**	(b, c, d)	**6.**	(a, b, c)
	7.	(a)	**8.**	(b, c)	**9.**	(a, b, d)	**10.**	(a, c)	**11.**	(a, b)	**12.**	(d)
	13.	(b, c)										
E	**1.**	$\frac{9}{8}$ sq. units	**4.**	$\frac{3}{\pi}+\frac{1}{\pi^2}$	**6.**	$\frac{1}{20}\log 3$	**7.**	$a=2\sqrt{2}$	**8.**	$\frac{6-\pi\sqrt{3}}{12}$	**9.**	$\log\frac{3}{2}$ sq. units
	11.	$\frac{\pi^2}{16}$	**12.**	$\frac{5\pi-2}{4}$ sq. units			**13.**	$\frac{\pi\alpha}{\sin\alpha}$	**14.**	$4+25\sin^{-1}\frac{4}{5}$		
	15.	$\frac{1}{2}\left[\log 2-\frac{1}{2}\right]$ sq. units					**16.**	$\frac{1}{2}\left[\log 2+\frac{\pi}{2}-1\right]$	**20.**	$\frac{e^2-5}{4e}$		
	21.	$\frac{4-\sqrt{2}}{\log 2}-\frac{5}{2}\log 2+\frac{3}{2}$			**23.**	$\frac{8}{\pi^2}$	**24.**	$\left(\pi-\frac{2}{3}\right)$ sq. units			**25.**	$n=3$
	26.	$\frac{1}{2}\log 6-\frac{1}{10}$	**27.**	$2n+1-\cos\gamma$	**28.**	121 : 4			**30.**	$\frac{\pi}{12}\left[\pi+3\log_e(2+\sqrt{3})-4\sqrt{3}\right]$		
	31.	$\frac{16\sqrt{2}-20}{3}$	**33.**	π^2	**34.**	$\frac{17}{27}$ sq. units	**35.**	log 2	**36.**	$f(x)=x^3-x^2$	**37.**	$\frac{\pi}{2}$
	38.	$\frac{257}{192}$ sq. units			**40.**	$\frac{\pi(1+e)}{1+\pi^2}\left(\frac{e^{n+1}-1}{e-1}\right)$			**41.**	$\left(\frac{20}{3}-4\sqrt{2}\right)$ sq. units	**43.**	2π
	44.	$\frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3-\frac{\pi}{4}\right]$			**45.**	$\frac{24}{5}\left[e\cos\left(\frac{1}{2}\right)+\frac{1}{2}e\sin\left(\frac{1}{2}\right)-1\right]$			**46.**	$\frac{1}{3}$ sq. units	**48.**	$\frac{125}{3}$ sq. units
	49.	5051										
F	**1.**	(A) - p ; (B) - s ; (C) - p ; (D) - r			**2.**	(A) - s ; (B) - s ; (C) - p ; (D) - r			**3.**	(d)		
G	**1.**	(a)	**2.**	(d)	**3.**	(b)	**4.**	(b)	**5.**	(a)	**6.**	(d)
	7.	(a)	**8.**	(a)	**9.**	(b)	**10.**	(c)	**11.**	(a)	**12.**	(b)
	13.	(a)	**14.**	(d)	**15.**	(a, b, c)	**16.**	(c, d)				
I	**1.**	0	**2.**	4	**3.**	2	**4.**	0	**5.**	3	**6.**	9
	7.	7	**8.**	1								

Section-B : JEE Main/ AIEEE

1.	(a)	**2.**	(b)	**3.**	(d)	**4.**	(b)	**5.**	(d)	**6.**	(a)
7.	(d)	**8.**	(c)	**9.**	(d)	**10.**	(d)	**11.**	(b)	**12.**	(d)
13.	(c)	**14.**	(b)	**15.**	(d)	**16.**	(d)	**17.**	(b)	**18.**	(a)
19.	(d)	**20.**	(d)	**21.**	(b)	**22.**	(b)	**23.**	(d)	**24.**	(c)
25.	(b)	**26.**	(c)	**27.**	(d)	**28.**	(a)	**29.**	(b)	**30.**	(d)
31.	(b)	**32.**	(c)	**33.**	(d)	**34.**	(a)	**35.**	(d)	**36.**	(b)
37.	(c)	**38.**	(b, c)	**39.**	(d)	**40.**	(a)	**41.**	(b)	**42.**	(c)
43.	(b)	**44.**	(a)	**45.**	(d)						

GP_3480

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. Given that,

$$f(x)=\begin{vmatrix}\sec x & \cos x & \sec^2 x+\cot x\operatorname{cosec} x\\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x\\ 1 & \cos^2 x & \cos^2 x\end{vmatrix}$$

Operating $R_1-\sec x.R_3$,

$$=\begin{vmatrix}0 & 0 & \sec^2 x+\cot x\operatorname{cosec} x-\cos x\\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x\\ 1 & \cos^2 x & \cos^2 x\end{vmatrix}$$

Expanding along R_1, we get

$$=(\sec^2 x+\cot x\operatorname{cosec} x-\cos x)(\cos^4 x-\cos^2 x)$$

$$=\left(\frac{1}{\cos^2 x}+\frac{\cos x}{\sin^2 x}-\cos x\right)\cos^2 x(\cos^2 x-1)$$

$$=-\sin^2 x-\cos^5 x$$

$$\therefore\quad \int_0^{\pi/2} f(x)dx=-\int_0^{\pi/2}(\sin^2 x+\cos^5 x)dx$$

Using

$$\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx=\frac{(n-1)(n-3)\ldots 2\,or\,1}{(n)(n-2)\ldots 2}$$

Multiply the above by $\pi/2$ when n is even. We get

$$=-\left[\frac{1}{2}.\frac{\pi}{2}+\frac{4}{5}.\frac{2}{3}\right]=-\left[\frac{\pi}{4}+\frac{8}{15}\right]=-\left(\frac{15\pi+32}{60}\right)$$

2. $\int_0^{1.5}[x^2]dx$,

We have $0<x<1.5\Rightarrow 0<x^2<2.25$

$\therefore\ [x^2]=0,\ 0<x^2<1=1,\ 1\le x^2<2=2,\ 2\le x^2<(1.5)^2$

or $[x^2]=0,\ 0<x<1\ =1,\ 1\le x<\sqrt{2}\ =2, \sqrt{2}\le x<1.5$

$$\therefore\quad I=\int_0^{1.5}[x^2]dx=\int_0^1 0dx+\int_1^{\sqrt{2}}1dx+\int_{\sqrt{2}}^{1.5}2dx$$

$$=0+[x]_1^{\sqrt{2}}+[2x]_{\sqrt{2}}^{1.5}$$

$$=\sqrt{2}-1+3-2\sqrt{2}\ =2-\sqrt{2}$$

3. Let $I=\int_{-2}^{2}|1-x^2|dx=2\int_0^2|1-x^2|\,dx$

$$\left[\because \int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx \text{ if } f \text{ is an even function}\right]$$

$$=2\int_0^1(1-x^2)dx+2\int_1^2(x^2-1)dx$$

$$=2\left[x-\frac{x^3}{3}\right]_0^1+2\left[\frac{x^3}{3}-x\right]_1^2=\frac{4}{3}+\frac{8}{3}=\frac{12}{3}=4$$

4. We have, $I=\int_{\pi/4}^{3\pi/4}\frac{\phi}{1+\sin\phi}d\phi$ …(1)

$$\Rightarrow\quad I=\int_{\pi/4}^{3\pi/4}\frac{\pi-\phi}{1+\sin(\pi-\phi)}d\phi$$

$$\left[\text{Using}\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow\quad I=\int_{\pi/4}^{3\pi/4}\frac{\pi-\phi}{1+\sin\phi}d\phi \quad\ldots(2)$$

Adding (1) and (2), we get $2I=\int_{\pi/4}^{3\pi/4}\frac{\pi}{1+\sin\phi}d\phi$

$$=\pi\int_{\pi/4}^{3\pi/4}\frac{1-\sin\phi}{1-\sin^2\phi}d\phi=\pi\int_{\pi/4}^{3\pi/4}\frac{1-\sin\phi}{\cos^2\phi}d\phi$$

$$=\pi\int_{\pi/4}^{3\pi/4}(\sec^2\phi-\sec\phi\tan\phi)\,d\phi$$

$$=\pi[\tan\phi-\sec\phi]_{\pi/4}^{3\pi/4}$$

$$=\pi[\tan 3\pi/4-\sec 3\pi/4-\tan\pi/4+\sec\pi/4]$$

$$=2\pi(\sqrt{2}-1)\ \Rightarrow\ I=\pi(\sqrt{2}-1)$$

5. Let $I=\int_2^3\frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ …(1)

$$I=\int_2^3\frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx \quad\ldots(2)$$

$$\left[\text{Using}\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

Adding (1) and (2), we get

$$2I=\int_2^3\frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$$

$$\Rightarrow\quad I=\frac{1}{2}\int_2^3 1dx=\frac{1}{2}(3-2)=\frac{1}{2}$$

6. $af(x)+bf\left(\frac{1}{x}\right)=\frac{1}{x}-5$ …(1)

Integrating both sides within the limits 1 to 2, we get

$$a\int_1^2 f(x)dx+b\int_1^2 f\left(\frac{1}{x}\right)dx=[\log x-5x]_1^2=\log 2-5 \quad\ldots(2)$$

Replacing x by $\frac{1}{x}$ in (1), we get $af\left(\frac{1}{x}\right)+bf(x)=x-5$

Integrating both sides within the limits 1 to 2, we get

$$a\int_1^2 f\left(\frac{1}{x}\right)dx+b\int_1^2 f(x)dx=\left[\frac{x^2}{2}-5x\right]_1^2=-\frac{7}{2}\quad\ldots(3)$$

Eliminate $\int_1^2 f\left(\frac{1}{x}\right)$ between (2) and (3) by multiplying (2) by a and (3) by b and subtracting

$\therefore \quad (a^2-b^2)\int_1^2 f(x)dx = a(\log 2-5)+b.\frac{7}{2}$

$\therefore \quad \int_1^2 f(x)dx = \frac{1}{(a^2-b^2)}\left[a(\log 2-5)+\frac{7b}{2}\right]$

7. Let $I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$\Rightarrow \quad I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}(\pi-x)}{\sin^{2n}(2\pi-x)+\cos^{2n}(2\pi-x)}dx$

$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ ...(2)

Adding (1) and (2) we get

$2I = \int_0^{2\pi} \frac{2\pi\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$\Rightarrow \quad I = \pi\int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$\Rightarrow \quad I = 2\pi\int_0^{\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

$\left[\text{Using} \int_0^{2a} f(x)dx = 2\int_0^a (x)dx \text{ if } f(2a-x)=f(x)\right]$

$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ ...(3)

[Using above property again]

$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x}dx$ ....(4)

$\left[\text{Using} \int_0^a f(x)dx = \int_0^a (a-x)dx\right]$

Adding (3) and (4) we get

$2I = 4\pi\int_0^{\pi/2} 1.dx = 4\pi\left(\frac{\pi}{2}-0\right) = 2\pi^2 \Rightarrow I = \pi^2$

8. Let $I = \int_1^{e^{37}} \frac{\pi\sin(\pi\ln x)}{x}dx$

Let $\pi\ln x = t$

$\Rightarrow \quad \frac{\pi}{x}dx = dt$ also as $x\to 1, t\to 0, x\to e^{37}, t\to 37\pi$

$\therefore \quad I = \int_0^{37\pi} \sin t\,dt = [-\cos t]_0^{37\pi} = -\cos 37\pi + 1$

$= -(-1)+1 = 2$

9. $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k)-F(1) = [F(x)]_1^k$

Put $x^2 = t$

$\therefore \quad 2xdx = dt$; At $x=1, t=1$ and at $x=4, t=16$

$\therefore \quad I = \int_1^{16} \frac{e^{\sin t}}{t}dt = F[(t)]_1^{16}$ $\quad \therefore \ k = 16.$

B. True/False

1. Let $I = \int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}dx$ ...(1)

$= \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f[2a-(2a-x)]}dx$

[Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$I = \int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)}$ ...(2)

Adding (1) and (2), we get

$2I = \int_0^{2a} \frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}dx = \int_0^{2a} 1.dx$

$= [x]_0^{2a} = 2a \Rightarrow I = a$

$\therefore$ The given statement is true.

C. MCQs with ONE Correct Answer

1. (d) $\int_0^1 (1+e^{-x^2})dx = \int_0^1 \left(1+1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots\infty\right)dx$

$= \left[2x - \frac{x^3}{3.1!}+\frac{x^5}{5.2!}-\frac{x^7}{7.3!}+\ldots\infty\right]_0^1$

$= \left[2-\frac{1}{3.1!}+\frac{1}{5.2!}-\frac{1}{7.3!}+\ldots\infty\right]$

2. (b) Given $\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx$

$= \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$

$= \int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx$

$+\int_1^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$

$\Rightarrow \quad \int_1^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$

Now we know that if $\int_\alpha^\beta f(x)dx = 0$ then it means that $f(x)$ is + ve on some part of (α, β) and – ve on other part of (α, β).

But here $1 + \cos^8 x$ is always + ve,

$\therefore \quad ax^2 + bx + c$ is + ve on some part of $[1, 2]$ and – ve on other part $[1, 2]$

$\therefore \quad ax^2+bx+c=0$ has at least one root in (1, 2).

$\Rightarrow \quad ax^2 + bx + c = 0$ has at least one root in (0,2).

3. **(c)** $ATQ\int_1^b f(x)dx = (b-1)\sin(3b+4)$

Differentiating both sides w.r.t b, we get

$f(b) = 3(b-1)\cos(3b+4) + \sin(3b+4)$

$\Rightarrow f(x) = 3(x-1)\cos(3x+4) + \sin(3x+4)$

4. **(a)** $I\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ ...(1)

$$= \int_0^{\pi/2} \frac{\sqrt{\cot(\pi/2-x)}}{\sqrt{\cot(\pi/2-x)}+\sqrt{\tan(\pi/2-x)}}dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} = dx \quad ...(2)$$

Adding (1) and (2) we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cot x}+\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}} = dx = \int_0^{\pi/2} 1.dx$$

$$= (x)_0^{\pi/2} = \pi/2 \quad \therefore I = \pi/4$$

5. **(c)** $I = \int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)xdx, n \in Z$ ...(1)

$$= \int_0^{\pi} e^{\cos^2(\pi-x)}\cos^3[(2n+1)(\pi-x)]dx$$

Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore \quad I = \int_0^{\pi} e^{\cos^2 x}\cos^3[(2n+1)\pi-(2n+1)x]dx$$

$$I = \int_0^{\pi} (-e^{\cos^2 x}\cos^3)(2n+1)x\,dx \quad ...(2)$$

Adding (1) and (2) we get

$2I = 0 \Rightarrow I = 0$

6. **(d)** We have,

$$I = \int_{-\pi/2}^{\pi/2} \{f(x)+f(-x)\}\{g(x)-g(-x)\}dx$$

Let $F(x) = (f(x)+f(-x))(g(x)-g(-x))$

then $F(-x) = (f(-x)+f(x))(g(-x)-g(x))$

$= -[f(x)+f(-x)][g(x)-g(-x)]$

$= -F(x)$

$\therefore$ $F(x)$ is an odd function, $\therefore$ we get $I = 0$

7. **(d)** Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^3 x} = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x+\cos^3 x}dx$ ... (1)

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^3\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right)+\cos^3\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^3 x+\sin^3 x}dx \quad ...(2)$$

Adding (1) and (2) we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x+\sin^3 x}{\sin^3 x+\cos^3 x}dx = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2}; \quad \therefore I = \frac{\pi}{4}$$

8. **(d)** $f(x) = A\sin(\pi x/2)+B$

$$\Rightarrow f'(x) = \frac{A\pi}{2}\cos\left(\frac{\pi x}{2}\right) \Rightarrow f'\left(\frac{1}{2}\right) = \frac{A\pi}{2}\cos\frac{\pi}{4} = \sqrt{2}$$

$$\Rightarrow A = 4/\pi \text{ and } \int_0^1 f(x)dx = \frac{2A}{\pi}$$

$$\Rightarrow \int_0^1\left[A\sin\left(\frac{\pi x}{2}\right)+B\right]dx = \frac{2A}{\pi}$$

$$\Rightarrow \left|-\frac{2A}{\pi}\cos\left(\frac{\pi x}{2}\right)+Bx\right|_0^1 = \frac{2A}{\pi}$$

$$\Rightarrow B+\frac{2A}{\pi} = \frac{2A}{\pi} \Rightarrow B = 0$$

9. **(a)** Let $I = \int_{\pi}^{2\pi}[2\sin x]dx$

$\pi \le x < 7\pi/6 \Rightarrow -1 \le 2\sin x < 0 \Rightarrow [2\sin x] = -1$

$7\pi/6 \le x < 11\pi/6 \Rightarrow -2 \le 2\sin x < -1$

$\Rightarrow [2\sin x] = -1$

$$\therefore \quad I = \int_{\pi}^{7\pi/6} -1dx + \int_{7\pi/6}^{11\pi/6} -2dx + \int_{11\pi/6}^{2\pi} -1dx$$

$$= \left(-\frac{7\pi}{6}+\pi\right)+2\left(-\frac{11\pi}{6}+\frac{7\pi}{6}\right)+\left(-2\pi+\frac{11\pi}{6}\right)$$

$$= -\frac{\pi}{6}-\frac{8\pi}{6}-\frac{\pi}{6} = -\frac{10\pi}{6} = \frac{-5\pi}{3}$$

10. **(a)** Given that $g(x) = \int_0^x \cos^4 t\,dt$

$$\therefore g(x+\pi) = \int_0^{x+\pi} \cos^4 t\,dt$$

$$= \int_0^{\pi} \cos^4 t\,dt + \int_{\pi}^{x+\pi} \cos^4 t\,dt$$

$g(x+\pi) = g(\pi)+I$, where $I = \int_{\pi}^{x+\pi} \cos^4 t\,dt$

Put $t = \pi + y, dt = dy$

also as $t \to \pi, y \to 0$

as $t \to x+\pi, y \to x$

$$\therefore \quad I = \int_0^x \cos^4(\pi+y)dy$$

$$= \int_0^x \cos^4 y\,dy = \int_0^x \cos^4 t\,dt = g(x)$$

$\therefore \quad g(x+\pi) = g(\pi)+g(x)$

11. **(a)** We have

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} \quad ...(1)$$

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi - x)}$$

$$\left[\text{Using the prop.} \int_a^b f(x)dx = \int_a^b (f(a+b-x)\,dx\right]$$

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x} \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right)dx$$

$$= \int_{\pi/4}^{3\pi/4} 2\text{cosec}^2 x\,dx = 2(-\cot x)_{\pi/4}^{3\pi/4}$$

$$= -2[\cot 3\pi/4 - \cot \pi/4] = -2(-1-1) = 4$$

$\Rightarrow$ I = 2

12. (c) In the range $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, we have to find the value of $[2\sin x]$.

$$[2\sin x] = \begin{cases} 2 & \text{if } x = \pi/2 \\ 1 & \text{if } \frac{\pi}{2} < x \le \frac{5\pi}{6} \\ 0 & \text{if } \frac{5\pi}{6} < x \le \pi \\ -1 & \text{if } \pi < x \le \frac{7\pi}{6} \\ -2 & \text{if } \frac{7\pi}{6} < x \le \frac{3\pi}{2} \end{cases}$$

Thus

$$I = \int_{\pi/2}^{5\pi/6} 1.dx + \int_{5\pi/6}^{\pi} 0.dx + \int_{\pi}^{7\pi/6} (-1)dx + \int_{7\pi/6}^{3\pi/2} (-2)dx$$

or $I = \left[\frac{5\pi}{6} - \frac{\pi}{2}\right] + 0 - 1\left[\frac{7\pi}{6} - \pi\right] - 2\left[\frac{3\pi}{2} - \frac{7\pi}{6}\right]$

$$= \frac{2\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6} = \frac{-3\pi}{6} = \frac{-\pi}{2}$$

13. (b) $g(x) = \int_0^x f(t)dt$

$$\Rightarrow \quad g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt$$

Now, $\frac{1}{2} \le f(t) \le 1$ for $t \in [0,1]$

We get $\int_0^1 \frac{1}{2}dt \le \int_0^1 f(t)dt \le \int_0^1 1dt$

(applying line integral on inequality)

$$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)dt \le 1 \qquad \ldots(1)$$

Again, $0 \le f(t) \le \frac{1}{2}$ for $t \in [1,2]$

We get $\int_1^2 0dt \le \int_1^2 f(t)dt \le \int_1^2 \frac{1}{2}dt$

(applying line integral on inequality)

$$\Rightarrow \quad 0 \le \int_1^2 f(t)dt \le \frac{1}{2} \qquad \ldots(2)$$

From (1) and (2), we get

$\frac{1}{2} \le \int_0^1 f(t)dt + \int_1^2 f(t)dt \le \frac{3}{2}$ or $\frac{1}{2} \le g(2) \le \frac{3}{2}$

$\Rightarrow$ $0 \le g(2) \le 2$ is the most appropriate solution.

14. (c) If $f(x) = \begin{Bmatrix} e^{\cos x}\sin x & for\,|x| \le 2 \\ 2 & \text{otherwise} \end{Bmatrix}$

$$= \begin{Bmatrix} e^{\cos x}\sin x & for -2 \le x \le 2 \\ 2 & \text{otherwise} \end{Bmatrix}$$

$$\int_{-2}^{3} f(x)dx = \int_{-2}^{2} f(x)dx + \int_{2}^{3} f(x)dx$$

$$= \int_{-2}^{2} e^{\cos x}\sin x dx + \int_2^3 2dx = 0 + 2[x]_2^3$$

[$\because$ $e^{\cos x}\sin x$ is an odd function.]

$= 2\,[3-2] = 2$

$$\therefore \quad \int_{-2}^{3} f(x)dx = 2$$

15. (b) Let $I = \int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$

We know that for $\frac{1}{e} < x < 1, \log_e x < 0$ and hence

$$\frac{\log_e x}{x} < 0$$

and for $1 < x < e^2$, $\log x > 0$ and hence $\frac{\log_e x}{x} > 0$

$$\therefore \quad I = \int_{1/e}^{1}\left[-\frac{\log_e x}{x}\right]dx + \int_1^{e^2} \frac{\log_e x}{x}dx$$

$$= -\frac{1}{2}\left[(\log_e x)^2\right]_{1/e}^{1} + \frac{1}{2}\left[(\log_e x)^2\right]_1^{e^2}$$

$$= \frac{1}{2} + 2 = \frac{5}{2}.$$

16. (c) $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$ .....(1)

Put x = – y then dx = – dy

$$I = \int_{\pi}^{-\pi} \frac{\cos^2 y}{1+a^{-y}} dy = \int_{-\pi}^{\pi} \frac{a^y \cos^2 y}{1+a^y} dy$$

$$I = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \left[\because \int_a^b f(y)dy = \int_a^b f(x)dx\right] \quad ...(2)$$

Adding (1) and (2),

$$2I = \int_{-\pi}^{\pi} \frac{(1+a^x)\cos^2 x}{(1+a^x)} dx = \int_{-\pi}^{\pi} \cos^2 x dx$$

$$2I = 2\int_0^{\pi} \cos^2 x dx \qquad \text{(even function)}$$

$$I = 2\int_0^{\pi/2} \cos^2 x dx \qquad ....(3)$$

$$[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x) = f(x)]$$

$$= 2\int_0^{\pi/2} \sin^2 x dx \qquad ....(4)$$

Adding (3) and (4)

$$2I = 2\int_0^{\pi/2} (\cos^2 x + \sin^2 x)dx = 2.\pi/2 = \pi$$

$\therefore I = \pi/2$

17. (b) The given lines are

$y = x - 1; y = -x - 1;$

$y = x + 1$ and $y = -x + 1$

which are two pairs of parallel lines and distance between the lines of each pair is $\sqrt{2}$. Also non parallel lines are perpendicular. Thus lines represents a square of side $\sqrt{2}$. Hence, area $= (\sqrt{2})^2 = 2$ sq. units.

18. (a) Here $f(x) = \int_1^x \sqrt{2-t^2} dt \Rightarrow f'(x) = \sqrt{2-x^2}$

Now the given equation $x^2 - f'(x) = 0$ becomes

$x^2 - \sqrt{2-x^2} = 0 \Rightarrow x^2 = \sqrt{2-x^2} \Rightarrow x = \pm 1$

19. (c) Given that T > 0 is a fixed real number. f is continuous $\forall x \in R$ such that $f(x+T) = f(x)$

$\Rightarrow$ f is a periodic function of period T

Also given $I = \int_0^T f(x)dx$

Then let $I_1 = \int_3^{3+3T} f(2x)dx$

Put $2x = z \Rightarrow dx = \frac{dz}{2}$

also as $x \to 3, z \to 6$; as $x \to 3+3T, z \to 6+6T$

$$I_1 = \frac{1}{2}\int_6^{6+6T} f(z)dz$$

$$= \frac{1}{2}\left[\int_6^T f(z)dz + \sum_{n=1}^{5}\int_{nT}^{(n+1)T} f(z)dz + \int_{6T}^{6T+6} f(z)dz\right]$$

Now, $\int_{nT}^{(n+1)T} f(z)dz = \int_0^T f(nT+u)du,$

where $z = nT + u$

$$= \int_0^T f(u)du = 1 \qquad [\because f(nT+u) = f(u)]$$

Similarly, we can show that

$$\int_{6T}^{6T+6} f(z)dz = \int_0^6 f(z)dz$$

$$\therefore I_1 = \frac{1}{2}\left[\int_6^T f(z)dz + 5I + \int_0^6 f(z)dz\right]$$

$$= \frac{1}{2}\left[\int_0^T f(z)dz + 5I\right] = \frac{1}{2}(6I) = 3\text{I}$$

20. (a) Let $I = \int_{-1/2}^{1/2} \left([x] + \ln\left(\frac{1+x}{1-x}\right)\right) dx$

$$= \int_{-1/2}^{1/2} [x]dx + \int_{-1/2}^{1/2} \ln\left(\frac{1+x}{1-x}\right) dx$$

$$= \int_{-1/2}^{0} -1dx + \int_0^{1/2} 0dx + 0$$

$$\left[\because \log\left(\frac{1+x}{1-x}\right) \text{is an odd function}\right]$$

$$= [-x]_{-1/2}^{0} = 0 - \left(\frac{1}{2}\right) = -1/2$$

21. (a) We have $I_{m,n} = \int_0^1 t^m (1+t)^n dt$

Intergrating by parts considering $(1+t)^n$ as first function, we get

$$I_{m,n} = \left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1 - \frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1} dt$$

$$I_{m,n} = \frac{2^n}{m+1} - \frac{n}{m+1} I_{m+1,n-1}$$

22. (d) We have $f(x)=\int_{x^2}^{x^2+1} e^{-t^2}dt$

Then $f'(x)=e^{-(x^2+1)^2}.2x-e^{-x^4}.2x$

[Using Leibnitz theorem, $\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)dt$

$=f[\psi(x)].\psi'(x)-f(\phi(x)).\phi'(x)]$

$=2x[e^{-(x^2+1)^2}-e^{-x^4}]$ $\qquad \because (x^2+1)^2>x^4$

$\Rightarrow \quad e^{+(x^2+1)^2}>e^{x^4} \Rightarrow e^{-(x^2+1)^2}<e^{-x^4}$

$\therefore e^{-(x^2+1)^2}-e^{-x^4}<0, \therefore f'(x)>0, \forall x<0$

$\therefore$ *f(x)* increases when $x<0$

23. (d) The curves given are

$y=\sqrt{x}$ ...(1)

$2y+3=x$ ...(2)

and x-axis $y=0$ ...(3)

Eqn. (1), $[y^2=x]$ represents right handed parabola but with +ve values of y i.e., part of curve lying above x-axis.

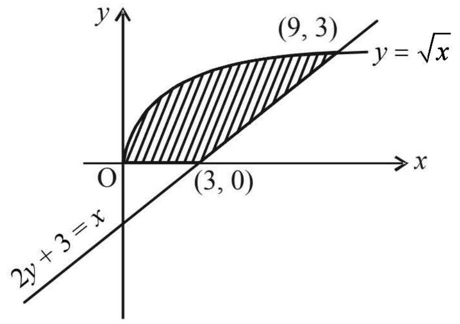
Solving (1) and (2) we get,

$2y+3=y^2$

$\Rightarrow \quad y^2-2y-3=0,\ (y-3)(y+1)=0$

$y=3$ (as $y\neq -ve$) $\Rightarrow x=9$

Also (2) meets x-axis at (3, 0)



Shaded area is the required area given by

$A=\int_0^9 \sqrt{x}dx-\int_3^9 \frac{x-3}{2}dx = \left[\frac{2x^{3/2}}{3}\right]_0^9-\frac{1}{2}\left[\frac{x^2}{2}-3x\right]_3^9$

$=\frac{2\times 27}{3}-\frac{1}{2}\left[\frac{81}{2}-27-\frac{9}{2}+9\right]$

$=\frac{54}{3}-\frac{1}{2}[18]=18-9=9$ sq. units

24. (a) $\int_0^{t^2} xf(x)dx=\frac{2}{5}t^5$ (Here, $t>0$)

Differentiating both sides w.r.t. t

[Using Leibnitz theorem]

$\Rightarrow \quad t^2f(t^2)\times 2t-0=\frac{2}{5}\times 5t^4 \Rightarrow f(t^2)=t$

Put $t=\frac{2}{5} \Rightarrow f\left(\frac{4}{25}\right)=\frac{2}{5}$

25. (b) $I=\int_0^1\sqrt{\frac{1-x}{1+x}}dx$

$=\int_0^1 \frac{1-x}{\sqrt{1-x^2}}dx=\sin^{-1}x\Big|_0^1\left(-\frac{1}{2}\right)\int_0^1\frac{2x}{\sqrt{1-x^2}}dx$

$=\frac{\pi}{2}+\frac{1}{2}\left[2\sqrt{1-x^2}\Big|_0^1\right]=\frac{\pi}{2}+(0-1)=\frac{\pi}{2}-1$

26. (a) $y=ax^2$ and $x=ay^2$

Points of intersection are $O\,(0,0)$ and $A\left(\frac{1}{a},\frac{1}{a}\right)$

$\therefore$ Area $=\int_0^{1/a}\left(\sqrt{\frac{x}{a}}-ax^2\right)dx=\frac{2}{3a^2}-\frac{1}{3a^2}$

$=\frac{1}{3a^2}=1 \Rightarrow a=\pm\frac{1}{\sqrt{3}}$

27. (c) $I=\int_{-2}^{0}[x^3+3x^2+3x+3+(x+1)\cos(x+1)]dx$

$=\left[\frac{x^4}{4}+x^3+\frac{3x^2}{2}+3x+(x+1)\sin(x+1)+\cos(x+1)\right]_{-2}^{0}$

$=(\sin 1+\cos 1)-(4-8+6-6+\sin 1+\cos 1)=4$

28. (d) The given curves are

$y=(x+1)^2$ ...(1)

upward parabola with vertex at (–1 ,0) meeting y–axis at (0, 1)

$y=(x-1)^2$ ...(2)

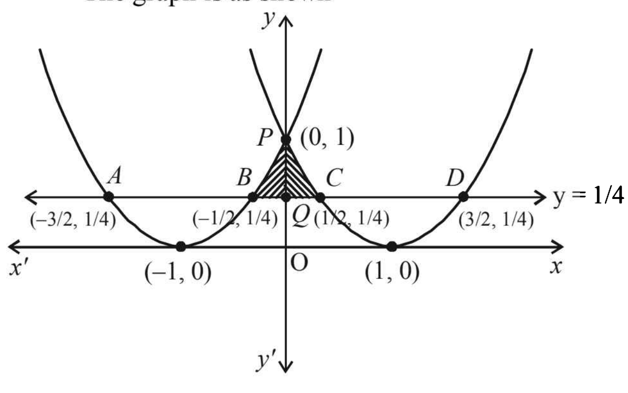
upward parabola with vertex at (1 ,0) meeting y–axis at (0, 1)

$y=1/4$ ... (3)

a line parallel to x–axis meeting (1) at $\left(-\frac{1}{2},\frac{1}{4}\right),\left(-\frac{3}{2},\frac{1}{4}\right)$

and meeting (2) at $\left(\frac{3}{2},\frac{1}{4}\right),\left(\frac{1}{2},\frac{1}{4}\right)$.

The graph is as shown



The required area is the shaded portion given by ar *(BPCQB)* =2 *Ar(PQCP)* (by symmetry)

$$=2\left[\int_0^{1/2}\left((x-1)^2-\frac{1}{4}\right)dx\right]=2\left[\left(\frac{(x-1)^3}{3}-\frac{x}{4}\right)_0^{1/2}\right]$$

$$=2\left[\left(-\frac{1}{24}-\frac{1}{8}\right)-\left(-\frac{1}{3}\right)\right]=2\left[\frac{-1-3+8}{24}\right]=\frac{1}{3}\text{ sq. units.}$$

29. (b) The given curves are

$$y=\sqrt{\frac{1+\sin x}{\cos x}}=\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} \quad ...(1)$$

and $$y=\sqrt{\frac{1-\sin x}{\cos x}}=\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}} \quad ...(2)$$

∴ The area bounded by the above curves, by the lines $x=0$ and $x=\frac{\pi}{4}$ is given by

$$A=\int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$=\int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

Let $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx=dt \Rightarrow dx=\frac{2}{1+t^2}dt$

Also when $x\to 0, t\to 0$ and when $x\to\frac{\pi}{4}$, $t\to\tan\frac{\pi}{8}$

$$\therefore A=\int_0^{\tan\frac{\pi}{8}}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt=\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$$

30. (c) Given that f is a non negative function defined on $[0, 1]$ and $\int_0^x\sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)dt,\ 0\le x\le 1$

Differentiating both sides with respect to x, we get

$$\sqrt{1-[f'(x)]^2}=f(x)$$

$$\Rightarrow 1-[f'(x)]^2=[f(x)]^2 \Rightarrow [f'(x)]^2=1-[f(x)]^2$$

$$\Rightarrow \frac{d}{dx}f(x)=\pm\sqrt{1-[f(x)]^2} \Rightarrow \pm\frac{d\,f(x)}{\sqrt{1-[f(x)]^2}}=dx$$

Integrating both sides with respect to x, we get

$$\pm\int\frac{d\,f(x)}{\sqrt{1-[f(x)]^2}}=\int dx \Rightarrow \pm\sin^{-1}f(x)=x+C$$

∵ Given that $f(0)=0 \Rightarrow C=0$

Hence $f(x)=\pm\sin x$

But as $f(x)$ is a non negative function on [0, 1]

∴ $f(x)=\sin x$.

Now $\sin x<x, \forall x>0$

$$\therefore\quad f\left(\frac{1}{2}\right)<\frac{1}{2}\text{ and } f\left(\frac{1}{3}\right)<\frac{1}{3}.$$

31. (b) $\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t\,ln(1+t)}{t^4+4}dt \quad \left[\frac{0}{0}\text{form}\right]$

Applying L' Hospital's rule, we get

$$\lim_{x\to 0}\frac{\frac{x\,ln(1+x)}{x^4+4}}{3x^2}=\lim_{x\to\infty}\frac{ln\,(1+x)}{x}\cdot\frac{1}{3(x^4+4)}$$

$$=1.\frac{1}{12}=\frac{1}{12}$$

32. (b) $e^{-x}f(x)=2+\int_0^x\sqrt{1+t^4}\,dt\ \forall\ x\in(-1,1)$

At $x=0$, $f(0)=2$

Now on differentiating, we get

$$-e^{-x}f(x)+e^{-x}f'(x)=0\sqrt{1+x^4}$$

$$\Rightarrow -f(0)+f'(0)=1 \Rightarrow f'(0)=3$$

Now $f^{-1}(f(x))=x$

$$\Rightarrow \left[\left(f^{-1}\right)'(f(x))\right]f'(x)=1$$

$$\Rightarrow \left(f^{-1}\right)'(f(0))f'(0)=1 \Rightarrow \left(f^{-1}\right)'(2)=\frac{1}{3}$$

33. (a) $I=\frac{1}{2}\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}}\frac{2x\sin x^2}{\sin x^2+\sin(\ell n6-x^2)}dx$

Let $x^2=t \Rightarrow 2x\,dx=dt$

Also, when $x=\sqrt{\ell n2}$, $t=\ell n2$

when $x=\sqrt{\ell n3}$, $t=\ell n3$

$$\therefore I=\frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin t\,dt}{\sin t+\sin(\ell n6-t)} \quad ...(1)$$

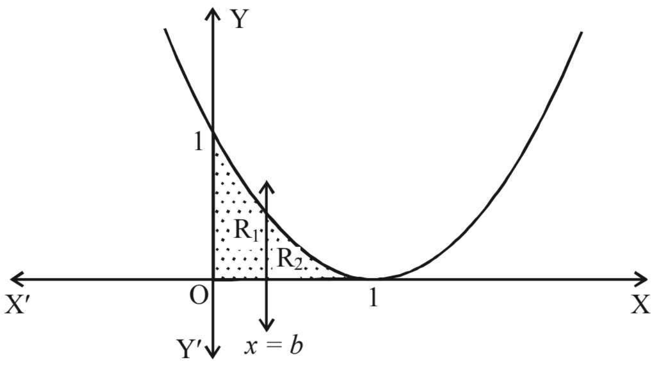
Using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

We get, $$I=\frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin(\ell n6-t)}{\sin t+\sin(\ell n6-t)}dt \quad ...(2)$$

Adding values of I in equation (1) and (2)

$$2I=\frac{1}{2}\int_{\ell n2}^{\ell n3}1dt=\frac{1}{2}(\ell n3-\ell n2)=\frac{1}{2}\ell n\frac{3}{2} \Rightarrow I=\frac{1}{4}\ell n\frac{3}{2}$$

34. (b) $R_1 = \int_0^b (x-1)^2 dx = \left[\frac{(x-1)^3}{3}\right]_0^b = \frac{(b-1)^3+1}{3}$



$R_2 = \int_b^1 (x-1)^2 dx = \left[\frac{(x-1)^3}{3}\right]_b^1 = -\frac{(b-1)^3}{3}$

As $R_1 - R_2 = \frac{1}{4} \Rightarrow \frac{2(b-1)^3}{3} + \frac{1}{3} = \frac{1}{4}$

or $(b-1)^3 = -\frac{1}{8}$ or $b - 1 = \frac{-1}{2}$ or $b = \frac{1}{2}$

35. (c) We have

$R_1 = \int_{-1}^{2} xf(x)dx = \int_{-1}^{2} (1-x)f(1-x)dx$

[Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$]

$\Rightarrow R_1 = \int_{-1}^{2} (1-x)f(x)dx$ [As $f(x) = f(1-x)$ on $[-1, 2]$]

$\therefore R_1 + R_1 = \int_{-1}^{2} xf(x)dx + \int_{-1}^{2} (1-x)f(x)dx$

$\Rightarrow 2R_1 = \int_{-1}^{2} f(x)dx = R_2$

36. (b) $\int_{-\pi/2}^{\pi/2} \left[x^2 + \ln\left(\frac{\pi+x}{\pi-x}\right)\right]\cos x\, dx$

$= \int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx + \int_{-\pi/2}^{\pi/2} \ln\left(\frac{\pi+x}{\pi-x}\right)\cos x\, dx$

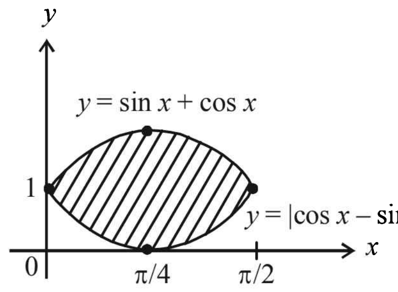
$= 2\int_0^{\pi/2} x^2 \cos x\, dx + 0$ [as $x^2 \cos x$ is an even function and $\ln\left(\frac{\pi+x}{\pi-x}\right) \cos x$ is an odd function]

$= 2[x^2 \sin x + 2x\cos x - 2\sin x]_0^{\pi/2}$

$= 2\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^2}{2} - 4$

37. (b) The rough graph of $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ suggest the required area is

$= \int_0^{\pi/2} [(\sin x + \cos x) - |\cos x - \sin x|]\, dx$



$= \int_0^{\pi/4} 2\sin x\, dx + \int_{\pi/4}^{\pi/2} 2\cos x\, dx$

$= 2\left[(-\cos x)_0^{\pi/4} + (\sin x)_{\pi/4}^{\pi/2}\right] = 2\sqrt{2}(\sqrt{2}-1)$

38. (d) We have $f'(x) - 2f(x) < 0$

$\Rightarrow e^{-2x} f'(x) - 2e^{-2x} f(x) < 0 \Rightarrow \frac{d}{dx}(e^{-2x} f(x)) < 0$

$\Rightarrow e^{-2x} f(x)$ is strictly decreasing function on $\left[\frac{1}{2}, 1\right]$

$\therefore\ e^{-2x} f(x) < e^{-1} f\left(\frac{1}{2}\right)$ or $f(x) < e^{2x-1}$

Also given that $f(x)$ is positive function so $f(x) > 0$

$\therefore\ 0 < f(x) < e^{2x-1}$

$\Rightarrow 0 < \int_{1/2}^{1} f(x)dx < \int_{1/2}^{1} e^{2x-1} dx$

$\Rightarrow 0 < \int_{1/2}^{1} f(x)dx < \left[\frac{e^{2x-1}}{2}\right]_{1/2}^{1}$

$\Rightarrow \int_{1/2}^{1} f(x)dx \in \left(0, \frac{e-1}{2}\right)$

39. (a) Let $I = \int_{\pi/4}^{\pi/2} (2\operatorname{cosec} x)^{17}\, dx$

$= \int_{\pi/4}^{\pi/2} (\operatorname{cosec} x + \cot x + \operatorname{cosec} x - \cot x)^{16}\, 2\operatorname{cosec} x\, dx$

$I = 2\int_{\pi/4}^{\pi/2} \left(\operatorname{cosec} x + \cot x + \frac{1}{\operatorname{cosec} x + \cot x}\right)^{16} . \operatorname{cosec} x\, dx$

Let $\operatorname{cosec} x + \cot x = e^u$

$\Rightarrow \left(-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x\right)dx = e^u du$

$\Rightarrow -\operatorname{cosec} x\, dx = du$

Also at $x = \frac{\pi}{4}$, $u = \ln\left(\sqrt{2}+1\right)$

at $x=\frac{\pi}{2}, u=ln\, 1=0$

$$\therefore\ I=-2\int_{ln(\sqrt{2}+1)}^{0}\left(e^{u}+e^{-u}\right)^{16}du$$

$$=\ 2\int_{0}^{ln(\sqrt{2}+1)}\left(e^{u}+e^{-u}\right)^{16}du$$

40. (a) $I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$ ....(i)

Applying $\int_a^b f(x)\,dx$

$=\int_a^b f(a+b-x)\,dx$, we get

$I=\int_{-\pi/2}^{\pi/2}\frac{e^x x^2\cos x}{1+e^x}dx$ ....(ii)

Adding (i) and (ii)

$$2I=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx=2\int_0^{\pi/2}x^2\cos x\,dx$$

$$I=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi^2}{4}-2$$

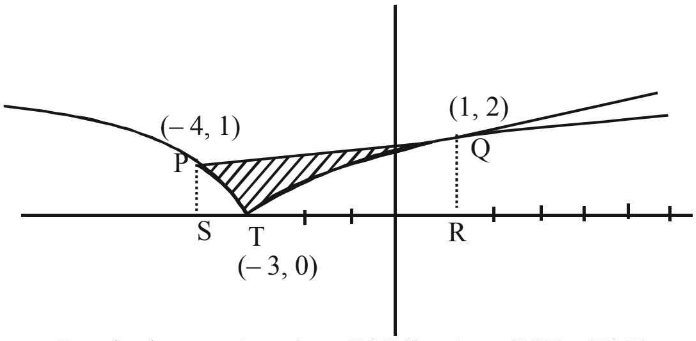
41. (c) $y\ge\sqrt{|x+3|}\Rightarrow y^2=|x+3|$

$\Rightarrow\ y^2=\begin{cases}-(x+3)\text{ if } x<-3\\ (x+3)\text{ if } x\ge-3\end{cases}$ ....(i)

Also $y\le\frac{x+9}{5}$ and $x\le 6$ ....(ii)

Solving (i) and (ii), we get intersection points as (1, 2), (6, 3), (–4, 1), (–39, –6)

The graph of given region is as follows-



Required area = Area (trap PQRS) – Area (PST + TQR)

$$=\frac{1}{2}\times(1+2)\times5-\left[\int_{-4}^{-3}\sqrt{-x-3}\,dx+\int_{-3}^{1}\sqrt{x+3}\,dx\right]$$

$$=\frac{15}{2}-\left[\left(\frac{2(-x-3)^{3/2}}{-3}\right)_{-4}^{-3}+\left(\frac{2(x+3)^{3/2}}{3}\right)_{-3}^{1}\right]$$

$$=\frac{15}{2}-\left[\frac{2}{3}+\frac{16}{3}\right]=\frac{15}{2}-6=\frac{3}{2}\text{ sq.units}$$

D. MCQs with ONE or MORE THAN ONE Correct

1. (a) $\int_0^x f(t)dt=x+\int_x^1 t\,f(t)dt$

Differentiating both sides w.r.t. x,

$f(x).1-f(0).0=1+1.f(1).0-xf(x).1$

$\therefore\ (x+1)f(x)=1,\quad\Rightarrow\ f(x)=\frac{1}{x+1};$

Put $x=1$ $\quad\therefore\ f(1)=\frac{1}{2}$

2. (a) $\int_{-1}^{1}f(x)dx=\int_{-1}^{1}x-[x]dx=\int_{-1}^{1}xdx-\int_{-1}^{1}[x]dx$

$$=0-\int_{-1}^{1}[x]dx \qquad ...(1)$$

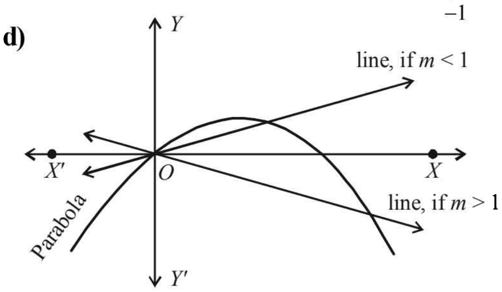
[$\because x$ is an odd function]

But $[x]=\begin{cases}-1, & \text{if } -1\le x<0\\ 0, & \text{if } 0\le x<1\\ 1, & \text{if } x=1\end{cases}$

$$\therefore\quad\int_{-1}^{1}[x]dx=\int_{-1}^{0}(-1)dx+\int_{0}^{1}0dx=-x]_{-1}^{0}+0=-1$$

Thus, putting value in equation (1) we get $\int_{-1}^{1}f(x)dx=1$

3. (b, d)



The two curves meet at

$mx=x-x^2$ or $x^2=x(1-m)$, $\quad\therefore\ x=0,1-m$

$\int(y_1-y_2)dx=\int(x-x^2-mx)dx$ (P, L)

Clearly m < 1 or m > 1, but m ≠ 1

Now, $\left[(1-m)\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{1-m}=\frac{9}{2}$, if m<1

or $(1-m)^3=27$, $\therefore\ m=-2$

But if m>1 then 1– m is – ive, then $\left[(1-m)\frac{x^2}{2}-\frac{x^3}{3}\right]_{1-m}^{0}=\frac{9}{2}$

$\therefore\quad(1-m)^3=-27$, or $1-m=-3$, $\therefore$ m = 4.

4. (a, b, c, d)

$\therefore f(x)$ is a non constant twice differentiable function such that $f(x)=f(1-x)\Rightarrow f'(x)=-f'(1-x)$ ...(1)

For $x=\frac{1}{2}$, we get $f'\left(\frac{1}{2}\right)=-f'\left(1-\frac{1}{2}\right)$

$\Rightarrow f'\left(\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)=0 \Rightarrow f'\left(\frac{1}{2}\right)=0$

$\Rightarrow$ (b) is correct

For $x=\frac{1}{4}$, we get $f'\left(\frac{1}{4}\right)=-f'\left(\frac{3}{4}\right)$

but given that $f'\left(\frac{1}{4}\right)=0 \quad \therefore f'\left(\frac{1}{4}\right)=f'\left(\frac{3}{4}\right)=0$

Hence, $f'(x)$ satisfies all conditions of Rolle's theorem for $x\in\left[\frac{1}{4},\frac{1}{2}\right]$ and $\left[\frac{1}{2},\frac{3}{4}\right]$. So there exists at least one point $C_1\in\left(\frac{1}{4},\frac{1}{2}\right)$ and at least one point $C_2\in\left(\frac{1}{2},\frac{3}{4}\right)$. Such that $f''(C_1)=0$ and $f''(C_2)=0$

$\therefore f''(x)$ varishes at least twice on [0, 1] $\Rightarrow$ (a) is correct.

Also using $f(x)=f(1-x)$

$\Rightarrow f\left(x+\frac{1}{2}\right)=f\left(1-x-\frac{1}{2}\right)=f\left(-x+\frac{1}{2}\right)$

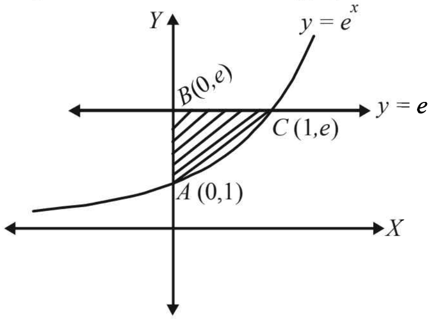
$\Rightarrow f\left(x+\frac{1}{2}\right)$ is an even function.

$\Rightarrow \sin x.\ f\left(x+\frac{1}{2}\right)$ is an odd function.

$\Rightarrow \int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx=0$, $\therefore$ (c) is correct.

5. **(b, c, d)**

The area bounded by the curve $y=e^x$ and lines $x=0$ and $y=e$ is as shown in the graph.



Required area $=\int_0^1 (e-e^x)dx=[ex]_0^1-\int_0^1 e^x dx$

$=e-\int_0^1 e^x dx.=1$

Also required area

$=\int_0^e x\,dy$ (where $e^x=y \Rightarrow x=\ln y$)

$=\int_1^e \ln y\,dy$

$=\int_1^e \ln(e+1-y)dy$ $\left[\text{Using the property } \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

6. **(a, b, c)** We have

$$I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx \quad \text{....(1)}$$

$$\Rightarrow I_n=\int_{-\pi}^{\pi}\frac{\sin n(-x)}{(1+\pi^{-x})\sin(-x)}dx$$

$$\left[\text{Using } \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right]$$

$$\Rightarrow I_n=\int_{-\pi}^{\pi}\frac{\pi^x\sin nx}{(1+\pi^x)\sin x}dx \quad \text{...(2)}$$

Adding equations (1) and (2), we get

$$2I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{\sin x}dx=2\int_0^{\pi}\frac{\sin nx}{\sin x}dx$$

[as integrand is an even function]

$$\Rightarrow I_n=\int_0^{\pi}\frac{\sin nx}{\sin x}dx$$

Now $I_{n+2}-I_n=\int_0^{\pi}\frac{\sin(n+2)x-\sin nx}{\sin x}dx$

$$=\int_0^{\pi}\frac{2\cos(n+1)x\sin x}{\sin x}dx=2\int_0^{\pi}\cos(n+1)x\,dx$$

$$=2\left[\frac{\sin(n+1)x}{n+1}\right]_0^{\pi}=0$$

$\therefore\ I_{n+2}=I_n$

Also $I_1=\int_0^{\pi}1dx=\pi$ and $I_0=0$

Hence $\sum_{m=1}^{10} I_{2m+1}=I_3+I_5+I_7+...I_{21}$

$=10\,I$ (using $I_{n+2}=I_n$) $=10\ \pi$

and $\sum_{m=1}^{10} I_{2m}=I_2+I_4+I_6+...+I_{20}$

$=20\times I_0$ (using $I_{n+2}=I_n$)

$=20\times 0=0$

7. **(a)** $\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx=\int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx$

$$=\left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x-4\tan^{-1}x\right]_0^1$$

$$=\frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-4.\frac{\pi}{4}=\frac{22}{7}-\pi$$

8. **(b, c)** We have

$$f(x) = lnx + \int_0^x \sqrt{1+\sin t}\, dt$$

$\Rightarrow f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$ which exists $\forall x \in (0,\infty)$

and $f'(x)$ has finite value $\forall x \in (0,\infty)$, *so* $f'(x)$ is continuous

Also $f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$

Which does not exist at the points where

$\sin x = -1$ like $x = \frac{3\pi}{2}, \frac{7\pi}{2}, ...$

$\therefore f'(x)$ is not differentiable.

$\therefore$ (a) is false but (b) is true

Now $\sqrt{1+\sin t} \geq 0 \Rightarrow \int_0^x \sqrt{1+\sin t}\,dt \geq 0 \forall x \in (0,\infty)$

And $lnx > 0 \forall x \in (1,\infty) \Rightarrow f(x) > 0 \forall x \in (1,\infty)$

For $x \geq e^3$

$$f(x) = ln\, x + \int_0^x \sqrt{1+\sin t}\, dt \geq 3$$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x} \leq \frac{1}{x} + \sqrt{2}, \forall x > 0$$

Now for $x \geq e^3$

$\Rightarrow 0 < f'(x) \leq \frac{1}{x} + \sqrt{2} < \frac{1}{e^3} + \sqrt{2} < 3 \;\forall x \in (e^3, \infty)$

$\Rightarrow |f'(x)| < |f(x)|$

$\therefore$ (c) is true.

Also $\lim_{x\to\infty} f(x) = \infty$

$\therefore |f(x)| + |f'(x)|$ is not bounded.

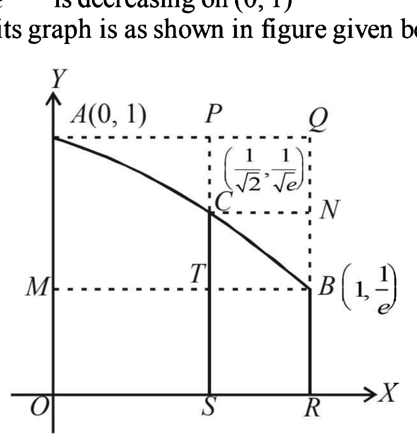
$\therefore$ (d) is wrong.

9. **(a, b, d)** First of all let us draw a rough sketch of $y = e^{-x}$.
At $x = 0, y = 1$ and at $x = 1, y = 1/e$

Also $\frac{dy}{dx} = -2xe^{-x^2} < 0 \;\; \forall\; x \in (0,1)$

$\therefore y = e^{-x^2}$ is decreasing on (0, 1)

Hence its graph is as shown in figure given below



Now, S = area exclosed by curve = ABRO

and area of rectangle ORBM $= \frac{1}{e}$

Clearly $S > \frac{1}{e}$ $\quad\therefore$ A is true.

Also $x^2 < x \;\; \forall\; x \in [0,1]$

$\Rightarrow -x^2 > -x \;\; \Rightarrow e^{-x^2} \geq e^{-x} \;\; \forall\; x \in [0,1]$

$\Rightarrow \int_0^1 e^{-x^2} dx > \int_0^1 e^{-x} dx = 1 - \frac{1}{e}$

$\Rightarrow S > 1 - \frac{1}{e}$ $\quad\therefore$ (b) is true.

Now S < area of rectangle APSO + area of rectangle CSRN

$\Rightarrow \quad S < \frac{1}{\sqrt{2}} \times 1 + \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{e}}$

$\therefore \quad S < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$ $\quad\because$ (d) is true

Also as $\frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) < 1 - \frac{1}{e}$ $\quad\therefore$ *(c)* is incorrect.

10. **(a, c)** Let $F(t) = e^t(\sin^6 at + \cos^6 at)$

Then $F(k\pi + t) = e^{k\pi + t}[\sin^6(k\pi + t)a + \cos^6(k\pi + t)a]$

$= e^{k\pi}.e^t[\sin^6 at + \cos^6 at]$ for even values of a.

$\therefore \quad F(k\pi + t) = e^{k\pi} F(t)$ ...(i)

Now $\int_0^{4\pi} F(t)dt = \int_0^{\pi} F(t)dt + \int_{\pi}^{2\pi} F(t)dt + \int_{2\pi}^{3\pi} F(t)dt + \int_{3\pi}^{4\pi} F(t)dt$

Also $\int_{\pi}^{2\pi} F(t)dt = \int_0^{\pi} F(\pi + x)dx$ (putting $t = \pi + x$)

$= \int_0^{\pi} e^{\pi} F(x)dx$ using eq^n(i) $= e^{\pi}\int_0^{\pi} F(t)dt$

Similarly $\int_{2\pi}^{3\pi} F(t)dt = e^{2\pi}\int_0^{\pi} F(t)dt$

$\int_{3\pi}^{4\pi} F(t)dt = e^{3\pi}\int_0^{\pi} F(t)dt$

$\therefore \quad \int_0^{4\pi} F(t)dt = (1 + e^{\pi} + e^{2\pi} + e^{3\pi})\int_0^{\pi} F(t)dt$

$\Rightarrow \quad \frac{\int_0^{4\pi} F(t)dt}{\int_0^{\pi} F(t)dt} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$, where *'a'* can take any even value.

11. **(a, b)** $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$

$= (7\tan^4 x - 3)(\tan^4 x + \tan^2 x)$

$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$

$$\int_0^{\pi/4} f(x)dx = \left[\tan^7 x - \tan^3 x\right]_0^{\pi/4} = 1 - 1 = 0$$

$\therefore \quad \int_0^{\pi/4} xf(x)dx = \left[x\left(\tan^7 x - \tan^3 x\right)\right]_0^{\pi/4}$

$\qquad - \int_0^{\pi/4}\left(\tan^7 x - \tan^3 x\right)dx$

$= \int_0^{\pi/4} \tan^3 x(1-\tan^2 x)\ \sec^2 x\, dx = \left[\frac{\tan^4 x}{4} - \frac{\tan^6 x}{6}\right]_0^{\pi/4}$

$= \frac{1}{12}$

12. **(d)** $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$

$\Rightarrow \frac{192x^3}{3} \le f'(x) \le \frac{192x^3}{2} \Rightarrow 64x^3 \le f'(x) \le 96x^3$

$\Rightarrow \int_{1/2}^{x} 64x^3 dx \le \int_{1/2}^{x} f'(x)dx \le \int_{1/2}^{x} 96x^3 dx$

$\Rightarrow \frac{64x^4}{4} - \frac{64}{4}\times\frac{1}{16} \le \int_{1/2}^{x} f'(x)dx \le \frac{96x^4}{4} - \frac{96}{4\times16}$

$\Rightarrow 16x^4 - 1 \le \int_{1/2}^{x} f'(x) \le 24x^4 - \frac{3}{2}$

$\Rightarrow 16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}$

$\Rightarrow \int_{1/2}^{1}(16x^4 - 1)dx \le \int_{1/2}^{1} f(x)dx \le \int_{1/2}^{1}\left(24x^4 - \frac{3}{2}\right)dx$

$\Rightarrow \left(\frac{16x^5}{5} - x\right)_{\frac{1}{2}}^{1} \le \int_{1/2}^{1} f(x)dx \le \left[\frac{24x^5}{5} - \frac{3}{2}x\right]_{\frac{1}{2}}^{1}$

$\Rightarrow 2.6 \le \int_{1/2}^{1} f(x)dx \le 3.9$

$\therefore$ Only (d) is the correct option.

13. **(b, c)**

$$f(x) = \lim_{n\to\infty}\left[\frac{n^n(x+n)\left(x+\frac{n}{2}\right)...\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)...\left(x^2+\frac{n^2}{n^2}\right)}\right]^{x/n}$$

$$= \lim_{n\to\infty}\left[\frac{n^{2n}\left(\frac{x}{n}+1\right)\left(\frac{x}{n}+\frac{1}{2}\right)....\left(\frac{x}{n}+\frac{1}{n}\right)}{n^{2n}\cdot n!\left(\frac{x^2}{n^2}+1\right)\left(\frac{x^2}{n^2}+\frac{1}{2^2}\right)....\left(\frac{x^2}{n^2}+\frac{1}{n^2}\right)}\right]^{x/n}$$

$$= \lim_{n\to\infty}\left[\frac{\left(\frac{x}{n}+1\right)\left(\frac{x}{n}+\frac{1}{2}\right)....\left(\frac{x}{n}+\frac{1}{n}\right)}{\left(1.\frac{x^2}{n^2}+1\right)\left(2.\frac{x^2}{n^2}+\frac{1}{2}\right)....\left(n.\frac{x^2}{n^2}+\frac{1}{n}\right)}\right]^{x/n}$$

$$\Rightarrow \ln f(x) = \lim_{n\to\infty}\frac{x}{n}\left[\sum_{r=1}^{n}\ln\left(\frac{x}{n}+\frac{1}{r}\right) - \sum_{r=1}^{n}\ln\left(\frac{rx^2}{n^2}+\frac{1}{r}\right)\right]$$

$$= \lim_{n\to\infty}\frac{x}{n}\left[\sum_{r=1}^{n}\left\{\ln\frac{1}{r}+\ln\left(\frac{rx}{n}+1\right)\right\} - \left\{\ln\frac{1}{r}+\ln\left(\frac{r^2x^2}{n^2}+1\right)\right\}\right]$$

$$= \lim_{n\to\infty}\frac{x}{n}\left[\sum_{r=1}^{n}\ln\left(1+\frac{rx}{n}\right) - \ln\left(1+\frac{r^2x^2}{n^2}\right)\right]$$

$$= x\int_0^1 \ln(1+xy)\,dy - x\int_0^1 \ln\left(1+x^2y^2\right)dy$$

Let $xy = t \Rightarrow x\,dy = dt$

$\therefore \quad \ln f(x) = \int_0^x \ln(1+t)\,dt - \int_0^x \ln\left(1+t^2\right)dt$

$\ln f(x) = \int_0^x \ln\left(\frac{1+t}{1+t^2}\right)dt$

$\Rightarrow \frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right)$

$\Rightarrow \frac{f'(2)}{f(2)} = \ln\left(\frac{3}{5}\right) < 0$

$\Rightarrow f'(2) < 0 \ \therefore$ (c) is correct

and $\frac{f'(3)}{f(3)} = \ln\left(\frac{2}{5}\right) < \frac{f'(2)}{f(2)}$ $\therefore$ (d) is not correct

Also $f'(x) = f(x)\ \ln\left(\frac{1+x}{1+x^2}\right) > 0, , \forall x \in (0, 1)$

$\therefore$ f is an increasing function.

$\therefore \quad \frac{1}{2} < 1 \Rightarrow f\left(\frac{1}{2}\right) \le f(1)$

$\therefore$ (a) is not correct

and $\frac{1}{3} < \frac{2}{3} \Rightarrow f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$

$\therefore$ (b) is correct

Hence (b) and (c) are the correct options.

E. Subjective Problems

1. To find the area bounded by

$x^2 = 4y$ ...(1)

which is an upward parabola with vertex at (0, 0).

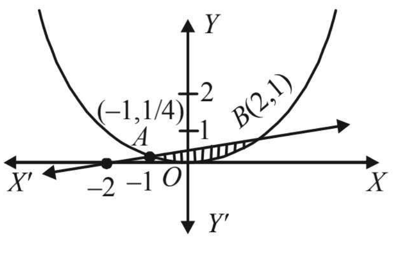
and $x - 4y = -2$ or $\frac{x}{-2} + \frac{y}{1/2} = 1a$ ...(2)

which is a st. line with its intercepts as –2 and 1/2 on axes. For Pt's of intersection of (1) and (2) putting value of 4y fom (2) in (1) we get

$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow \quad x = 2, -1 \quad \Rightarrow \quad y = 1, 1/4$

$\therefore \quad A(-1, 1/4)B(2, 1).$



Shaded region in the fig is the req area.

$\therefore$ Required area $= \int_{-1}^{2}(y_{line} - y_{parabola})dx$

$$= \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] = 9/8 \text{ sq. units}$$

2. We know that in integration as a limit sum

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f(r/n)$$

Similarly the given series can be written as

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{6n}\right) = \lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+\frac{r}{n}}$$

$$= \int_0^5 \frac{1}{1+x}dx = [\log|1+x|]_0^5 = \log 6 - \log 1 = \log 6$$

3. Let $I = \int_0^{\pi} xf(\sin x)dx$ ...(1)

$$\Rightarrow \quad I = \int_0^{\pi}(\pi - x)f(\sin x)dx$$

Adding (1) and (2), we get, $2I = \int_0^{\pi}\pi f(\sin x)dx$

$$I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$ Hence Proved.

4. $\int_{-1}^{3/2} |x\sin\pi x|\,dx$

For $-1 \le x < 0 \Rightarrow -\pi < px < 0 \Rightarrow \sin\pi x < 0$

$\Rightarrow x\sin\pi x > 0$

For $1 < x < 3/2 \Rightarrow \pi < \pi x < 3\pi/2 \Rightarrow \sin\pi x < 0$

$\Rightarrow x\sin\pi x < 0$

$$\therefore \int_{-1}^{3/2}|x\sin\pi x|\,dx = \int_{-1}^{1} x\sin\pi x\,dx + \int_{1}^{3/2}(-x\sin\pi x)ax$$

$$= 2\int_0^1 x\sin\pi x\,dx - \int_1^{3/2} x\sin\pi x\,dx$$

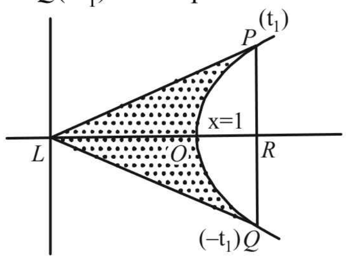
$$= 2\left[\frac{-x\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_0^1 - \left[\frac{-x\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$= 2\left[\left(\frac{-\cos\pi}{\pi} + 0\right) - (0+0)\right]$$

$$-\left[\left(\frac{-3/2\cos 3\pi/2}{\pi} + \frac{\sin 3\pi/2}{\pi^2}\right) - \left(\frac{-\cos\pi}{\pi} + \frac{\sin\pi}{\pi^2}\right)\right]$$

$$= 2\left[\frac{1}{\pi}\right] - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{2}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi} = \frac{3}{\pi} + \frac{1}{\pi^2}$$

5. Let $P(t_1)$ and $Q(-t_1)$ be two points on the hyperbola.



$$\text{Area (PRQOP)} = \int_{-t_1}^{t_1} ydx = \int_{-t_1}^{t_1}\left(\frac{e^t + e^{-t}}{2}\right)\left(\frac{dx}{dt}\right)dt$$

$$= \int_{-t_1}^{t_1}\left(\frac{e^t - e^{-t}}{2}\right)\frac{d}{dt}\left(\frac{e^t + e^{-t}}{2}\right).dt$$

$$= \int_{-t_1}^{t_1}\left(\frac{e^t - e^{-t}}{2}\right)dt = \int_{-t_1}^{t_1}\frac{e^{2t} + e^{-2t} - 2}{4}dt$$

$$= \left[\frac{e2t}{8} - \frac{e^{-2t}}{8} - \frac{2t}{4}\right]_{-t_1}^{t_1} = \frac{2}{8}(e^{2t_1} - e^{-2t_1} - 4t_1)$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - t_1 \quad ...(1)$$

Area of $\Delta LPR = \frac{1}{2}LR \times PQ = LR \times PR$

$$= \frac{e^{t_1} + e^{-t_1}}{2} \times \frac{e^{t_1} - e^{-t_1}}{2} = \frac{e^{2t_1} - e^{-2t_1}}{4} \quad ...(2)$$

$\therefore$ The required area $= Ar(\Delta LPQ) - Ar(PRQOP)$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{e^{2t_1} - e^{-2t_1}}{4} + t_1 = t_1$$

6. $I = \int_0^{\pi/4}\frac{\sin x + \cos x}{9 + 16\sin 2x}dx$

Let $\sin x - cox x = t \Rightarrow$ as $x \to 0, t \to -1$ as $x \to \pi/4, t \to 0$

$\Rightarrow \quad (\cos x + \sin x)dx = dt$

Also, $t^2 = 1 - \sin 2x \Rightarrow \sin 2x = 1 - t^2$

$$I = \int_{-1}^{0}\frac{dt}{9 + 16(1-t^2)} = \int_{-1}^{0}\frac{dt}{25 - 16t^2}$$

$$= \frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2 - t^2} = \frac{1}{16}.\frac{1}{2.\frac{5}{4}}\log\left[\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{40}\left[\log 1-\log\frac{1}{9}\right]=\frac{\log 9}{40}=\frac{2\log 3}{40}=\frac{1}{20}\log 3$$

7. $y=1+\frac{8}{x^2}$

Req. area $=\int_2^4 y\,dx=\int_2^4\left(1+\frac{8}{x^2}\right)dx=\left[x-\frac{8}{x}\right]_2^4=4$

If $x = 4a$ bisects the area then we have

$$\int_2^a\left(1+\frac{8}{x^2}\right)dx=\left[x-\frac{8}{x}\right]_2^a=\left[a-\frac{8}{a}-2+4\right]=\frac{4}{2}$$

$\Rightarrow\ a-\frac{8}{a}=0\Rightarrow a^2=0\ \Rightarrow\ a=\pm 2\sqrt{2}$

Since $2<a<4$ $\quad\therefore a=2\sqrt{2}$

8. Let $I=\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Put $x=\sin\theta\Rightarrow dx=\cos\theta\,d\theta$
Also when $x = 0, \theta = 0$
and when $x=1/2, \theta=\pi/6$

Thus, $I=\int_0^{\pi/6}\frac{\sin\theta\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$

$\Rightarrow\ I=\int_0^{\pi/6}\theta\sin\theta\,d\theta$

Intergrating the above by parts, we get

$$I=[\theta(-\cos\theta)]_0^{\pi/6}+\int_0^{\pi/6}1.\cos\theta d\theta$$

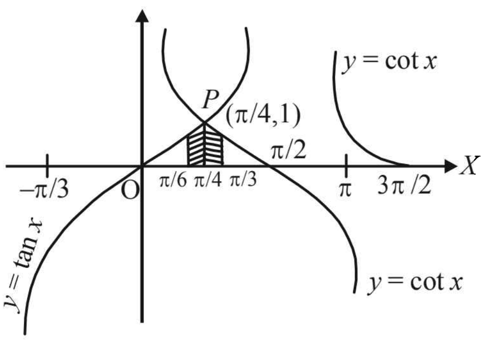
$$=[-\theta\cos\theta+\sin\theta]_0^{\pi/6}=\frac{-\pi}{6}.\frac{\sqrt{3}}{2}+\frac{1}{2}=\frac{6-\pi\sqrt{3}}{12}$$

9. To find the area bold by x - axis and curves

$y=\tan x, -\pi/3\le x\le\pi/3$ ...(1)

and $y=\cot x, \pi/6\le x\le 3\pi/2$ ...(2)

The curves intersect at P, where $\tan x = \cot x$, which is satisfied at $x=\pi/4$ within the given domain of x.



The required area is shaded area

$$A=\int_{\pi/6}^{\pi/4}\tan x\,dx+\int_{\pi/4}^{\pi/3}\cot x\,dx$$

$$=[\log\sec x]_{\pi/6}^{\pi/4}+[\log\sin x]_{\pi/4}^{\pi/3}$$

$$=\left(\log\sqrt{2}-\log\frac{2}{\sqrt{3}}\right)+\left(\log\frac{\sqrt{3}}{2}-\log\frac{1}{\sqrt{2}}\right)$$

$$=2\left(\log\sqrt{2}.\frac{\sqrt{3}}{2}\right)=2\log\sqrt{\frac{3}{2}}=\log 3/2\text{ sq. units}$$

10. Let $\int f(x)dx=F(x)+c$

Then $F'(x)=f(x)$ ...(1)

Now $I=\int_a^{a+t}f(x)dx=F(a+t)-F(a)$

$\therefore\ \frac{dI}{da}=F'(a+t)-F(a)=f(a+t)-f(a)$

[Using eq. (1)]

$=f(a)-f(a)$ [Using given condition]
$=0$

This shows that I is independent of a.

11. Let $I=\int_0^{\pi/2}\frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ ...(1)

$$I=\int_0^{\pi/2}\frac{(\pi/2-x)\sin(\pi/2-x)\cos(\pi/2-x)}{\cos^4(\pi/2-x)+\sin^4(\pi/2-x)}dx$$

[Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$]

$\Rightarrow\ I=\int_0^{\pi/2}\frac{(\pi/2-x)\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ ...(2)

Adding (1) and (2), we get

$$2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$\Rightarrow\ I=\frac{\pi}{4}\int_0^{\pi/2}\frac{\sec^2 x\tan x}{\tan^4 x+1}dx$$

(Dividing Nr and Dr by $\cos^4 x$)

$$=\frac{\pi}{2\times 4}\int_0^{\pi/2}\frac{2\tan x\sec^2 x\,dx}{1+(\tan^2 x)^2}$$

Put $\tan^2 x=t\ \Rightarrow\ 2\tan x\sec^2 x\,dx=dt$
Also as $x\to 0, t\to 0$; as $x\to\pi/2, t\to\infty$

$$\therefore\ I=\frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2}=\frac{\pi}{8}[\tan^{-1}t]_0^{\infty}=\frac{\pi}{8}[\pi/2-0]=\pi^2/16$$

12. The given curves are

$$y = \sqrt{5-x^2} \quad \ldots(1)$$

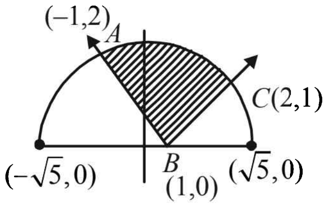
$$y = |x-1| \quad \ldots(2)$$

We can clearly see that (on squaring both sides of (1)) eq. (1) represents a circle. But as y is + ve sq. root, $\therefore$ (1) represents upper half of circle with centre (0, 0) and radius $\sqrt{5}$.

Eq. (2) represents the curve

$$y = \begin{cases} -x+1 & \text{if } x<1 \\ x-1 & \text{if } x \ge 1 \end{cases}$$

Graph of these curves are as shown in figure with point of intersection of $y=\sqrt{5-x^2}$ and $y=-x+1$ as $A(-1,2)$ and of $y=\sqrt{5-x^2}$ and $y=x-1$ as $C(2,1)$.



The required area = Shaded area

$$= \int_{-1}^{2} (y_{(1)} - y_{(2)})dx = \int_{-1}^{2} \sqrt{5-x^2}dx - \int_{-1}^{2} |x-1|\,dx$$

$$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2} - \int_{-1}^{1}\{-(x-1)\}\,dx - \int_{1}^{2}(x-1)dx$$

$$= \left(\frac{2}{2}\sqrt{5-4} + \frac{5}{4}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(\frac{-1}{2}\sqrt{5-1} + \frac{5}{2}\sin{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$$

$$- \left(\frac{-x^2}{2}+x\right)_{-1}^{1} - \left(\frac{x^2}{2}-x\right)_{1}^{2}$$

$$= 1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} + 1 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$-\left[\left(\frac{-1}{2}+1\right) - \left(\frac{-1}{2}-1\right)\right] - \left[(2-2) - \left(\frac{1}{2}-1\right)\right]$$

$$= 2 + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right] - 2 - \frac{1}{2}$$

$$= \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \cos^{-1}\frac{2}{\sqrt{5}}\right] - \frac{1}{2} = \frac{5}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

$$= \frac{5\pi-2}{4} \text{ square units.}$$

13. Let $I = \int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\sin x}$ ...(1)

$$I = \int_0^{\pi} \frac{(\pi-x)dx}{1+\cos\alpha(\sin(\pi-x))}$$

[Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi-x)dx}{1+\cos\alpha\sin x} \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x+\pi-x}{1+\cos\alpha\sin x}dx = \int_0^{\pi} \frac{\pi}{1+\cos\alpha\sin x}dx$$

$$\therefore \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos\alpha\sin x}dx = \frac{\pi}{2}.2\int_0^{\pi/2} \frac{1}{1+\cos\alpha\sin x}dx$$

$$= \pi\int_0^{\pi/2} \frac{1}{1+\cos\alpha.\dfrac{2\tan x/2}{1+\tan^2 x/2}}dx$$

$$= \pi\int_0^{\pi/2} \frac{\sec^2}{1+\tan^2 x/2 + 2\cos\alpha\tan x/2}dx$$

Put $\tan x/2 = t$, $\frac{1}{2}\sec^2\frac{x}{2}dx = dt \Rightarrow \sec^2 x/2dx = 2dt$

Also when $x \to 0, t \to 0$ as $x \to \pi 2, t \to 1$

$$\therefore \quad I = \pi\int_0^1 \frac{2dt}{t^2 + (2\cos\alpha)t + 1}$$

$$= 2\pi\int_0^1 \frac{dt}{(t+\cos\alpha)^2 + 1 - \cos^2\alpha} = 2\pi\int_0^1 \frac{dt}{(t+\cos\alpha)^2 + \sin^2\alpha}$$

$$= 2\pi.\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^1$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{2\cos^2\alpha/2}{2\sin\alpha/2\cos\alpha/2}\right) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\cot\alpha/2) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\tan^{-1}(\pi/2-\alpha/2)) - \tan^{-1}(\tan(\pi/2-\alpha))\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2} + \alpha\right] = \frac{2\pi}{\sin\alpha}\left[\frac{\alpha}{2}\right] = \frac{\pi\alpha}{\sin\alpha}$$

14. We have to find the area bounded by the curves

$$x^2 + y^2 = 25 \quad ...(1)$$

$$4y = \left|4 - x^2\right| \quad ...(2)$$

$$x = 0 \quad ...(3)$$

and above x-axis.

Now, $4y = |4x - x^2| = \begin{cases} 4 - x^2, \text{if } x^2 < 4 \\ x^2 - 4, \text{if } x^2 \geq 4 \end{cases}$

$$4y = \begin{cases} 4 - x^2, \text{if } -2 < x < 2 \\ x^2 - 4, \text{if } x \geq 2 \text{ or } \leq -2 \end{cases}$$

Thus we have three curves

(I) Circle $x^2 + y^2 = 25$

(II) P_1 : Parabola, $x^2 = -4(y-1), -2 \leq x \leq 2$

(III) P_2 : Parabola, $x^2 = 4(y+1), x \geq 2 \text{ or } x \leq -2$

(I) and (II) intersect at $-4y + 4 + y^2 = 25$

or $(y-2)^2 = 5^2 \therefore y - 2 = \pm 5$

$y = 7, y = -3$

$y = -3, 7$ are rejected since.

$y = -3$ is below x-axis and

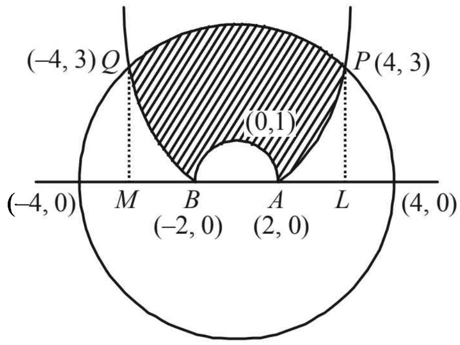
$y = 7$ gives imaginary value of x. So, (I) and (II) do not intersect but II intersects x-axis at (2, 0) and (–2, 0).

(I) and (III) intersect at

$$4y + 4 + y^2 = 25 \quad \text{or} \quad (y+2)^2 = 5^2$$

$\therefore \quad y + 2 = \pm 5 \qquad \therefore \quad y = 3, -7.$

$y = -7$ is rejected, $y = 3$ gives the points above x-axis. When $y = 3$, $x = \pm 4$. Hence the points of intersection of (I) and (III) are (4, 3) and (– 4, 3). Thus we have the shape of the curve as given in figure.



Required area is

$$= 2\left[\int_0^4 y_{circle}dx - \int_0^2 y_{P_1}dx - \int_2^4 y_{P_2}dx\right]$$

$$= 2\left[\int_0^4 \sqrt{25 - x^2}\,dx - \frac{1}{4}\int_0^2 (4 - x^2)dx - \frac{1}{4}\int_0^4 (x^2 - 4)dx\right]$$

$$= 2\left[\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^4 - \frac{1}{4}\left(4x - \frac{x^3}{3}\right)_0^2 - \frac{1}{4}\left(\frac{x^3}{3} - 4x\right)_2^4\right.$$

$$= 2\left[6 + \frac{25}{2}\sin^{-1}\frac{4}{5} - \frac{4}{3} - \frac{4}{3} - \frac{4}{3}\right]$$

$$= 12 + 25\sin^{-1}\frac{4}{5} - 8 = 4 + 25\sin^{-1}\frac{4}{5}$$

15. The given curve is $y = \tan x$ ...(1)

Let P be the point on (1) where $x = \pi/4$

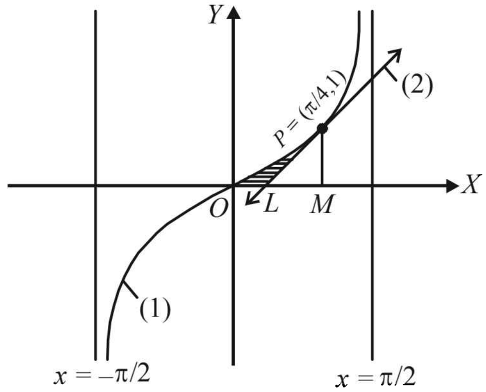
$\therefore \quad y = \tan \pi/4 = 1$

i.e. co-ordinates of P are $(\pi/4, 1)$

$\therefore$ Equation of tangent at P is $y - 1 = 2(x - \pi/4)$

or $y = 2x + 1 - \pi/2$ ...(2)

The graph of (1) and (2) are as shown in the figure.



Tangent (2) meets x-axis at, $L\left(\frac{\pi - 2}{4}, 0\right)$

Now the required area = shaded area

= Area $OPMO - Ar(\Delta PLM)$

$$= \int_0^{\pi/4} \tan x\,dx - \frac{1}{2}(OM - OL)PM$$

$$= [\log \sec x]_0^{\pi/4} - \frac{1}{2}\left\{\frac{\pi}{4} - \frac{\pi - 2}{4}\right\}.1 = \frac{1}{2}\left[\log 2 - \frac{1}{2}\right] \text{ sq.units.}$$

16. Let $I = \int_0^1 1.\log[\sqrt{1-x} + \sqrt{1+x}]dx$

Intergrating by parts, we get

$$I = [x\log(\sqrt{1-x} + \sqrt{1+x})]_0^1$$

$$-\int_0^1 x.\frac{1}{\sqrt{1-x} + \sqrt{1+x}}.\left[\frac{-1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}}\right]dx$$

$$= \log\sqrt{2} - \int_0^1 x\frac{(\sqrt{1+x} - \sqrt{1-x})}{(\sqrt{1+x} + \sqrt{1-x})(\sqrt{1+x} - \sqrt{1-x})}.\frac{(\sqrt{1-x} - \sqrt{1+x})}{2\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1 \frac{x(\sqrt{1+x} - \sqrt{1-x})^2}{(1 + x - 1 + x)\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1 \frac{1 + x + 1 - x - 2\sqrt{1-x^2}}{2\sqrt{1-x^2}}dx$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}}dx - \frac{1}{2}\int_0^1 1dx$$

$$= \frac{1}{2}\left[\log 2 + \left(\sin^{-1}x\right)_0^1 - (x)_0^1\right] = \frac{1}{2}[\log 2 + \pi/2 - 1]$$

17. Let $I = \int_0^a f(x)g(x)dx = \int_0^a f(a-x)g(a-x)dx$

[Using the prop. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$= \int_0^a f(x)(2-g(x))dx$

As given that $f(a-x) = f(x)$ and $g(a-x)+g(x) = 2$

$= 2\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$; $\therefore\ I = 2\int_0^a f(x)dx - I$

$\Rightarrow\ 2I = 2\int_0^a f(x)dx \Rightarrow I = \int_0^a f(x)dx$

Hence the result.

18. We have, $I = \int_0^{\pi/2} f(\sin 2x)\cos x dx$ ...(1)

$I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$ ...(2)

[Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

Adding (1) and (2), we get

$2I = \int_0^{\pi/2} f(\sin 2x)(\cos x + \sin x)dx$

$\Rightarrow\ 2I = 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$

[Using the property,

$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ when $f(2a-x) = f(x)$]

$\Rightarrow\ I = \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$

$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin(\pi/4 + x)dx$

$= \sqrt{2}\int_0^{\pi/4} f\left[\sin\left(2\left(\frac{\pi}{4} - x\right)\right)\right]\sin(\pi/4 + \pi/4 - x)dx$

[Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x dx$ Hence Proved.

19. To prove : $\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + ... + \cos(2k-1)x]$

It is equivalent to prove that

$\sin 2kx = 2\sin x\cos x + 2\cos 3x\sin x + ... + 2\cos(2k-1)x\sin x$

Now, R.H.S.

$= (\sin 2x) + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + ... + (\sin 2kx - \sin(2k-2)x)$

$= \sin 2kx$ = L.H.S. Hence Proved.

Now $\int_0^{\pi/2} \sin 2kx \cot x dx = \int_0^{\pi/2} \frac{\sin 2kx}{\sin x}.\cos x dx$

$= \int_0^{\pi/2} 2(\cos x + \cos 3x + ... + \cos(2k-1)x)\cos x dx$

[Using the identity proved above]

$= \int_0^{\pi/2} [2\cos^2 x + 2\cos 3x\cos x + 2\cos 5x\cos x + ... + 2\cos(2k-1)x\cos x]dx$

$= \int_0^{\pi/2} [(1+\cos 2x) + (\cos 4x + \cos 2x) + (\cos 6x + \cos 4x) + ... + \{(\cos 2kx) + \cos(2k-2)x\}]dx$

$= \int_0^{\pi/2} 1 + 2[\cos 2x + \cos 4x + \cos 6x + ... + \cos(2k-2)x] + \cos 2kx\, dx$

$= \left[x + 2\left\{\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + ... + \frac{\sin(2k-2)x}{2k-2}\right\} + \frac{\sin 2kx}{2k}\right]_0^{\pi/2}$

$= \pi/2$ [$\because \sin n\pi = 0, \forall n \in N$]

Hence Proved

20. The given curves are

$y = ex\log_e x$ ...(1)

and $y = \frac{\log_e x}{ex}$ ...(2)

The two curves intersect where $ex\log x = \frac{\log x}{ex}$

$\Rightarrow\ \left(ex - \frac{1}{ex}\right)\log x = 0 \Rightarrow x = \frac{1}{e}$ or $x = 1$

At $x = 1/e$ or $ex = 1$, $\log x = -\log e = -1, y = -1$

So that $\left(\frac{1}{e}, -1\right)$ is one point of intersection and at $x = 1$,

$\log 1 = 0\ \therefore\ y = 0$

$\therefore$ (1, 0) is the other common point of intersection of the curves. Now in between these two points, $\frac{1}{e} < x < 1$

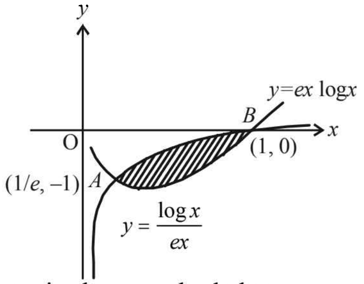
or $\log\left(\frac{1}{e}\right) < \log x < \log 1$, or $-1 < \log x < 0$

i.e. $\log x$ is – ve, throughout

$\therefore \quad y_1 = ex\log_e x, y_2 = \frac{\log_e x}{ex}$

Clearly under the condition stated above $y_1 < y_2$ both being –ve in the interval $\frac{1}{e} < x < 1$.

The rough sketch of the two curves is as shown in fig. and shaded area is the required area.



$\therefore$ The required area = shaded area

$$= \left|\int_{1/e}^{1}(y_1 - y_2)\,dx\right| = \left|\int_{1/e}^{1}\left[ex\log x - \frac{\log x}{ex}\right]dx\right|$$

$$= \left|e\int_{1/e}^{1} x\log x - \frac{1}{e}\int_{1/e}^{1}\frac{\log x}{x}dx\right|$$

$$= \left|e\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_{1/e}^{1} - \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1}\right|$$

$$= \left|e\left[\left(-\frac{1}{4}\right) - \left(-\frac{1}{2e^2} - \frac{1}{4e^2}\right)\right] - \frac{1}{e}\left[0 - \frac{1}{2}\right]\right|$$

$$= \left|e\left[-\frac{1}{4} + \frac{3}{4e^2}\right] + \frac{1}{2e}\right| = \left|\frac{-e}{4} + \frac{3}{4e} + \frac{1}{2e}\right|$$

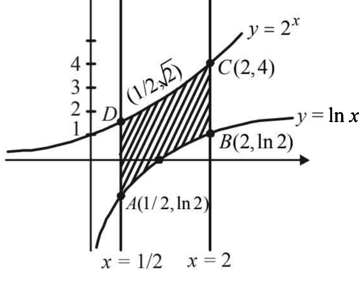
$$= \left|\frac{5}{4e} - \frac{e}{4}\right| = \left|\frac{5 - e^2}{4e}\right| = \frac{e^2 - 5}{4e}$$

21. The given curves are

$x = \frac{1}{2}$ ... (1), $x = 2$...(2), $y = \ln x$...(3), $y = 2^x$ ...(4)

Clearly (1) and (2) represent straight lines parallel to y - axis at distances 1/2 and 2 units from it, respectively. Line $x = \frac{1}{2}$ meets (3) at $\left(\frac{1}{2}, -\ln 2\right)$ and (4) at $\left(\frac{1}{2}, \sqrt{2}\right)$. Line $x = 2$ meets (3) at (2, ln 2) and (4) at (2, 4).

The graph of curves are as shown in the figure.



Required area = ABCDA

$$= \int_{1/2}^{1}(-\ln x)dx + \int_{1/2}^{2} 2^x\,dx - \int_{1}^{2}\ln x\,dx$$

$$= \int_{1/2}^{2} 2^x\,dx - \int_{1/2}^{2}\ln x\,dx = \int_{1/2}^{2}(2^x - \ln x)\,dx$$

$$= \left[\frac{2^x}{\log 2} - (x\log x - x)\right]_{1/2}^{2}$$

$$= \left(\frac{4}{\log 2} - 2\log 2 + 2\right) - \left(\frac{\sqrt{2}}{\log 2} + \frac{1}{2}\log 2 - \frac{1}{2}\right)$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}\right)$$

22. We are given that f is a continuous function and

$$\int_0^x f(t)dt \to \infty \, as\,|x| \to \infty$$

To show that every line $y = mx$ intersects the curve

$$y^2 + \int_0^x f(t)dt = 2.$$

If possible, let $y = mx$ intersects the given curve, then Substituting $y = mx$ in the equation of the curve we get

$$m^2x^2 + \int_0^x f(t)dt = 2 \qquad \text{...........(1)}$$

Consider $F(x) = m^2x^2 + \int_0^x f(t)dt - 2$

Then $F(x)$ is a continuous function as $f(x)$ is given to be continuous.

Also $F(x) \to \infty \, as\,|x| \to \infty$

But $F(0) = -2$

Thus $F(0) =$ –ve and F(b) = +ve where b is some value of x, and $F(x)$ is continuous.

Therefore $F(x) = 0$ for some value of $x \in (0, b)$ or eq. (1) is solvable for x.

Hence $y = mx$ intersects the given curve.

23. Let $I = \int_0^{\pi}\frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi}dx$

Consider, $2x - \pi = y$ n$\Rightarrow$ $dx = \frac{dy}{2}$, Also, $x = \left(\frac{\pi}{2} + \frac{y}{2}\right)$

When $x \to 0, y \to -\pi$ when $x \to \pi, y \to \pi$

$\therefore$ We get

$$I = \int_{-\pi}^{\pi}\frac{\left(\frac{\pi + y}{2}\right)\sin(\pi + y)\sin\left[\frac{\pi}{2}\cos\left(\frac{\pi}{2} + \frac{y}{2}\right)\right]}{y}\cdot\frac{dy}{2}$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\left(\frac{\pi}{y} + 1\right)(-\sin y)\sin\left(\frac{-\pi}{2}\sin\frac{y}{2}\right)dy$$

$$=\frac{\pi}{4}\int_{-\pi}^{\pi}\frac{\sin y\sin(\pi/2\sin y/2)}{y}dy$$

$$+\frac{1}{4}\int_{-\pi}^{\pi}\sin y\sin\left(\frac{\pi}{2}\sin\frac{y}{2}\right)dy$$

$$=0+\frac{2}{4}\int_{0}^{\pi}\sin y\sin(\pi/2\sin y/2)\,dy$$

[Using $\int_{-a}^{a} f(x)\,dx = 0$ if f is odd function

$= 2\int_0^a f(x)\,dx$ if f *is* an even function]

$$\therefore I=\frac{1}{2}\int_0^{\pi}2\sin y/2\cos y/2\sin(\pi/2\sin y/2)\,dy$$

Let $\sin y/2 = u \Rightarrow \frac{1}{2}\cos y/2\,dy = du$

$\Rightarrow \cos y/2\,dy = 2du$

Also as $y\to 0, u\to 0$ and as $y\to\pi, u\to 1$

$$\therefore I=\int_0^1 2u\sin\left(\frac{\pi u}{2}\right)du$$

$$=\left[2u\frac{-\cos\frac{\pi u}{2}}{\pi/2}\right]_0^1+\int_0^1 2.\frac{2}{\pi}\cos\left(\frac{\pi u}{2}\right)du$$

$$=0+\frac{4}{\pi}\left.\frac{\sin\left(\frac{\pi u}{2}\right)}{\pi/2}\right|_0^1=\frac{8}{\pi^2}\left(\sin\frac{\pi}{2}-0\right)=\frac{8}{\pi^2}$$

24. The given curves are $y = x^2$ and $y=\frac{2}{1+x^2}$. Here $y = x^2$ is upward parabola with vertex at origin.

Also, $y=\frac{2}{1+x^2}$ is a curve symm. with respect to y-axis.

At $x = 0, y = 2$,

$\frac{dy}{dx}=\frac{-4x}{(1+x^2)^2}<0$ for $x > 0$

$\therefore$ Curve is decreasing on $(0,\infty)$

Moreover $\frac{dy}{dx}=0$ at $x = 0$

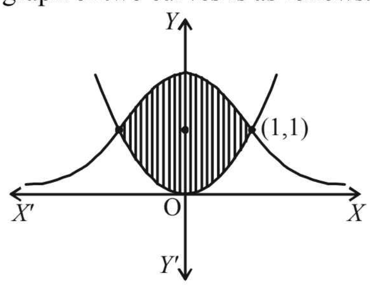
$\Rightarrow$ At (0,2) tangent to curve is parallel to x – axis.

As $x\to\infty$, $y\to 0$

$\therefore$ y = 0 is asymptote of the given curve.

For the given curves, point of intersection : solving their equations we get $x = 1, y = 1$, i.e., (1,1).

Thus the graph of two curves is as follows:



$$\therefore \text{The required area} = 2\int_0^1\left(\frac{2}{1+x^2}-x^2\right)dx$$

$$=\left(4\tan^{-1}x-\frac{2x^3}{3}\right)_0^1=4.\frac{\pi}{4}-\frac{2}{3}=\pi-\frac{2}{3}\text{ sq. units.}$$

25. Given that $\int_0^1 e^x(x-1)^n\,dx = 16-6e$

where $n\in N$ and $n\le 5$

To find the value of n.

Let $I_n=\int_0^1 e^x(x-1)^n\,dx$

$$=[(x-1)^n e^x]_0^1-\int_0^1 n(x-1)^{n-1}e^x\,dx$$

$$=-(-1)^n-\int_0^1 n(x-1)^{n-1}e^x\,dx$$

$$I_n=(-1)^{n+1}-n\,I_{n-1} \quad ....(1)$$

Also, $I_1=\int_0^1 e^x(x-1)\,dx$

$$=[e^x(x-1)]_0^1-\int_0^1 e^x dx=-(-1)-(e^x)_0^1$$

$$=-(e-1)=2-e$$

Using eq. (1), $I_2=(-1)^3-2I_1=-1-2(2-e)=2e-5$

Similarly, $I_3=(-1)^4-3I_2=1-3(2e-5)=16-6e$

$\therefore n = 3$

26. $$I=\int_2^3\frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$$

$$=\int_2^3\frac{2x^5-2x^3+x^4+2x^2+1}{(x^2+1)^2(x^2-1)}dx$$

$$=\int_2^3\frac{2x^3(x^2-1)+(x^2+1)^2}{(x^2+1)^2(x^2-1)}dx$$

$$=\int_2^3\frac{2x^3}{(x^2+1)^2}+\int_2^3\frac{1}{x^2-1}dx$$

$$=\int_2^3\frac{x^2.2x}{(x^2+1)^2}+\left[\frac{1}{2}\log\frac{x-1}{x+1}\right]_2^3$$

$$=\int_5^{10}\frac{t-1}{t^2}dt+\frac{1}{2}\left(\log\frac{2}{4}-\log\frac{1}{3}\right)$$

Put $x^2+1=t,\ 2x\,dx=dt$

when $x\to 2, t\to 5$, $x\to 3, t\to 10$

$$=\int_5^{10}\left(\frac{1}{t}-\frac{1}{t^2}\right)dt+\frac{1}{2}\log\frac{3}{2}=\left(\log|t|+\frac{1}{t}\right)_5^{10}+\frac{1}{2}\log\frac{3}{2}$$

$$=\log 10-\log 5+\frac{1}{10}-\frac{1}{5}+\frac{1}{2}\log\frac{3}{2}$$

$$= \log 2 + \left(-\frac{1}{10}\right) + \frac{1}{2}\log\frac{3}{2} = \frac{1}{2}\left[2\log 2 + \log\frac{3}{2}\right] - \frac{1}{10}$$

$$= \frac{1}{2}\log 6 - \frac{1}{10}$$

27. To prove that $\int_0^{n\pi+v} |\sin x|\, dx = 2n + 1 - \cos v$

Let $I = \int_0^{n\pi+v} |\sin x|\, dx$

$$= \int_0^{v} |\sin x|\, dx + \int_v^{n\pi+v} |\sin x|\, dx$$

Now we know that $|\sin x|$ is a periodic function of period π, So using the property.

$$= \int_a^{a+nT} f(x)dx = n\int_0^{T} f(x)dx$$

where $n \in I$ and $f(x)$ is a periodic function of period T

We get, $I = \int_0^{v} \sin x\, dx + n\int_0^{\pi} \sin x\, dx$

[$\therefore$ $|\sin x| = \sin x$ for $0 \le x \le v$]

$$= (-\cos x)_0^v + n(-\cos x)_0^\pi = -\cos v + 1 + n(1+1)$$

$$= 2n + 1 - \cos v = \text{R.H.S.}$$

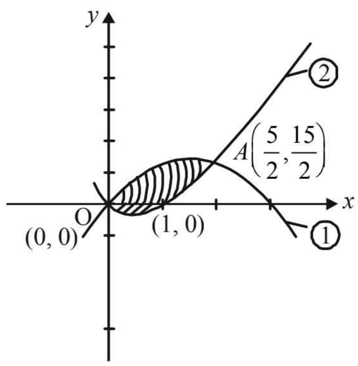
28. The given equations of parabola are

$y = 4x - x^2$ or $(x-2)^2 = -(y-4)$ ......(1)

and $y = x^2 - x$ or $\left(x - \frac{1}{2}\right)^2 = \left(y + \frac{1}{4}\right)$ ......(2)

Solving the equations of two parabolas we get their points of intersection as $O(0,0), A\left(\frac{5}{2}, \frac{15}{4}\right)$

Here the area below x – axis,



$$A_1 = \int_0^1 (-y_2)\, dx = \int_0^1 (x - x^2)dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units.}$$

Area above x – axis,

$$A_2 = \int_0^{5/2} y_1\, dx - \int_1^{5/2} y_2\, dx$$

$$= \int_0^{5/2} (4x - x^2)\, dx - \int_1^{5/2} (x^2 - x)\, dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^{5/2} - \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^{5/2}$$

$$= \left(\frac{25}{2} - \frac{125}{24}\right) - \left[\left(\frac{125}{24} - \frac{25}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{25}{2} - \frac{125}{12} + \frac{25}{8} - \frac{1}{6} = \frac{300 - 250 + 75 - 4}{24} = \frac{121}{24}$$

$\therefore$ Ratio of areas above x – axis and below x – axis.

$$A_2 : A_1 = \frac{121}{24} : \frac{1}{6} = \frac{121}{4} = 121 : 4$$

29. Given $I_m = \int_0^{\pi} \frac{1 - \cos mx}{1 - \cos x} dx$

To prove: $I_m = m\pi$, $m = 0, 1, 2,......$

For m = 0

$$I_0 = \int_0^{\pi} \frac{1 - \cos 0}{1 - \cos x} dx = \int_0^{\pi} \frac{1 - 1}{1 - \cos x} dx = 0$$

$\because$ Result is true for m = 0

For m = 1,

$$I_1 = \int_0^{\pi} \frac{1 - \cos x}{1 - \cos x} dx = \int_0^{\pi} 1.dx$$

$$(x)_0^\pi = \pi - 0 = \pi$$

$\therefore$ Result is true for $m = 1$

Let the result be true for $m \le k$ i.e. $I_k = k\pi$ ....(1)

Consider $I_{k+1} = \int_0^{\pi} \frac{1 - \cos(k+1)x}{1 - \cos x} dx$

Now, $1 - \cos(k+1)x$

$= 1 - \cos kx \cos x + \sin kx \sin x$

$= 1 + \cos kx \cos x + \sin kx \sin x - 2 \cos kx \cos x$

$= 1 + \cos(k-1)x - 2\cos kx \cos x$

$= 2 - (1 - \cos(k-1)x) - 2\cos kx \cos x$

$= 2(1 - \cos kx \cos x) - (1 - \cos(k-1)x)$

$= 2 - 2\cos kx + 2\cos kx - 2\cos kx \cos x - [1 - \cos(k-1)x]$

$= 2(1 - \cos kx) + 2\cos kx(1 - \cos x) - (1 - \cos(k-1)x)$

$$\therefore I_{k+1} = \int_0^{\pi} \frac{2(1-\cos kx) + 2\cos kx(1-\cos x) - (1-\cos(k-1)x)}{1-\cos x} dx$$

$$= 2\int_0^{\pi} \frac{1-\cos kx}{1-\cos x} dx + 2\int_0^{\pi} \cos kx\, dx - \int_0^{\pi} \frac{1-\cos(k-1)x}{1-\cos x} dx$$

$$= 2I_k + 2\left(\frac{\sin kx}{k}\right)_0^\pi - I_{k-1}$$

$$= 2(k\pi) + 2(0) - (k-1)\pi \quad \text{[Using (i)]}$$

$$= 2k\pi - k\pi + \pi = (k+1)\pi$$

Thus result is true for $m = k + 1$ as well. Therefore by the principle of mathematical induction, given statement is true for all $m = 0, 1, 2,$ ....

30. Let $I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

We know that $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$

Also $\sin^{-1}y+\cos^{-1}y=\frac{\pi}{2}$

$\therefore$ We get $\frac{\pi}{2}-\cos^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$

$\Rightarrow \cos^{-1}\left(\frac{2x}{1+x^2}\right)=\frac{\pi}{2}-2\tan^{-1}x$

$\therefore I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\left[\frac{\pi}{2}-2\tan^{-1}x\right]dx$

$=\frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx-2\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4\tan^{-1}x}{1-x^4}dx$

$=2.\frac{\pi}{2}\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx-2\times 0$

$=$ [Using $\int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx$ if f is even]

$=0$ if f is odd

$=\pi\int_0^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx$

$\therefore I=-\pi\int_0^{1/\sqrt{3}}\frac{(1-x^4)-1}{1-x^4}dx$

$=-\pi\int_0^{1/\sqrt{3}}1-\frac{1}{1-x^4}dx=-\pi\int_0^{1/\sqrt{3}}\left[1-\frac{1}{2}\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)\right]dx$

$=-\pi\left[x-\frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+x}{1+x}\right|+\tan^{-1}x\right)\right]_0^{1/\sqrt{3}}$

$=-\pi\left[\frac{1}{\sqrt{3}}-\frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\right|-\tan^{-1}\frac{1}{\sqrt{3}}\right)-0\right]$

$=-\pi\left[\frac{1}{\sqrt{3}}-\frac{1}{4}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)-\frac{\pi}{12}\right]$

$=\pi\left[\frac{\pi}{12}+\frac{1}{4}\log(2+\sqrt{3})-\frac{\sqrt{3}}{3}\right]$

$=\frac{\pi}{12}[\pi+3\log(2+\sqrt{3})-4\sqrt{3}]$

31. Let us consider any point $P(x, y)$ inside the square such that its distance from origin $\le$ its distance from any of the edges say AD

$\therefore OP\le PM$ or $\sqrt{(x^2+y^2)}<1-x$

or $y^2\le -2\left(x-\frac{1}{2}\right)$ ....(1)

Above represents all points within and on the parabola 1. If we consider the edges BC then $OP<PN$ will imply

$y^2\le 2\left(x+\frac{1}{2}\right)$ ....(2)

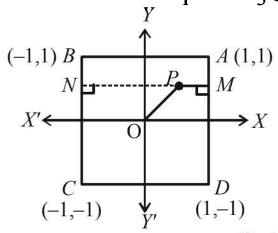
Similarly if we consider the edges AB and CD, we will have

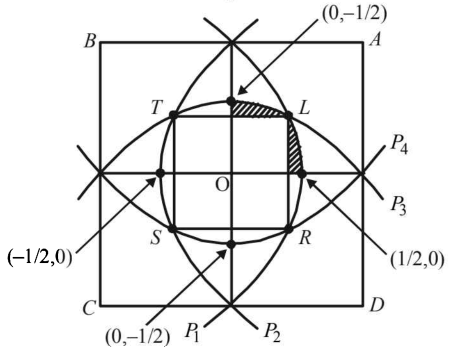
$x^2\le -2\left(y-\frac{1}{2}\right)$ ....(3)

$x^2\le 2\left(y+\frac{1}{2}\right)$ ....(4)

Hence S consists of the region bounded by four parabolas meeting the axes at $\left(\pm\frac{1}{2},0\right)$ and $\left(0,\pm\frac{1}{2}\right)$

The point L is intersection of P_1 and P_3 given by (1) and (3).





$y^2-x^2=-2(x-y)=2(y-x)$

$\therefore$ $y-x=0$ $\therefore y=x$

$\therefore$ $x^2+2x-1=0\Rightarrow (x+1)^2=2$

$\therefore$ $x=\sqrt{2}-1$ as x is +ve

$\therefore$ L is $(\sqrt{2}-1,\sqrt{2}-1)$

$\therefore$ Total area $=4\left[\text{square of side}(\sqrt{2}-1)+2\int_{\sqrt{2}-1}^{1/2}y\,dx\right]$

$=4\left\{(\sqrt{2}-1)^2+2\int_{\sqrt{2}-1}^{1/2}\sqrt{(1-2x)}dx\right\}$

$=4\left[3-2\sqrt{2}-\frac{2}{2}.\frac{2}{3}\{(1-2x)^{3/2}\}_{\sqrt{2}-1}^{1/2}\right]$

$=4\left[3-2\sqrt{2}-\frac{2}{3}\{0-(1-2\sqrt{2}+2)^{3/2}\right]$

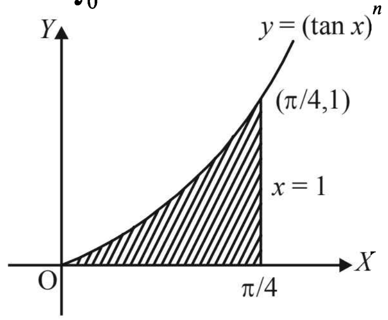
$=4\left[3-2\sqrt{2}+\frac{2}{3}(3-2\sqrt{2})^{3/2}\right]$

$=4(3-2\sqrt{2})\left[1+\frac{2}{3}\sqrt{(3-2\sqrt{2})}\right]$

$=4(3-2\sqrt{2})\left[1+\frac{2}{3}(\sqrt{2}-1)\right]$

$=\frac{4}{3}(3-2\sqrt{2})(1+2\sqrt{2})=\frac{4}{3}[(4\sqrt{2}-5)]=\frac{16\sqrt{2}-20}{3}$

32. We have $A_n = \int_0^{\pi/4} (\tan x)^n \, dx$



Since $0 < \tan x < 1$, when $0 < x < \pi/4$, we have

$0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$

$$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1} dx < \int_0^{\pi/4} (\tan x)^n dx$$

$\Rightarrow A_{n+1} < A_n$

Now, for $n > 2$

$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}] dx$$

$$= \int_0^{\pi/4} (\tan x)^n + (1 + \tan^2 x) \, dx$$

$$= \int_0^{\pi/4} (\tan x)^n + (\sec^2 x) \, dx$$

$$= \left[\frac{1}{(n+1)} (\tan x)^{n+1} \right]_0^{\pi/4}$$

$$\left[\because \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \frac{1}{(n+1)} (1 - 0)$$

Since $A_{n+2} < A_{n+1} < A_n$. we get, $A_n + A_{n+2} < 2A_n$

$$\Rightarrow \frac{1}{n+1} < 2A_n \Rightarrow \frac{1}{2n+2} < A_n \quad \text{........(1)}$$

Also for $n > 2, A_n + A_n < A_n + A_{n-2} = \dfrac{1}{n-1}$

$$\Rightarrow 2A_n < \frac{1}{n-1}$$

$$\Rightarrow A_n < \frac{1}{2n-2} \quad \text{........(2)}$$

Combining (1) and (2) we get

$$\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$$

Hence Proved.

33. $\displaystyle\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx = I$ (say)

$$\text{or } I = \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x} dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1 + \cos^2 x} dx$$

$$I = 0 + 2\int_0^{\pi} \frac{2x \sin x}{1 + \cos^2 x} dx \quad \left[\because \frac{2x}{1 + \cos^2 x} \text{is an odd function} \right]$$

$$\text{or } I = 4\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \text{.....(1)}$$

$$\text{or } I = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \{\cos(\pi - x)\}^2} dx = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$$

$$\text{or } I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$\text{or } I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - 1 \quad \text{[from (1)]}$$

$$\text{or } 2I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

Putting $\cos x = t$, $-\sin x \, dx = dt$

When $x \to 0, t \to 1$ and when $x \to \pi, t \to -1$

$$\Rightarrow I = 2\pi \int_1^{-1} \frac{-dt}{1 + r^2} = 2\pi \int_{-1}^{1} \frac{dt}{1 + t^2} = 4\pi \int_0^1 \frac{dt}{1 + t^2}$$

$$\Rightarrow I = 4\pi \left(\tan^{-1} t \right)_0^1 = 4\pi \{\tan^{-1}(1) - \tan^{-1}(0)\}$$

$$\Rightarrow I = 4\pi \left\{ \frac{\pi}{4} - 0 \right\} = \pi^2$$

34. We draw the graph of $y = x^2$, $y = (1-x)^2$ and $y = 2x(1-x)$ in figure.

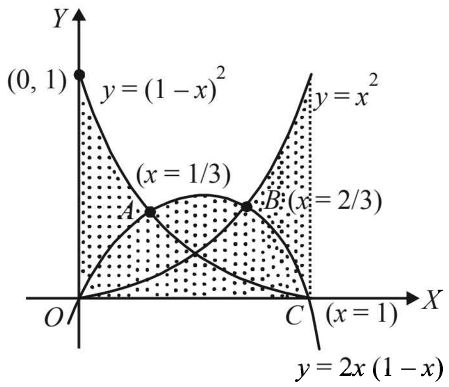
Let us find the point of intersection of $y = x^2$ and $y = 2x(1-x)$

The x – coordinate of the point of intersection satisfies the equation $x^2 = 2x(1-x)$, $\Rightarrow 3x^2 = 2x \Rightarrow 0$ or $x = 2/3$

$\therefore$ At B, $x = 2/3$

Similarly, we find the x coordinate of the points of intersection of $y = (1-x)^2$ and $y = 2x(1-x)$ are $x = 1/3$ and $x = 1$

$\therefore$ At A, $x = 1/3$ and at C $x = 1$



From the figure it is clear that

$$f(x) = \begin{cases} (1-x)^2 & \text{for } 0 \le x \le 1/3 \\ 2x(1-x) & \text{for } 1/3 \le x \le 2/3 \\ x^2 & \text{for } 2/3 \le x \le 1 \end{cases}$$

The required area A is given by

$$A = \int_0^1 f(x) dx$$

$$= \int_0^{1/3} (1-x)^2 dx + \int_{1/3}^{2/3} 2x(1-x) dx + \int_{2/3}^{1} x^2 dx$$

$$= \left[-\frac{1}{3}(1-x)^3\right]_0^{1/3} + \left[x^2 - \frac{2x^2}{3}\right]_{1/3}^{2/3} + \left[\frac{1}{3}x^3\right]_{2/3}^{1}$$

$$= -\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3}\left(\frac{2}{3}\right)^2 - \frac{2}{3}\left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \frac{2}{3}\left(\frac{1}{3}\right)^3$$

$$+\frac{1}{3}(1) - \frac{1}{3}\left(\frac{2}{3}\right)^3 = \frac{17}{27} \text{ sq. units}$$

35. $\therefore I = \int_0^1 y dx = \int_0^1 \tan^{-1} x dx - \int_0^1 \tan^{-1}(x-1)dx$

$$= \int_0^1 \tan^{-1} x dx - \int_0^1 \tan^{-1}\{(1-x)-1\}$$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \tan^{-1} x\, dx - \int_0^1 (-\tan^{-1} x)\, dx = 2\int_0^1 \tan^{-1} x dx \text{ (Proved)}$$

$$= 2\left[x \tan^{-1} dx - \frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= \frac{\pi}{2} - \log 2 \qquad \text{...... (1)}$$

Now, $\int_0^1 \tan^{-1}(1-x+x^2)dx$

$$= \int_0^1 \cot^{-1}\frac{1}{1-x+x^2} dx = \int_0^1 \left(\frac{\pi}{2} - \tan^{-1}\frac{1}{1-x+x^2}\right) dx$$

$$= \left[\frac{\pi}{2}x\right]_0^1 - I = \frac{\pi}{2} - \left(\frac{\pi}{2} - \log 2\right) = \log 2 \text{ by (1)}$$

36. $f(x) = x^3 - x^2$

Let P be on C_1, $y = x^2$ be (t, t^2)

$\therefore$ ordinate of Q is also t^2.

Now Q lies on $y = 2x$, and $y = t^2$

$\therefore$ $x = t^2/2$

$\therefore$ $Q\left(\frac{t^2}{2}, t^2\right)$

For point R, $x = t$ and it is on $y = f(x)$

$\therefore$ R is $[t, f(t)]$

$$\text{Area } OPQ = \int_0^{t^2} (\underset{C_1}{x_1} - \underset{C_2}{x_2})dy = \int_0^{t^2}\left(\sqrt{y} - \frac{y}{2}\right)dy$$

$$= \frac{2}{3}t^3 - \frac{t^4}{4} \qquad \text{...(1)}$$

$$\text{Area } OPR = \int_{0_{C_1}}^{t} ydx + \left|\int_{0_{C_3}}^{t} ydx\right|$$

$$= \int_0^t x^2 dx + \left|\int_0^t f(x)dx\right| = \frac{t^3}{3} + \left|\int_0^t f(x)dx\right| \qquad \text{....(2)}$$

Equating (1) and (2), we get,

$$\frac{t^3}{3} - \frac{t^4}{4}\left|\int_0^t f(x)dx\right|$$

Differentiating both sides, we get,

$$t^2 - t^3 = -f(t)$$

$\therefore$ $f(t) = x^3 - x^2$.

37. $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$

$$\Rightarrow I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}} \quad \Rightarrow I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}$$

Adding, $2I = \int_0^{\pi} dx = \pi \Rightarrow I = \pi/2$

38. $f(x) = \begin{cases} x^2 + ax + b; \ x < -1 \\ 2x \qquad ; -1 \le x \le 1 \\ x^2 + ax + b; \ x > 1 \end{cases}$

$\because f(x)$ is continuous at $x = -1$ and $x = 1$

$\therefore (-1)^2 + a(-1) + b = -2$

and $2 = (1)^2 + a.1 + b$

i.e. $a - b = 3$ and $a + b = 1$

On solving we get $a = 2, b = -1$

$\therefore$ $f(x) = \begin{cases} x^2 + 2x - 1; \ x < -1 \\ 2x \qquad ; -1 \le x \le 1 \\ x^2 + 2x - 1; \ x > 1 \end{cases}$

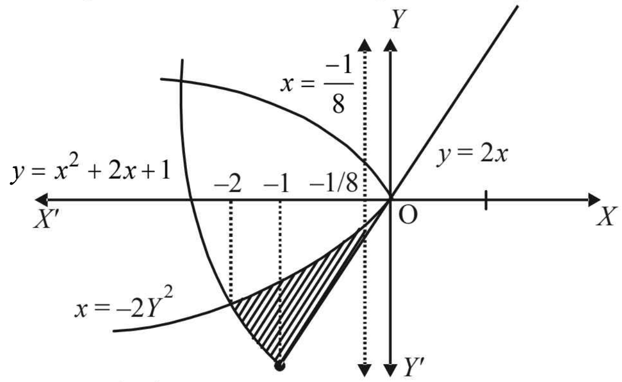
Given curves are $y = f(x)$, $x = -2y^2$ and $8x + 1 = 0$

Solving $x = -2y^2$, $y = x^2 + 2x - 1$ $(x < -1)$ we get $x = -2$

Also $y = 2x$, $x = -2y^2$ meet at $(0, 0)$

$y = 2x$ and $x = -1/8$ meet at $\left(-\frac{1}{8}, \frac{-1}{4}\right)$

The required area is the shaded region in the figure.



$\therefore$ Required area

NOTE THIS STEP:

$$= \int_{-2}^{-1}\left[\sqrt{\frac{-x}{2}} - (x^2 + 2x - 1)\right]dx + \int_{-1}^{-1/8}\left[\sqrt{\frac{-x}{2}} - 2x\right]dx$$

$$= \left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3} - \frac{x^3}{3} - x^2 + x\right]_{-2}^{-1} + \left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3} - x^2\right]_{-1}^{-1/8}$$

$$= \left(\frac{\sqrt{2}}{3} + \frac{1}{3} - 1 - 1\right) - \left(\frac{4}{3} + \frac{8}{3} - 4 - 2\right)$$

$$+\left(\frac{\sqrt{2}}{3}.\frac{1}{16\sqrt{2}} - \frac{1}{64}\right) - \left(\frac{\sqrt{2}}{3} - 1\right)$$

$$= \left(\frac{\sqrt{2}-5}{3}\right) - \left(\frac{4+8-18}{3}\right) + \left(\frac{4-3}{192}\right) - \left(\frac{\sqrt{2}-3}{3}\right)$$

$$= \frac{257}{192} \text{ sq. units}$$

39. $f(x) = \int_1^x \frac{\ln t}{1+t} dt$ for $x > 0$ (given)

Now $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1+t} dt$: Put $t = \frac{1}{u}$, so that

$dt = -\frac{1}{u^2} du$

Therefore $f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln(1/u)}{1+\frac{1}{u}} \cdot \frac{(-1)}{u^2} du$

$$= \int_1^x \frac{\ln u}{u(u+1)} du = \int_1^x \frac{\ln t}{t(t+1)} dt$$

Now, $f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{1+t} dt + \int_1^x \frac{\ln t}{t(1+t)} dt$

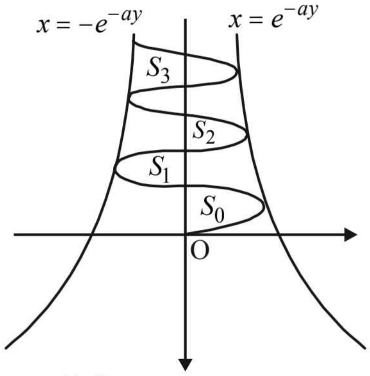
$$= \int_1^x \frac{(1+t)\ln t}{t(1+t)} dt = \int_1^x \frac{\ln t}{t} dt = \frac{1}{2}(\ln t)^2 \Big]_1^x = \frac{1}{2}(\ln x)^2$$

Put $x = e$, hence $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}(\ln e)^2 = \frac{1}{2}$

Hence Proved.

40. Given that $x = \sin by.\, e^{-ay} \Rightarrow -e^{-ay} \le x \le e^{-ay}$

The figure is drawn taking a and b both +ve. The given curve oscillates between $x = e^{-ay}$ and $x = -e^{-ay}$



Clearly, $S_j = \int_{\frac{j\pi}{b}}^{\frac{(j+1)\pi}{b}} \sin by.e^{-ay}.dy$

Integrating by parts, $I = \int \sin by.e^{-ay}.dy$

We get $I = -\frac{e^{-ay}}{a^2+b^2}(a\sin by + b\cos by)$

So, $S_j = \left| -\frac{1}{a^2+b^2}\left[e^{-a\frac{(j+1)\pi}{b}} \{a\sin(j+1)\pi \right.\right.$

$\left.\left. + b\cos(j+1)\pi - e^{\frac{-aj\pi}{b}}(a\sin j\pi + b\cos j\pi) \right] \right|$

$$\Rightarrow S_j = \left| -\frac{1}{a^2+b^2}\left[e^{-\frac{a}{b}(j+1)\pi} b(-1)^{j+1} - e^{-\frac{a}{b}j\pi} b(-1)^j \right] \right|$$

$$= \left| \frac{b.(-1)^j e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi} + 1 \right) \right| = b.\frac{e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi} + 1 \right)$$

Now, $\frac{S_j}{S_{j-1}} = \frac{e^{-\frac{a}{b}j\pi}}{e^{-\frac{a}{b}(j-1)\pi}} = e^{-\frac{a}{b}\pi}$ = constant

$\Rightarrow S_o, S_1, S_2, \ldots\ldots\ldots S_j$ form a G.P.

For $a = -1$ and $b = \pi$ $S_j = \frac{\pi e^j}{(1+\pi^2)}(1+e)$

$$\Rightarrow \sum_{j=0}^{n} S_j = \frac{\pi(1+e)}{(1+\pi^2)} \cdot \frac{(e^{(n+1)}-1)}{(e-1)}.$$

41. The given curves are $y = x^2$
which is an upward parabola with vertex at (0, 0)

$y = |2 - x^2|$

or $y = \begin{cases} 2 - x^2 \text{ if } & -\sqrt{2} \le x \le \sqrt{2} \\ x^2 - 2 \text{ if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \end{cases}$

or $x^2 = -(y-2);\ -\sqrt{2} < x < \sqrt{2}$ ....(2)

a downward parabola with vertex at (0, 2)

$x^2 = y + 2;\ x < -\sqrt{2},\ x > \sqrt{2}$ ....(3)

An upward parabola with vertex at (0, – 2)

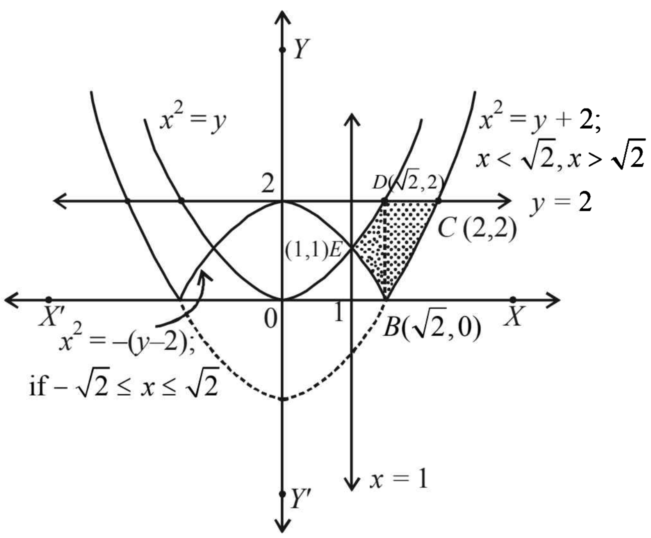
$y = 2$ ....(4)

A straight line parallel to x – axis

$x = 1$ ....(5)

A straight line parallel to y – axis

The graph of these curves is as follows.



∴ Required area = $BCDEB$

$$= \int_1^{\sqrt{2}} [Y_{(1)} - Y_{(2)}]dx + \int_{\sqrt{2}}^{2} [Y_{(4)} - Y_{(3)}]dx \quad \text{..........(1)}$$

$$= \int_1^{\sqrt{2}} [x^2 - (2 - x^2)]dx + \int_{\sqrt{2}}^{2} [2 - (x^2 - 2)]dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)dx + \int_{\sqrt{2}}^{2} (4 - x^2)dx$$

$$=\left[\frac{2x^3}{3}-2x\right]_1^{\sqrt{2}}+\left[4x-\frac{x^3}{3}\right]_{\sqrt{2}}^2$$

$$=\left(\frac{4\sqrt{2}}{3}-2\sqrt{2}-\frac{2}{3}+2\right)+\left(8-\frac{8}{3}-4\sqrt{2}+\frac{2\sqrt{2}}{3}\right)$$

$$=-\frac{2}{3}\sqrt{2}+\frac{4}{3}+\frac{16}{3}-\frac{10\sqrt{2}}{3}$$

$$=\frac{20-12\sqrt{2}}{3}=\left(\frac{20}{3}-4\sqrt{2}\right)\text{ sq. units.}$$

42. Given that $f(x)$ is an even function, then to prove

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx=\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$$

Let $I=\int_0^{\pi/2} f(\cos 2x)\cos x\,dx$ ....(1)

$$=\int_0^{\pi/2} f\left[\cos 2\left(\frac{\pi}{2}-x\right)\right]\cos\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\text{Using}\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{\pi/2} f(-\cos 2x)\sin x\,dx$$

$I=\int_0^{\pi/2} f(\cos 2x)\sin x\,dx$ ....(2)

[As f is an even function]

Adding two values of I in (1) and (2) we get

$$2I=\int_0^{\pi/2} f(\cos 2x)(\sin x+\cos x)dx$$

$$\Rightarrow I=\frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]dx$$

$$I=\frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\cos(x-\pi/4)dx$$

Let $x-\pi/4=t$ so that $dx=dt$

as $x\to 0,\ t\to -\pi/4$

and as $x\to\pi/4$, $t\to\pi/2-\pi/4=\pi/4$

$$\therefore\quad I=\frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[\cos 2(t+\pi/4)]\cos t\,dx$$

$$=\frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[-\sin 2t]\cos t\,dt$$

$$=\frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cos t\,dt$$

[$\because f$ is an even function]

$$=\frac{2}{\sqrt{2}}\int_0^{\pi/4} f(\sin 2t)\cos t\,dt$$

$$=\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx$$

R.H.S. *Hence proved.*

43. We have,

$$y(x)=\int_{\pi^2/16}^{x^2}\frac{\cos x\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

$$=\cos x\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

[$\because$ $\cos x$ is independent of θ]

$$\Rightarrow\frac{dy}{dx}=-\sin x\int_{\pi^2/16}^{x^2}\left[\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\right]d\theta$$

$$+\cos x\frac{d}{dx}\left[\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta\right]$$

$$=-\sin x\int_{\frac{\pi^2}{16}}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

$$+\cos x\left[\int_{\pi^2/16}^{x^2}\frac{\cos x}{1+\sin^2 x}.2x-0\right]\text{ (Using Leibnitz thm.)}$$

$$\Rightarrow\left.\frac{dy}{dx}\right|_{x=\pi}=-\sin x\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}.d\theta+\frac{(\cos^2\pi).2\pi}{1+\sin^2\pi}$$

$=0+2\pi=2\pi$

44. Let $I=\int_{-\pi/3}^{\pi/3}\frac{\pi+4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx$

$$=\int_{-\pi/3}^{\pi/3}\frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx+\int_{-\pi/3}^{\pi/3}\frac{4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx$$

The second integral becomes zero integrand being an odd function of x.

$$=2\pi\int_0^{\pi/3}\frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$

{ using the prop. of even function and also $|x|=x$ for $0\le x\le\pi/3$ }

Let $x+\pi/3=y\Rightarrow dx=dy$

also as $x\to 0,\ y\to\pi/3$ as $x\to\pi/3,\ y\to 2\pi/3$

$\therefore$ The given integral becomes

$$=2\pi\int_{\pi/3}^{2\pi/3}\frac{dy}{2-\cos y}=2\pi\int_{\pi/3}^{2\pi/3}\frac{dy}{2-\frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$$=2\pi\int_{\pi/3}^{2\pi/3}\frac{\sec^2 y/2}{3\tan^2 y/2+1}dy$$

$$=\frac{2\pi}{3}\int_{\pi/3}^{2\pi/3}\frac{\sec^2 y/2}{\tan^2 y/2+(1/\sqrt{3})^2}dy$$

$$=\frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}(\sqrt{3}\tan y/\sqrt{2})\right]_{\pi/3}^{2\pi/3}$$

$$=\frac{4\pi}{3}\left[\tan^{-1}3-\tan^{-1}1\right]=\frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3-\pi/4\right]$$

45. Let

$$I=\int_0^{\pi}e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$$

$= \int_0^{\pi} e^{|\cos x|} 2\sin\left(\frac{1}{2}\cos x\right)\sin x\,dx$

$+\int_0^{\pi} e^{|\cos x|} 3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$

$= I_1 + I_2$

Now using the property that

$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a-x)=f(x)$

$= 0$ if $f(2a-x)=-f(x)$

We get, $I_1 = 0$

and $I_2 = 2\int_0^{\pi/2} e^{|\cos x|} 3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$

$= 6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$,

$\therefore \quad I_2 = 6\int_0^1 e^t \cos t/2\,dt$

Integrating by parts, we get

$I_2 = 6[(e^t \cos t/2)_0^1 + \frac{1}{2}\int_0^1 e^t \sin t/2\,dt]$

$= 6\left\{e\cos(1/2) - 1 + \frac{1}{2}\left\{(e^t \sin t/2)_0^1 - \frac{1}{2}\int_0^1 e^t \cos t/2\,dt\right\}\right]$

$I_2 = 6\left[e\cos\left(\frac{1}{2}\right) - 1 + \frac{1}{2}\left\{e\sin\left(\frac{1}{2}\right) - \frac{1}{2}\cdot\frac{1}{6}I_2\right\}\right]$

$I_2 = 6\left[e\cos(1/2) - 1 + \frac{1}{2}(e\sin t/2) - \frac{1}{24}I_2\right]$

$I_2 + \frac{1}{4}I_2 = 6\left[e\cos(1/2) + \frac{1}{2}e\sin(1/2) - 1\right]$

$\frac{5I_2}{4} = 6\left[e\cos(1/2) + \frac{1}{2}e\sin(1/2) - 1\right]$

$\Rightarrow I_2 = \frac{24}{5}\left[e\cos(1/2 + \frac{1}{2}e\sin\left(\frac{1}{2}\right) - 1\right]$

46. The given curves are, $x^2 = y$ ....(i)

$x^2 = -y$ ....(ii)

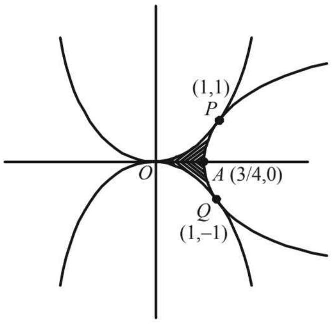
$y^2 = 4x - 3$ ....(iii)

Clearly point of intersection of (i) and (ii) is (0, 0). For point of intersection of (i) and (iii), solving them as follows

$x^4 - 4x + 3 = 0 \quad (x-1)(x^3 + x^2 + x - 3) = 0$

or $(x-1)^2(x^2 + 2x + 3) = 0$; $\Rightarrow x = 1$ and then $y = 1$

$\therefore$ Req. point is (1, 1). Similarly point of intersection of (ii) and (iii) is (1, – 1). The graph of three curves is as follows:



We also observe that at $x = 1$ and $y = 1$

$\frac{dy}{dx}$ for (i) and (iii) is same and hence the two curves touch each other at (1, 1).

Same is the case with (ii) and (iii) at (1, –1).

Required area = Shaded region in figure = 2 (*Ar OPA*)

$= 2\left[\int_0^1 x^2 dx - \int_{3/4}^1 \sqrt{4x-3}\,dx\right]$

$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{2(4x-3)^{3/2}}{4\times 3}\right)_{3/4}^1\right] = 2\left[\frac{1}{3} - \frac{1}{6}\right]$

$= 2\times\frac{1}{6} = \frac{1}{3}$ sq. units

47. Given that $f(x)$ is a differentiable function such that $f'(x) = g(x)$, then

$\int_0^3 g(x)dx = \int_0^3 f'(x)dx = [f(x)]_0^3 = f(3) - f(0)$

But $|f(x)| < 1 \Rightarrow -1 < f(x) < 1, \forall x \in R$

$\therefore \quad f(3) = f(0) \in (-1, 1)$

Similarly

$\int_{-3}^0 g(x)dx = \int_{-3}^0 f'(x)dx = [f(0) - f(3)] \in (-2, 2)$

Also given $[f(0)]^2 + [g(0)]^2 = 9$

$\Rightarrow [g(0)]^2 = 9 - [f(0)]^2$

$\Rightarrow |g(0)|^2 > 9 - 1 \qquad [\because |f(x)| < 1]$

$\Rightarrow |g(0)| > 2\sqrt{2} \Rightarrow g(0) > 2\sqrt{2}$ or $g(0) < -2\sqrt{2}$

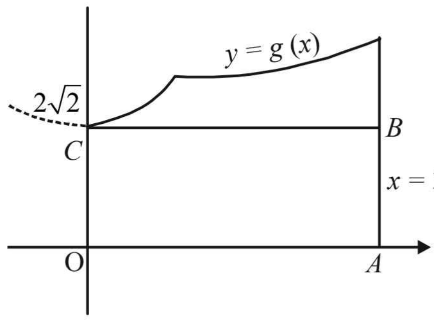
First let us consider $g(0) > 2\sqrt{2}$

Let us suppose that $g''(x)$ be positive for all $x \in (-3, 3)$.

Then $g''(x) > 0 \Rightarrow$ the curve $y = g(x)$ is open upwards.

Now one of the two situations are possible.

(i) $g(x)$ is increasing



$\therefore \left|\int_0^3 g(x)dx\right| >$ area of rect. *OABC*

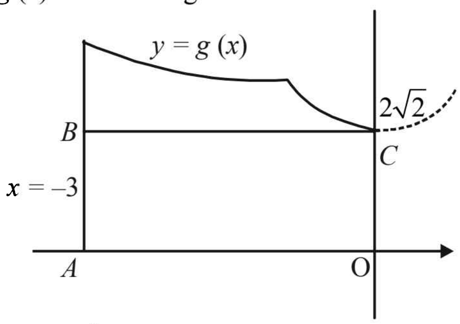
i.e. $\left|\int_0^3 g(x)dx\right| > 6\sqrt{2} > 2$

a contradiction as $\int_0^1 g(x)dx \in (-2, 2)$

$\therefore$ at least at one of the point $c \in (-3,3)$, $g''(x) < 0$. But $g(x) > 0$ on $(-3, 3)$

Hence $g(x)\,g''(x) < 0$ at some $x \in (-3, 3)$.

(ii) $g(x)$ is decreasing



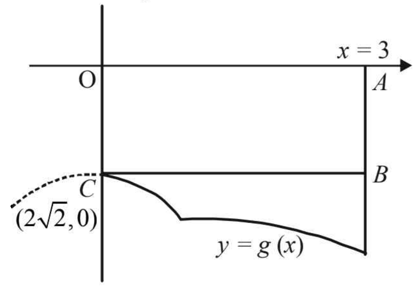
$\therefore \left|\int_{-3}^{0} g(x)dx\right| >$ area of rect. $OABC$

i.e. $\left|\int_{-3}^{0} g(x)dx\right| > 3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as $\int_{-3}^{0} g(x)dx \in (-2,2)$

$\therefore$ at least at one of point $c \in (-3,3)\ g''(x)$ should be – ve. But $g(x) > 0$ on $(-3, 3)$.
Hence $g(x)\,g''(x) < 0$ at some $x \in (-3, 3)$.

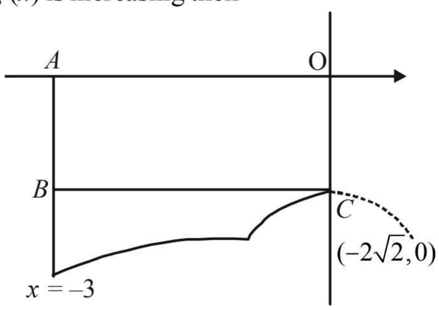
Secondly let us consider $g(0) < -2\sqrt{2}$.
Let us suppose that $g''(x)$ be – ve on $(-3, 3)$.
then $g''(x) < 0 \Rightarrow$ the curve $y = g(x)$ is open downward.
Again one of the two situations are possible
(i) $g(x)$ is decreasing then



$\left|\int_{0}^{3} g(x)dx\right| >$ Ar or rect. $OABC = 3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as $\int_{0}^{3} g(x)dx \in (-2,2)$

$\therefore$ At least at one of the point $c \in (-3, 3)$, $g''(x)$ is + ve. But $g(x) < 0$ on $(-3, 3)$.
Hence $g(x)\,g''(x) < 0$ for some $x \in (-3, 3)$.
(ii) $g(x)$ is increasing then



$\left|\int_{-3}^{0} g(x)dx\right| >$ Ar of rect. $OABC = 3.2\sqrt{2} = 6\sqrt{2} > 2$

a contradiction as as $\int_{-3}^{0} g(x)dx \in (-2,2)$

$\therefore$ At least at one of the point $c \in (-3, 3)\ g''(x)$ is + ve. But $g(x) < 0$ on $(-3, 3)$.
Hence $g(x)\,g''(x) < 0$ for some $x \in (-3,3)$.
Combining all the cases, discussed aboe, we can conclude that at least at one point in $(-3, 3)$, $g(x)\,g''(x) < 0$.

48. We have, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$

$\Rightarrow 4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a$

$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3a$

$4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$

Consider the equation

$4x^2 f(-1) + 4xf(1) + f(2) = 3x^2 + 3x$

or $[4f(-1) - 3]x^2 + [4f(1) - 3]x + f(2) = 0$

Then clearly this eqn. is satisfied by $x = a,b,c$
A quadratic eqn. satisfied by more than two values of x means it is an identity and hence

$4f(-1) - 3 = 0 \Rightarrow \quad f(-1) = 3/4$

$4f(1) - 3 = 0 \quad\quad f(1) = 3/4$

$f(2) = 0 \quad\quad f(2) = 0$

Let $f(x) = px^2 + qx + r$ [$f(x)$ being a quadratic eqn.]

$f(-1) = \frac{3}{4} \Rightarrow p - q + r = \frac{3}{4}$

$f(1) = \frac{3}{4} \Rightarrow p + q + r = \frac{3}{4}$

$f(2) = 0 \Rightarrow 4p + 2q + r = 0$

Solving the above we get $q = 0,\ p = -\frac{1}{4},\ r = 1$

$\therefore \quad f(x) = -\frac{1}{4}x^2 + 1$

It's maximum value occur at $f'(x) = 0$
i.e., $x = 0$ then $f(x) = 1$, $\therefore\ V(0, 1)$
Let $A(-2, 0)$ be the point where curve meet x–axis.

Let B be the point $\left(h. \frac{4-h^2}{4}\right)$

As $\angle AVB = 90^0$, $m_{AV} \times m_{BV} = -1$

$\Rightarrow \left(\frac{0-1}{-2-1}\right) \times \left(\frac{\frac{4-h^2}{4} - 1}{h - 0}\right) = -1$

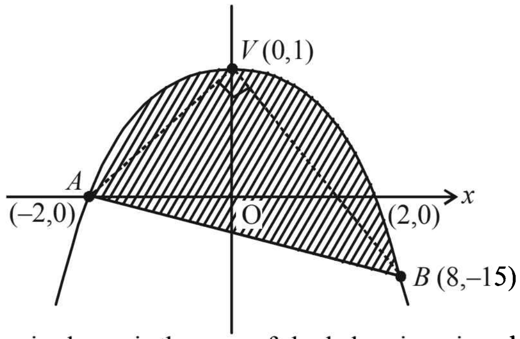
$\Rightarrow \frac{1}{2} \times \left(\frac{-h}{4}\right) = -1 \Rightarrow h = 8$

$\therefore \quad B(8, -15)$

Equation of chord AB *is*

$y + 15 = \frac{0-(-15)}{-2-8}(x-8) \Rightarrow y + 15 = -\frac{3}{2}(x-8)$

$\Rightarrow 2y + 30 = -3x + 24 \Rightarrow 3y + 2y + 6 = 0$



Required area is the area of shaded region given by

$$=\int_{-2}^{2}\left(-\frac{x^2}{4}+1\right)dx+\int_{-2}^{8}\left\{-\left(\frac{-6-3x}{2}\right)\right\}dx-\int_{2}^{8}\left\{-\left(-\frac{x^2}{4}+1\right)\right\}dx$$

$$=2\int_{0}^{2}\left(-\frac{x^2}{4}+1\right)dx+\frac{1}{2}\int_{-2}^{8}(6+3x)\,dx+\frac{1}{4}\int_{2}^{8}\left(-x^2+4\right)dx$$

$$=2\left[\left(\frac{-x^3}{12}+x\right)_0^2\right]+\frac{1}{2}\left[6x+\frac{3x^2}{2}\right]_{-2}^{8}+\frac{1}{4}\left[\frac{-x^3}{3}+4x\right]_2^8$$

$$=2\left[\frac{-8}{12}+2\right]+\frac{1}{2}[(48+3\times 32)-(-12+6)]$$

$$+\left[\frac{1}{4}\left(\frac{-512}{3}+32\right)-\left(\frac{-8}{3}+8\right)\right]$$

$$=2\left[\frac{4}{3}\right]+\frac{1}{2}[150]+\frac{1}{4}\left[\frac{-432}{3}\right]=\frac{125}{3}\text{ sq. units.}$$

49. Let $I=\int_0^1(1-x^{50})^{100}dx$ and $I'=\int_0^1(1-x^{50})^{101}dx$

Then, $I'=\int_0^1 1.(1-x^{50})^{101}dx$

$$=\left[x(1-x^{50})^{101}\right]_0^1+101\int_0^1 50x^{50}(1-x^{50})^{100}dx$$

$$=+5050\int_0^1 x^{50}(1-x^{50})^{100}dx$$

$$-I'=+5050\int_0^1 -x^{50}(1-x^{50})^{100}dx$$

$$\Rightarrow 5050\,I-I'=5050\int_0^1(1-x^{50})^{100}dx$$

$$+5050\int_0^1\left[-x^{50}(1-x^{50})^{100}\right]dx$$

$$\Rightarrow 5050\int_0^1(1-x^{50})^{101}dx=5050\,I'$$

$$\Rightarrow 5050\,I=5051\,I'\Rightarrow 5050\,\frac{I}{I'}=5051$$

F. Match the Following

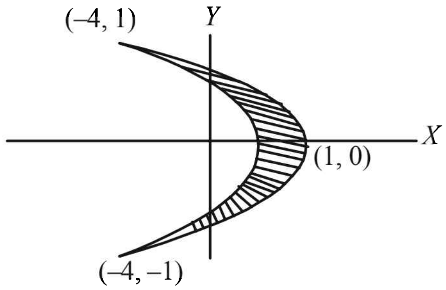
1. **(A) → p, (B) → s, (C) → p, (D) → r**

(A) $\int_0^{\pi/2}(\sin x)^{\cos x}(\cos x\cot x-\log(\sin x)^{\sin x})dx$

$=\int_0^1 du$ where $(\sin x)^{\cos x}=u=1$

(A) → (p)

(B)



Solving $y^2=-\frac{1}{4}x$ and $y^2=-\frac{1}{5}(x-1)$, we get intersection points as $(-4,\pm 1)$

∴ Required area

$$=\int_{-1}^{1}[(1-5y^2)+4y^2]dy=2\int_0^1(1-y^2)dy=\frac{4}{3},$$

(B) → (s)

(C) By inspection, the point of intersection of two curves $y=3^{x-1}\log x$ and $y=x^x-1$ is (1, 0)

For first curve $\frac{dy}{dx}=\frac{3^{x-1}}{x}+3^{x-1}\log 3\log x$

$\Rightarrow\left(\frac{dy}{dx}\right)_{(1,0)}=1=m_1$

For second curve $\frac{dy}{dx}=x^x(1+\log x)$

$\Rightarrow\left(\frac{dy}{dx}\right)_{(1,0)}=1=m_2$

∵ $m_1=m_2$ ⇒ Two curves touch each other

⇒ Angle between them is 0°

∴ $\cos\theta=1$,

(C) → (p)

(D) $\frac{dy}{dx}=\frac{6}{x+y}\Rightarrow\frac{dx}{dy}-\frac{1}{6}x=\frac{y}{6}$

I.F. $=e^{-y/6}$

⇒ Solution is $xe^{-y/6}=-ye^{-y/6}-6e^{-y/6}+c$

$\Rightarrow x+y+6=ce^{y/6}$

$\Rightarrow x+y+6=6e^{y/6}$ ∴ $(y(0)=0)$

$\Rightarrow 12=6e^{y/6}$ (using $x+y=6$)

$\Rightarrow y=6\ln 2$ (D) → (r)

2. **(A) → s; (B) → s; (C) → p; (D) → r**

(A) $\int_{-1}^{1}\frac{dx}{1+x^2}=[\tan^{-1}x]_{-1}^{1}=\tan^{-1}(1)-\tan^{-1}(-1)$

$$=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{2\pi}{4}=\frac{\pi}{2}$$

(B) $\int_0^1\frac{dx}{\sqrt{1-x^2}}=\left[\sin^{-1}x\right]_0^1=\sin^{-1}(1)-\sin^{-1}(0)$

$$=\frac{\pi}{2}-0=\frac{\pi}{2}$$

(C) $\int_2^3\frac{dx}{1-x^2}=\left[\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right]_2^3=\frac{1}{2}[\log 2-\log 3]$

$$=\frac{1}{2}\log 2/3$$

(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \left[\sec^{-1} x\right]_1^2 = \sec^{-1} 2 - \sec^{-1} 1$

$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$

3. **(d)** $P(2)$ Let $f(x) = ax^2 + bx + c$

where $a, b, c \geq 0$ and a, b, c are integers.

$\because f(0) = 0 \Rightarrow c = 0$

$\therefore f(x) = ax^2 + bx$

Also $\int_0^1 f(x)dx = 1$

$\Rightarrow \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = 1 \Rightarrow \frac{a}{3} + \frac{b}{2} = 1 \Rightarrow 2a + 3b = 6$

$\because$ a and b are integers

$a = 0$ and $b = 2$

or $a = 3$ and $b = 0$

$\therefore$ There are only 2 solutions.

$Q(3)$ $f(x) = \sin x^2 + \cos x^2$

$f(x)$ is max. $\sqrt{2}$ at $x^2 = \frac{\pi}{4}$ or $\frac{9\pi}{4}$

$\Rightarrow x = \pm\frac{\sqrt{\pi}}{2}$ or $\pm\frac{3\sqrt{\pi}}{2} \in \left[-\sqrt{13}, \sqrt{13}\right]$

$\therefore$ There are four points.

$R(1)$ $I = \int_{-2}^{2} \frac{3x^2}{1+e^x} dx = \int_{-2}^{2} \frac{3x^2}{1+e^{-x}} dx$

$\left[\text{Using} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$= \int_{-2}^{2} \frac{3x^2 e^x}{1+e^x} dx$

$2I = \int_{-2}^{2} \frac{3x^2\left(1+e^x\right)}{1+e^x} dx = \int_{-2}^{2} 3x^2 dx$

$2I = \left(x^3\right)_{-2}^{2} = 8 - (-8) = 16 \Rightarrow I = 8$

$S(4)$ $\dfrac{\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx}{\int_0^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx} = 0$

$\because$ Numerator = 0, function being odd.

Hence option (d) is correct sequence.

G. Comprehension Based Questions

1. **(a)** $\int_0^{\pi/2} \sin x\, dx = \frac{\left(\frac{\pi}{2} - 0\right)}{4}\left(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}\right)$

$= \frac{\pi}{8}(1 + \sqrt{2})$

2. **(d)** $\lim_{x\to a} \dfrac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x) + f(a))}{(x-a)^3} = 0$

$\lim_{h\to 0} \dfrac{\int_a^{a+h} f(x)dx - \frac{h}{2}(f(a+h) + f(a))}{h^3} = 0$

$\Rightarrow \lim_{h\to 0} \dfrac{f(a+h) - \frac{1}{2}[f(a) + f(a+h)] - \frac{h}{2}(f'(a+h))}{3h^2} = 0$

[Using L'Hospital rule]

$\Rightarrow \lim_{h\to 0} \dfrac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$

$\Rightarrow \lim_{h\to 0} \dfrac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$

[Using L'Hospital rule]

$\Rightarrow \lim_{h\to 0} -\frac{f''(a+h)}{12} = 0 \Rightarrow f''(x) = 0, \forall a \in R$

$\Rightarrow$ $f(x)$ must be of max. degree 1.

3. **(b)** $f''(x) < 0, \forall x \in (a, b)$, for $c \in (a, b)$

$F(c) = \frac{c-a}{2}(f(a) + f(c)) + \frac{b-c}{2}(f(b) + f(c))$

$= \frac{b-a}{2}f(c) + \frac{c-a}{2}f(a) + \frac{b-c}{2}f(b)$

$\Rightarrow F'(c) = \frac{b-a}{2}f'(c) + \frac{1}{2}f(a) - \frac{1}{2}f(b)$

$= \frac{1}{2}[(b-a)f'(a) - f(b)]$

$F''(c) = \frac{1}{2}(b-a)f''(c) < 0$

$[\because f''(x) < 0, \forall x \in (a,b) \text{ and } b > a]$

$\therefore$ $F(c)$ is max. at the point $(c, f(c))$ where $F'(c) = 0$

$\Rightarrow f'(c) = \frac{1}{2}\left(\frac{f(b) - f(a)}{b-a}\right)$.

(For 4-6). Given the implicit function $y^3 - 3y + x = 0$

For $x \in (-\infty, -2) \cup (2, \infty)$ it is $y = f(x)$ real valued differentiable function, and for $x \in (-2, 2)$ it is $y = g(x)$ real valued differentiable function.

4. **(b)** We have $y^3 - 3y + x = 0 \Rightarrow 3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3(1-y^2)}$ or $f'(x) = \frac{1}{3\left[1 - [f(x)]^2\right]}$

Also $3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 - 3\frac{d^2y}{dx^2} = 0$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2y}{1-y^2}\left(\frac{dy}{dx}\right)^2 \Rightarrow f''(x) = \frac{2f(x)}{9\left[1 - [f(x)]^2\right]^3}$

$\therefore$ $f''(-10\sqrt{2}) = \frac{2 \times 2\sqrt{2}}{9[1-8]^3} = \frac{-4\sqrt{2}}{3^2 \times 7^3}$

5. **(a)** For $x < -2$

we have, $3y - y^3 < -2 \Rightarrow y^3 - 3y - 2 > 0$

$\Rightarrow (y+1)^2(y-2) > 0 \Rightarrow y > 2 \; \forall \; x < -2$

$\Rightarrow f(x)$ is positive $\forall \; x < -2$

Hence required area $= \int_a^b f(x)dx = \int_a^b 1.f(x)dx$

$= x f(x)]_a^b - \int_a^b x f'(x)dx$

$= b f(b) - a f(a) - \int_a^b \frac{x.1}{3[1-(f(x))^2]}dx$

$= \int_a^b \frac{x}{3[(f(x))^2 - 1]} + b f(b) - a f(a)$

6. **(d)** For $y = g(x)$, we have $y^3 - 3y + x = 0$

$\Rightarrow [g(x)]^3 - 3[g(x)] + x = 0 \quad ...(1)$

Putting $x = -x$, we get

$\Rightarrow [g(-x)]^3 - 3[g(-x)] - x = 0 \quad ...(2)$

Adding equations (1) and (2) we get

$[g(x)]^3 + [g(-x)]^3 - 3\{[g(x)] + [g(-x)]\} = 0 \frac{n!}{r!(n-r)!}$

$\Rightarrow [g(x) + g(-x)]$

$[(g(x))^2 + (g(-x))^2 - g(x)g(-x) - 3] = 0$

For $g(0) = 0$, we should have $g(x) + g(-x) = 0$

[$\because$ From other factor we get $g(0) = \pm\sqrt{3}$]

$\Rightarrow g(x)$ is an odd function

$\therefore \int_{-1}^{1} g'(x)dx = [g(x)]_{-1}^{1} = g(1) - g(-1)$

$= g(1) + g(1) = 2\,g(1).$

7. **(a)** We have $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}; 0 < a < 2$

$\Rightarrow f'(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2}$

$\Rightarrow (x^2 + ax + 1)^2 f'(x) = 2a(x^2 - 1)$

$\Rightarrow (x^2 + ax + 1)^2 f''(x) + 2(x^2 + ax + 1)$

$(2x + a) f'(x) = 4ax \quad ...(1)$

Putting $x = -1$ in equation (1), we get

$(2 - a^2) f''(-1) = -4a \quad ...(2)$

Putting $x = 1$ in equation (1), we get

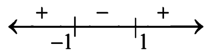
$(2 + a)^2 f''(1) = 4a \quad ...(3)$

Adding equations (2) and (3), we get

$(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

8. **(a)** We have $f'(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2}$

$f'(x) = 0 \Rightarrow x = -1, 1$ are the critical points.



$\therefore f(x)$ is decreasing on $(-1, 1)$

Also using equation (1), $f''(-1) = \frac{-4a}{(2-a)^2} < 0$

and $f''(1) = \frac{4a}{(2+a)^2} > 0$

$\therefore x = -1$ is a point of local maximum and $x = 1$ is a point of local minimum.

9. **(b)** $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2}dt \Rightarrow g'(x) = \frac{f'(e^x)}{1+e^{2x}}$

$= \frac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})} = \frac{2ae^x}{1+e^{2x}} \cdot \frac{e^{2x}-1}{(e^{2x}+ae^x+1)^2}$

Now $g'(x) > 0$ for $e^{2x} - 1 > 0 \Rightarrow x > 0$

and $g'(x) < 0$ for $e^{2x} - 1 < 0 \Rightarrow x < 0$

$\therefore g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

10 **(c)** $f(x) = 4x^3 + 3x^2 + 2x + 1$

$\because f(x)$ is a cubic polynomial

$\therefore$ It has at least one real root.

Also $f'(x) = 12x^2 + 6x + 2 = 2(6x^2 + 3x + 1) > 0 \, \forall \, x \in R$

$\therefore f(x)$ is strictly increasing function

$\Rightarrow$ There is only one real root of $f(x) = 0$

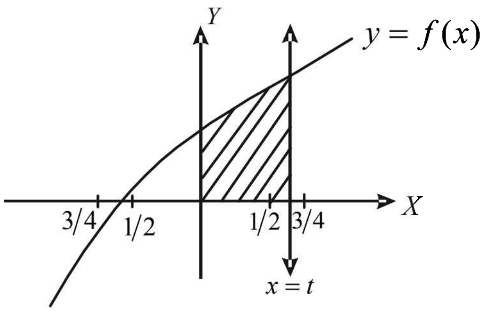
Also $f(-1/2) = 1 - 1 + \frac{3}{4} - \frac{1}{2} > 0$

and $f(-3/4) = 1 - \frac{3}{2} + \frac{27}{16} - \frac{27}{16} < 0$

$\therefore$ Real root lies between $-\frac{3}{4}$ and $-\frac{1}{2}$ and hence

$s \in \left(-\frac{3}{4}, -\frac{1}{2}\right)$

11. **(a)** $y = f(x)$, $x = 0$, $y = 0$ and $x = t$ bounds the area as shown in the figure



$\therefore$ Required area is given by

$A = \int_0^t dx = \int_0^t (4x^3 + 3x^2 + 2x + 1)dx$

$= t^4 + t^3 + t^2 + t = t(t+1)(t^2+1)$

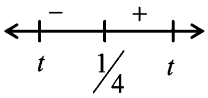
New $\frac{1}{2} < t < \frac{3}{4}$; $\frac{3}{2} < t+1 < \frac{7}{4}$; $\frac{5}{4} < t^2+1 < \frac{25}{16}$

$\therefore \frac{1}{2}\times\frac{3}{2}\times\frac{5}{4} < A < \frac{3}{4}\times\frac{7}{4}\times\frac{25}{16}$ or $A \in \left(\frac{15}{16}, \frac{525}{256}\right) \subset \left(\frac{3}{4}, 3\right)$

12. **(b)** $f'(x) = 2(6x^2+3x+1)$

$f''(x) = 6(4x+1) \Rightarrow$ Critical point $x = -1/4$

$\therefore$ decreasing in $\left(-t, -\frac{1}{4}\right)$



and increasing in $\left(-\frac{1}{4}, t\right)$

13. **(a)** $g(a) = \lim_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$

$\therefore g\left(\frac{1}{2}\right) = \lim_{h\to 0^+} \int_h^{1-h} t^{-1/2}(1-t)^{-1/2}\,dt$

$= \lim_{h\to 0^+} \int_h^{1-h} \frac{1}{\sqrt{t(1-t)}}\,dt = \lim_{h\to 0^+} \int_h^{1-h} \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(t-\frac{1}{2}\right)^2}}\,dt$

$= \lim_{h\to 0^+} \left[\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_h^{1-h} = \lim_{h\to 0^+} \sin^{-1}(2t-1)\Big]_h^{1-h}$

$= \lim_{h\to 0^+} \left[\sin^{-1}(1-2h) - \sin^{-1}(2h-1)\right]$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

14. **(d)** $g(a) = \lim_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$

$g(a) = \lim_{h\to 0^+} \int_h^{1-h} (1-t)^{-a}\, t^{a-1}\,dt$

$\left(\text{using} \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right)$

Also $g(1-a) = \lim_{h\to 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a}\,dt$

Thus $g(a) = g(1-a)$

$\Rightarrow g'(a) = -g'(1-a) \Rightarrow g'(a) + g'(1-a) = 0$

Putting $a = \frac{1}{2}$ we get $g'\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right) = 0$

or $g'\left(\frac{1}{2}\right) = 0$

15. **(a, b, c)** $f(x) = xF(x) \Rightarrow f'(x) = F(x) + xF'(x)$

$\therefore f'(1) = F(1) + F'(1) = F'(1) < 0 \left(\because F'(x) < 0, x \in \left(\frac{1}{2}, 3\right)\right)$

$f(2) = 2F(2) < 0,$

$\left(\because F'(x) < 0 \Rightarrow F \text{ is decreasing on } \left(\frac{1}{2}, 3\right) \text{ and } F'(1) = 0,\right.$

$F(3) = -4)$

$f'(x) = F(x) + xF'(x)$

For the same reason given above and $F'(x) < 0$ given.

$F(x) < 0 \ \forall\ x \in (1, 3)$

$\therefore \ f'(x) \neq 0, x \in (1, 3)$

16. **(c, d)** $\int_1^3 x^2 F'(x)\,dx = -12$

$\Rightarrow \left[x^2 F(x)\right]_1^3 - \int_1^3 2x\, F(x)\,dx = -12$

$\Rightarrow 9F(3) - F(1) - 2\int_1^3 xF(x)\,dx = -12$

$\Rightarrow \int_1^3 xF(x)\,dx = -12 \Rightarrow \int_1^3 f(x)\,dx = -12$ ...(i)

Also $\int_1^3 x^3 F''(x)\,dx = 40$

$\Rightarrow \left[x^3 F'(x)\right]_1^3 - 3\int_1^3 x^2 F'(x)\,dx = 40$

$\Rightarrow \left[x^2\left(f'(x) - F(x)\right)\right]_1^3 - 3\times(-12) = 40$

$\left\{\begin{array}{l}\text{Using } xF'(x) = f'(x) - F(x) \\ \text{and} \int_1^3 x^2 F'(x)\,dx = -12\end{array}\right.$

$\Rightarrow 9(f'(3) - F(3)) - (f'(1) - F(1)) = 4$

$\Rightarrow 9f'(3) - 9\times(-4) - f'(1) + 0 = 4$

$\Rightarrow 9f'(3) - f'(1) + 32 = 0$

I. Integer Value Correct Type

1. **(0)** Given that $f(x) = \int_0^x f(t)\,dt$

Clearly $f(0) = 0$. Also $f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1$

Integrating both sides with respect to x, we get

$\int \frac{f'(x)}{f(x)}\,dx = \int 1\,dx$

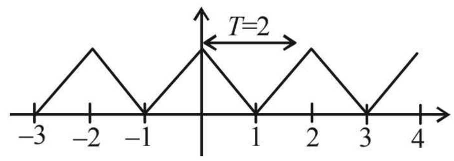
$\Rightarrow \ln f(x) = x + \ln C \quad \Rightarrow f(x) = Ce^x$

Now $f(0) = 0 \Rightarrow Ce^x = 0 \Rightarrow C = 0$

$\therefore \quad f(x) = 0\ \forall\ x \Rightarrow \quad f(\ln 5) = 0$

2. **(4)** Given function is $f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$

The graph of this function is as below



Clearly $f(x)$ is periodic with period 2

Also $\cos \pi x$ is periodic with period 2

$\therefore f(x)\cos \pi x$ is periodic with period 2

$$\therefore I = \frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$$

$$= \frac{\pi^2}{10}\times 10\int_0^2 f(x)\cos \pi x\, dx$$

$$= \pi^2\left[\int_0^1 (1-x)\cos \pi x\, dx + \int_1^2 (x-1)\cos \pi x\, dx\right]$$

$$= \pi^2\left[\left\{(1-x)\frac{\sin \pi x}{\pi}\Big|_0^1 + \int_0^1 \frac{\sin \pi x}{\pi}dx\right\} + \left\{(x-1)\frac{\sin \pi x}{\pi}\Big|_1^2 - \int_1^2 \frac{\sin \pi x}{\pi}dx\right\}\right]$$

$$= \pi^2\left[\left(-\frac{1}{\pi^2}\cos \pi x\right)\Big|_0^1 - \left(-\frac{1}{\pi^2}\cos \pi x\right)^2_1\right]$$

$$= \left[(-\cos\pi + \cos 0) - (-\cos 2\pi + \cos \pi)\right] = [2+2] = 4$$

3. **(2)** $\int_0^1 4x^3\left[\frac{d^2}{dx^2}\left(1-x^2\right)^5\right]dx$

$$= 4x^3\left[\frac{d}{dx}\left(1-x^2\right)^5\right]\Big|_0^1 - \int_0^1\left[\frac{d}{dx}\left(1-x^2\right)^5\right].12x^2dx$$

$$= -12x^2\left(1-x^2\right)^5\Big|_0^1 + \int_0^1\left(1-x^2\right)^5.24x dx$$

$$= -12\int_0^1\left(1-x^2\right)^5.(-2x)dx$$

$$= -12\left(\frac{\left(1-x^2\right)^6}{6}\right)_0^1 = -12\left(0-\frac{1}{6}\right) = 2$$

4. **(0)** $I = \int_{-1}^{2}\frac{xf(x^2)}{2+f(x+1)}dx$

$-1 < x < 2 \Rightarrow 0 < x^2 < 4$

Also $0 < x^2 < 1 \Rightarrow f(x^2) = [x^2] = 0$

$1 \le x^2 < 2 \Rightarrow f(x^2) = [x^2] = 1$

$2 \le x^2 < 3 \Rightarrow f(x^2) = 0$ (using definition of f)

$3 \le x^2 < 4 \Rightarrow f(x^2) = 0$ (using definition of f)

Also $1 \le x^2 < 2 \Rightarrow 1 \le x < \sqrt{2}$

$\Rightarrow 2 \le x+1 < \sqrt{2}+1$

$\Rightarrow f(x+1) = 0$

$$\therefore I = \int_1^{\sqrt{2}}\frac{x\times 1}{2+0}dx = \left[\frac{x^2}{4}\right]_1^{\sqrt{2}} = \frac{2}{4}-\frac{1}{4} = \frac{1}{4}$$

$\Rightarrow 4I = 1$ or $4I - 1 = 0$

5. **(3)** $F(x) = \int_x^{x^2+\pi/6} 2\cos^2 t\, dt$

$$F'(\alpha) = 2\cos^2\left(\alpha^2+\frac{\pi}{6}\right).2\alpha - 2\cos^2\alpha$$

$$F'(\alpha) + 2 = \int_0^\alpha f(x)dx$$

$\Rightarrow F''(\alpha) = f(\alpha)$

$$\therefore f(\alpha) = 4\alpha.2\cos\left(\alpha^2+\frac{\pi}{6}\right).\left[-\sin\left(\alpha^2+\frac{\pi}{6}\right)\right].2\alpha + 4\cos^2\left(\alpha^2+\frac{\pi}{6}\right) - 4\cos\alpha\,(-\sin\alpha)$$

$$\therefore f(0) = 4\cos^2\frac{\pi}{6} = 4\times\frac{3}{4} = 3$$

6. **(9)** $\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Let $9x + 3\tan^{-1}x = t \Rightarrow \frac{12+9x^2}{1+x^2}dx = dt$

$$\therefore \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\therefore \log_e\left|1+e^{9+\frac{3\pi}{4}}-1\right| - \frac{3\pi}{4} = 9$$

7. **(7)** $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14} \Rightarrow \lim_{x\to 1}\frac{\int_{-1}^x f(t)dt}{\int_{-1}^x t|f(f(t))|dt}$

$\because \int_{-1}^1 f(t)dt = 0$ and $\int_{-1}^1 t|f(f(t))|dt = 0$

$f(t)$ being odd function

$\therefore$ Using *L* Hospital's rule, we get

$$\lim_{x\to 1}\frac{f(x)}{x|f(f(x))|} = \frac{1}{14}$$

$$\Rightarrow \frac{f(1)}{|f(f(1))|} = \frac{1}{14} \Rightarrow \frac{1/2}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

$$\Rightarrow \left|f\left(\frac{1}{2}\right)\right| = 7 \Rightarrow f\left(\frac{1}{2}\right) = 7$$

8. **(1)** Let $f(x) = \int_0^x \frac{t^2}{1+t^4}dt - 2x + 1$

$\Rightarrow f'(x) = \frac{x^2}{1+x^4} - 2 < 0\ \forall\ x \in [0,1]$

$\therefore$ f is decreasing on [0, 1]

Also f(0) = 1

and $f(1) = \int_0^1 \frac{t^2}{1+t^4}dt - 1$

For $0 \le t \le 1 \Rightarrow 0 \le \frac{t^2}{1+t^4} < \frac{1}{2}$

$\therefore \int_0^1 \frac{t^2}{1+t^4}dt < \frac{1}{2}$

$\Rightarrow f(1) < 0$

$\therefore$ f(x) crosses *x*-axis exactly once in [0, 1]

$\therefore$ f(x) = 0 has exactly one root in [0, 1]

Section-B JEE Main/ AIEEE

1. **(a)** $I=\int_0^{10\pi}|\sin x|\,dx=10\int_0^{\pi}|\sin x|\,dx=10\int_0^{\pi}\sin x\,dx$

[$\because$ $|\sin x|$ is periodic with period π and $\sin x>0$ if $0<x<\pi$]

$$I=20\int_0^{\pi/2}\sin x\,dx=20[-\cos x]_0^{\pi/2}=20$$

2. **(b)** $I_n+I_{n+2}=\int_0^{\pi/4}\tan^n x(1+\tan^2 x)dx$

$$=\int_0^{\pi/4}\tan^n x\sec^2 x\,dx=\left[\frac{\tan^{n+1}x}{n+1}\right]_0^{\pi/4}$$

$$=\frac{1-0}{n+1}=\frac{1}{n+1}$$

$$\therefore\ I_n+I_{n+2}=\frac{1}{n+1}\Rightarrow\lim_{n\to\infty}n\,[I_n+I_{n+2}]$$

$$=\lim_{n\to\infty}n.\frac{1}{n+1}=\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{n}{n\left(1+\frac{1}{n}\right)}=1$$

3. **(d)** $\int_0^2\left[x^2\right]dx=\int_0^1\left[x^2\right]dx+\int_1^{\sqrt2}\left[x^2\right]dx+\int_{\sqrt2}^{\sqrt3}\left[x^2\right]+\int_{\sqrt3}^{2}\left[x^2\right]dx$

$$=\int_0^1 0dx+\int_1^{\sqrt2}1dx+\int_{\sqrt2}^{\sqrt3}2dx+\int_{\sqrt3}^{2}3dx$$

$$=[x]_1^{\sqrt2}+[2x]_{\sqrt2}^{\sqrt3}+[3x]_{\sqrt3}^{2}$$

$$=\sqrt2-1+2\sqrt3-2\sqrt2+6-3\sqrt3=5-\sqrt3-\sqrt2$$

4. **(b)** $\int_{-\pi}^{\pi}\frac{2x\,(1+\sin x)}{1+\cos^2 x}dx$

$$=\int_{-\pi}^{\pi}\frac{2x\,dx}{1+\cos^2 x}+2\int_{-\pi}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

$=0+4\int_0^{\pi}\frac{x\sin x\,dx}{1+\cos^2 x}$; $\left[\because\int_{-a}^{a}f(x)\,dx=0\right.$ if f(x) is odd

$=2\int_0^a f(x)\,dx$ if $f(x)$ is even.

$$I=4\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$I=4\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow I=4\pi\int_0^{\pi}\frac{\sin x\,dx}{1+\cos^2 x}-4\int\frac{x\sin x\,dx}{1+\cos^2 x}$$

$$\Rightarrow 2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

put $\cos x=t\Rightarrow-\sin x\,dx=dt$

$$\therefore\ I=-2\pi\int_1^{-1}\frac{1}{1+t^2}dt=2\pi\int_{-1}^{1}\frac{1}{1+t^2}dt$$

$$=2\pi\left[\tan^{-1}t\right]_{-1}^{1}=2\pi\left[\tan^{-1}1-\tan^{-1}(-1)\right]$$

$$=2\pi\left[\frac{\pi}{4}-\left(\frac{-\pi}{4}\right)\right]=2\pi.\frac{\pi}{2}=\pi^2$$

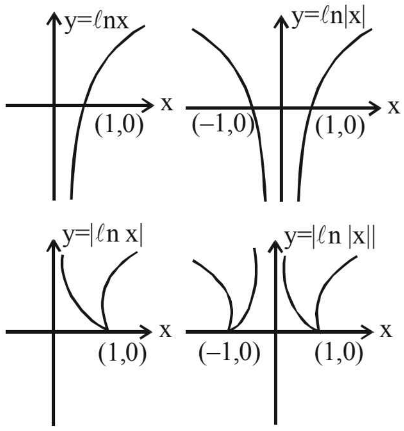
5. **(d)** We have $\int_0^2 f(x)dx=\frac{3}{4}$; Now,

$$\int_0^2 xf'(x)dx=x\int_0^2 f'(x)dx-\int_0^2 f(x)dx$$

$$=[x\ f(x)]_0^2-\frac{3}{4}=2f(2)-\frac{3}{4}$$

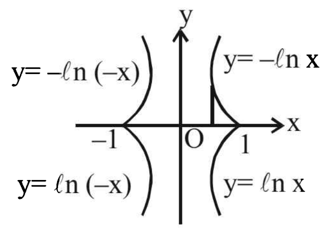
$$=0-\frac{3}{4}\ (\because f(2)=0)=-\frac{3}{4}.$$

6. **(a)** First we draw each curve as separate graph



NOTE : Graph of $y=|f(x)|$ can be obtained from the graph of the curve $y=f(x)$ by drawing the mirror image of the portion of the graph below x-axis, with respect to x-axis.

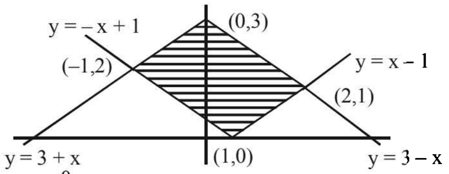
Clearly the bounded area is as shown in the following figure.



Required area $=4\int_0^1(-\ell nx)dx$

$$=-4[x\ \ell n\ x-x]_0^1=4\ \text{sq. units}$$

7. (d)



$A = \int_{-1}^{0} \{(3+x)-(-x+1)\}\, dx +$

$\int_{0}^{1}\{(3-x)-(-x+1)\}\, dx + \int_{1}^{2}\{(3-x)-(x-1)\}\, dx$

$= \int_{-1}^{0} (2+2x)dx + \int_{0}^{1} 2dx + \int_{1}^{2} (4-2x)dx$

$= \left[2x-x^2\right]_{-1}^{0} + \left[2x\right]_0^1 + \left[4x-x^2\right]_1^2$

$= 0-(-2+1)+(2-0)+(8-4)-(4-1)$

$= 1+2+4-3 = 4$ sq. units

8. (c) $I = \int_a^b xf(x)dx = \int_a^b (a+b-x)f(a+b-x)dx$

$= (a+b)\int_a^b f(a+b-x)dx - \int_a^b xf(a+b-x)dx$

$= (a+b)\int_a^b f(x)\, dx - \int_a^b xf(x)dx$

[$\because$ given that $f(a+b-x) = f(x)$]

$2I = (a+b)\int_a^b f(x)dx \Rightarrow I = \frac{(a+b)}{2}\int_a^b f(x)dx$

9. (d) Given $f'(x) = f(x) \Rightarrow \frac{f'(x)}{f(x)} = 1$

Integrating $\log f(x) = x + c \Rightarrow f(x) = e^{x+c}$

$f(0) = 1 \Rightarrow f(x) = e^x$

$\therefore \int_0^1 f(x)g(x)dx = \int_0^1 e^x(x^2 - e^x)dx$

$= \int_0^1 x^2e^x dx - \int_0^1 e^{2x}dx$

$= \left[x^2e^x\right]_0^1 - 2\left[xe^x - e^x\right]_0^1 - \frac{1}{2}\left[e^{2x}\right]_0^1$

$= e - \left[\frac{e^2}{2} - \frac{1}{2}\right] - 2\left[e - e + 1\right] = e - \frac{e^2}{2} - \frac{3}{2}$

10. (d) $I = \int_0^1 x(1-x)^n\, dx = \int_0^1 (1-x)(1-1+x)^n\, dx$

$= \int_0^1 (1-x)x^n dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2}$

11. (b) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ [Using definite integrals as limit of sum]

$= \int_0^1 e^x dx = e - 1$

12. (d) $\int_{-2}^{3} |1-x^2|\, dx = \int_{-2}^{3} |x^2-1|\, dx$

Now $|x^2-1| = \begin{cases} x^2-1 & if \quad x \le -1 \\ 1-x^2 & if \quad -1 \le x \le 1 \\ x^2-1 & if \quad x \ge 1 \end{cases}$

$\therefore$ Integral is $\int_{-2}^{-1} (x^2-1)dx + \int_{-1}^{1} (1-x^2)dx + \int_{1}^{3} (x^2-1)dx$

$\left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_1^3$

$= \left(-\frac{1}{3}+1\right) - \left(-\frac{8}{3}+2\right) + \left(2 - \frac{2}{3}\right) + \left(\frac{27}{3} - 3\right) - \left(\frac{1}{3} - 1\right)$

$= \frac{2}{3} + \frac{2}{3} + \frac{4}{3} + 6 + \frac{2}{3} = \frac{28}{3}$

13. (c) $I = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sqrt{1+\sin 2x}} dx$

We know $[(\sin x + \cos x)^2 = 1 + \sin 2x]$, so

$I = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{(\sin x + \cos x)} dx = \int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$

$\left[\because \sin x + \cos x > 0 \; if \; 0 < x < \frac{\pi}{2}\right]$

or $I = \left[-\cos x + \sin x\right]_0^{\frac{\pi}{2}} = 2$

14. (b) Let $I = \int_0^{\pi} xf(\sin x)dx = \int_0^{\pi} (\pi - x)f(\sin x)dx$

$\therefore 2I = \pi \int_2^{\pi} f(\sin x)dx = \pi . 2\int_0^{\frac{\pi}{2}} f(\sin x)dx$

$\therefore I = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx \Rightarrow A = \pi$

15. (d) $f(x) = \frac{e^x}{1+e^x} \Rightarrow f(-x) = \frac{e^{-x}}{1+e^{-x}} = \frac{1}{e^x+1}$

$\therefore f(x) + f(-x) = 1 \;\forall\; x$

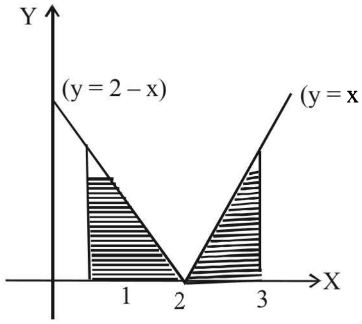
Now $I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx$

$= \int_{f(-a)}^{f(a)} (1-x)g\{x(1-x)\}dx$

$\left[\text{using} \int_a^b f(x)\, dx\, a = \int_a^b f(a+b-x)dx\right]$

$= I_2 - I_1 \Rightarrow 2I_1 = I_2 \quad \therefore \frac{I_2}{I_1} = 2$

16. **(d)** The required area is shown by shaded region



Required Area

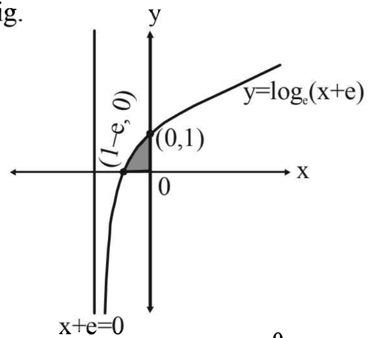
$$A = \int_1^3 |x-2|dx = 2\int_2^3 (x-2)dx = 2\left[\frac{x^2}{2} - 2x\right]_2^3 = 1$$

17. **(b)** $I_1 = \int_0^1 2^{x^2} dx,\ I_2 = \int_0^1 2^{x^3} dx,$

$$= I_3 = \int_0^1 2^{x^2} dx,\ I_4 = \int_0^1 2^{x^3} dx\ \forall 0 < x < 1,\ x^2 > x^3$$

$$\Rightarrow \int_0^1 2^{x^2} dx > \int_0^1 2^{x^3} dx \Rightarrow I_1 > I_2$$

18. **(a)** The graph of the curve $y = \log_e(x + e)$ is as shown in the fig.



Required area $A = \int_{1-e}^{0} ydx = \int_{1-e}^{0} \log_e(x+e)dx$

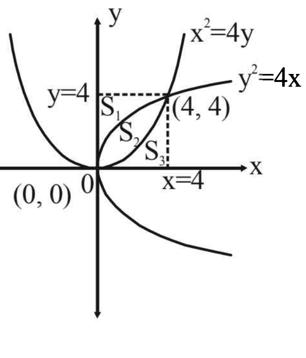
put $x + e = t \Rightarrow dx = dt$ also At $x = 1 - e, t = 1$

At $x = 0, t = e$ $\therefore A = \int_1^e \log_e tdt = [t\log_e t - t]_1^e$

$e - e - 0 + 1 = 1$

Hence the required area is 1 square unit.

19. **(d)** Intersection points of $x^2 = 4y$ and $y^2 = 4x$ are (0, 0) and (4, 4). The graph is as shown in the figure.



By symmetry, we observe

$$S_1 = S_3 = \int_0^4 ydx = \int_0^4 \frac{x^2}{4}dx = \left[\frac{x^3}{12}\right]_0^4 = \frac{16}{3} \text{ sq. units}$$

$$\text{Also } S_2 = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \frac{4}{3} \times 8 - \frac{16}{3} = \frac{16}{3} \text{ sq. units}$$

$$\therefore S_1 : S_2 : S_3 = 1:1:1$$

20. **(d)** Given that $\int_{\pi/4}^{\beta} f(x)dx = \beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$

Differentiating w. r . t β

$$f(\beta) = \beta\cos\beta + \sin\beta - \frac{\pi}{4}\sin\beta + \sqrt{2}$$

$$f\left(\frac{\pi}{2}\right) = \left(1 - \frac{\pi}{4}\right)\sin\frac{\pi}{2} + \sqrt{2} = 1 - \frac{\pi}{4} + \sqrt{2}$$

21. **(b)** Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}dx$ ....(1)

$$= \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}}dx$$

$$\left[\text{Using} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}dx \quad ....(2)$$

Adding equations (1) and (2) we get

$$2I = \int_{-\pi}^{\pi} \cos^2 x\left(\frac{1+a^x}{1+a^x}\right)dx = \int_{-\pi}^{\pi} \cos^2 x\, dx$$

$$= 2\int_0^{\pi} \cos^2 x\, dx = 2 \times 2\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = 4\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = 2\int_0^{\frac{\pi}{2}} \left(1 - \cos^2 x\right)dx$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} dx - 2\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

$$\Rightarrow I + I = 2\left(\frac{\pi}{2}\right) = \pi \Rightarrow I = \frac{\pi}{2}$$

22. **(b)** $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx$ …(1)

$$I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}dx \quad \ldots(2)$$

[using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$]

Adding equation (1) and (2)

$$2I = \int_3^6 dx = 3 \Rightarrow I = \frac{3}{2}$$

23. (d) $I = \int_0^\pi xf(\sin x)dx = \int_0^\pi (\pi - x)f(\sin x)dx$

$$= \pi\int_0^\pi f(\sin x)dx - I \Rightarrow 2I = \pi\int_0^\pi f(\sin x)dx$$

$$I = \frac{\pi}{2}\int_0^\pi f(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)dx$$

$$= \pi\int_0^{\pi/2} f(\cos x)dx$$

24. (c) $I = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} [(x+\pi)^3 + \cos^2(x+3\pi)]dx$

Put $x + \pi = t$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [t^3 + \cos^2 t)dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt$$

[using the property of even and odd function]

$$= \int_0^{\frac{\pi}{2}} (1+\cos 2t)dt = \frac{\pi}{2} + 0$$

25. (b) Let $a = k + h$ where k is an integer such that $[a] = k$ and $0 \le h < 1$

$$\therefore \int_1^a [x]f'(x)dx = \int_1^2 1f'(x)dx + \int_2^3 2f'(x)dx + \ldots$$

$$\int_{k-1}^{k} (k-1)dx + \int_k^{k+h} kf'(x)dx$$

$\{f(2)-f(1)\} + 2\{f(3)-f(2)\} + 3\{f(4)-f(3)\}$
$+ \ldots\ldots + (k-1)\{f(k)-f(k-1)\}$
$+ k\{f(k+h) - f(k)\}$

$= -f(1) - f(2) - f(3) \ldots\ldots\ldots - f(k) + kf(k+h)$

$= [a]f(a) - \{f(1) + f(2) + f(3) + \ldots\ldots\ldots f([a])\}$

26. (c) Given $f(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x) = \int_1^x \frac{\log t}{1+t}dt$

$$\therefore F(e) = f(e) + f\left(\frac{1}{e}\right)$$

$$\Rightarrow F(e) = \int_1^e \frac{\log t}{1+t}dt + \int_1^{1/e} \frac{\log t}{1+t}dt \quad \ldots(A)$$

Now for solving, $I = \int_1^{1/e} \frac{\log t}{1+t}dt$

$\therefore$ Put $\frac{1}{t} = z \Rightarrow -\frac{1}{t^2}dt = dz \Rightarrow dt = -\frac{dz}{z^2}$

and limit for $t = 1 \Rightarrow z = 1$ and for $t = 1/e \Rightarrow z = e$

$$\therefore I = \int_1^e \frac{\log\left(\frac{1}{z}\right)}{1+\frac{1}{z}}\left(-\frac{dz}{z^2}\right) = \int_1^e \frac{(\log 1 - \log z).z}{z+1}\left(-\frac{dz}{z^2}\right)$$

$$= \int_1^e -\frac{\log z}{(z+1)}\left(-\frac{dz}{z}\right) \qquad [\because \log 1 = 0]$$

$$= \int_1^e \frac{\log z}{z(z+1)}dz$$

$$\therefore I = \int_1^e \frac{\log t}{t(t+1)}dt$$

[By property $\int_a^b f(t)dt = \int_a^b f(x)dx$]

Equation (A) becomes

$$F(e) = \int_1^e \frac{\log t}{1+t}dt + \int_1^e \frac{\log t}{t(1+t)}dt$$

$$= \int_1^e \frac{t.\log t + \log t}{t(1+t)}dt = \int_1^e \frac{(\log t)(t+1)}{t(1+t)}dt$$

$$\Rightarrow F(e) = \int_1^e \frac{\log t}{t}dt$$

Let $\log t = x$ $\qquad \therefore \frac{1}{t}dt = dx$

[for limit $t = 1, x = 0$ and $t = e, x = \log e = 1$]

$$\therefore F(e) = \int_0^1 x\,dx \quad F(e) = \left[\frac{x^2}{2}\right]_0^1 \Rightarrow F(e) = \frac{1}{2}$$

27. (d) $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$

$$\therefore \left[\sec^{-1} t\right]_{\sqrt{2}}^{x} = \frac{\pi}{2} \qquad \left[\because \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\right]$$

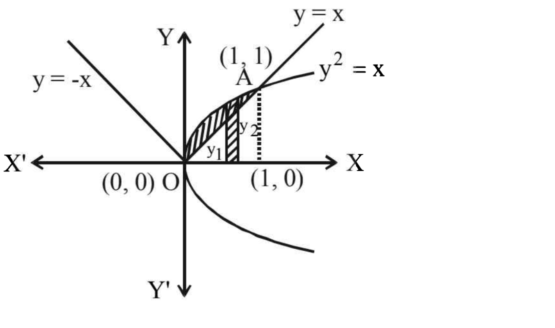
$$\Rightarrow \sec^{-1} x - \sec^{-1}\sqrt{2} = \frac{\pi}{2}$$

$$\Rightarrow \sec^{-1} x - \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow \sec^{-1} x = \frac{\pi}{2} + \frac{\pi}{4}$$

$$\Rightarrow \sec^{-1} x = \frac{3\pi}{4} \Rightarrow x = \sec\frac{3\pi}{4} \Rightarrow x = -\sqrt{2}$$

28. (a) The area enclosed between the curves $y^2 = x$ and $y = |x|$

From the figure, area lies between $y^2 = x$ and $y = x$



$\therefore$ Required area $= \int_0^1 (y_2 - y_1)dx$

$$= \int_0^1 (\sqrt{x} - x)dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$$

$$\therefore \text{Required area} = \frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{2}\left[x^2\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

29. (b) We know that $\frac{\sin x}{x} < 1$, for $x \in (0, 1)$

$$\Rightarrow \frac{\sin x}{\sqrt{x}} < \sqrt{x} \text{ on } x \in (0,1)$$

$$\Rightarrow \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \sqrt{x} dx = \left[\frac{2x^{3/2}}{3}\right]_0^1$$

$$\Rightarrow \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \frac{2}{3} \Rightarrow I < \frac{2}{3} \text{ Also } \frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}}$$

for $x \in (0, 1)$

$$\Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 x^{-1/2} dx = \left[2\sqrt{x}\right]_0^1 = 2$$

$$\Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} dx < 2 \Rightarrow J < 2$$

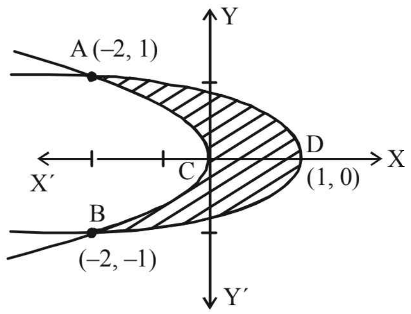
30. (d) $x + 2y^2 = 0 \Rightarrow y^2 = -\frac{x}{2}$

[Left handed parabola with vertex at (0, 0)]

$x + 3y^2 = 1 \Rightarrow y^2 = -\frac{1}{3}(x-1)$

[Left handed parabola with vertex at (1, 0)]

Solving the two equations we get the points of intersection as (–2, 1), (–2, –1)



The required area is ACBDA, given by

$$= \left|\int_{-1}^{1} (1 - 3y^2 - 2y^2)dy\right| = \left|\left[y - \frac{5y^3}{3}\right]_{-1}^{1}\right|$$

$$= \left|\left(1 - \frac{5}{3}\right) - \left(-1 + \frac{5}{3}\right)\right| = 2 \times \frac{2}{3} = \frac{4}{3} \text{ sq. units.}$$

31. (b) The given parabola is $(y-2)^2 = x - 1$

Vertex (1, 2) and it meets x–axis at (5, 0)

Also it gives $y^2 - 4y - x + 5 = 0$

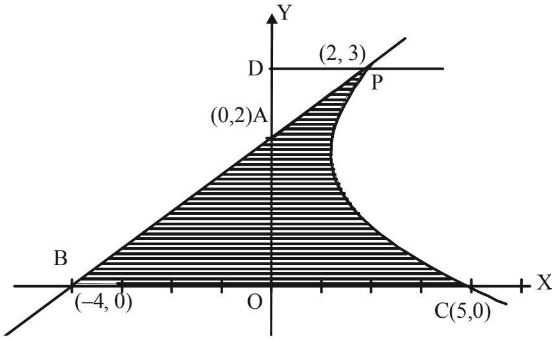
So, that equation of tangent to the parabola at (2, 3) is

$$y.3 - 2(y+3) - \frac{1}{2}(x+2) + 5 = 0 \text{ or } x - 2y + 4 = 0$$

which meets x-axis at (– 4, 0).

In the figure shaded area is the required area.

Let us draw PD perpendicular to y – axis.



Then required area = Ar ΔBOA + Ar (OCPD) – Ar (ΔAPD)

$$= \frac{1}{2} \times 4 \times 2 + \int_0^3 x dy - \frac{1}{2} \times 2 \times 1$$

$$= 3 + \int_0^3 (y-2)^2 + 1\, dy = 3 + \left[\frac{(y-2)^3}{3} + y\right]_0^3$$

$$= 3 + \left[\frac{1}{3} + 3 + \frac{8}{3}\right] = 3 + 6 = 9 \text{ Sq. units}$$

32. (c) Let $I = \int_0^\pi [\cot x]\, dx$ ....(1)

$$= \int_0^\pi [\cot(\pi - x)]\, dx = \int_0^\pi [-\cot x]\, dx \quad \text{....(2)}$$

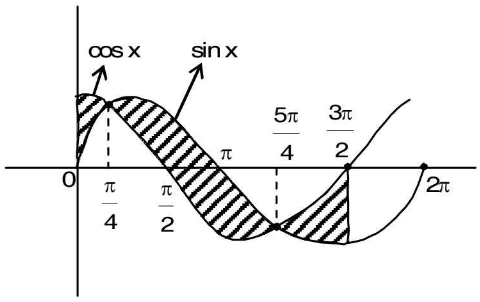
Adding two values of I in eq^n s (1) & (2), We get

$$2I = \int_0^\pi ([\cot x] + [-\cot x])dx = \int_0^\pi (-1)dx$$

$[\because [x] + [-x] = -1, \text{ if } x \notin z \text{ and } [x] + [-x] = 0, \text{ if } x \in z]$

$$= [-x]_0^\pi = -\pi \Rightarrow I = -\frac{\pi}{2}$$

33. (d)



Area above x-axis = Area below x-axis

$\therefore$ Required area

$$= 2\left[\int_0^{\frac{\pi}{4}}(\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin x - \cos x)dx + \int_{\frac{\pi}{2}}^{\pi}\sin x\, dx\right]$$

$= 4\sqrt{2} - 2$

34. (a) $p'(x) = p'(1-x) \Rightarrow p(x) = -p(1-x) + c$

at $x = 0$

$p(0) = -p(1) + c \Rightarrow 42 = c$

Now, $p(x) = -p(1-x) + 42 \Rightarrow p(x) + p(1-x) = 42$

$\Rightarrow I = \int_0^1 p(x)dx$ ...(i)

$\Rightarrow I = \int_0^1 p(1-x)dx$ ...(ii)

on adding (i) and (ii), $2I = \int_0^1 (42)dx \Rightarrow I = 21$

35. (d) $I = \int_0^1 \frac{8\log(1+x)}{1+x^2}dx$

Put $x = \tan\theta$, $\therefore \frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta d\theta$

$\therefore I = 8\int_0^{\pi/4}\frac{\log(1+\tan\theta)}{1+\tan^2\theta}.\sec^2\theta d\theta$

$I = 8\int_0^{\pi/4}\log(1+\tan\theta)d\theta$ ...(i)

$= 8\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$

$= 8\int_0^{\pi/4}\log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta = 8\int_0^{\pi/4}\log\left[\frac{2}{1+\tan\theta}\right]d\theta$

$= 8\int_0^{\pi/4}[\log 2 - \log(1+\tan\theta)]d\theta$

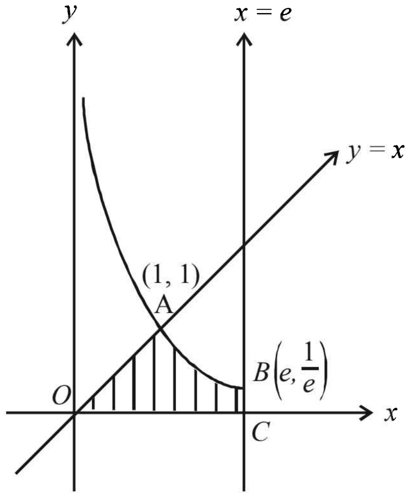
$I = 8.(\log 2)[x]_0^{\pi/4} - 8\int_0^{\pi/4}\log(1+\tan\theta)d\theta$

$I = 8.\frac{\pi}{4}.\log 2 - I$ [From equation (i)]

$\Rightarrow 2I = 2\pi\log 2$, $\therefore I = \pi\log 2$

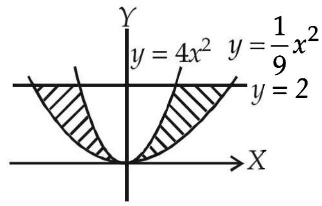
36. (b) Area of required region $AOBC$

$= \int_0^1 x dx + \int_1^e \frac{1}{x}dx = \frac{1}{2} + 1 = \frac{3}{2}$ sq. units



37. (c) Given curves $x^2 = \frac{y}{4}$ and $x^2 = 9y$ are the parabolas whose equations can be written as $y = 4x^2$ and $y = \frac{1}{9}x^2$.

Also, given $y = 2$.
Now, shaded portion shows the required area which is symmetric.



$\therefore$ Area $= 2\int_0^2\left(\sqrt{9y} - \sqrt{\frac{y}{4}}\right)dy$

Area $= 2\int_0^2\left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right)dy$

$= 2\left[\frac{2}{3}\times 3.y^{\frac{3}{2}} - \frac{1}{2}\times\frac{2}{3}.y^{\frac{3}{2}}\right]_0^2$

$= 2\left[2y^{\frac{3}{2}} - \frac{1}{3}y^{\frac{3}{2}}\right] = 2\times\frac{5}{3}y^{\frac{3}{2}}\Big|_0^2 = 2.\frac{5}{3}2\sqrt{2} = \frac{20\sqrt{2}}{3}$

38. (b, c) $g(x+\pi) = \int_0^{x+\pi}\cos 4t\, dt$

$= \int_0^{\pi}\cos 4t\, dt + \int_{\pi}^{\pi+x}\cos 4t\, dt = g(\pi) + \int_0^x \cos 4t\, dt$

Putting $t = \pi + y$ in second integral, we get

$\int_x^{\pi+x}\cos 4t\, dt = \int_0^{\pi}\cos 4t\, dt$

$= g(\pi) + g(x) = g(x) - g(\pi)$

$\therefore g(\pi) = 0)$

39. (d) Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}$ ...(i)

Also, Given, I

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}$ ...(2)

By adding (1) and (2), we get

$2I = \int_{\pi/6}^{\pi/3} dx$

$\Rightarrow \quad I = \frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\pi}{12}$, statement-1 is false

$\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$

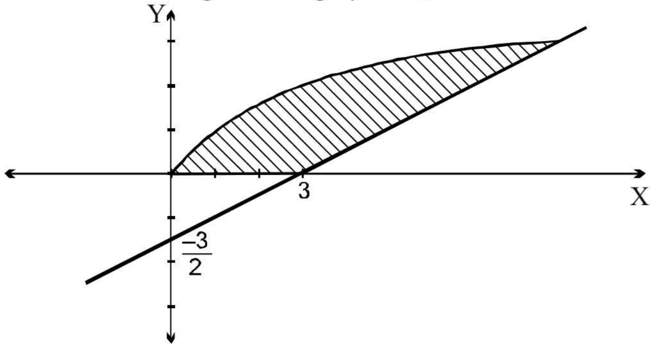
It is fundamental property.

40. (a) Given curves are

$y = \sqrt{x}$ ...(1)

and $2y - x + 3 = 0$ ...(2)

On solving both we get $y = -1, 3$



Required area $= \int_0^3 \left\{(2y+3) - y^2\right\} dy$

$= y^2 + 3y - \frac{y^3}{3}\Big|_0^3 = 9.$

41. (b) Let $I = \int_0^{\pi} \sqrt{1+4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\,dx$

$= \int_0^{\pi} \left|2\sin\frac{x}{2} - 1\right| dx$

$= \int_0^{\pi/3} \left(1 - 2\sin\frac{x}{2}\right)dx + \int_{\pi/3}^{\pi} \left(2\sin\frac{x}{2} - 1\right)dx$

$\left[\because \ \sin\frac{x}{2} = \frac{1}{2} \Rightarrow \frac{x}{2} = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{3}, \frac{x}{2} = \frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{3}\right]$

$= \left[x + 4\cos\frac{x}{2}\right]_0^{\pi/3} + \left[-4\cos\frac{x}{2} - x\right]_{\pi/3}^{\pi}$

$= \frac{\pi}{3} + 4\frac{\sqrt{3}}{2} - 4 + \left(0 - \pi + 4\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) = 4\sqrt{3} - 4 - \frac{\pi}{3}$

42. (c) Given curves are $x^2 + y^2 = 1$ and $y^2 = 1 - x$. Intersecting points are $x = 0, 1$

Area of shaded portion is the required area.

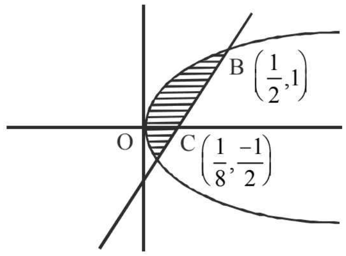
So, Required Area = Area of semi-circle + Area bounded by parabola

$= \frac{\pi r^2}{2} + 2\int_0^1 \sqrt{1-x}dx = \frac{\pi}{2} + 2\int_0^1 \sqrt{1-x}\,dx$

($\because$ radius of circle = 1)

$= \frac{\pi}{2} + 2\left[\frac{(1-x)^{3/2}}{-3/2}\right]_0^1 = \frac{\pi}{2} - \frac{4}{3}(-1) = \frac{\pi}{2} + \frac{4}{3}$ Sq. unit

43. (b) Required area



= Area of ABCD – ar (ABOCD)

$= \int_{-1/2}^{1} \frac{y+1}{4}dy - \int_{-1/2}^{1} \frac{y^2}{2}dy = \frac{1}{4}\left[\frac{y^2}{2} + y\right]_{-1/2}^{1} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-1/2}^{1}$

$= \frac{1}{4}\left[\frac{3}{2} + \frac{3}{8}\right] - \frac{9}{48} = \frac{15}{32} - \frac{9}{48} = \frac{27}{96} = \frac{9}{32}$

44. (a) $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}$

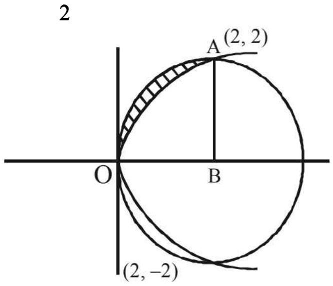
$I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(6-x)^2}$ ...(i)

$I = \int_2^4 \frac{\log(6-x)^2}{\log(6-x)^2 + \log x^2}$ ...(ii)

Adding (1) and (2)

$2I = \int_2^4 dx = [x]_2^4 = 2 \Rightarrow I = 1$

45. (d)



Points of intersection of the two curves are (0, 0), (2, 2) and (2, –2)

Area = Area (OAB) – area under parabola (0 to 2)

$= \frac{\pi \times (2)^2}{4} - \int_0^2 \sqrt{2}\sqrt{x}\,dx = \pi - \frac{8}{3}$