CHAPTER

1

# Introduction to Vectors

- Coordinate Axes and Coordinate Planes in Three-Dimensional Space
- Evolution of Vector Concept
- Types of Vectors
- Addition of Vectors
- Components of a Vector
- Multiplication of a Vector by a Scalar
- Vector Joining Two Points
- Section Formula
- Vector Along the Bisector of Given Two Vectors
- Linear Combination, Linear Independence and Linear Dependence

## COORDINATE AXES AND COORDINATE PLANES IN THREE-DIMENSIONAL SPACE

Consider three planes intersecting at a point $O$ such that these three planes are mutually perpendicular to each other as shown in the following figure.

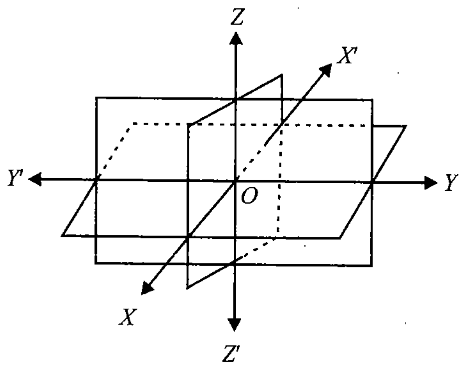



**Fig. 1.1**

These three planes intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$, called the $x$-, $y$- and $z$-axes, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes $XOY$, $YOZ$ and $ZOX$, called respectively, the $XY$-plane, the $YZ$-plane and the $ZX$-plane, are known as the three coordinate planes. We take the $XOY$ plane as the plane of the paper and the line $Z'OZ$ as perpendicular to the plane $XOY$. If the plane of the paper is considered to be horizontal, then the line $Z'OZ$ will be vertical. The distances measured from $XY$-plane upwards in the direction of $OZ$ are taken as positive and those measured downwards in the direction of $OZ$ are taken as negative. Similarly, the distances measured to the right of $ZX$-plane along $OY$ are taken as positive, to the left of $ZX$-plane and along $OY'$ as negative, in front of the $YZ$-plane along $OX$ as positive and to the back of it along $OX'$ as negative. The point $O$ is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants can be named as $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$ and are denoted by I, II, III, ..., VIII, respectively.

### Coordinates of a Point in Space

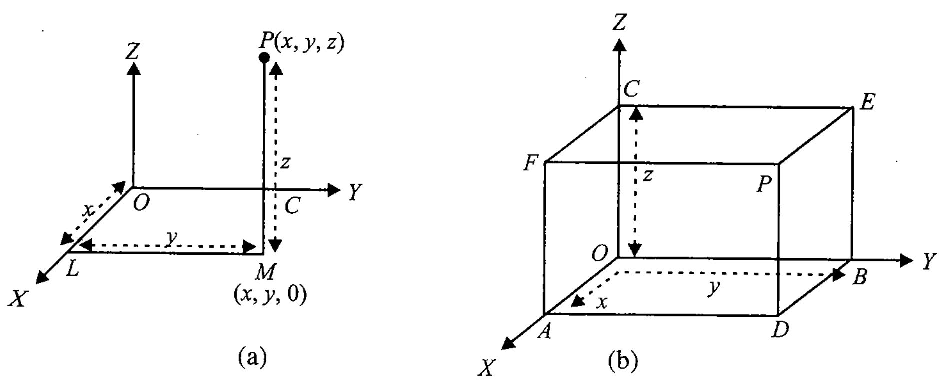



**Fig. 1.2**

Consider a point $P$ in space, we drop a perpendicular $PM$ on the $XY$-plane with $M$ as the foot of this perpendicular. Then, from point $M$, we draw a perpendicular $ML$ to the $x$-axis, meeting it at $L$. Let $OL$ be $x$, $LM$ be $y$ and $MP$ be $z$. Then $x$, $y$ and $z$ are called the $x$-, $y$- and $z$-coordinates, respectively, of point $P$ in the space. In Fig. 1.2, we may note that the point $P(x, y, z)$ lies in the octant $XOYZ$ and so all $x, y, z$ are positive. If $P$ was in any other octant, the signs of $x$, $y$ and $z$ would change accordingly. Thus, to each point $P$ in the space, there corresponds an ordered triplet $(x, y, z)$ of real numbers.

We observe that if P $(x, y, z)$ is any point in the space, then $x$, $y$ and $z$ are perpendicular distances from $YZ$, $ZX$ and $XY$ planes, respectively.

**Note:** *The coordinates of the origin O are* (0, 0, 0). *The coordinates of any point on the x-axis will be* $(x, 0, 0)$ *and the coordinates of any point in the YZ-plane will be* $(0, y, z)$.

The sign of the coordinates of a point determines the octant in which the point lies. The following table shows the signs of the coordinates in the eight octants:

| Octant Coordinates | I | II | III | IV | V | VI | VII | VIII |
|---|---|---|---|---|---|---|---|---|
| $x$ | + | − | − | + | + | − | − | + |
| $y$ | + | + | − | − | + | + | − | − |
| $z$ | + | + | + | + | − | − | − | − |

## Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points referred to a system of rectangular axes $OX$, $OY$ and $OZ$. Through the points $P$ and $Q$ draw planes parallel to the coordinate planes so as to form a rectangular parallelopiped with one diagonal $PQ$.

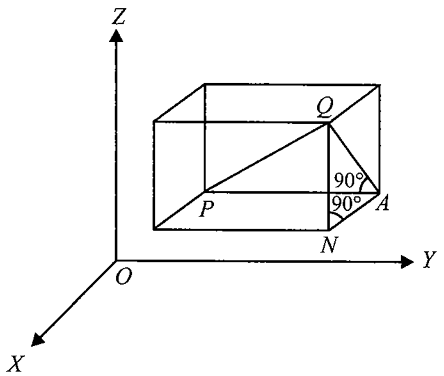



**Fig. 1.3**

Now, since $\angle PAQ$ is a right angle, it follows that in triangle $PAQ$,

$$PQ^2 = PA^2 + AQ^2 \quad \text{(i)}$$

Also, triangle $ANQ$ is right-angled with $\angle ANQ$ being the right angle. Therefore,

$$AQ^2 = AN^2 + NQ^2 \quad \text{(ii)}$$

From (i) and (ii), we have

$PQ^2 = PA^2 + AN^2 + NQ^2$

Now $PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Therefore, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

This gives us the distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$.

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point $P$ is origin $O$, then $OQ = \sqrt{x_2^2 + y_2^2 + z_2^2}$, which gives the distance between the origin $O$ and any point $Q$ $(x_2, y_2, z_2)$.

## Section Formula

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let point $R$ $(x, y, z)$ divide $PQ$ in the given ratio $m : n$ internally. Draw $PL$, $QM$ and $RN$ perpendicular to the $XY$-plane. Obviously $PL \parallel RN \parallel OM$ and feet of these perpendiculars lie in the $XY$-plane. Through point $R$ draw a line $ST$ parallel to line $LM$. Line $ST$ will intersect line $LP$ externally at point $S$ and line $MQ$ at $T$, as shown in the following figure.

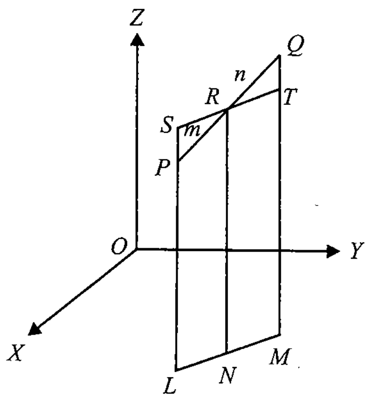



**Fig. 1.4**

Also note that quadrilaterals $LNRS$ and $NMTR$ are parallelograms.

The triangles $PSR$ and $QTR$ are similar. Therefore,

$$\frac{m}{n} = \frac{PR}{QR} = \frac{SP}{QT} = \frac{SL - PL}{QM - TM} = \frac{NR - PL}{QM - NR} = \frac{z - z_1}{z_2 - z}$$

This implies $z = \dfrac{mz_2 + nz_1}{m + n}$

Hence, the coordinates of the point $R$ which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q$ $(x_2, y_2, z_2)$ internally in the ratio $m : n$ are $\dfrac{mx_2 + nx_1}{m + n}$, $\dfrac{my_2 + ny_1}{m + n}$ and $\dfrac{mz_2 + nz_1}{m + n}$.

If point $R$ divides $PQ$ externally in the ratio $m : n$, then its coordinates are obtained by replacing $n$ with $-n$ so that the coordinates become $\frac{mx_2 - nx_1}{m-n}$, $\frac{my_2 - ny_1}{m-n}$ and $\frac{mz_2 - nz_1}{m-n}$.

**Notes:**

1. *If R is the midpoint of PQ, then m : n =* 1:1; *so* $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$, $z = \frac{z_1 + z_2}{2}$.
   *These are the coordinates of the midpoint of the segment joining* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$

2. *The coordinates of the point R which divides PQ in the ratio k : 1 are obtained by taking* $k = \frac{m}{n}$, *which are given by* $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$

3. *If vertices of triangle are* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *and* $C(x_3, y_3, z_3)$, *and* $AB = c$, $BC = a$, $AC = b$, *then centroid of the triangle is* $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ *and its incenter is*
   $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

## EVOLUTION OF VECTOR CONCEPT

In our day-to-day life, we come across many queries such as 'What is your height?' and 'How should a football player hit the ball to give a pass to another player of his team?' Observe that a possible answer to the first query may be 1.5 m, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors*. In mathematics, physics and engineering, we frequently come across with both types of quantities, namely scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density and resistance and vector quantities such as displacement, velocity, acceleration, force, momentum and electric field intensity.

Let '$l$' be a straight line in plane or three-dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a directed line (Fig.1.5 (i), (ii)).

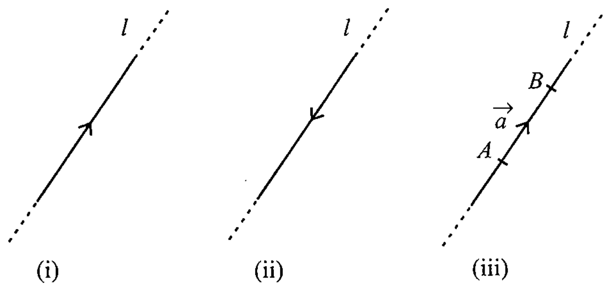



**Fig. 1.5**

Now observe that if we restrict the line $l$ to the line segment $AB$, then a magnitude is prescribed on line (i) with one of the two directions, so that we obtain a directed line segment, Fig. 1.5 (iii). Thus, a directed line segment has magnitude as well as direction.

## Definition

A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig. 1.5 (iii), denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

Point $A$ from where vector $\overrightarrow{AB}$ starts is called its initial point, and point $B$ where it ends is called its terminal point. The distance between initial and terminal points of a vector is called the magnitude (or length) of the vector, denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$ or $a$. The arrow indicates the direction of the vector.

## Position Vector

Consider a point $P$ in space, having coordinates $(x, y, z)$ with respect to the origin $O$ (0, 0, 0). Then, the vector $\overrightarrow{OP}$ having $O$ and $P$ as its initial and terminal points, respectively, is called the position vector of the point $P$ with respect to $O$. Using distance formula, the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.

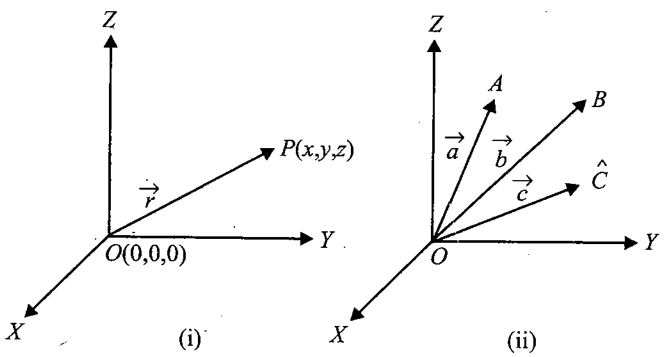



**Fig. 1.6**

In practice, the position vectors of points $A$, $B$, $C$, etc., with respect to origin $O$ are denoted by $\vec{a}, \vec{b}, \vec{c}$, etc., respectively (Fig.1.6 (ii)).

## Direction Cosines

Consider the position vector $\overrightarrow{OP}$ (or $\vec{r}$) of a point $P(x, y, z)$. The angles $\alpha$, $\beta$ and $\gamma$ made by the vector $\vec{r}$ with the positive directions of $x$-, $y$- and $z$-axes, respectively, are called its direction angles. The cosine values of these angles, i.e., $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the vector $\vec{r}$ and are usually denoted by $l$, $m$ and $n$, respectively.

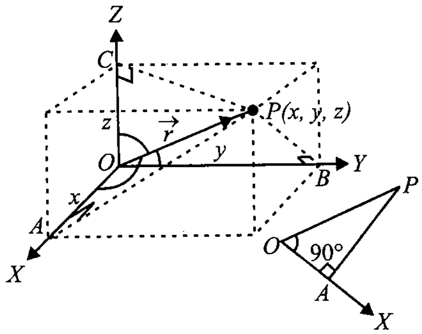



Fig. 1.7

From the figure, one may note that triangle $OAP$ is right angled, and in it, we have $\cos\alpha = x/r$ ($r$ stands for $|\vec{r}|$). Similarly, from the right-angled triangles $OBP$ and $OCP$, we may write $\cos\beta = y/r$ and $\cos\gamma = z/r$. Thus, the coordinates of point $P$ may also be expressed as $(lr, mr, nr)$. The numbers $lr$, $mr$ and $nr$, proportional to the direction cosines, are called the direction ratios of vector $\vec{r}$ and denoted as $a$, $b$ and $c$, respectively (see this topic in detail in Chapter 3).

## TYPES OF VECTORS

### Zero Vector

A vector whose initial and terminal points coincide is called a zero vector (or null vector) and is denoted as $\vec{0}$. A zero vector cannot be assigned a definite direction as it has zero magnitude or, alternatively, it may be regarded as having any direction. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

### Unit Vector

A vector of unit magnitude is called a unit vector. Unit vectors are denoted by small letters with a cap on them.

Thus, $\hat{a}$ is unit vector of $\vec{a}$, where $|\hat{a}| = 1$, i.e., if vector $\vec{a}$ is divided by magnitude $|\vec{a}|$, then we get a unit vector in the direction of $\vec{a}$. Thus, $\dfrac{\vec{a}}{|\vec{a}|} = \hat{a} \Leftrightarrow \vec{a} = |\vec{a}|\,\hat{a}$, where $\hat{a}$ is the unit vector in the direction of $\vec{a}$.

### Coinitial Vectors

Two or more vectors having the same initial point are called coinitial vectors.

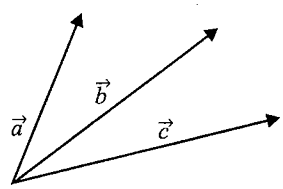



Fig. 1.8

## Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction regardless of the positions of their initial points. They are written as $\vec{a} = \vec{b}$.

## Negative of a Vector

A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but the direction is opposite to that of it, is called negative of the given vector. For example, vector $\overrightarrow{BA}$ is negative of vector $\overrightarrow{AB}$ and is written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

## Free Vectors

Vectors whose initial points are not specified are called free vectors.

## Localised Vectors

A vector drawn parallel to a given vector but through a specified point as the initial point is called a localised vector.

## Parallel Vectors

Two or more vectors are said to be parallel if they have the same support or parallel support.

Parallel vectors may have equal or unequal magnitudes and their directions may be same or opposite as shown in Fig. 1.9.

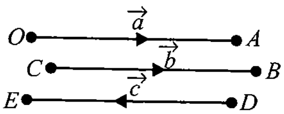



**Fig. 1.9**

## Like and Unlike Vectors

Two parallel vectors having the same direction are called **like vectors** (see Fig. 1.10 (a)).

Two parallel vectors having opposite directions are called **unlike vectors** (see Fig. 1.10 (b)).

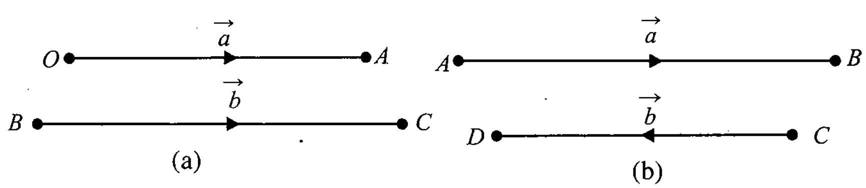



**Fig. 1.10**

## Collinear Vectors

Vectors $\vec{a}$ and $\vec{b}$ are collinear if they have same direction or are parallel or anti-parallel. Since their magnitudes are different, we can find some scalar $\lambda$ for which $\vec{a} = \lambda\vec{b}$. If $\lambda > 0$, $\vec{a}$ and $\vec{b}$ are in the same direction; if $\lambda < 0$, $\vec{a}$ and $\vec{b}$ are in opposite directions. Collinear vectors are often called dependent vectors.

## Non-Collinear Vectors

Two vectors acting in different directions are called non-collinear vectors. Non-collinear vectors are often called independent vectors. Here we cannot write vector $\vec{a}$ in terms of $\vec{b}$, though they have same magnitude. However we can find component of one vector in the direction of the other. Two non-collinear vectors describe plane.

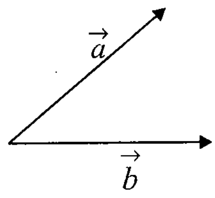



**Fig. 1.11**

## Coplanar Vectors

Two parallel vectors or non-collinear vectors are always coplanar or two vectors $\vec{a}$ and $\vec{b}$ in different directions determine unique plane in space. Now if vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. Generally more than two vectors are coplanar if all are in the same plane.

Three non-coplanar vectors describe space.

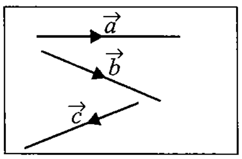



**Fig. 1.12**

# ADDITION OF VECTORS

A vector $\overrightarrow{AB}$ simply means the displacement from point $A$ to point $B$. Now consider a situation where a boy moves from $A$ to $B$ and then from $B$ to $C$. The net displacement made by the boy from point $A$ to point $C$ is given by vector $\overrightarrow{AC}$ and expressed as

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

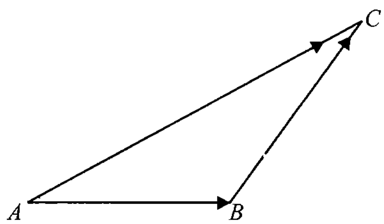



**Fig. 1.13**

This is known as the triangle law of vector addition.

In general, if we have two vectors $\vec{a}$ and $\vec{b}$ (Fig. 1.14 (i)), then to add them, they are positioned such that the initial point of one coincides with the terminal point of the other (Fig. 1.14 (ii)).

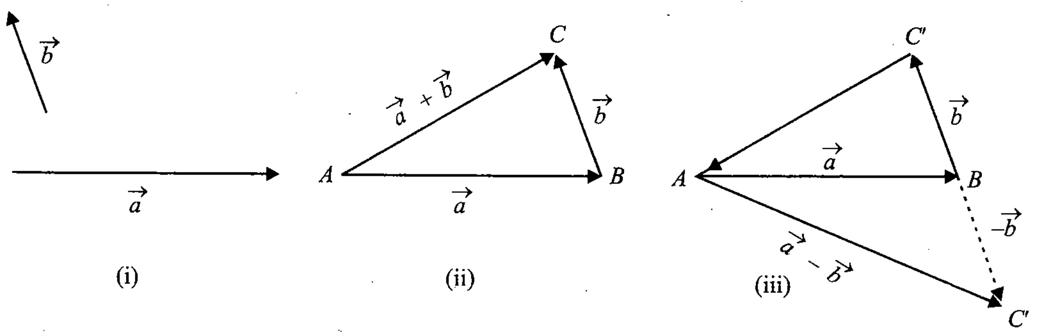



**Fig. 1.14**

For example, in Fig. 1.14 (ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction, so that its initial point coincides with the terminal point of $\vec{a}$. Then the vector $\vec{a}+\vec{b}$, represented by the third side $AC$ of the triangle $ABC$, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$, i.e., in triangle $ABC$ (Fig. 1.14 (ii)), we have

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

Since $\overrightarrow{AC}=-\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AA}=\vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig. 1.14 (iii)).

Now, construct a vector $\overrightarrow{BC}$ so that its magnitude is same as that of vector $\overrightarrow{BC}$, but the direction is opposite to that of $\overrightarrow{BC}$ (Fig. 1.14 (iii)), i.e.,

$$\overrightarrow{BC}=-\overrightarrow{BC'}$$

Then, on applying triangle law from Fig. 1.14 (iii), we have

$$\overrightarrow{AC'}=\overrightarrow{AB}+\overrightarrow{BC'}=\overrightarrow{AB}+(-\overrightarrow{BC})=\vec{a}-\vec{b}$$

Vector $\overrightarrow{AC'}$ is said to represent the difference of $\vec{a}$ and $\vec{b}$

Now, consider a boat going from one bank of a river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors—one is the velocity imparted to the boat by its engine and the other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat actually starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

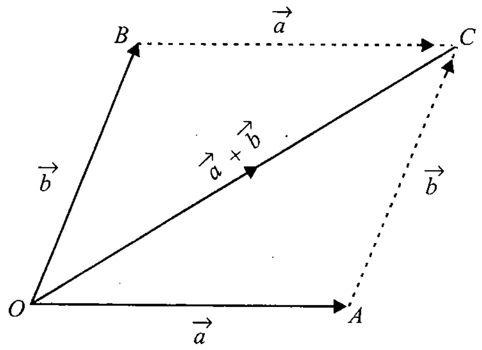



**Fig. 1.15**

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig. 1.15), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the parallelogram law of vector addition.

**Notes:**

**1.** *From figure using the triangle law, one may note that*

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

*or* $$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \qquad (\text{since } \overrightarrow{AC} = \overrightarrow{OB})$$

*which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.*

**2.** *If $\overrightarrow{OA}$ and $\overrightarrow{AC}$ are collinear, their sum is still $\overrightarrow{OC}$. Although in this case we do not have a triangle or a parallelogram in their usual sense.*

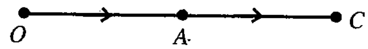



**Fig. 1.16**

**3.** *As from the figure:*

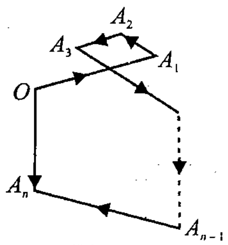



**Fig. 1.17**

$\overrightarrow{OA} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n}$ *by the polygon law of addition.*

## Properties of Vector Addition

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative property)

2. $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associative property)
3. $\vec{a}+\vec{0} = \vec{a}$ (additive identity)
4. $\vec{a}+(-\vec{a}) = \vec{0}$ (additive inverse)
5. $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ and $|\vec{a}-\vec{b}| \ge |\vec{a}|-|\vec{b}|$

**Example 1.1** **If vector $\vec{a}+\vec{b}$ bisects the angle between $\vec{a}$ and $\vec{b}$, then prove that $|\vec{a}| = |\vec{b}|$.**

**Sol.**

We know that vector $\vec{a}+\vec{b}$ is along the diagonal of the parallelogram whose adjacent sides are vectors $\vec{a}$ and $\vec{b}$. Now if $\vec{a}+\vec{b}$ bisects the angle between vectors $\vec{a}$ and $\vec{b}$, then the parallelogram must be a rhombus, hence $|\vec{a}| = |\vec{b}|$.

**Example 1.2** **If $\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$, then prove that $B$ is the midpoint of $AC$.**

**Sol.**

$\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$

$\Rightarrow$ $\overrightarrow{AB}=\overrightarrow{BC}$

$\Rightarrow$ Vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear

$\Rightarrow$ Points $A$, $B$, $C$ are collinear

Also $|\overrightarrow{AB}|=|\overrightarrow{BC}|$

$\Rightarrow$ $B$ is the midpoint of $AC$

**Example 1.3** **$ABCDE$ is a pentagon. Prove that the resultant of forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.**

**Sol.**

$$\vec{R} = \overrightarrow{AB}+\overrightarrow{AE}+\overrightarrow{BC}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}$$

$$= (\overrightarrow{AB}+\overrightarrow{BC}) + (\overrightarrow{AE}+\overrightarrow{ED}+\overrightarrow{DC}) + \overrightarrow{AC}$$

$$= \overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC} = 3\overrightarrow{AC}$$

**Example 1.4** **Prove that the resultant of two forces acting at point $O$ and represented by $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is given by $2\overrightarrow{OD}$, where $D$ is the midpoint of $BC$.**

**Sol.**

$$\vec{R} = \overrightarrow{OB}+\overrightarrow{OC}$$

$$= (\overrightarrow{OD}+\overrightarrow{DB})+(\overrightarrow{OD}+\overrightarrow{DC})$$

$$= 2\overrightarrow{OD}+(\overrightarrow{DB}+\overrightarrow{DC}) = 2\overrightarrow{OD}+\vec{0} = 2\overrightarrow{OD}$$

(Since $D$ is the midpoint of $BC$, we have $\overrightarrow{DB} = -\overrightarrow{DC}$)

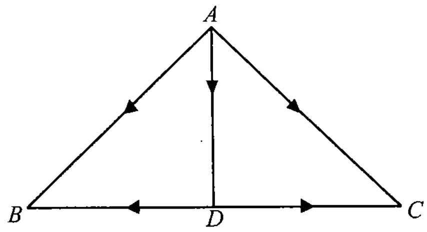


**Fig. 1.18**

**Example 1.5** **Prove that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.**

**Sol.**

$$\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$$
$$\overrightarrow{BC} + \overrightarrow{BA} = 2\overrightarrow{BE}$$
$$\overrightarrow{CA} + \overrightarrow{CB} = 2\overrightarrow{CF}$$

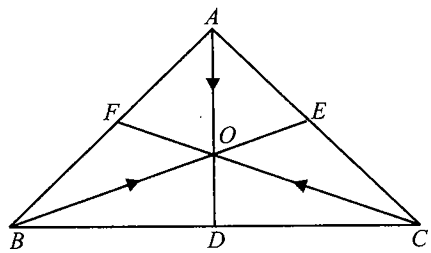


**Fig. 1.19**

Adding, we get

$$(\overrightarrow{AB} + \overrightarrow{BA}) + (\overrightarrow{AC} + \overrightarrow{CA}) + (\overrightarrow{BC} + \overrightarrow{CB}) = 2\,(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF})$$

or $\vec{0} + \vec{0} + \vec{0} = 2\,(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF})$

or $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$

**Example 1.6** ***ABC* is a triangle and *P* any point on *BC*. If $\overrightarrow{PQ}$ is the sum of $\overrightarrow{AP}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$, show that *ABQC* is a parallelogram and *Q*, therefore, is a fixed point.**

**Sol.**

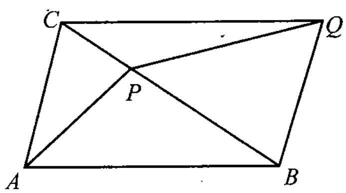


**Fig. 1.20**

Here $\overrightarrow{PQ} = \overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PC}$

$$\overrightarrow{PQ} - \overrightarrow{PC} = \overrightarrow{AP} + \overrightarrow{PB}$$

$$\overrightarrow{PQ} + \overrightarrow{CP} = \overrightarrow{AP} + \overrightarrow{PB}$$

$$\overrightarrow{CQ} = \overrightarrow{AB} \Rightarrow CQ = AB \text{ and } CQ \parallel AB$$

$\therefore$ $ABQC$ is a parallelogram.

But $A$, $B$ and $C$ are given to be fixed points and $ABQC$ is a parallelogram

Therefore, $Q$ is a fixed point.

**Example 1.7** **Two forces $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are acting at the vertex $A$ of a quadrilateral $ABCD$ and two forces $\overrightarrow{CB}$ and $\overrightarrow{CD}$ at $C$. Prove that their resultant is given by $4\overrightarrow{EF}$, where $E$ and $F$ are the midpoints of $AC$ and $BD$, respectively.**

**Sol.** $\overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AF}$, where $F$ is the midpoint of $BD$.

$\overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{CF}$

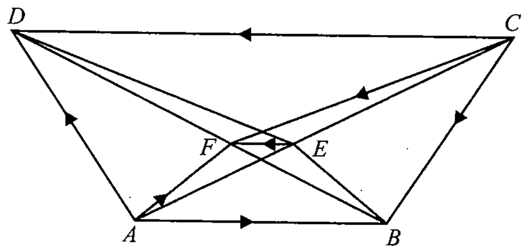



**Fig. 1.21**

$$\therefore \quad \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 2(\overrightarrow{AF} + \overrightarrow{CF})$$

$$= -2(\overrightarrow{FA} + \overrightarrow{FC})$$

$$= -2[2\overrightarrow{FE}], \text{ where } E \text{ is the midpoint of } AC$$

$$= 4\overrightarrow{EF}$$

**Example 1.8** **If $O(\vec{0})$ is the circumcentre and $O'$ the orthocentre of a triangle $ABC$, then prove that**

i. $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OO'}$

ii. $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{O'O}$

iii. $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO} = \overrightarrow{AP}$

**where $AP$ is the diameter through $A$ of the circumcircle.**

**Sol.**

$O$ is the circumcentre, which is the intersection of the right bisectors of the sides of the triangle, and $O'$ is the orthocenter, which is the point of intersection of altitudes drawn from the vertices. Also, from geometry, we know that

$2OD = AO'$. Therefore,

$2\overrightarrow{OD} = \overrightarrow{AO'}$ (i)

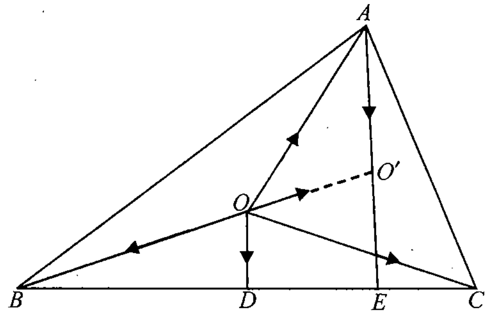



**Fig. 1.22**

i. To prove: $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OO'}$

Now $\overrightarrow{OB}+\overrightarrow{OC}=2\overrightarrow{OD}=\overrightarrow{AO'}$

$\Rightarrow \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AO'}=\overrightarrow{OO'}$ (by (i))

ii. To prove: $\overrightarrow{O'A}+\overrightarrow{O'B}+\overrightarrow{O'C}=2\overrightarrow{OO'}$

L.H.S. $=2\overrightarrow{DO}+2\overrightarrow{O'D}$ (by (i))

$=2(\overrightarrow{O'D}+\overrightarrow{DO})=2\overrightarrow{O'O}$

iii. To prove: $\overrightarrow{AO'}+\overrightarrow{O'B}+\overrightarrow{O'C}=2\overrightarrow{AO}=\overrightarrow{AP}$

L.H.S. $=2\overrightarrow{AO'}-\overrightarrow{AO'}+\overrightarrow{O'B}+\overrightarrow{O'C}$

$=2\overrightarrow{AO'}+(\overrightarrow{O'A}+\overrightarrow{O'B}+\overrightarrow{O'C}$

$=2\overrightarrow{AO'}+2\overrightarrow{O'O}=2\overrightarrow{AO}$

$=\overrightarrow{AP}$ (where $AP$ is the diameter through $A$ of the circumcircle).

## COMPONENTS OF A VECTOR

Let us take the points $A$ (1, 0, 0), $B$ (0, 1, 0) and $C$ (0, 0, 1) on the $x$-axis, $y$-axis and $z$-axis, respectively.

Then, clearly $|\overrightarrow{OA}|=1$, $|\overrightarrow{OB}|=1$ and $|\overrightarrow{OB}|=1$

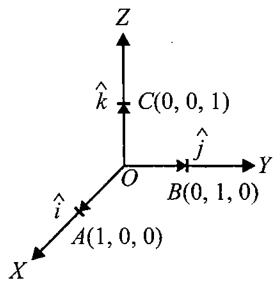



**Fig. 1.23**

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called unit vectors along the axes $OX$, $OY$ and $OZ$, respectively, and are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.

Now, consider the position vector $\overrightarrow{OP}$ of a point $P(x, y, z)$ as shown in the following figure. Let $P_1$ be the foot of the perpendicular from $P$ on the plane $XOY$.

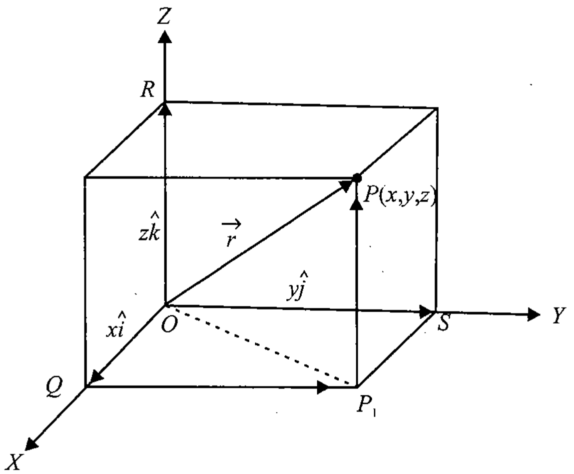



**Fig. 1.24**

We, thus, see that $P_1P$ is parallel to $z$-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the $x$-, $y$- and $z$-axes, respectively, and by the definition of the coordinates of $P$, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{OP_1} = x\hat{i} + y\hat{j}$ and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of $P$ with reference to $O$ is given by

$|\overrightarrow{OP}|$ (or $\vec{r}$) $= x\hat{i} + y\hat{j} + z\hat{k}$

This form of any vector is called its component form. Here, $x$, $y$ and $z$ are called the scalar components of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components of $\vec{r}$ along the respective axes. Sometimes $x$, $y$ and $z$ are also called rectangular components.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is readily determined by applying the Pythagoras theorem twice. We note that in the right-angled triangle $OQP_1$,

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2}$$

And in the right-angled triangle $OP_1P$, we have

$$|\overrightarrow{OP}| = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

**Notes:**

*If* $\vec{a}$ *and* $\vec{b}$ *are any two vectors given in the component form* $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ *and* $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, *respectively, then*

1. *The sum (or resultant) of vectors* $\vec{a}$ *and* $\vec{b}$ *is given by*

$$\vec{a}+\vec{b}=(a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$$

2. *The difference between vectors* $\vec{a}$ *and* $\vec{b}$ *is given by*

$$\vec{a}-\vec{b}=(a_1-b_2)\hat{i}+(a_2-b_2)\hat{j}+(a_3-b_3)\hat{k}$$

3. *Vectors* $\vec{a}$ *and* $\vec{b}$ *are equal if and only if*

$$\vec{b}=\lambda\vec{a}=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$$

*The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:*

*Let* $\vec{a}$ *and* $\vec{b}$ *be any two vectors, and k and m be any scalars. Then*

i. $k\vec{a}+m\vec{a}=(k+m)\vec{a}$

ii. $k(m\vec{a})=(km)\vec{a}$

iii. $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$

**Remarks**

i. One may observe that whatever be the value of $\lambda$, vector $\lambda\vec{a}$ is always collinear to vector $\vec{a}$. In fact, two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a non-zero scalar $\lambda$ such that $\vec{b}=\lambda\vec{a}$. If the vectors $\vec{a}$ and $\vec{b}$ are given in the component form, i.e., $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=\lambda(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$$

$$\Leftrightarrow b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$$

$$\Leftrightarrow b_1=\lambda a_1,\ b_2=\lambda a_2,\ b_3=\lambda a_3$$

$$\Leftrightarrow \frac{b_1}{a_1}=\frac{b_2}{a_2}=\frac{b_3}{a_3}=\lambda$$

ii. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then $a_1$, $a_2$, $a_3$ are also called direction ratios of $\vec{a}$.

iii. In case it is given that $l$, $m$, $n$ are direction cosines of a vector, then $l\hat{i}+m\hat{j}+n\hat{k}$ $=(\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector where $\alpha$, $\beta$ and $\gamma$ are the angles which the vector makes with the $x$-, $y$- and $z$-axes, respectively.

**Example 1.9** **A unit vector of modulus 2 is equally inclined to $x$- and $y$-axes at an angle $\frac{\pi}{3}$. Find the length of projection of the vector on $z$-axis.**

**Sol.**

Given that the vector is inclined at an angle $\frac{\pi}{3}$ with both $x$- and $y$-axes.

$\Rightarrow \quad \cos\alpha = \cos\beta = \frac{1}{2}$

Also we know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad \cos^2\gamma = \frac{1}{2}$

$\Rightarrow \quad \cos\gamma = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow$ Given vector is $2(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$

$$= 2\left(\frac{\hat{i}}{2} + \frac{\hat{j}}{2} \pm \frac{\hat{k}}{\sqrt{2}}\right) = \hat{i} + \hat{j} \pm \sqrt{2}\hat{k}$$

$\Rightarrow$ Length of projection of vector on z-axis is $\sqrt{2}$ units.

**Example 1.10** **If the projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively, find the angle at which vector $\vec{a}$ is inclined to z-axis.**

**Sol.**

Since projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively,

Vector $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$

$|\vec{a}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

Then $\cos\gamma = \frac{2}{3}$ (where $\gamma$ is the angle of vector $\vec{a}$ with z-axis)

$\Rightarrow \quad \gamma = \cos^{-1}\frac{2}{3}$

## MULTIPLICATION OF A VECTOR BY A SCALAR

Let $\vec{a}$ be a vector and $\lambda$ a scalar. Then the product of vector $\vec{a}$ by scalar $\lambda$, denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar $\lambda$. Note that $\lambda\vec{a}$ is also a vector, collinear to vector $\vec{a}$. Vector $\lambda\vec{a}$ has the direction same (or opposite) as that of vector $\vec{a}$ if the value of $\lambda$ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualization of multiplication of a vector by a scalar is given in the following figure.

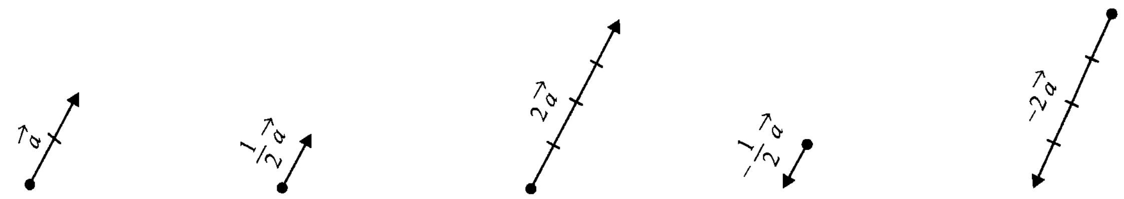



**Fig. 1.25**

When $\lambda = -1$, $\lambda \vec{a} = -\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$.

Vector $-\vec{a}$ is called the negative (or additive inverse) of vector $\vec{a}$ and we always have $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

Also, if $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$, i.e., $\vec{a}$ is not a null vector, then

$$|\lambda \vec{a}| = |\lambda|\,|\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

**Example 1.11** **Find the vector of magnitude 9 units in the direction of vector $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$.**

**Sol.**

Given vector $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{a}$ is $\hat{a} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$

Now vector of magnitude 9 in the direction of $\vec{a}$ is $9\frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3} = 3(\hat{i} + 2\hat{j} + 2\hat{k})$

## VECTOR JOINING TWO POINTS

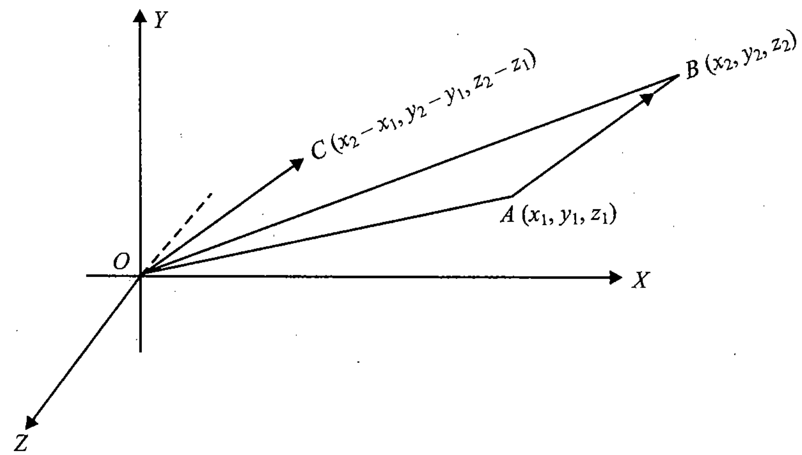



**Fig. 1.26**

In the figure, vector $\vec{AB}$ is shifted without rotation and placed at origin.

Now vector $\vec{AB} = \vec{OC}$

Since $|\vec{AB}| = |\vec{OC}|$, coordinates of point $C$ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Hence vector $\vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Thus $\vec{AB} = \vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= \vec{OB} - \vec{OA}$$

= Position vector of $B$ – position vector of $A$

Also from above, we have $\vec{OB} = \vec{OA} + \vec{AB}$ which describes triangle rule of vector addition.

Further $\vec{OB} = \vec{OA} + \vec{AB} = \vec{OA} + \vec{OC}$ ($\because \vec{OC} = \vec{AB}$), which describes parallelogram rule of vector addition.

**Example 1.12** **If $2\overrightarrow{AC} = 3\overrightarrow{CB}$, then prove that $2\overrightarrow{OA} + 3\overrightarrow{OB} = 5\overrightarrow{OC}$, where $O$ is the origin.**

**Sol.**

$$2\overrightarrow{AC} = 3\overrightarrow{CB} \Rightarrow 2(\overrightarrow{OC} - \overrightarrow{OA}) = 3(\overrightarrow{OB} - \overrightarrow{OC})$$

$$\Rightarrow \quad 2\overrightarrow{OA} + 3\overrightarrow{OB} = 5\overrightarrow{OC}$$

**Example 1.13** **Prove that points $\hat{i} + 2\hat{j} - 3\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $2\hat{i} + 5\hat{j} - \hat{k}$ form a triangle in space.**

**Sol.**

Given points are $A(\hat{i} + 2\hat{j} - 3\hat{k}), B(2\hat{i} - \hat{j} + \hat{k}), C(2\hat{i} + 5\hat{j} - \hat{k})$

Vectors $\vec{AB} = \hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{AC} = \hat{i} + 3\hat{j} + 2\hat{k}$

Clearly vectors $\vec{AB}$ and $\vec{AC}$ are non-collinear as there does not exist any real $\lambda$ for which $\vec{AB} = \lambda\vec{AC}$.

Hence, vectors $\vec{AB}$ and $\vec{AC}$ or given three points form a triangle.

## SECTION FORMULA

### Internal Division

Let $A$ and $B$ be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and $C$ be a point dividing $AB$ internally in the ratio $m : n$. Then the position vector of $C$ is given by $\overrightarrow{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.

**Proof:**

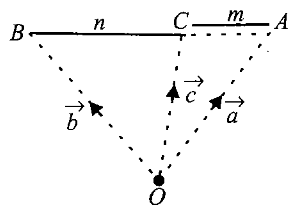



**Fig. 1.27**

Let $O$ be the origin. Then $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of $C$ which divides $AB$ internally in the ratio $m : n$. Then $\dfrac{AC}{CB} = \dfrac{m}{n}$.

$\Rightarrow \quad n\overrightarrow{AC} = m\overrightarrow{CB}$

$\Rightarrow \quad n(\text{P.V. of } C - \text{P.V. of } A) = m(\text{P.V. of } B - \text{P.V. of } C)$

$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$

$\Rightarrow \quad n\vec{c} - n\vec{a} = m\vec{b} - m\vec{c}$

$\Rightarrow \quad \vec{c}(n+m) = m\vec{b} + n\vec{a}$

$\Rightarrow \quad \vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$ or $\overrightarrow{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

### External Division

Let $A$ and $B$ be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and $C$ be a point dividing $\overrightarrow{AB}$ externally in the ratio $m : n$. Then the position vector of $\vec{C}$ is given by $\overrightarrow{OC} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$.

**Proof:**

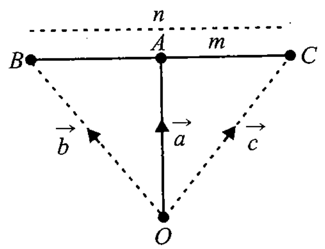



**Fig. 1.28**

Let $O$ be the origin. Then $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of point $C$ dividing $AB$ externally in the ratio $m : n$.

Then, $\dfrac{AC}{BC} = \dfrac{m}{n}$

$\Rightarrow \quad nAC = mBC$

$\Rightarrow \quad n\overrightarrow{AC} = m\overrightarrow{BC}$

$\Rightarrow \quad n(\text{P.V. of } C - \text{P.V. of } A) = m(\text{P.V. of } C - \text{P.V. of } B)$

$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{c} - \vec{b})$

$\Rightarrow \quad n\vec{c} - n\vec{a} = m\vec{c} - m\vec{b}$

$\Rightarrow \quad \vec{c}(m-n) = m\vec{b} - n\vec{a}$

$\Rightarrow \quad \vec{c} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$ or $\overrightarrow{OC} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

**Notes:**

**1.** *If C is the midpoint of AB, then it divides AB in the ratio* 1 : 1.

*Therefore*, the *P.V. of C is* $\dfrac{1\cdot\vec{a} + 1\cdot\vec{b}}{1+1} = \dfrac{\vec{a}+\vec{b}}{2}$. *Thus, the position vector of the midpoint of AB is* $\dfrac{1}{2}(\vec{a}+\vec{b})$.

**2.** *We have* $\vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n} = \dfrac{m}{m+n}\vec{b} + \dfrac{n}{m+n}\vec{a}$. *Therefore,*

$\vec{c} = \lambda\vec{a} + \mu\vec{b}$, *where* $\lambda = \dfrac{n}{m+n}$ *and* $\mu = \dfrac{m}{m+n}$

*Thus, position vector of any point C on* $\overrightarrow{AB}$ *can always be taken as* $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, *where* $\lambda + m = 1$.

**3.** *We have* $\vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$. *Therefore,*

$(m+n)\,\vec{c} = m\vec{b} + n\vec{a}$

$n\overrightarrow{OA} + m\overrightarrow{OB} = (m+n)\overrightarrow{OC}$, *where* $\vec{C}$ *is a point on* $\overrightarrow{AB}$ *dividing it in the ratio m : n.*

*In* $\Delta ABC$, *having vertices* $A(\vec{a}), B(\vec{b})$ *and* $C(\vec{a})$

*Centroid is* $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

*Incentre is* $\dfrac{BC\,\vec{a} + AC\,\vec{b} + AB\,\vec{c}}{AB + AC + AB}$

*Orthocentre is* $\dfrac{\tan A\,\vec{a} + \tan B\,\vec{b} + \tan C\,\vec{c}}{\tan A + \tan B + \tan C}$

*Circumcentre is* $\dfrac{\sin 2A\,\vec{a} + \sin 2B\,\vec{b} + \sin 2C\,\vec{c}}{\sin 2A + \sin 2B + \sin 2C}$

**Example 1.14** **If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of points $A, B, C$ and $D$, respectively referred to same origin $O$ such that no three of these points are collinear and $\vec{a}+\vec{c}=\vec{b}+\vec{d}$, then prove that the quadrilateral $ABCD$ is a parallelogram.**

**Sol.**

Since $\vec{a}+\vec{c}=\vec{b}+\vec{d}$

$\Rightarrow \quad \dfrac{\vec{a}+\vec{c}}{2}=\dfrac{\vec{b}+\vec{d}}{2}$

$\Rightarrow$ Midpoint of $AC$ and $BD$ coincide

$\Rightarrow$ Quadrilateral $ABCD$ is a parallelogram

**Example 1.15** **Find the point of intersection of $AB$ and $CD$, where $A(6, -7, 0)$, $B(16, -19, -4)$, $C(0, 3, -6)$ and $D(2, -5, 10)$.**

**Sol.**

Let $AB$ and $CD$ intersect at $P$.

Let $P$ divides $AB$ in ratio $\lambda:1$ and $CD$ in ratio $\mu:1$

Then coordinates of $P$ are $\left(\dfrac{16\lambda+6}{\lambda+1}, \dfrac{-19\lambda-7}{\lambda+1}, \dfrac{-4\lambda}{\lambda+1}\right)$ or $\left(\dfrac{2\mu}{\mu+1}, \dfrac{-5\mu+3}{\mu+1}, \dfrac{10\mu-6}{\mu+1}\right)$

Comparing we have $\lambda=-\dfrac{1}{3}$ or $\mu=1$.

Using these values, we get point of intersection as $(1, -1, 2)$

Here it is also proved that lines $AB$ and $CD$ intersect or points $A$, $B$, $C$ and $D$ are coplanar.

**Example 1.16** **Find the angle of vector $\vec{a}=6\hat{i}+2\hat{j}-3\hat{k}$ with $x$-axis.**

**Sol.**

$\vec{a}=6\hat{i}+2\hat{j}-3\hat{k}$

$\Rightarrow \quad |\vec{a}|=\sqrt{(6)^2+(2)^2+(-3)^2}=7$

$\Rightarrow$ Angle of vector with $x$-axis is $\cos^{-1}\dfrac{6}{7}$

**Example 1.17**
**a. Show that the lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.**
**b. Show that the joins of the midpoints of the opposite edges of a tetrahedron intersect and bisect each other.**

**Sol.**

a. $G_1$, the centroid of $\Delta BCD$, is $\dfrac{\vec{b}+\vec{c}+\vec{d}}{3}$ and $A$ is $\vec{a}$. The position vector of point $G$ which divides $AG_1$ in the ratio $3:1$ is

$$\frac{3\cdot\frac{\vec{b}+\vec{c}+\vec{d}}{3}+1\cdot\vec{a}}{3+1}=\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

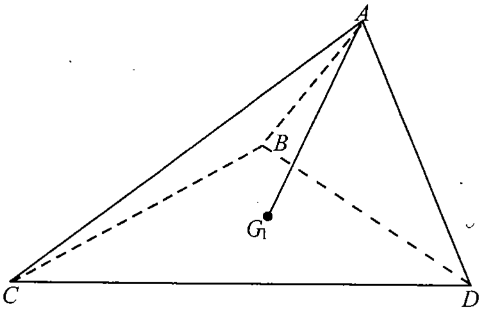



**Fig. 1.29**

The symmetry of the result shows that this point will also lie on $BG_2$, $CG_3$ and $DG_4$ (where $G_2$, $G_3$, $G_4$ are centroids of faces ACD, ABD and ABC, respectively). Hence, these four lines are concurrent at point $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$, which is called the centroid of the tetrahedron.

b. The midpoint of $DA$ is $\frac{\vec{a}+\vec{d}}{2}$ and that of $BC$ is $\frac{\vec{b}+\vec{c}}{2}$ and the midpoint of these midpoints is $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$ and symmetry of the result proves the fact.

**Example 1.18** **The midpoints of two opposite sides of a quadrilateral and the midpoints of the diagonals are the vertices of a parallelogram. Prove this using vectors.**

**Sol.**

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be the position vectors of vertices $A$, $B$, $C$ and $D$, respectively.

Let $E$, $F$, $G$ and $H$ be the midpoints of $AB$, $CD$, $AC$ and $BD$, respectively.

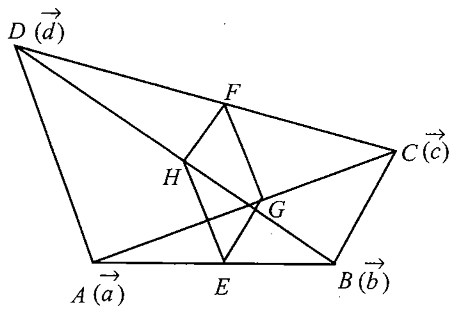



**Fig. 1.30**

P.V. of $E=\frac{\vec{a}+\vec{b}}{2}$

P.V. of $F = \frac{\vec{c}+\vec{d}}{2}$

P.V. of $G = \frac{\vec{a}+\vec{c}}{2}$

P.V. of $H = \frac{\vec{b}+\vec{d}}{2}$

$\overrightarrow{EG}$ = P.V. of G − P.V. of $E = \frac{\vec{a}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}}{2} = \frac{\vec{c}-\vec{b}}{2}$

$\overrightarrow{HF}$ = P.V. of F − P.V. of $H = \frac{\vec{c}+\vec{d}}{2} - \frac{\vec{b}+\vec{d}}{2} = \frac{\vec{c}-\vec{b}}{2}$

$\therefore \quad \overrightarrow{EG} = \overrightarrow{HF} \Rightarrow EG \parallel HF$ and $EG = HF$

$\Rightarrow$ *EGHF* is a parallelogram.

## Some More Solved Examples

**Example 1.19** **Check whether the three vectors $2\hat{i}+2\hat{j}+3\hat{k}, -3\hat{i}+3\hat{j}+2\hat{k}$ and $3\hat{i}+4\hat{k}$ form a triangle or not.**

**Sol.**

If vectors $\vec{a} = 2\hat{i}+2\hat{j}+3\hat{k}, \vec{b} = -3\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{c} = 3\hat{i}+4\hat{k}$ form a triangle, then we must have $\vec{a}+\vec{b}+\vec{c} = 0$.

But for given vectors, $\vec{a}+\vec{b}+\vec{c} \neq 0$. Hence these vectors do not form a triangle.

**Example 1.20** **Find the resultant of vectors $\vec{a} = \hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-4\hat{k}$. Find the unit vector in the direction of the resultant vector.**

**Sol.**

The resultant vector of $\vec{a}$ and $\vec{b}$ is $\vec{a}+\vec{b} = 2\hat{i}+\hat{j}-2\hat{k} = \vec{c}$ (let)

Now unit vector in the direction of $\vec{c}$ is $\hat{c} = \frac{2\hat{i}+\hat{j}-2\hat{k}}{\sqrt{(2)^2+(1)^2+(-2)^2}} = \frac{1}{3}(2\hat{i}+\hat{j}-2\hat{k})$

**Example 1.21** **If in parallelogram ABCD, diagonal vectors are $\overrightarrow{AC} = 2\hat{i}+3\hat{j}+4\hat{k}$ and $\overrightarrow{BD} = -6\hat{i}+7\hat{j}-2\hat{k}$, then find the adjacent side vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.**

**Sol.**

Let $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$

Then $\vec{AC} = \vec{a} + \vec{b}$ and $\vec{BD} = \vec{b} - \vec{a}$

$\Rightarrow \quad \vec{b} = \dfrac{\overrightarrow{AC} + \overrightarrow{BD}}{2}$ and $\vec{a} = \dfrac{\overrightarrow{AC} - \overrightarrow{BD}}{2}$

$\Rightarrow \quad \vec{AB} = -2\hat{i} + 35 + \hat{k}$ and $\vec{AD} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

**Example 1.22** **If two sides of a triangle are $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{k}$, then find the length of the third side.**

**Sol.**

Given sides of the triangle are $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = \hat{i} + \hat{k}$

If vector along the third side is $\vec{c}$, then we must have $\vec{a} + \vec{b} + \vec{c} = 0$

Then $\vec{c} = -(\hat{i} + 2\hat{j}) - (\hat{i} + \hat{k}) = -2\hat{i} - 2\hat{j} - \hat{k}$

Therefore, the length of the third side $|\vec{c}|$ is $\sqrt{(-2)^2 + (-2)^2 + (-1)^2} = 3$

**Example 1.23** **Three coinitial vectors of magnitudes $a$, $2a$ and $3a$ meet at a point and their directions are along the diagonals of three adjacent faces of a cube. Determine their resultant $R$. Also prove that sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through the same corner, the vectors being directed from the corner, is twice the vector determined by the diagonal of the cube.**

**Sol.**

Let the length of an edge of the cube be taken as unity and the vectors represented by $OA$, $OB$ and $OC$ (let the three coterminous edges of unit be $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively). $OR$, $OS$ and $OT$ are the three diagonals of the three adjacent faces of the cube along which act the forces of magnitudes $a$, $2a$ and $3a$, respectively. To find the vectors representing these forces, we shall first find unit vectors in these directions and then multiply them by the corresponding given magnitudes of these forces.

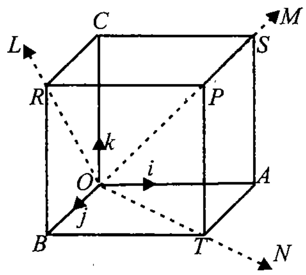



**Fig. 1.31**

Since $\overrightarrow{OR} = \hat{j} + \hat{k}$, the unit vector along $OR$ is $\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$.

Hence, force $\vec{F}_1$ of magnitude $a$ along $OR$ is given by

$$\vec{F}_1 = \frac{a}{\sqrt{2}}(\hat{i}+\hat{k})$$

Similarly, force $\vec{F}_1$ of magnitude $2a$ along $OS$ is $\frac{2a}{\sqrt{2}}(\hat{k}+\hat{i})$ and force $\vec{F}_3$ of magnitude $3a$ along $OT$ is $\frac{3a}{\sqrt{2}}(\hat{i}+\hat{j})$.

If $\vec{R}$ be their resultant, then $\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$= \frac{a}{\sqrt{2}}(\hat{j}+\hat{k}) + \frac{2a}{\sqrt{2}}(\hat{k}+\hat{i}) + \frac{3a}{\sqrt{2}}(\hat{i}+\hat{j})$$

$$= \frac{5a}{\sqrt{2}}\hat{i} + \frac{4a}{\sqrt{2}}\hat{j} + \frac{3a}{\sqrt{2}}\hat{k}$$

Again, $\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = \hat{j}+\hat{k}+\hat{i}+\hat{k}+\hat{i}+\hat{j}$

$$= 2(\hat{i}+\hat{j}+\hat{k})$$

Also $\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{TP} = (\hat{i}+\hat{j}+\hat{k})$ $(\because \overrightarrow{OT} = \hat{i}+\hat{j}$ and $\overrightarrow{TP} = \overrightarrow{OC} = \hat{k})$

$\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = 2\overrightarrow{OP}$

**Example 1.24** **The axes of coordinates are rotated about the $z$-axis through an angle of $\pi/4$ in the anticlockwise direction and the components of a vector are $2\sqrt{2}$, $3\sqrt{2}$, 4. Prove that the components of the same vector in the original system are –1, 5, 4.**

**Sol.**

If $\hat{i}, \hat{j}, \hat{k}$ are the new unit vectors along the coordinate axes, then

$$\vec{a} = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j} + 4\hat{k} \quad \text{(i)}$$

$\hat{i}, \hat{j}, \hat{k}$ are obtained by rotating by 45° about the $z$-axis.

Then $\hat{i}$ is replaced by $\hat{i}\cos 45° + \hat{j}\sin 45° = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

and

$\hat{j}$ is replaced by $-\hat{i}\cos 45° + \hat{j}\sin 45° = \frac{-\hat{i}+\hat{j}}{\sqrt{2}}$

$\hat{k} = \hat{k}$,

$$\vec{a} = 2\sqrt{2}\left[\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right] + 3\sqrt{2}\left[\frac{-\hat{i}+\hat{j}}{\sqrt{2}}\right] + 4\hat{k}$$

$$\vec{a} = (2-3)\hat{i} + (2+3)\hat{j} + 4\hat{k}$$

$$\vec{a} = -\hat{i} + 5\hat{j} + 4\hat{k}$$

**Example 1.25** **If the resultant of two forces is equal in magnitude to one of the components and perpendicular to it in direction, find the other component using the vector method.**

**Sol.**

Let $P$ be horizontal in the direction of unit vector $\hat{i}$. The resultant is also $P$ but perpendicular to it in the direction of unit vector $\hat{j}$. If $Q$ be the other force making an angle $\theta$ (obtuse) as the resultant is perpendicular to $P$, then the two forces are $P\hat{i}$ and $Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j}$. Their resultant is $P\hat{j}$.

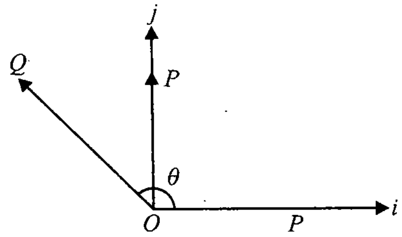



**Fig. 1.32**

$\therefore P\hat{j} = P\hat{i} + (Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j})$

Comparing the coefficients of $\hat{i}$ and $\hat{j}$, we get

$P + Q\cos\theta = 0$ and $Q\sin\theta = P$

or $Q\cos\theta = -P$ and $Q\sin\theta = P$

Squaring and adding $Q = P\sqrt{2}$ and dividing

$\tan\theta = -1$

$\theta = 135°$

**Example 1.26** **A man travelling towards east at 8 km/h finds that the wind seems to blow directly from the north. On doubling the speed, he finds that it appears to come from the north-east. Find the velocity of the wind.**

**Sol.**

The velocity of wind relative to man = Actual velocity of wind – Actual velocity of man (i)

Let $\hat{i}$ and $\hat{j}$ represent unit vectors along east and north. Let the actual velocity of wind be given by $x\hat{i} + y\hat{j}$.

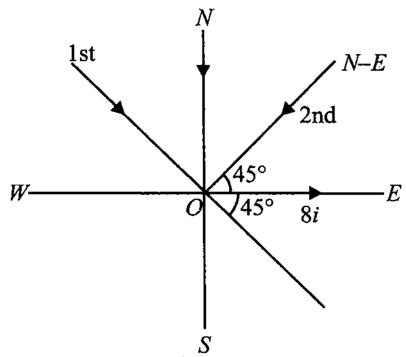



**Fig. 1.33**

In the first case the man's velocity is $8\hat{i}$ and that of the wind blowing from the north relative to the man is $-p\hat{j}$. Therefore,

$-p\hat{j} = (x\hat{i} + y\hat{j}) - 8\hat{i}$ [from Eq. (i)]

Comparing coefficients, $x - 8 = 0$, $y = -p$ (ii)

In the second case when the man doubles his speed, wind seems to come from the north–east direction

$-q(\hat{i} + \hat{j}) = (x\hat{i} + y\hat{j}) - 16\hat{i}$

$\therefore\ x - 16 = -q, y = -q$ (iii)

Putting $x = 8$, we get $q = 8$

$y = -8$

Hence, the velocity of wind is $x\hat{i} + y\hat{j} = 8\ (\hat{i} - \hat{j})$

Its magnitude is $\sqrt{(8^2 + 8^2)} = 8\sqrt{2}$ and $\tan\theta = -1$

$\theta = -45°$.

Hence, its direction is from the north–west.

## Concept Application Exercise 1.1

1. If $ABCD$ is a rhombus whose diagonals cut at the origin $O$, then prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{O}$.
2. Let $D$, $E$ and $F$ be the middle points of the sides $BC$, $CA$ and $AB$, respectively, of a triangle $ABC$. Then prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$.
3. Let $ABCD$ be a parallelogram whose diagonals intersect at $P$ and let $O$ be the origin. Then prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OP}$.
4. If $A$, $B$, $C$ and $D$ be any four points and $E$ and $F$ be the middle points of $AC$ and $BD$, respectively, then prove that $\overrightarrow{CB} + \overrightarrow{CD} + \overrightarrow{AD} + \overrightarrow{AB} = 4\overrightarrow{EF}$.
5. If $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$, then $A$, $B$ and $C$ are (where $O$ is the origin)

   **a.** coplanar **b.** collinear **c.** non-collinear **d.** none of these
6. If the sides of an angle are given by vectors $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, then find the internal bisector of the angle.
7. $ABCD$ is a parallelogram. If $L$ and $M$ be the middle points of $BC$ and $CD$, respectively, express $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Also show that $\overrightarrow{AL} + \overrightarrow{AM} = (3/2)\overrightarrow{AC}$.

**8.** $ABCD$ is a quadrilateral and $E$ the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OD}$ is equal to $4\overrightarrow{OE}$, where $O$ is any point.

**9.** What is the unit vector parallel to $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$? What vector should be added to $\vec{a}$ so that the resultant is the unit vector $\hat{i}$?

**10.** The position vectors of points $A$ and $B$ w.r.t. an origin are $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 2\hat{k}$, respectively. Determine vector $\overrightarrow{OP}$ which bisects angle $AOB$, where $P$ is a point on $AB$.

**11.** If $\vec{r_1}, \vec{r_2}, \vec{r_3}$ are the position vectors of three collinear points and scalars $p$ and $q$ exist such that $\vec{r_3} = p \cdot \vec{r_1} + q\vec{r_2}$, then show that $p + q = 1$.

## VECTOR ALONG THE BISECTOR OF GIVEN TWO VECTORS

We know that the diagonal in a parallelogram is not necessarily the bisector of the angle formed by two adjacent sides. However, the diagonal in a rhombus bisects the angle between two adjacent sides.

Consider vectors $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$ forming a parallelogram $ABCD$ as shown in the following figure.

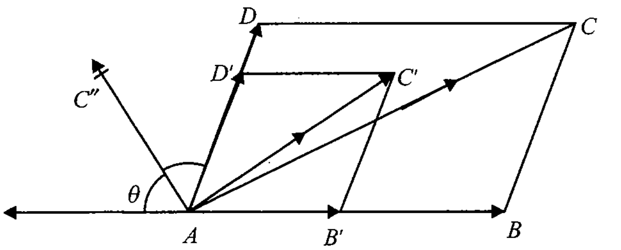



**Fig. 1.34**

Consider the two unit vectors along the given vectors, which form a rhombus $AB'C'D'$.

Now $\overrightarrow{AB'} = \dfrac{\vec{a}}{|\vec{a}|}$ and $\overrightarrow{AD'} = \dfrac{\vec{b}}{|\vec{b}|}$. Therefore,

$$\overrightarrow{AC'} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$$

So any vector along the bisector is $\lambda\left(\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}\right)$.

Similarly, any vector along the external bisector is $\overrightarrow{AC''} = \lambda\left(\dfrac{\vec{a}}{|\vec{a}|} - \dfrac{\vec{b}}{|\vec{b}|}\right)$

**Example 1.27** **If $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$, determine vector $\vec{c}$ along the internal bisector of the angle between vectors $\vec{a}$ and $\vec{b}$, such that $|\vec{c}| = 5\sqrt{6}$.**

**Sol.**

$$\hat{a} = \frac{1}{9}(7\hat{i} - 4\hat{j} - 4\hat{k})$$

$$\hat{b} = \frac{1}{3}(-2\hat{i} - \hat{j} + 2\hat{k})$$

$$\vec{c} = \lambda[\hat{a} + \hat{b}] = \lambda\frac{1}{9}(\hat{i} - 7\hat{j} + 2\hat{k}) \qquad \text{(i)}$$

$$|\vec{c}| = 5\sqrt{6}$$

$$\Rightarrow \frac{\lambda^2}{81}(1 + 49 + 4) = 25 \times 6$$

$$\lambda^2 = \frac{25 \times 6 \times 81}{54} = 225$$

$\lambda = \pm 15$

Putting the value of $\lambda$ in (i), we get

$$\vec{c} = \pm\frac{5}{3}(\hat{i} - 7\hat{j} + 2\hat{k})$$

**Example 1.28** **Find a unit vector $\vec{c}$ if $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between vectors $\vec{c}$ and $3\hat{i} + 4\hat{j}$.**

**Sol.**

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$, where $x^2 + y^2 + z^2 = 1$. (i)

Unit vector along $3\hat{i} + 4\hat{j}$ is $\frac{3\hat{i} + 4\hat{j}}{5}$.

The bisector of these two is $-\hat{i} + \hat{j} - \hat{k}$ (given). Therefore,

$$-\hat{i} + \hat{j} - \hat{k} = \lambda\left(x\hat{i} + y\hat{j} + z\hat{k} + \frac{3\hat{i} + 4\hat{j}}{5}\right)$$

$$-\hat{i} + \hat{j} - \hat{k} = \frac{1}{5}\lambda[(5x + 3)\hat{i} + (5y + 4)\hat{j} + 5z\hat{k}] \qquad \text{(ii)}$$

$$\frac{\lambda}{5}(5x + 3) = -1, \quad \frac{\lambda}{5}(5y + 4) = 1, \quad \frac{\lambda}{5}5z = -1$$

$$x = -\frac{5 + 3\lambda}{5\lambda}, \quad y = \frac{5 - 4\lambda}{5\lambda}, \quad z = -\frac{1}{\lambda}$$

Putting these values in (i), i.e., $x^2 + y^2 + z^2 = 1$, we get

$(5 + 3\lambda)^2 + (5 - 4\lambda)^2 + 25 = 25\lambda^2$

$25\lambda^2 - 10\lambda + 75 = 25\lambda^2$

$\lambda = 15/2$

$\therefore \quad \vec{c} = \frac{1}{15}(-11\hat{i}+10\hat{j}-2\hat{k})$

## LINEAR COMBINATION, LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

### Linear Combination

A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ if there exist scalars $m_1, m_2, \cdots, m_n$ such that $\vec{r} = m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n$.

### Linearly Independent

A system of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ is said to be linearly independent if

$m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n = \vec{0} \Rightarrow m_1 = m_2 = \cdots = m_n = 0$

It can be easily verified that

i. A pair of non-collinear vectors is linearly independent.

*Proof:*

Let $\overline{a_1}$ and $\overline{a_2}$ are non-collinear vectors such that $m_1\overline{a_1} + m_2\overline{a_2} = \vec{O}$

Let $m_1, m_2 \neq 0$

$\Rightarrow \overline{a_1} = -\frac{m_2}{m_1}\overline{a_2}$

$\Rightarrow \overline{a_1}$ and $\overline{a_2}$ are collinear, which contradicts the given fact.

Hence $m_1, m_2 = 0$

ii. A triad of non-coplanar vector is linearly independent.

### Linearly Dependent

A set of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$ is said to be linearly dependent if there exist scalars $m_1, m_2, \cdots, m_n$, not all zero, such that $m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n = \vec{0}$.

It can be easily verified that

i. A pair of collinear vectors is linearly dependent.

ii. A triad of coplanar vectors is linearly dependent.

### Theorem 1.1

If $\vec{a}$ and $\vec{b}$ be two non-collinear vectors, then every vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be expressed in one and only one way as a linear combination $x\vec{a} + y\vec{b}$; $x$ and $y$ being scalars.

**Proof:**

i.

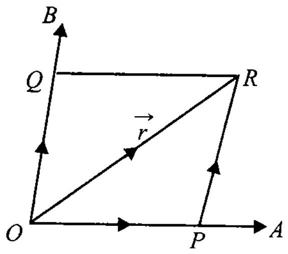



**Fig. 1.35**

Let $O$ be any point such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

As $\vec{r}$ is coplanar with $\vec{a}$ and $\vec{b}$, the lines $OA$, $OB$ and $OR$ are coplanar.

Through $R$, draw lines parallel to $OA$ and $OB$, meeting them at $P$ and $Q$, respectively.

Clearly, $\overrightarrow{OP} = x\,\overrightarrow{OA} = x\vec{a}$ ($\because \overrightarrow{OP}$ and $\overrightarrow{OA}$ are collinear vectors)

Also $\overrightarrow{OQ} = y\,\overrightarrow{OB} = y\vec{b}$ ($\because \overrightarrow{OQ}$ and $\overrightarrow{OB}$ are collinear vectors)

$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \overrightarrow{OQ}$ ($\because \overrightarrow{OQ}$ and $\overrightarrow{PR}$ are equal)

$= x\vec{a} + y\vec{b}$ (i)

Thus, $\vec{r}$ can be expressed in one way as a linear combination $x\vec{a} + y\vec{b}$.

ii. To prove that this resolution is unique, let $\vec{r} = x'\vec{a} + y'\vec{b}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

Then, $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b}$

or $(x - x')\vec{a} + (y - y')\vec{b} = \vec{0}$

Since $\vec{a}$ and $\vec{b}$ are non-collinear vectors, we must have

$x - x' = 0,\ y - y' = 0$

i.e., $x = x',\ y = y'$

Thus the representation is unique.

**Note:**

*If OA and OB are perpendicular, then these two lines can be taken as the x- and the y-axes, respectively. Let $\hat{i}$ be the unit vector along the x-axis and $\hat{j}$ be the unit vector along the y-axis. Therefore, we have*

$\vec{r} = x\vec{i} + y\vec{j}$

*Also* $r = \sqrt{x^2 + y^2}$

## Theorem 1.2

If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, then any vector $\vec{r}$ can be uniquely expressed as a linear combination $x\vec{a} + y\vec{b} + z\vec{c}$; $x$, $y$ and $z$ being scalars.

**Proof:**

i.

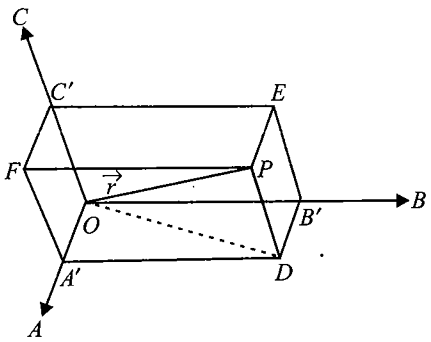



**Fig. 1.36**

Take any point $O$ so that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OP} = \vec{r}$.

On $OP$ as diagonal, construct a parallelpiped having edges $OA'$, $OB'$ and $OC'$ along $OA$, $OB$ and $OC$, respectively. Then there exist three scalars $x$, $y$ and $z$ such that

$$\overrightarrow{OA'} = x\,\overrightarrow{OA} = x\vec{a},\ \overrightarrow{OB'} = y\,\overrightarrow{OB} = y\vec{b},\ \overrightarrow{OC'} = z\,\overrightarrow{OC} = z\vec{c}$$

$\therefore$ $\vec{r} = \overrightarrow{OP}$

$= \overrightarrow{OA'} + \overrightarrow{A'P}$

$= \overrightarrow{OA'} + \overrightarrow{A'D} + \overrightarrow{DP}$ (by definition of addition of vectors)

$= \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}$

$= x\vec{a} + y\vec{b} + z\vec{c}$ (i)

Thus $\vec{r}$ can be represented as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

ii To prove that this representation is unique, let, if possible, $\vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$. (ii)

Then from (i) and (ii), we have

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$$

or

$$(x - x')\vec{a} + (y - y')\vec{b} + (z - z')\vec{c} = \vec{0}$$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are independent, $x - x' = 0$, $y - y' = 0$ and $z - z' = 0$, or $x = x'$, $y = y'$ and $z = z'$. Hence proved.

## Theorem 1.3

If vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are coplanar, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

**Proof:**

If vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then there exist scalars $\lambda$ and $\mu$ such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$. Hence,

$$c_1\hat{i} + c_2\hat{j} + c_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \mu(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

Now $\hat{i}, \hat{j}$ and $\hat{k}$ are non-coplanar and hence independent. Then,

$c_1 = \lambda a_1 + \mu b_1$, $c_2 = \lambda a_2 + \mu b_2$ and $c_3 = \lambda a_3 + \mu b_3$

The above system of equations in terms of $\lambda$ and $\mu$ is consistent.

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Similarly, if vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$ and $x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$ are coplanar (where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar). Then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$ can be proved with the same arguments.

To prove that four points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are coplanar, it is just sufficient to prove that vectors $\overrightarrow{AB}$, $\overrightarrow{BD}$ and $\overrightarrow{CD}$ are coplanar.

**Notes:**

1. *Two collinear vectors are always linearly dependent.*
2. *Two non-collinear non-zero vectors are always linearly independent.*
3. *Three coplanar vectors are always linearly dependent.*
4. *Three non-coplanar non-zero vectors are always linearly independent.*
5. *More than three vectors are always linearly dependent.*
6. *Three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear if and only if there exist scalars x, y and z not all zero such that (i) $x\vec{a} + y\vec{b} + z\vec{c} = 0$ and (ii) $x + y + z = 0$.*

   ***Proof:***

   *Let us suppose that points A, B and C are collinear and their position vectors are $\vec{a}, \vec{b}$ and $\vec{c}$, respectively. Let C divide the join of $\vec{a}$ and $\vec{b}$ in the ratio y : x. Then,*

$$\vec{c} = \frac{x\vec{a} + y\vec{b}}{x+y}$$

or $x\vec{a} + y\vec{b} - (x+y)\vec{c} = \vec{0}$

or $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, *where* $z = -(x+y)$

*Also*, $x + y + z = x + y - (x + y) = 0$.

*Conversely, let* $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, *where* $x + y + z = 0$. Therefore,

$x\vec{a} + y\vec{b} = -z\vec{c} = (x+y)\vec{c}, \because x + y = -z$

or $\vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x+y}$

*This relation shows that* $\vec{c}$ *divides the join of* $\vec{a}$ *and* $\vec{b}$ *in the ratio* $y : x$. *Hence the three points A, B and C are collinear.*

**7.** *Four points with position vectors* $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ *are coplanar if there exist scalars x, y, z and w (sum of any two is not zero) such that* $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$ *with* $x + y + z + w = 0$.

***Proof:***

$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$

$\Rightarrow x\vec{a} + y\vec{b} = -(z\vec{c} + w\vec{d})$ (i)

$x + y + z + w = 0$

$\Rightarrow x + y = -(w + z)$ (ii)

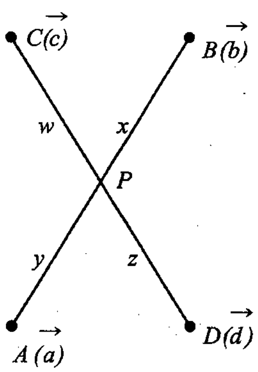



**Fig. 1.37**

*From (i) and (ii), we have* $\dfrac{x\vec{a} + y\vec{b}}{x+y} = \dfrac{z\vec{c} + w\vec{d}}{z+w}$

*Thus there is point P*

$$\Rightarrow \frac{x\vec{a} + y\vec{b}}{x+y} = \frac{z\vec{c} + w\vec{d}}{z+w} \quad \text{(iii)}$$

$\dfrac{x\vec{a} + y\vec{b}}{x+y}$ *is the position vector of a point on AB which divides it in the ratio y : x.*

$\dfrac{z\vec{c}+w\vec{d}}{z+w}$ *is the position vector of a point on CD which divides it in the ratio w : z.*

*From (iii), these points are coincident; hence the points are coplanar.*

**Example 1.29** **The vectors $2\hat{i}+3\hat{j}$, $5\hat{i}+6\hat{j}$ and $8\hat{i}+\lambda\hat{j}$ have their initial points at (1, 1). Find the value of $\lambda$ so that the vectors terminate on one straight line**

**Sol.**

Since the vectors $2\hat{i}+3\hat{j}$ and $5\hat{i}+6\hat{j}$ have (1, 1) as the initial point, therefore their terminal points are (3, 4) and (6, 7), respectively. The equation of the line joining these two points is $x - y + 1 = 0$. The terminal point of $8\hat{i}+\lambda\hat{j}$ is $(9, \lambda + 1)$. Since the vectors terminate on the same straight line, $(9, \lambda + 1)$ lies on $x - y + 1 = 0$. Therefore,

$9 - \lambda - 1 + 1 = 0$

$\Rightarrow \lambda = 9$

**Example 1.30** **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-zero vectors, no two of which are collinear, $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$, then find the value of $|\vec{a}+2\vec{b}+6\vec{c}|$.**

**Sol.**

Given $\vec{a}+2\vec{b}=\lambda\vec{c}$ (i)

and $\vec{b}+3\vec{c}=\mu\vec{a}$, (ii)

where no two of $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear vectors.

Eliminating $\vec{b}$ from the above relations, we have

$\vec{a}-6\vec{c}=\lambda\vec{c}-2\mu\vec{a}$

$\vec{a}(1+2\mu)=(\lambda+6)\vec{c}$

$\Rightarrow \quad \mu=-\frac{1}{2}$ and $\lambda=-6$ as $\vec{a}$ and $\vec{c}$ are non-collinear.

Putting $\mu=-\frac{1}{2}$ in (ii) or $\lambda=-6$ in (i), we get

$\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$

$\Rightarrow \quad |\vec{a}+2\vec{b}+3\vec{c}|=0$

**Example 1.31** **a. Prove that the points $\vec{a}-2\vec{b}+3\vec{c}$, $2\vec{a}+3\vec{b}-4\vec{c}$ and $-7\vec{b}+10\vec{c}$ are collinear, where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar.**

**b. Prove that the points $A(1, 2, 3)$, $B(3, 4, 7)$ and $C(-3, -2, -5)$ are collinear. Find the ratio in which point $C$ divides $AB$.**

**Sol.**

a. Let the given points be $A$, $B$ and $C$. Therefore,

$\overrightarrow{AB}$ = P.V. of $B$ – P.V. of $A$

$=(2\vec{a}+3\vec{b}-4\vec{c})-(\vec{a}-2\vec{b}+3\vec{c})$

$=\vec{a}+5\vec{b}-7\vec{c}$

$\overrightarrow{AC}$ = P.V. of $C$ − P.V. of $A$

$= (-7\vec{b} + 10\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$

$= -\vec{a} - 5\vec{b} + 7\vec{c} = -\overrightarrow{AB}$

Since $\overrightarrow{AC} = -\overrightarrow{AB}$, it follows that the points $A$, $B$ and $C$ are collinear.

b. Let $C$ divide $AB$ in the ratio $k : 1$; then $C(-3, -2, -5) \equiv \left(\frac{3k+1}{k+1}, \frac{4k+2}{k+1}, \frac{7k+3}{k+1}\right)$

$\Rightarrow \frac{3k+1}{k+1} = -3, \frac{4k+2}{k+1} = -2$ and $\frac{7k+3}{k+1} = -5$

$\Rightarrow k = -\frac{2}{3}$ from all relations

Hence, $C$ divides $AB$ externally in the ratio 2:3.

**Example 1.32** **Check whether the given three vectors are coplanar or non-coplanar:**
$-2\hat{i} - 2\hat{j} + 4\hat{k}, -2\hat{i} + 4\hat{j} - 2\hat{k}, 4\hat{i} - 2\hat{j} - 2\hat{k}$

**Sol.**

Given vectors are $-2\hat{i} - 2\hat{j} + 4\hat{k}, -2\hat{i} + 4\hat{j} - 2\hat{k}, 4\hat{i} - 2\hat{j} - 2\hat{k}$

$$\Rightarrow \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = 16 + 16 + 16 - 64 + 8 + 8 = 0$$

Hence the vectors are coplanar.

**Example 1.33** **Prove that the four points $6\hat{i} - 7\hat{j}, 16\hat{i} - 19\hat{j} - 4\hat{k}, 3\hat{j} - 6\hat{k}$ and $2\hat{i} + 5\hat{j} + 10\hat{k}$ form a tetrahedron in space.**

**Sol.**

Given points are $A(6\hat{i} - 7\hat{j}), B(16\hat{i} - 19\hat{j} - 4\hat{k}), C(3\hat{j} - 6\hat{k}), D(2\hat{i} + 5\hat{j} + 10\hat{k})$

Hence vectors $\overrightarrow{AB} = 10\hat{i} - 12\hat{j} - 4\hat{k}$, $\overrightarrow{AC} = -6\hat{i} + 10\hat{j} - 6\hat{k}$ and $\overrightarrow{AD} = -4\hat{i} + 12\hat{j} + 10\hat{k}$

Now determinant of coefficients of $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is

$$\begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix} = 10(100 + 72) + 12(-60 - 24) - 4(-72 + 40) \neq 0$$

Hence, the given points are non-coplanar and therefore form a tetrahedron in space.

**Example 1.34** **If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, show that points $l_1\vec{a}+m_1\vec{b}$, $l_2\vec{a}+m_2\vec{b}$ and $l_3\vec{a}+m_3\vec{b}$ are collinear if** $\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$

**Sol.**

We know that three points having P.V.s $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear if there exists a relation of the form $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$.

Now $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ gives

$x(l_1\vec{a} + m_1\vec{b}) + y(l_2\vec{a} + m_2\vec{b}) + z(l_3\vec{a} + m_3\vec{b}) = \vec{0}$

or $(xl_1 + yl_2 + zl_3)\vec{a} + (xm_1 + ym_2 + zm_3)\vec{b} = \vec{0}$

Since $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, it follows that

$xl_1 + yl_2 + zl_3 = 0$ (i)

$xm_1 + ym_2 + zm_3 = 0$ (ii)

Because otherwise one is expressible as a scalar multiple of the other which would mean that $\vec{a}$ and $\vec{b}$ are collinear.

Also $x + y + z = 0$. (iii)

Eliminating $x$, $y$ and $z$ from (i), (ii) and (iii), we get

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

**Alternative method:**

$A(l_1\vec{a} + m_1\vec{b})$, $B(l_2\vec{a} + m_2\vec{b})$ and $C(l_3\vec{a} + m_3\vec{b})$ are collinear.

$\Rightarrow$ Vectors $= (l_2 - l_3)\vec{a} + (m_2 - m_3)\vec{b}$ and $\overrightarrow{AB} = (l_1 - l_2)\vec{a} + (m_1 - m_2)\vec{b}$ are collinear.

$$\Rightarrow \frac{l_1 - l_2}{l_2 - l_3} = \frac{m_1 - m_2}{m_2 - m_3}$$

$$\Rightarrow \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

**Example 1.35** **Vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find for what value of $x$ vectors $\vec{c} = (x-2)\vec{a}+\vec{b}$ and $\vec{d} = (2x+1)\vec{a}-\vec{b}$ are collinear?**

**Sol.**

Both the vectors $\vec{c}$ and $\vec{d}$ are non-zero as the coefficients of $\vec{b}$ in both are non-zero.

Two vectors $\vec{c}$ and $\vec{d}$ are collinear if one of them is a linear multiple of the other. Therefore,

$\vec{d} = \lambda \vec{c}$

or $(2x+1)\vec{a} - \vec{b} = \lambda\{(x-2)\vec{a} + \vec{b}\}$ (i)

or $\{(2x+1) - \lambda(x-2)\}\vec{a} - (1+\lambda)\vec{b} = 0$

The above expression is of the form $p\vec{a} + q\vec{b} = 0$, where $\vec{a}$ and $\vec{b}$ are non-collinear, and hence we have $p = 0$ and $q = 0$. Therefore,

$2x + 1 - \lambda(x-2) = 0$ (ii)

and $1 + \lambda = 0$ (iii)

From (iii), $\lambda = -1$ and putting this value in (i), we get $x = \frac{1}{3}$

**Alternative method:**

$\vec{c} = (x-2)\vec{a} + \vec{b}$ and $\vec{d} = (2x+1)\vec{a} - \vec{b}$ are collinear.

If $\frac{x-2}{2x+1} = \frac{1}{-1} \Rightarrow x = \frac{1}{3}$

**Example 1.36** **The median $AD$ of the triangle $ABC$ is bisected at $E$ and $BE$ meets $AC$ at $F$. Find $AF : FC$.**

**Sol.**

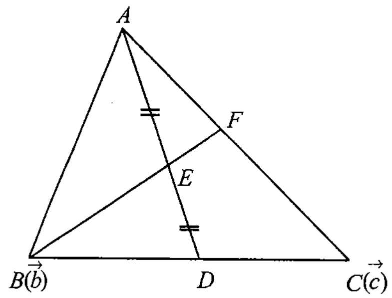



**Fig. 1.38**

Taking $A$ at the origin

Let P.V. of $B$ and $C$ be $\vec{b}$ and $\vec{c}$, respectively.

P.V. of $D$ is $\frac{\vec{b}+\vec{c}}{2}$ and P.V. of $E$ is $\frac{\vec{b}+\vec{c}}{4}$

Let $AF:FC = p:1$.

Then position vector of $F$ is $\frac{p\vec{c}}{p+1}$ (i)

Let $BF:EF = q:1$.

The position vector of $F$ is $\frac{q\frac{(\vec{b}+\vec{c})}{4} - \vec{b}}{q-1}$ (ii)

Comparing P.V. of $F$ in (i) and (ii), we have

$$\frac{p\vec{c}}{p+1} = \frac{q\frac{(\vec{b}+\vec{c})}{4} - \vec{b}}{q-1}$$

Since vectors $\vec{b}$ and $\vec{c}$ are independent, we have

$$\frac{p}{p+1} = \frac{q}{4(q-1)} \text{ and } \frac{q-4}{4(q-1)} = 0$$

$\Rightarrow$ $p = 1/4$ and $q = 4$

$\Rightarrow$ $AF : FC = 1:2$

**Example 1.37** **Prove that the necessary and sufficient condition for any four points in three-dimensional space to be coplanar is that there exists a linear relation connecting their position vectors such that the algebraic sum of the coefficients (not all zero) in it is zero.**

**Sol.**

Let us suppose that the points $A, B, C$ and $D$ whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, are coplanar. In that case the lines $AB$ and $CD$ will intersect at some point $P$ (it being assumed that $AB$ and $CD$ are not parallel, and if they are, then we will choose any other pair of non-parallel lines formed by the given points). If $P$ divides $AB$ in the ratio $q : p$ and $CD$ in the ratio $n : m$, then the position vector of $P$ written from $AB$ and $CD$ is

$$\frac{p\vec{a}+q\vec{b}}{p+q} = \frac{m\vec{c}+n\vec{d}}{m+n}$$

or $$\frac{p}{p+q}\vec{a}+\frac{q}{p+q}\vec{b}-\frac{m}{m+n}\vec{c}-\frac{n}{m+n}\vec{d}=\vec{0}$$

or $$L\vec{a}+M\vec{b}+N\vec{c}+P\vec{d}=\vec{0}$$

where $L+M+N+P = \frac{p}{p+q}+\frac{q}{p+q}-\frac{m}{m+n}-\frac{n}{m+n} = 1-1=0$

Hence the condition is necessary.

**Converse:** Let $l\vec{a} + m\vec{b} + n\vec{c} + p\vec{d} = \vec{0}$

where $l + m + n + p = 0$ (i)

We will show that the points $A, B, C$ and $D$ are coplanar.

Now of the three scalars $l+m$, $l+n$ and $l+p$, one at least is not zero, because if all of them are zero, then $l+m=0, l+n=0, l+p=0$. Therefore,

$m=n=p=-l$

Hence $l+m+n+p=0 \quad \Rightarrow \quad l-3l=0 \quad \Rightarrow \quad l=0$

Hence $m=n=p=-l=0$

Thus $l=0, m=0, n=0, p=0$, which is against the hypothesis.

Let us suppose that $l + m$ is not zero.

$l+m=-(n+p)\neq 0,$ [From (i)] (ii)

Also from the given relation, we have

$$l\vec{a}+m\vec{b}=-(n\vec{c}+p\vec{d})$$

or $$\frac{l\vec{a}+m\vec{b}}{l+m}=\frac{n\vec{c}+p\vec{d}}{n+p}$$ [From (ii)] (iii)

L.H.S. represents a point which divides $AB$ in the ratio $m:l$ and R.H.S. represents a point which divides $CD$ in the ratio $p:n$. These points being the same, it follows that a point on $AB$ is the same as a point on $CD$, showing that the lines $AB$ and $CD$ intersect. Hence the four points $A$, $B$, $C$ and $D$ are coplanar.

**Example 1.38** **a. If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that vectors $3\vec{a}-7\vec{b}-4\vec{c}$, $3\vec{a}-2\vec{b}+\vec{c}$ and $\vec{a}+\vec{b}+2\vec{c}$ are coplanar.**

**b. If the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar, then prove that $a=4$.**

**Sol.**

a. If the given vectors are coplanar, then we should be able to express one of them as a linear combination of the other two.

Let us assume that $3\vec{a}-7\vec{b}-4\vec{c}=x(3\vec{a}-2\vec{b}+\vec{c})+y(\vec{a}+\vec{b}+2\vec{c})$,

where $x$ and $y$ are scalars. Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar, equating the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$3x+y=3,\ -2x+y=-7,\ x+2y=-4$

Solving the first two, we find that $x=2$ and $y=-3$. These values of $x$ and $y$ satisfy the third equation as well.

Hence the given vectors are coplanar.

b. Given vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar. Then $\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & -3\\ 3 & a & 5\end{vmatrix}=0$

$\Rightarrow\ 3-7a+25=0$

$\Rightarrow\ a=4$

**Example 1.39** **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that the four points $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.**

**Sol.**

Let the given points be $A$, $B$, $C$ and $D$. If they are coplanar, then the three coterminous vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ should be coplanar.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{a} - 5\vec{b} + 4\vec{c}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{a} + \vec{b} - \vec{c}$

and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -\vec{a} - 9\vec{b} + 7\vec{c}$

Since the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar, we must have $\begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ -1 & -9 & 7 \end{vmatrix} = 0$, which is true.

Hence proved.

**Example 1.40** **Points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are related as $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ and $x + y + z + w = 0$, where $x$, $y$, $z$ and $w$ are scalars (sum of any two of $x$, $y$, $z$ and $w$ is not zero).**

**Prove that if $A$, $B$, $C$ and $D$ are concyclic, then $|xy||\vec{a} - \vec{b}|^2 = |wz||\vec{c} - \vec{d}|^2$**

**Sol.**

From the given conditions, it is clear that points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are coplanar.

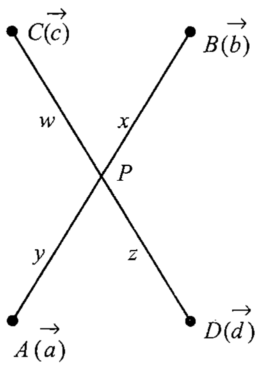



**Fig. 1.39**

Now, $A$, $B$, $C$ and $D$ are concyclic. Therefore,

$AP \times BP = CP \times DP$

$$\left|\frac{y}{x+y}\right| |\vec{a} - \vec{b}| \left|\frac{x}{x+y}\right| |\vec{a} - \vec{b}| = \left|\frac{w}{w+z}\right| |\vec{c} - \vec{d}| \left|\frac{z}{w+z}\right| |\vec{c} - \vec{d}|$$

$|xy||\vec{a} - \vec{b}|^2 = |wz||\vec{c} - \vec{d}|^2$

## Concept Application Exercise 1.2

1. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors in three-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that terminals $A$, $B$, $C$ and $D$ of these vectors are coplanar. Find the point at which $AC$ and $BD$ meet. Find the ratio in which $P$ divides $AC$ and $BD$.
2. Show that the vectors $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$ are non-coplanar vectors (where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors).

3. Examine the following vectors for linear independence:

i. $\vec{i}+\vec{j}+\vec{k}, 2\vec{i}+3\vec{j}-\vec{k}, -\vec{i}-2\vec{j}+2\vec{k}$

ii. $3\vec{i}+\vec{j}-\vec{k}, 2\vec{i}-\vec{j}+7\vec{k}, 7\vec{i}-\vec{j}+13\vec{k}$

4. If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A}=(p+4q)\vec{a}+(2p+q+1)\vec{b}$ and $\vec{B}=(-2p+q+2)\vec{a}+(2p-3q-1)\vec{b}$, and if $3\vec{A}=2\vec{B}$, then determine $p$ and $q$.

5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three non-coplanar vectors, then prove that points $l_1\vec{a}+m_1\vec{b}+n_1\vec{c}, l_2\vec{a}+m_2\vec{b}+n_2\vec{c}, l_3\vec{a}+m_3\vec{b}+n_3\vec{c}, l_4\vec{a}+m_4\vec{b}+n_4\vec{c}$ are coplanar if

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero, non-coplanar vectors, then find the linear relation between the following four vectors: $\vec{a}-2\vec{b}+3\vec{c}, 2\vec{a}-3\vec{b}+4\vec{c}, 3\vec{a}-4\vec{b}+5\vec{c}, 7\vec{a}-11\vec{b}+15\vec{c}$.

# Exercises

## Subjective Type

*Solutions on page 1.57*

1. The position vectors of the vertices $A$, $B$ and $C$ of a triangle are $\hat{i}+\hat{j}, \hat{j}+\hat{k}$ and $\hat{i}+\hat{k}$, respectively. Find a unit vector $\hat{r}$ lying in the plane of $ABC$ and perpendicular to $IA$, where $I$ is the incentre of the triangle.
2. A ship is sailing towards the north at a speed of 1.25 m/s. The current is taking it towards the east at the rate of 1 m/s and a sailor is climbing a vertical pole on the ship at the rate of 0.5 m/s. Find the velocity of the sailor in space.
3. Given four points $P_1, P_2, P_3$ and $P_4$ on the coordinate plane with origin $O$ which satisfy the condition $\overrightarrow{OP}_{n-1}+\overrightarrow{OP}_{n+1}=\frac{3}{2}\overrightarrow{OP}_n$.

   i. If $P_1$ and $P_2$ lie on the curve $xy = 1$, then prove that $P_3$ does not lie on the curve.

   ii. If $P_1$, $P_2$ and $P_3$ lie on the circle $x^2 + y^2 = 1$, then prove that $P_4$ also lies on this circle.
4. $ABCD$ is a tetrahedron and $O$ is any point. If the lines joining $O$ to the vertices meet the opposite faces at $P$, $Q$, $R$ and $S$, prove that $\frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}+\frac{OS}{DS}=1$.
5. A pyramid with vertex at point $P$ has a regular hexagonal base $ABCDEF$. Position vectors of points $A$ and $B$ are $\hat{i}$ and $\hat{i}+2\hat{j}$, respectively. Centre of the base has the position vector $\hat{i}+\hat{j}+\sqrt{3}\hat{k}$. Altitude drawn from $P$ on the base meets the diagonal $AD$ at point $G$. Find all possible position vectors of $G$. It is given that the volume of the pyramid is $6\sqrt{3}$ cubic units and $AP = 5$ units.

6. A straight line $L$ cuts the lines $AB$, $AC$ and $AD$ of a parallelogram $ABCD$ at points $B_1$, $C_1$ and $D_1$, respectively. If $\overrightarrow{AB_1} = \lambda_1 \overrightarrow{AB}$, $\overrightarrow{AD_1} = \lambda_2 \overrightarrow{AD}$ and $\overrightarrow{AC_1} = \lambda_3 \overrightarrow{AC}$, then prove that $\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$.

7. The position vectors of the points $P$ and $Q$ are $5\hat{i}+7\hat{j}-2\hat{k}$ and $-3\hat{i}+3\hat{j}+6\hat{k}$, respectively. Vector $\vec{A}=3\hat{i}-\hat{j}+\hat{k}$ passes through point $P$ and vector $\vec{B}=-3\hat{i}+2\hat{j}+4\hat{k}$ passes through point $Q$. A third vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersects vectors $A$ and $B$. Find the position vectors of points of intersection.

8. Show that $x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$ are non-coplanar if $|x_1| > |y_1| + |z_1|$, $|y_2| > |x_2| + |z_2|$ and $|z_3| > |x_3| + |y_3|$.

9. If $\vec{A}$ and $\vec{B}$ be two vectors and $k$ be any scalar quantity greater than zero, then prove that $|\vec{A}+\vec{B}|^2 \le (1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2$.

10. Consider the vectors $\hat{i}+\cos(\beta-\alpha)\hat{j}+\cos(\gamma-\alpha)\hat{k}$, $\cos(\alpha-\beta)\hat{i}+\hat{j}+\cos(\gamma-\beta)\hat{k}$ and $\cos(\alpha-\gamma)\hat{i}+\cos(\beta-\gamma)\hat{j}+a\hat{k}$, where $\alpha$, $\beta$ and $\gamma$ are different angles. If these vectors are coplanar, show that $a$ is independent of $\alpha$, $\beta$ and $\gamma$.

11. In a triangle $PQR$, $S$ and $T$ are points on $QR$ and $PR$, respectively, such that $QS = 3SR$ and $PT = 4TR$. Let $M$ be the point of intersection of $PS$ and $QT$. Determine the ratio $QM : MT$ using the vector method.

12. A boat moves in still water with a velocity which is $k$ times less than the river flow velocity. Find the angle to the stream direction at which the boat should be rowed to minimize drifting.

13. If $D$, $E$ and $F$ are three points on the sides $BC$, $CA$ and $AB$, respectively, of a triangle $ABC$ such that the lines $AD$, $BE$ and $CF$ are concurrent, then show that

$$\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF} = -1$$

14. In a quadrilateral $PQRS$, $\overrightarrow{PQ} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$, $\overrightarrow{SP} = \vec{a} - \vec{b}$, $M$ is the midpoint of $\overrightarrow{QR}$ and $X$ is a point on $SM$ such that $SX = \frac{4}{5} SM$. Prove that $P$, $X$ and $R$ are collinear.

## Objective Type

Solutions on page 1.65

**Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* answer is correct. Find the correct answer.**

1. Four non-zero vectors will always be
   - **a.** linearly dependent
   - **b.** linearly independent
   - **c.** either a or b
   - **d.** none of these

2. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three units vectors such that $3\vec{a}+4\vec{b}+5\vec{c}=0$. Then which of the following statements is true?
   - **a.** $\vec{a}$ is parallel to $\vec{b}$
   - **b.** $\vec{a}$ is perpendicular to $\vec{b}$
   - **c.** $\vec{a}$ is neither parallel nor perpendicular to $\vec{b}$
   - **d.** none of these

3. Let $ABC$ be a triangle, the position vectors of whose vertices are respectively $\hat{i}+2\hat{j}+4\hat{k}$, $-2\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+4\hat{j}-3\hat{k}$. Then $\Delta ABC$ is

a. isosceles  b. equilateral  c. right angled  d. none of these

4. If $|\vec{a}+\vec{b}|<|\vec{a}-\vec{b}|$, then the angle between $\vec{a}$ and $\vec{b}$ can lie in the interval

a. $(-\pi/2, \pi/2)$  b. $(0, \pi)$  c. $(\pi/2, 3\pi/2)$  d. $(0, 2\pi)$

5. A point $O$ is the centre of a circle circumscribed about a triangle $ABC$. Then $\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C$ is equal to

a. $(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})\sin 2A$

b. $3\overrightarrow{OG}$, where $G$ is the centroid of triangle $ABC$

c. $\vec{0}$

d. none of these

6. If $G$ is the centroid of a triangle $ABC$, then $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}$ is equal to

a. $\vec{0}$  b. $3\overrightarrow{GA}$  c. $3\overrightarrow{GB}$  d. $3\overrightarrow{GC}$

7. If $\vec{a}$ is a non-zero vector of modulus $a$ and $m$ is a non-zero scalar, then $m\vec{a}$ is a unit vector if

a. $m=\pm 1$  b. $a=|m|$  c. $a=1/|m|$  d. $a=1/m$

8. $ABCD$ a parallelogram, and $A_1$ and $B_1$ are the midpoints of sides $BC$ and $CD$, respectively. If $\overrightarrow{AA_1}+\overrightarrow{AB_1}=\lambda\overrightarrow{AC}$, then $\lambda$ is equal to

a. $\frac{1}{2}$  b. 1  c. $\frac{3}{2}$  d. 2

9. The position vectors of the points $P$ and $Q$ with respect to the origin $O$ are $\vec{a}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-2\hat{k}$, respectively. If $M$ is a point on $PQ$, such that $OM$ is the bisector of $POQ$, then $\overrightarrow{OM}$ is

a. $2(\hat{i}-\hat{j}+\hat{k})$  b. $2\hat{i}+\hat{j}-2\hat{k}$  c. $2(-\hat{i}+\hat{j}-\hat{k})$  d. $2(\hat{i}+\hat{j}+\hat{k})$

10. $ABCD$ is a quadrilateral. $E$ is the point of intersection of the line joining the midpoints of the opposite sides. If $O$ is any point and $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=x\overrightarrow{OE}$, then $x$ is equal to

a. 3  b. 9  c. 7  d. 4

11. If vectors $\overrightarrow{AB}=-3\hat{i}+4\hat{k}$ and $\overrightarrow{AC}=5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a $\Delta ABC$, then the length of the median through $A$ is

a. $\sqrt{14}$  b. $\sqrt{18}$  c. $\sqrt{29}$  d. 5

12. $A, B, C$ and $D$ have position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively, such that $\vec{a}-\vec{b}=2(\vec{d}-\vec{c})$. Then

a. $AB$ and $CD$ bisect each other  b. $BD$ and $AC$ bisect each other

c. $AB$ and $CD$ trisect each other  d. $BD$ and $AC$ trisect each other

13. If $\vec{a}$ and $\vec{b}$ are two unit vectors and $\theta$ is the angle between them, then the unit vector along the angular bisector of $\vec{a}$ and $\vec{b}$ will be given by

a. $\dfrac{\vec{a}-\vec{b}}{2\cos(\theta/2)}$  b. $\dfrac{\vec{a}+\vec{b}}{2\cos(\theta/2)}$  c. $\dfrac{\vec{a}-\vec{b}}{\cos(\theta/2)}$  d. none of these

**14.** Let us define the length of a vector $a\hat{i}+b\hat{j}+c\hat{k}$ as $|a|+|b|+|c|$. This definition coincides with the usual definition of length of a vector $a\hat{i}+b\hat{j}+c\hat{k}$ if and only if

**a.** $a=b=c=0$ **b.** any two of $a$, $b$ and $c$ are zero

**c.** any one of $a$, $b$ and $c$ is zero **d.** $a+b+c=0$

**15.** Given three vectors $\vec{a}=6\hat{i}-3\hat{j}$, $\vec{b}=2\hat{i}-6\hat{j}$ and $\vec{c}=-2\hat{i}+21\hat{j}$ such that $\vec{\alpha}=\vec{a}+\vec{b}+\vec{c}$. Then the resolution of the vector $\vec{\alpha}$ into components with respect to $\vec{a}$ and $\vec{b}$ is given by

**a.** $3\vec{a}-2\vec{b}$ **b.** $3\vec{b}-2\vec{a}$ **c.** $2\vec{a}-3\vec{b}$ **d.** $\vec{a}-2\vec{b}$

**16.** If $\vec{\alpha}+\vec{\beta}+\vec{\gamma}=a\vec{\delta}$ and $\vec{\beta}+\vec{\gamma}+\vec{\delta}=b\vec{\alpha}$, $\vec{\alpha}$ and $\vec{\delta}$ are non-collinear, then $\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta}$ equals

**a.** $a\vec{\alpha}$ **b.** $b\vec{\delta}$ **c.** 0 **d.** $(a+b)\vec{\gamma}$

**17.** In triangle $ABC$, $\angle A=30°$, $H$ is the orthocentre and $D$ is the midpoint of $BC$. Segment $HD$ is produced to $T$ such that $HD=DT$. The length $AT$ is equal to

**a.** $2BC$ **b.** $3BC$ **c.** $\frac{4}{3}BC$ **d.** none of these

**18.** Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \cdots, \vec{r}_n$ be the position vectors of points $P_1, P_2, P_3, \cdots, P_n$ relative to the origin $O$. If the vector equation $a_1\vec{r}_1+a_2\vec{r}_2+\cdots+a_n\vec{r}_n=0$ holds, then a similar equation will also hold w.r.t. to any other origin provided

**a.** $a_1+a_2+\cdots+a_n=n$ **b.** $a_1+a_2+\cdots+a_n=1$

**c.** $a_1+a_2+\cdots+a_n=0$ **d.** $a_1=a_2=a_3=\cdots=a_n=0$

**19.** Given three non-zero, non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$. $\vec{r}_1=p\vec{a}+q\vec{b}+\vec{c}$ and $\vec{r}_2=\vec{a}+p\vec{b}+q\vec{c}$. If the vectors $\vec{r}_1+2\vec{r}_2$ and $2\vec{r}_1+\vec{r}_2$ are collinear, then $(p, q)$ is

**a.** (0, 0) **b.** (1, –1) **c.** (–1, 1) d. (1, 1)

**20.** If the vectors $\vec{a}$ and $\vec{b}$ are linearly independent satisfying $(\sqrt{3}\tan\theta+1)\vec{a}+(\sqrt{3}\sec\theta-2)\vec{b}=0$, then the most general values of $\theta$ are

**a.** $n\pi-\frac{\pi}{6}, n\in Z$ **b.** $2n\pi\pm\frac{11\pi}{6}, n\in Z$ **c.** $n\pi\pm\frac{\pi}{6}, n\in Z$ **d.** $2n\pi+\frac{11\pi}{6}, n\in Z$

**21.** In a trapezium, vector $\overrightarrow{BC}=\alpha\overrightarrow{AD}$. We will then find that $\vec{p}=\overrightarrow{AC}+\overrightarrow{BD}$ is collinear with $\overrightarrow{AD}$. If $\vec{p}=\mu\overrightarrow{AD}$, then which of the following is true?

**a.** $\mu=\alpha+2$ **b.** $\mu+\alpha=1$ **c.** $\alpha=\mu+1$ **d.** $\mu=\alpha+1$

**22.** Vectors $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ ; $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+4\hat{k}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are

**a.** not coplanar **b.** coplanar but cannot form a triangle

**c.** coplanar and form a triangle **d.** coplanar and can form a right-angled triangle

23. Vectors $\vec{a} = -4\hat{i} + 3\hat{k}$; $\vec{b} = 14\hat{i} + 2\hat{j} - 5\hat{k}$ are laid off from one point. Vector $\vec{d}$, which is being laid off from the same point dividing the angle between vectors $\vec{a}$ and $\vec{b}$ in equal halves and having the magnitude $\sqrt{6}$, is

a. $\hat{i} + \hat{j} + 2\hat{k}$ b. $\hat{i} - \hat{j} + 2\hat{k}$ c. $\hat{i} + \hat{j} - 2\hat{k}$ d. $2\hat{i} - \hat{j} - 2\hat{k}$

24. If $\hat{i} - 3\hat{j} + 5\hat{k}$ bisects the angle between $\hat{a}$ and $-\hat{i} + 2\hat{j} + 2\hat{k}$, where $\hat{a}$ is a unit vector, then

a. $\hat{a} = \frac{1}{105}(41\hat{i} + 88\hat{j} - 40\hat{k})$
b. $\hat{a} = \frac{1}{105}(41\hat{i} + 88\hat{j} + 40\hat{k})$
c. $\hat{a} = \frac{1}{105}(-41\hat{i} + 88\hat{j} - 40\hat{k})$
d. $\hat{a} = \frac{1}{105}(41\hat{i} - 88\hat{j} - 40\hat{k})$

25. If $4\hat{i} + 7\hat{j} + 8\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$ are the position vectors of the vertices *A*, *B* and *C*, respectively, of triangle *ABC*, the position vector of the point where the bisector of angle *A* meets *BC*, is

a. $\frac{2}{3}(-6\hat{i} - 8\hat{j} - 6\hat{k})$ b. $\frac{2}{3}(6\hat{i} + 8\hat{j} + 6\hat{k})$ c. $\frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$ d. $\frac{1}{3}(5\hat{j} + 12\hat{k})$

26. If $\vec{b}$ is a vector whose initial point divides the join of $5\hat{i}$ and $5\hat{j}$ in the ratio $k : 1$ and whose terminal point is the origin and $|\vec{b}| \le \sqrt{37}$, then *k* lies in the interval

a. $[-6, -1/6]$
b. $(-\infty, -6] \cup [-1/6, \infty)$
c. $[0, 6]$
d. none of these

27. Find the value of $\lambda$ so that the points *P*, *Q*, *R* and *S* on the sides *OA*, *OB*, *OC* and *AB*, respectively, of a regular tetrahedron *OABC* are coplanar. It is given that $\frac{OP}{OA} = \frac{1}{3}$, $\frac{OQ}{OB} = \frac{1}{2}$, $\frac{OR}{OC} = \frac{1}{3}$ and $\frac{OS}{AB} = \lambda$.

a. $\lambda = \frac{1}{2}$ b. $\lambda = -1$ c. $\lambda = 0$ d. for no value of $\lambda$

28. '*I*' is the incentre of triangle *ABC* whose corresponding sides are *a*, *b*, *c*, respectively. $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC}$ is always equal to

a. $\vec{0}$
b. $(a+b+c)\overrightarrow{BC}$
c. $(\vec{a} + \vec{b} + \vec{c})\overrightarrow{AC}$
d. $(a+b+c)\overrightarrow{AB}$

29. Let $x^2 + 3y^2 = 3$ be the equation of an ellipse in the *x*-*y* plane. *A* and *B* are two points whose position vectors are $-\sqrt{3}\hat{i}$ and $-\sqrt{3}\hat{i} + 2\hat{k}$. Then the position vector of a point *P* on the ellipse such that $\angle APB = \pi/4$ is

a. $\pm\hat{j}$ b. $\pm(\hat{i} + \hat{j})$ c. $\pm\hat{i}$ d. none of these

30. Locus of the point *P*, for which $\overrightarrow{OP}$ represents a vector with direction cosine $\cos\alpha = \frac{1}{2}$ ('*O*' is the origin) is

a. a circle parallel to the $y$-$z$ plane with centre on the $x$-axis
b. a cone concentric with the positive $x$-axis having vertex at the origin and the slant height equal to the magnitude of the vector
c. a ray emanating from the origin and making an angle of 60° with the $x$-axis
d. a disc parallel to the $y$-$z$ plane with centre on the $x$-axis and radius equal to $|\overrightarrow{OP}| \sin 60°$

**31.** If $\vec{x}$ and $\vec{y}$ are two non-collinear vectors and $ABC$ is a triangle with side lengths $a$, $b$ and $c$ satisfying $(20a-15b)\vec{x} + (15b-12c)\vec{y} + (12c-20a)(\vec{x} \times \vec{y}) = \vec{0}$, then triangle $ABC$ is

**a.** an acute-angled triangle
**b.** an obtuse-angled triangle
**c.** a right-angled triangle
**d.** an isosceles triangle

**32.** A uni-modular tangent vector on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$ at $t = 2$ is

**a.** $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$
**b.** $\frac{1}{3}(\hat{i} - \hat{j} - \hat{k})$
**c.** $\frac{1}{6}(2\hat{i} + \hat{j} + \hat{k})$
**d.** $\frac{2}{3}(\hat{i} + \hat{j} + \hat{k})$

**33.** If $\vec{x}$ and $\vec{y}$ are two non-collinear vectors and $a$, $b$ and $c$ represent the sides of a $\Delta ABC$ satisfying $(a-b)\vec{x} + (b-c)\vec{y} + (c-a)(\vec{x} \times \vec{y}) = 0$, then $\Delta ABC$ is (where $\vec{x} \times \vec{y}$ is perpendicular to the plane of $\vec{x}$ and $\vec{y}$)

**a.** an acute-angled triangle
**b.** an obtuse-angled triangle
**c.** a right-angled triangle
**d.** a scalene triangle

**34.** $\vec{A}$ is a vector with direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Assuming the $y$-$z$ plane as a mirror, the direction cosines of the reflected image of $\vec{A}$ in the $y$-$z$ plane are

**a.** $\cos\alpha, \cos\beta, \cos\gamma$
**b.** $\cos\alpha, -\cos\beta, \cos\gamma$
**c.** $-\cos\alpha, \cos\beta, \cos\gamma$
**d.** $-\cos\alpha, -\cos\beta, -\cos\gamma$

## Multiple Correct Answers Type

*Solutions on page 1.74*

**Each question has four choices *a*, *b*, *c*, and d, out of which *one or more* are correct.**

**1.** The vectors $x\hat{i} + (x+1)\hat{j} + (x+2)\hat{k}$, $(x+3)\hat{i} + (x+4)\hat{j} + (x+5)\hat{k}$ and $(x+6)\hat{i} + (x+7)\hat{j} + (x+8)\hat{k}$ are coplanar if $x$ is equal to

**a.** 1
**b.** –3
**c.** 4
**d.** 0

**2.** The sides of a parallelogram are $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$. The unit vector parallel to one of the diagonals is

**a.** $\frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$
**b.** $\frac{1}{7}(3\hat{i} - 6\hat{j} - 2\hat{k})$
**c.** $\frac{1}{\sqrt{69}}(\hat{i} + 2\hat{j} + 8\hat{k})$
**d.** $\frac{1}{\sqrt{69}}(-\hat{i} - 2\hat{j} + 8\hat{k})$

3. A vector $\vec{a}$ has the components $2p$ and 1 w.r.t. a rectangular Cartesian system. This system is rotated through a certain angle about the origin in the counterclockwise sense. If, with respect to a new system, $\vec{a}$ has components $(p + 1)$ and 1, then $p$ is equal to

**a.** $-1$ **b.** $-1/3$ **c.** 1 **d.** 2

4. If points $\hat{i} + \hat{j}, \hat{i} - \hat{j}$ and $p\hat{i} + q\hat{j} + r\hat{k}$ are collinear, then

**a.** $p = 1$ **b.** $r = 0$ **c.** $q \in R$ **d.** $q \neq 1$

5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors and $\lambda$ is a real number, then the vectors $\vec{a} + 2\vec{b} + 3\vec{c}, \lambda\vec{b} + \mu\vec{c}$ and $(2\lambda - 1)\vec{c}$ are coplanar when

**a.** $\mu \in R$ **b.** $\lambda = \frac{1}{2}$ **c.** $\lambda = 0$ **d.** no value of $\lambda$

6. If the resultant of three forces $\vec{F_1} = p\hat{i} + 3\hat{j} - \hat{k}, \vec{F_2} = 6\hat{i} - \hat{k}$ and $\vec{F_3} = -5\hat{i} + \hat{j} + 2\hat{k}$ acting on a particle has a magnitude equal to 5 units, then the value of $p$ is

**a.** $-6$ **b.** $-4$ **c.** 2 **d.** 4

7. If the vectors $\hat{i} - \hat{j}, \hat{j} + \hat{k}$ and $\vec{a}$ form a triangle, then $\vec{a}$ may be

**a.** $-\hat{i} - \hat{k}$ **b.** $\hat{i} - 2\hat{j} - \hat{k}$ **c.** $2\hat{i} + \hat{j} + \hat{k}$ **d.** $\hat{i} + \hat{k}$

8. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle $\theta$ and doubled in magnitude. It now becomes $4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}$. The values of $x$ are

**a.** 1 **b.** $-2/3$ **c.** 2 **d.** 4/3

9. $\vec{a}, \vec{b}$ and $\vec{c}$ are three coplanar unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$. If three vectors $\vec{p}, \vec{q}$ and $\vec{r}$ are parallel to $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, and have integral but different magnitudes, then among the following options, $|\vec{p} + \vec{q} + \vec{r}|$ can take a value equal to

**a.** 1 **b.** 0 **c.** $\sqrt{3}$ **d.** 2

10. If non-zero vectors $\vec{a}$ and $\vec{b}$ are equally inclined to coplanar vector $\vec{c}$, then $\vec{c}$ can be

**a.** $\frac{|\vec{a}|}{|\vec{a}| + 2|\vec{b}|}\vec{a} + \frac{|\vec{b}|}{|\vec{a}| + |\vec{b}|}\vec{b}$ **b.** $\frac{|\vec{b}|}{|\vec{a}| + |\vec{b}|}\vec{a} + \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}\vec{b}$

**c.** $\frac{|\vec{a}|}{|\vec{a}| + 2|\vec{b}|}\vec{a} + \frac{|\vec{b}|}{|\vec{a}| + 2|\vec{b}|}\vec{b}$ **d.** $\frac{|\vec{b}|}{2|\vec{a}| + |\vec{b}|}\vec{a} + \frac{|\vec{a}|}{2|\vec{a}| + |\vec{b}|}\vec{b}$

11. If $A(-4, 0, 3)$ and $B(14, 2, -5)$, then which one of the following points lie on the bisector of the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ ($O$ is the origin of reference)?

**a.** (2, 2, 4) **b.** (2, 11, 5) **c.** (−3, −3, −6) **d.** (1, 1, 2)

12. In a four-dimensional space where unit vectors along the axes are $\hat{i}, \hat{j}, \hat{k}$ and $\hat{l}$, and $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda - 1)\ (\vec{a}_1 - \vec{a}_2) + \mu (\vec{a}_2 + \vec{a}_3) + \gamma (\vec{a}_3 + \vec{a}_4 - 2\vec{a}_2) + \vec{a}_3 + \delta \vec{a}_4 = \vec{0}$, then

   **a.** $\lambda = 1$ **b.** $\mu = -2/3$ **c.** $\gamma = 2/3$ **d.** $\delta = 1/3$

13. Let $ABC$ be a triangle, the position vectors of whose vertices are $7\hat{j} + 10\hat{k}$, $-\hat{i} + 6\hat{j} + 6\hat{k}$ and $-4\hat{i} + 9\hat{j} + 6\hat{k}$. Then $\Delta ABC$ is

   **a.** isosceles **b.** equilateral **c.** right angled **d.** none of these

## Reasoning Type

*Solutions on page 1.77*

**Each question has four choices *a, b, c,* and *d*, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.**

**a.** Both the statements are true, and Statement 2 is the correct explanation for Statement 1.
**b.** Both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
**c.** Statement 1 is true and Statement 2 is false.
**d.** Statment 1 is false and Statement 2 is true.

1. A vector has components $p$ and 1 with respect to a rectangular Cartesian system. The axes are rotated through an angle $\alpha$ about the origin in the anticlockwise sense.
   **Statement 1:** If the vector has component $p + 2$ and 1 with respect to the new system, then $p = -1$
   **Statement 2:** Magnitude of the original vector and the new vector remains the same.

2. **Statement 1:** If three points $P$, $Q$ and $R$ have position vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, and $2\vec{a} + 3\vec{b} - 5\vec{c} = 0$, then the points $P$, $Q$ and $R$ must be collinear.
   **Statement 2:** If for three points $A$, $B$ and $C$; $\overrightarrow{AB} = \lambda\ \overrightarrow{AC}$, then points $A$, $B$ and $C$ must be collinear.

3. **Statement 1:** If $\vec{u}$ and $\vec{v}$ are unit vectors inclined at an angle $\alpha$ and $\vec{x}$ is a unit vector bisecting the angle between them, then $\vec{x} = (\vec{u} + \vec{v})/(2\sin(\alpha/2))$.
   **Statement 2:** If $\Delta ABC$ is an isosceles triangle with $AB = AC = 1$, then the vector representing the bisector of angle $A$ is given by $\overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{AC})/2$.

4. **Statement 1:** If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of any line segment, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
   **Statement 2:** If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of a line segment, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.

5. **Statement 1:** The direction cosines of one of the angular bisectors of two intersecting lines having direction cosines as $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$ are proportional to $l_1 + l_2$, $m_1 + m_2$, $n_1 + n_2$.
   **Statement 2:** The angle between the two intersecting lines having direction cosines as $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$ is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

6. **Statement 1:** In $\Delta ABC$, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$

**Statement 2:** If $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$, then $\overrightarrow{AB}=\vec{a}+\vec{b}$

7. **Statement 1:** $\vec{a}=3\vec{i}+p\vec{j}+3\vec{k}$ and $\vec{b}=2\vec{i}+3\vec{j}+q\vec{k}$ are parallel vectors if $p=9/2$ and $q=2$.

**Statement 2:** If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ are parallel, $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$.

8. **Statement 1:** If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, then $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

**Statement 2 :** If the diagonals of a parallelogram are equal in magnitude, then the parallelogram is a rectangle.

9. **Statement 1:** Let $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ be three points such that $\vec{a}=2\hat{i}+\hat{k}$, $\vec{b}=3\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=-\hat{i}+7\hat{j}-5\hat{k}$. Then $OABC$ is a tetrahedron.

**Statement 2:** Let $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ be three points such that vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar. Then $OABC$ is a tetrahedron, where $O$ is the origin.

10. **Statement 1:** Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of four points $A$, $B$, $C$ and $D$ and $3\vec{a}-2\vec{b}+5\vec{c}-6\vec{d}=\vec{0}$. Then points $A$, $B$, $C$ and $D$ are coplanar.

**Statement 2:** Three non-zero, linearly dependent coinitial vectors ($\overrightarrow{PQ}, \overrightarrow{PR}$ and $\overrightarrow{PS}$) are coplanar. Then $\overrightarrow{PQ}=\lambda\overrightarrow{PR}+\mu\overrightarrow{PS}$, where $\lambda$ and $\mu$ are scalars.

11. **Statement 1:** If $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{a}+\vec{b}|=5$, then $|\vec{a}-\vec{b}|=5$.

**Statement 2:** The length of the diagonals of a rectangle is the same.

## Linked Comprehension Type

*Solutions on page 1.79*

**Based on each paragraph, some multiple choice questions have to be answered. Each question has four choices *a, b, c,* and *d*, out of which *only one* is correct.**

**For Problems 1–3**

*ABCD* is a parallelogram. *L* is a point on *BC* which divides *BC* in the ratio 1 : 2. *AL* intersects *BD* at *P*. *M* is a point on *DC* which divides *DC* in the ratio 1 : 2 and *AM* intersects *BD* in *Q*.

1. Point *P* divides *AL* in the ratio

**a.** 1 : 2 **b.** 1 : 3 **c.** 3 : 1 **d.** 2 : 1

2. Point *Q* divides *DB* in the ratio

**a.** 1 : 2 **b.** 1 : 3 **c.** 3 : 1 **d.** 2 : 1

3. *PQ* : *DB* is equal to

**a.** 2/3 **b.** 1/3 **c.** 1/2 **d.** 3/4

**For Problems 4–5**

Let *OABCD* be a pentagon in which the sides *OA* and *CB* are parallel and the sides *OD* and *AB* are parallel. Also *OA* : *CB* = 2 : 1 and *OD* : *AB* = 1 : 3.

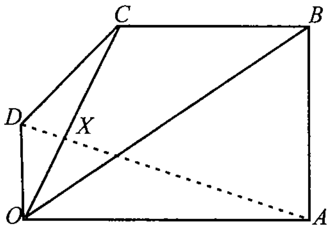



**Fig. 1.40**

**4.** The ratio $\dfrac{OX}{XC}$ is

**a.** 3/4 **b.** 1/3 **c.** 2/5 **d.** 1/2

**5.** The ratio $\dfrac{AX}{XD}$ is

**a.** 5/2 **b.** 6 **c.** 7/3 **d.** 4

**For Problems 6–7**

Consider the regular hexagon *ABCDEF* with centre at *O* (origin).

**6.** $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC}$ is equal to

**a.** $2\overrightarrow{AB}$ **b.** $3\overrightarrow{AB}$ **c.** $4\overrightarrow{AB}$ **d.** none of these

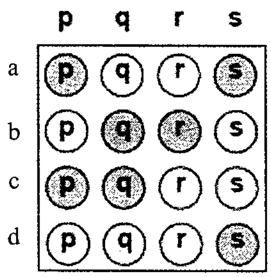
**7.** Five forces $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$ act at the vertex *A* of a regular hexagon *ABCDEF*. Then their resultant is

**a.** $6\overrightarrow{AO}$ **b.** $2\overrightarrow{AO}$ **c.** $4\overrightarrow{AO}$ **d.** $6\overrightarrow{AO}$

## Matrix-Match Type

*Solutions on page 1.82*

**Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:**



1. Refer to the following diagram:

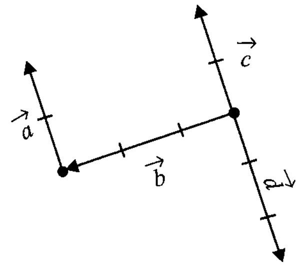



**Fig. 1.41**

| Column I | Column II |
|---|---|
| a. Collinear vectors | p. $\vec{a}$ |
| b. Coinitial vectors | q. $\vec{b}$ |
| c. Equal vectors | r. $\vec{c}$ |
| d. Unlike vectors (same initial point) | s. $\vec{d}$ |

2. $\vec{a}$ and $\vec{b}$ form the consecutive sides of a regular hexagon $ABCDEF$.

| Column I | Column II |
|---|---|
| a. If $\overrightarrow{CD} = x\vec{a} + y\vec{b}$, then | p. $x = -2$ |
| b. If $\overrightarrow{CE} = x\vec{a} + y\vec{b}$, then | q. $x = -1$ |
| c. If $\overrightarrow{AE} = x\vec{a} + y\vec{b}$, then | r. $y = 1$ |
| d. $\overrightarrow{AD} = -x\vec{b}$, then | s. $y = 2$ |

## Integer Answer Type

*Solutions on page 1.83*

1. Let $ABC$ be a triangle whose centroid is $G$, orhtocentre is $H$ and circumcentre is the origin '$O$'. If $D$ is any point in the plane of the triangle such that no three of $O, A, C$ and $D$ are collinear satisfying the relation $\vec{AD} + \vec{BD} + \vec{CH} + 3\vec{HG} = \lambda\,\vec{HD}$, then what is the value of the scalar '$\lambda$'?
2. If the resultant of three forces $\vec{F_1} = p\hat{i} + 3\hat{j} - \hat{k}$, $\vec{F_2} = -5\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{F_3} = 6\hat{i} - \hat{k}$ acting on a particle has a magnitude equal to 5 units, then what is difference in the values of $p$?
3. Let $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} - \vec{c} = 0$. If the area of triangle formed by vectors $\vec{a}$ and $\vec{b}$ is $A$, then what is the value of $4A^2$?

4. Find the least positive integral value of $x$ for which the angle between vectors $\vec{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ is acute.

5. Vectors along the adjacent sides of parallelogram are $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$. Find the length of the longer diagonal of the parallelogram.

6. If vectors $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + 2\hat{k}$ are coplanar, then find the value of $(\lambda - 4)$.

**Archives**

*Solutions on page 1.84*

## Subjective Type

1. Find all values of $\lambda$ such that $x, y, z \neq (0, 0, 0)$ and $(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z = \lambda(x\hat{i}+y\hat{j}+z\hat{k})$, where, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the coordinate axes. **(IIT-JEE, 1998)**

2. A vector has components $A_1$, $A_2$ and $A_3$ in a right-handed rectangular Cartesian coordinate system $OXYZ$. The coordinate system is rotated about the $x$-axis through an angle $\pi/2$. Find the components of $A$ in the new coordinate system in terms of $A_1$, $A_2$ and $A_3$. **(IIT-JEE, 1983)**

3. The position vectors of the point $A$, $B$, $C$ and $D$ are $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$, respectively. If the points $A$, $B$, $C$ and $D$ lie on a plane, find the value of $\lambda$. **(IIT-JEE, 1986)**

4. Let $OACB$ be a parallelogram with $O$ at the origin and $OC$ a diagonal. Let $D$ be the midpoint of $OA$. Using vector methods prove that $BD$ and $CO$ intersect in the same ratio. Determine this ratio. **(IIT-JEE, 1988)**

5. In a triangle $ABC$, $D$ and $E$ are points on $BC$ and $AC$, respectively, such that $BD = 2DC$ and $AE = 3EC$. Let $P$ be the point of intersection of $AD$ and $BE$. Find $BP/PE$ using the vector method. **(IIT-JEE,1993)**

6. Prove, by vector method or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the midpoint of the parallel sides (you may assume that the trapezium is not a parallelogram.) **(IIT-JEE, 1998)**

7. Show, by vector method, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices. **(IIT-JEE, 2001)**

8. Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}$, $t \in [0,1]$, where $f_1$, $f_2$, $g_1$, $g_2$ are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all and $\vec{A}(0) = 2\hat{i}+3\hat{j}$, $\vec{A}(1) = 6\hat{i}+2\hat{j}$, $\vec{B}(0) = 3\hat{i}+2\hat{j}$ and $\vec{B}(1) = 2\hat{i}+6\hat{j}$, then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some $t$. **(IIT-JEE, 2001)**

9. In a triangle $OAB$, $E$ is the midpoint of $BO$ and $D$ is a point on $AB$ such that $AD : DB = 2 : 1$. If $OD$ and $AE$ intersect at $P$, determine the ratio $OP : PD$ using the vector method. **(IIT-JEE, 1989)**

## Objective Type

### *Fill in the blanks*

1. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\vec{A} = (1, a, a^2)$, $\vec{B} = (1, b, b^2)$, $\vec{C} = (1, c, c^2)$ are non-coplanar, then the product $abc =$ ____________. **(IIT-JEE,1985)**

2. If the vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k}$ $(a, b, c \neq 1)$ are coplanar, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=$ ____________. **(IIT-JEE,1987)**

### *True or false*

1. The points with position vectors $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$ and $\vec{a}+k\vec{b}$ are collinear for all real values of $k$. **(IIT-JEE, 1984)**

### *Multiple choice questions with one correct answer*

1. The points with position vectors $60\hat{i}+3\hat{j}$, $40\hat{i}-8\hat{j}$, $a\hat{i}-52\hat{j}$ are collinear if
   **a.** $a=-40$ **b.** $a=40$
   **c.** $a=20$ **d.** none of these **(IIT-JEE, 1983)**

2. Let $a$, $b$ and $c$ be distinct non-negative numbers. If vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar, then $c$ is
   **a.** the arithmetic mean of $a$ and $b$ **b.** the geometric mean of $a$ and $b$
   **c.** the harmonic mean of $a$ and $b$ **d.** equal to zero **(IIT-JEE, 1993)**

3. Let $\vec{a}=\vec{i}-\vec{k}$, $\vec{b}=x\vec{i}+\vec{j}+(1-x)\vec{k}$ and $\vec{c}=y\vec{i}+x\vec{j}+(1+x-y)\vec{k}$. Then $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar for
   **a.** some values of $x$ **b.** some values of $y$
   **c.** no values of $x$ and $y$ **d.** for all values of $x$ and $y$ **(IIT-JEE, 2000)**

4. Let $\alpha$, $\beta$ and $\gamma$ be distinct and real numbers. The points with position vectors $\alpha i+\beta\hat{j}+\gamma\hat{k}$, $\beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}$ and $\gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$
   **a.** are collinear **b.** form an equilateral triangle
   **c.** form a scalene triangle **d.** form a right-angled triangle
   **(IIT-JEE, 1994)**

5. The number of distinct real values of $\lambda$, for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar is
   **a.** zero **b.** one **c.** two **d.** three
   **(IIT-JEE, 2007)**

6. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$, then
   **a.** $a=1, b=-1$ **b.** $a=1, b=\pm 1$ **c.** $\alpha=-1, \beta=\pm 1$ **d.** $\alpha=\pm 1, \beta=1$
   **(IIT-JEE, 1998)**

# ANSWERS AND SOLUTIONS

## Subjective Type

**1.** Since $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$, the incentre is same as the circumcentre, and hence $IA$ is perpendicular to $BC$. Therefore, $\vec{r}$ is parallel to $BC$.

$\vec{r} = \lambda(\hat{i} - \hat{j})$

Hence, unit vector $\vec{r} = \pm \frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

**2.** We take the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ parallel to the east, north and vertically upwards in the direction of pole, respectively. Then the velocity vectors of the current, ship and the sailor are, respectively, $\hat{i}, 1.25\hat{j}$ and $0.5\hat{k}$. Velocity $\vec{v}$ of the sailor in space is the resultant of these vectors.

Hence $\vec{v} = \hat{i} + 1.25\hat{j} + 0.5\hat{k}$

Then $|\vec{v}| = \sqrt{1 + (1.25)^2 + (0\cdot 5)^2}$

$= \sqrt{1 + 1.5625 + .25}$

$= \sqrt{2.8125} = 1.677$ m/s

**3.** (i) Put $n = 2$ in $\overrightarrow{OP}_{n-1} + \overrightarrow{OP}_{n+1} = \frac{3}{2}\overrightarrow{OP}_n$

$$\overrightarrow{OP}_3 = \frac{3}{2}\overrightarrow{OP}_2 - \overrightarrow{OP}_1 \qquad \text{(i)}$$

$$\left.\begin{aligned} \overrightarrow{OP}_1 &= a\hat{i} + \frac{1}{a}\hat{j} \\ \overrightarrow{OP}_2 &= b\hat{i} + \frac{1}{b}\hat{j} \end{aligned}\right\} ab \neq 0$$

$\therefore\ \overrightarrow{OP}_3 = \frac{3}{2}\left(b\hat{i} + \frac{1}{b}\hat{j}\right) - \left(a\hat{i} + \frac{1}{a}\hat{j}\right) = \left(\frac{3b}{2} - a\right)\hat{i} + \left(\frac{3}{2b} - \frac{1}{a}\right)\hat{j}$

If $P_3$ lies on $xy = 1$

$\left(\frac{3b}{2} - a\right)\left(\frac{3}{2b} - \frac{1}{a}\right) = 1$

$\Rightarrow (3b - 2a)(3a - 2b) = 4ab$

$\Rightarrow 9ab - 6b^2 - 6a^2 + 4ab = 4ab$

$\Rightarrow 2a^2 - 3ab + 2b^2 = 0$

which is not possible as Discriminant < 0 ($a = 0$ and $b = 0$ not possible)

(ii) $\overrightarrow{OP} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ and $\overrightarrow{OP}_2 = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$

$\therefore\ \overrightarrow{OP}_3 = \frac{3}{2}(\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) - (\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

$= \left(\frac{3}{2}\cos\beta - \cos\alpha\right)\hat{i} + \left(\frac{3}{2}\sin\beta - \sin\alpha\right)\hat{j}$

Since $P_3$ lies on $x^2 + y^2 = 1$

$\Rightarrow \left(\frac{3}{2}\cos\beta - \cos\alpha\right)^2 + \left(\frac{3}{2}\sin\beta - \sin\alpha\right)^2 = 1$

$\Rightarrow \frac{9}{4} + 1 - 3\,(\cos\beta\cos\alpha + \sin\beta\sin\alpha) = 1$

$\Rightarrow \frac{9}{4} - 3\cos(\beta - \alpha) = 0 \Rightarrow \cos(\beta - \alpha) = \frac{3}{4}$ (ii)

Put $n = 3$ in the given relation.

$\overrightarrow{OP_2} + \overrightarrow{OP_4} = \frac{3}{2}\overrightarrow{OP_3}$; $\overrightarrow{OP_4} = \frac{3}{2}\overrightarrow{OP_3} - \overrightarrow{OP_2}$

$\Rightarrow \overrightarrow{OP_4} = \frac{3}{2}\left(\frac{3}{2}\overrightarrow{OP_2} - \overrightarrow{OP_1}\right) - \overrightarrow{OP_2} = \frac{5}{4}\overrightarrow{OP_2} - \frac{3}{2}\overrightarrow{OP_1}$

$\Rightarrow \overrightarrow{OP_4} = \frac{5}{4}(\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) - \frac{3}{2}(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$, which lies on $x^2 + y^2 = 1$

**4.**

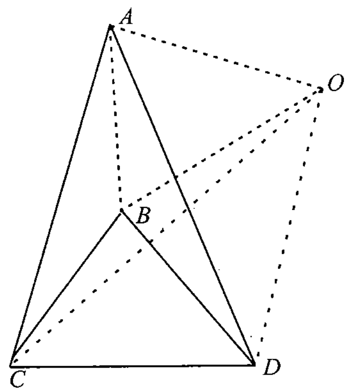



**Fig. 1.42**

Here $ABCD$ is a tetrahedron. Let $O$ be the origin and the P.V. of $A, B, C$ and $D$ be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively. We know that four linearly dependent vectors can be expressed as

$x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ (where $x, y, z$ and $t$ are scalars)

or $y\vec{b} + z\vec{c} + t\vec{d} = -x\vec{a}$

$\Rightarrow \frac{y\vec{b} + z\vec{c} + t\vec{d}}{y+z+t} = -\frac{x\vec{a}}{y+z+t}$

where L.H.S. is P.V. of a point in the plane $BCD$ and R.H.S. is a point on $\overrightarrow{AO}$
Therefore, there must be a point common to both the plane and the straight line. That is

$$\overrightarrow{OP} = \frac{-x\vec{a}}{y+z+t}$$

But, $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = -\frac{x\vec{a}}{y+z+t} - \vec{a} = -\left(\frac{x+y+z+t}{y+z+t}\right)\vec{a}$

$$\overrightarrow{OP} = \frac{x}{y+z+t}\left(\frac{y+z+t}{x+y+z+t}\right)\overrightarrow{AP}$$

$$\overrightarrow{OP} = \left(\frac{x}{x+y+z+t}\right)\overrightarrow{AP}$$

$$\Rightarrow \frac{OP}{AP} = \frac{x}{x+y+z+t}$$

Similarly, $\frac{OQ}{BQ} = \frac{y}{x+y+z+t}$

$$\frac{OR}{CR} = \frac{z}{x+y+z+t} \text{ and } \frac{OS}{DS} = \frac{t}{x+y+z+t}$$

$$\Rightarrow \frac{OP}{AP} + \frac{OQ}{BQ} + \frac{OR}{CR} + \frac{OS}{DS} = 1$$

**5.** Let the centre of the base be (0). Therefore,

$|\overrightarrow{AB}| = 2$

$$\Delta OAB = \frac{1}{4} \times 4 \times \sqrt{3} = \sqrt{3}$$

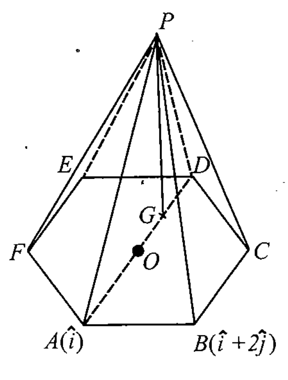



**Fig. 1.43**

Base area = $6\sqrt{3}$ sq. units

Let height of the pyramid be $h$. Therefore,

$$\frac{1}{3}\times 6\sqrt{3}h = 6\sqrt{3} \Rightarrow h = 3 \text{ units}$$

It is given that $|\overrightarrow{AP}| = 5$. Therefore,

$AG = \sqrt{25-9} = 4$ units

$\Rightarrow |\overrightarrow{AG}| = 4$ units

Now $|\overrightarrow{AG}|$ and $|\overrightarrow{AO}|$ are collinear. Therefore,

$$\overrightarrow{AG} = \lambda \overrightarrow{AO} \Rightarrow |\overrightarrow{AG}| = |\lambda||\overrightarrow{AO}| \Rightarrow 2|\lambda| = 4 \Rightarrow |\lambda| = 2$$

$$\Rightarrow \overrightarrow{AG} = \pm 2(\hat{i}+\hat{j}+\sqrt{3}\hat{k}) \Rightarrow \overrightarrow{OG} = \pm 2(\hat{i}+\hat{j}+\sqrt{3}\hat{k})+\hat{i}$$

$$\overrightarrow{OG} = -(\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}),\ 3\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}$$

6. Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$; then $\overrightarrow{AC} = \vec{a}+\vec{b}$.

Given $\overrightarrow{AB_1} = \lambda_1 \vec{a}$, $\overrightarrow{AD_1} = \lambda_2 \vec{b}$, $\overrightarrow{AC_1} = \lambda_3(\vec{a}+\vec{b})$

$$\overrightarrow{B_1D_1} = \overrightarrow{AD_1} - \overrightarrow{AB_1} = \lambda_2 \vec{b} - \lambda_1 \vec{a}$$

Since vectors $\overrightarrow{D_1C_1}$ and $\overrightarrow{B_1D_1}$ are collinear, we have

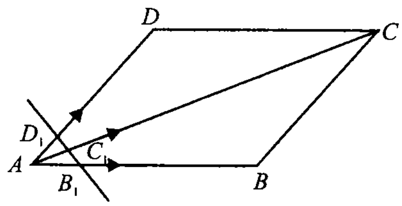



**Fig. 1.44**

$$\overrightarrow{D_1C_1} = k\overrightarrow{B_1D_1} \text{ for some } k \in R$$

$$\Rightarrow \overrightarrow{AC_1} - \overrightarrow{AD_1} = k\overrightarrow{B_1D_1}$$

$$\Rightarrow \lambda_3(\vec{a}+\vec{b}) - \lambda_2 \vec{b} = k(\lambda_2 \vec{b} - \lambda_1 \vec{a})$$

$$\Rightarrow \lambda_3 \vec{a} + (\lambda_3 - \lambda_2)\vec{b} = k\lambda_2 \vec{b} - k\lambda_1 \vec{a}$$

Hence, $\lambda_3 = -k\lambda_1$ and $\lambda_3 - \lambda_2 = k\lambda_2$

$$\Rightarrow k = -\frac{\lambda_3}{\lambda_1} = \frac{\lambda_3 - \lambda_2}{\lambda_2}$$

$$\Rightarrow \lambda_1\lambda_2 = \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$\Rightarrow \frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

**7.** Let vector $2\hat{i}+7\hat{j}-5\hat{k}$ intersect vectors $\vec{A}$ and $\vec{B}$ at points $L$ and $M$, respectively, which have to be determined. Take them to be $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, respectively.

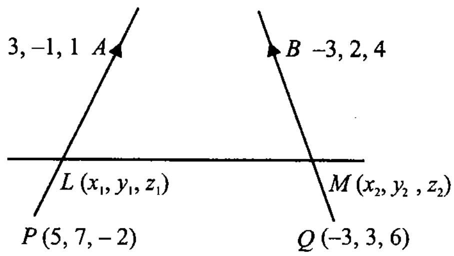



**Fig. 1.45**

$PL$ is collinear with vector $\vec{A}$. Therefore,

$\therefore\ \overrightarrow{PL} = \lambda\vec{A}$

Comparing the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get $\dfrac{x_1-5}{3} = \dfrac{y_1-7}{-1} = \dfrac{z_1+2}{1} = \lambda$ (say)

$L$ is $3\lambda+5, -\lambda+7, \lambda-2$

Similarly, $\overrightarrow{QM} = \mu\vec{B}$. Therefore,

$$\frac{x_3+3}{-3} = \frac{y_2-3}{2} = \frac{z_2-6}{4} = \mu \text{ (say)}$$

$\therefore\ M$ is $-3\mu-3, 2\mu+3, 4\mu+6$

Again $LM$ is collinear with vector $2\hat{i}+7\hat{j}-5\hat{k}$. Therefore,

$$\frac{x_2-x_1}{2} = \frac{y_2-y_1}{7} = \frac{z_2-z_1}{-5} = v \text{ (say)}$$

$$\frac{-3\mu-3\lambda-8}{2} = \frac{2\mu+\lambda-4}{7} = \frac{4\mu-\lambda+8}{-5} = v$$

$$3\mu+3\lambda+2v = -8$$
$$2\mu+\lambda-7v = 4$$
$$4\mu-\lambda+5v = -8$$

Solving, we get

$\lambda = \mu = v = -1$

Therefore, point $L$ is $(2, 8, -3)$ or $2\hat{i}+8\hat{j}-3\hat{k}$

and $M$ is $(0, 1, 2)$ or $\hat{j}+2\hat{k}$

**8.** If the given vectors are coplanar, then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

or the set of equations

$x_1x + y_1y + z_1z = 0$,

$x_2x + y_2y + z_2z = 0$ and

$x_3x + y_3y + z_3z = 0$ has a non-trivial solution.

Let the given set has a non-trivial solution $x$, $y$, $z$ without the loss of generality; we can assume that $x \geq y \geq z$.

For the given equation $x_1x + y_1y + z_1z = 0$, we have $x_1x = -y_1y - z_1z$. Therefore,

$|x_1x| = |y_1y + z_1z| \leq |y_1y| + |z_1z| \Rightarrow |x_1x| \leq |y_1x| + |z_1x| \Rightarrow |x_1| \leq |y_1| + |z_1|$,

which is a contradiction to the given inequality, i.e., $|x_1| > |y_1| + |z_1|$.

Similarly, the other inequalities rule out the possibility of a non-trivial solution.

Hence the given equation has only a trivial solution. Hence the given vectors are non-coplanar.

**9.** We know: $(1+k)\,|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2 = |\vec{A}|^2 + k|\vec{A}|^2 + |\vec{B}|^2 + \frac{1}{k}|\vec{B}|^2$ (i)

Also,

$$k|\vec{A}|^2 + \frac{1}{k}|\vec{B}|^2 \geq 2\left(k \cdot |\vec{A}|^2 \cdot \frac{1}{k}|\vec{B}|^2\right)^{1/2} = 2|\vec{A}|\cdot|\vec{B}| \qquad \text{(ii)}$$

(Since arithmetic mean ≥ geometric mean)

$$\therefore\ (1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2 \geq |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}|\cdot|\vec{B}| = (|\vec{A}| + |\vec{B}|)^2 \qquad \text{(Using (i) and (ii))}$$

And also $|\vec{A}| + |\vec{B}| \geq |\vec{A} + \vec{B}|$

Hence, $(1+k)\,|\vec{A}|^2 + \left(1+\frac{1}{k}\right)^2 |\vec{B}|^2 \geq |\vec{A}+\vec{B}|^2$

**10.** Since the vectors are coplanar

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & a \end{vmatrix} = 0$$

$$\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & a-1 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix} = 0$$

$\Rightarrow a = 1$

**11.**

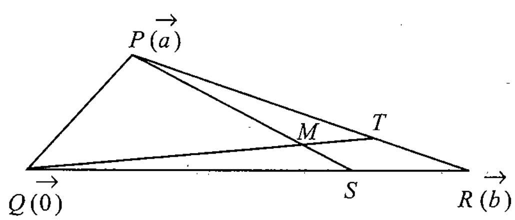



**Fig. 1.46**

Let $QM : MT = \lambda : 1$ and $PM : MS = \mu : 1$

and $\overrightarrow{QP} = \vec{a},\ \overrightarrow{QR} = \vec{b}$

$$\Rightarrow \overrightarrow{QT} = \frac{4\vec{b} + \vec{a}}{5}$$

$$\text{and } \overrightarrow{QM} = \frac{\lambda}{\lambda+1}\left(\frac{4\vec{b} + \vec{a}}{5}\right) \quad \text{(i)}$$

$$\overrightarrow{QS} = \frac{3}{4}\vec{b},\ \overrightarrow{QM} = \frac{\mu\left(\frac{3}{4}\vec{b}\right) + \vec{a}}{\mu+1} \quad \text{(ii)}$$

From (i) and (ii), $\dfrac{1}{\mu+1} = \dfrac{\lambda}{5(\lambda+1)}$ and $\dfrac{4\lambda}{5(\lambda+1)} = \dfrac{3\mu}{4(\mu+1)}$

$\Rightarrow \lambda = 15/4$ and $\mu = 16/3$

$\therefore QM : MT = 15 : 4$

**12.** Let the flow velocity of river be $u$ and the velocity of boat in still water be $v$.

Thus, $v = \dfrac{u}{K}$

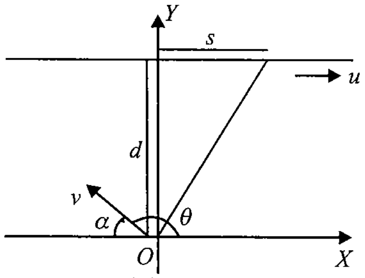



**Fig. 1.47**

Also, let the boat move at an angle $\theta$ with the stream direction.

Now the velocity of boat in the river is the vector resultant of the velocity of boat and flow velocity of river, which can be written as

$$\vec{v}_B = (u - v\cos\alpha)\hat{i} + (u\sin\alpha)\hat{j} = (u + v\cos\theta)\hat{i} + (u\sin\theta)\hat{j}$$

Hence, the time taken to cross the river = $\dfrac{d}{u\sin\theta}$ ($d$ = width of the river)

Thus, the drift $s = (u + v\cos\theta)\cdot d$

$$\Rightarrow s = d\left(\operatorname{cosec}\theta + \frac{v}{u}\cot\theta\right)$$

$$\Rightarrow \frac{ds}{d\theta} = d\left(\operatorname{cosec}\theta\cot\theta - \frac{v}{\mu}\operatorname{cosec}^2\theta\right) = 0$$

$$\Rightarrow \frac{v}{u}\operatorname{cosec}^2\theta = \operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \cos\theta = \frac{1}{k} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{k}\right)$$

**13.** $x\vec{a}+y\vec{b}+z\vec{c}+t\vec{h}=0$ such that

$x+y+z+t=0$ (i)

$x\vec{a}+y\vec{b}=-(z\vec{c}+t\vec{h})$

and $x+y=-(z+t)$

$\therefore \dfrac{x\vec{a}+y\vec{b}}{x+y}=\dfrac{z\vec{c}+t\vec{h}}{z+t}$ (ii)

Position vector of $F=\dfrac{x\vec{a}+y\vec{b}}{x+y}$

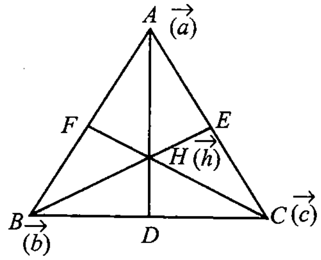



**Fig. 1.48**

Hence, $F$ divides $AB$ in the ratio $y/x$.

$\dfrac{AF}{FB}=\dfrac{y}{x}$

Similarly, $\dfrac{BD}{CD}=\dfrac{z}{y}$ and $\dfrac{CE}{AE}=\dfrac{x}{z}$

$\Rightarrow \dfrac{AF}{FB}\cdot\dfrac{BD}{CD}\cdot\dfrac{CE}{AE}=-1$

**14.** $\overrightarrow{OM}=\dfrac{\vec{b}}{2} \Rightarrow \overrightarrow{PM}=\vec{a}+\dfrac{\vec{b}}{2}$

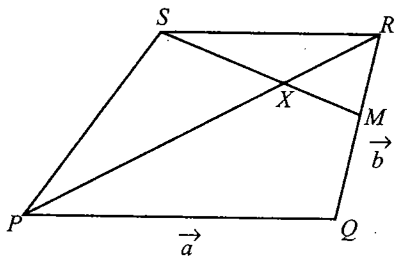



**Fig. 1.49**

$\overrightarrow{SM}=\overrightarrow{PM}-\overrightarrow{PS}=2\vec{a}-\dfrac{1}{2}\vec{b}$

$\overrightarrow{SX}=\dfrac{4}{5}\overrightarrow{SM}=\dfrac{8}{5}\vec{a}-\dfrac{2}{5}\vec{b}$

$\overrightarrow{PX} = \overrightarrow{PS} + \overrightarrow{SX}$

$= -\vec{a} + \vec{b} + \frac{8}{5}\vec{a} - \frac{2}{5}\vec{b} = \frac{3}{5}(\vec{a} + \vec{b})$

Also $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \vec{a} + \vec{b} = \frac{5}{3}\ \overrightarrow{PX}$

Hence $P$, $X$ and $R$ are collinear.

## Objective Type

1. **a.** Four or more than four non-zero vectors are always linearly dependent.
2. **d.** $3\vec{a} + 4\vec{b} + 5\vec{c} = 0$

   $\Rightarrow$ $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

   No other conclusion can be derived from it.
3. **c.** $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 4\hat{i} + 2\hat{j} - 4\hat{k}$

   $\overrightarrow{AB} = -3\hat{i} - 3\hat{k}, \overrightarrow{AC} = \hat{i} + 2\hat{j} - 7\hat{k}$

   $BC^2 = 36, AB^2 = 18, AC^2 = 54$

   Clearly, $AC^2 = BC^2 + AB^2$

   $\therefore \angle B = 90°$
4. **c.** $|\vec{a} + \vec{b}| < |\vec{a} - \vec{b}|$

   $\Rightarrow \frac{\pi}{2} < \theta < \frac{3\pi}{2}$
5. **c.** The position vector of the point $O$ with respect to itself is

   $$\frac{\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

   $$\Rightarrow \frac{\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C}{\sin 2A + \sin 2B + \sin 2C} = \vec{0}$$

   $$\Rightarrow \overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C = \vec{0}$$
6. **a.** We have $\overrightarrow{GB} + \overrightarrow{GC} = (1+1)\overrightarrow{GD} = 2\overrightarrow{GD}$, where $D$ is the midpoint of $BC$.

   $\therefore \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{GA} + 2\overrightarrow{GD} = \overrightarrow{GA} - \overrightarrow{GA} = 0$

   ($\because$ $G$ divides $AC$ in the ratio 2 : 1, $\therefore 2\overrightarrow{GD} = -\overrightarrow{GA}$)
7. **c.** $m\vec{a}$ is a unit vector if and only if

   $|m\vec{a}| = 1 \Rightarrow |m|\ |\vec{a}| = 1 \Rightarrow |m|\ a = 1 \Rightarrow a = \frac{1}{|m|}$

**8. c.**

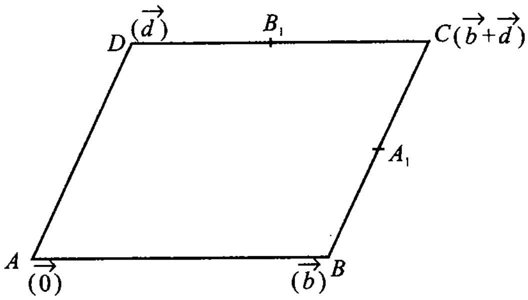



**Fig. 1.50**

Let P.V. of $A$, $B$ and $D$ be $\vec{0}$, $\vec{b}$ and $\vec{d}$, respectively.

Then P.V. of $C$, $\vec{c} = \vec{b} + \vec{d}$

Also P.V. of $A_1 = \vec{b} + \dfrac{\vec{d}}{2}$

and P.V. of $B_1 = \vec{d} + \dfrac{\vec{b}}{2}$

$\Rightarrow \overrightarrow{AA_1} + \overrightarrow{AB_1} = \dfrac{3}{2}(\vec{b} + \vec{d}) = \dfrac{3}{2}\overrightarrow{AC}$

**9. b.** Since $|\overrightarrow{OP}| = |\overrightarrow{OQ}| \neq \sqrt{14}$, $\Delta OPQ$ is isosceles.

Hence the internal bisector $OM$ is perpendicular to $PQ$ and $M$ is the midpoint of $P$ and $Q$.

$\therefore \overrightarrow{OM} = \dfrac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}) = 2\hat{i} + \hat{j} - 2\hat{k}$

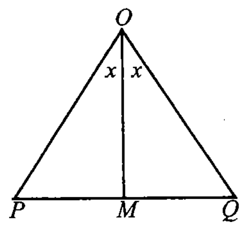



**Fig. 1.51**

**10. d.**

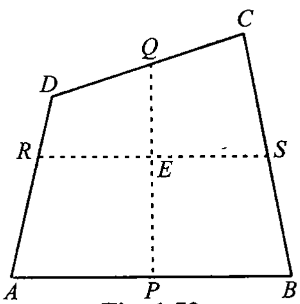



**Fig. 1.52**

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OD} = \vec{d}$. Therefore,

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$$

$P$, the midpoint of $AB$, is $\dfrac{\vec{a}+\vec{b}}{2}$

$Q$, the midpoint of $\overrightarrow{CD}$, is $\dfrac{\vec{c}+\vec{d}}{2}$

Therefore, the midpoint of $\overrightarrow{PQ}$ is $\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$.

Similarly the midpoint of $RS$ is $\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$, i.e., $\overrightarrow{OE} = \dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4} \Rightarrow x = 4$.

**11. b.**

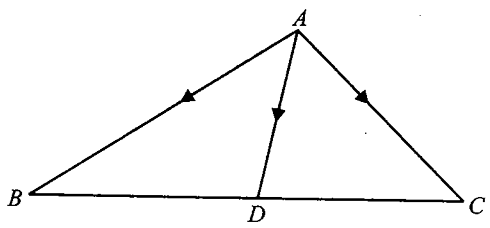



**Fig. 1.53**

$$\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$$

$$\therefore \overrightarrow{AD} = \frac{1}{2}\{(-3\hat{i}+4\hat{k}) + (5\hat{i}-2\hat{j}+4\hat{k})\}$$

$$= \hat{i} - \hat{j} + 4\hat{k}$$

Length of $AD = \sqrt{1+1+16} = \sqrt{18}$

**12. d.** $\vec{a} - \vec{b} = 2(\vec{d} - \vec{c})$

$$\therefore \frac{\vec{a}+2\vec{c}}{2+1} = \frac{\vec{b}+2\vec{d}}{2+1}$$

$\Rightarrow$ *AC* and *BD* trisect each other as L.H.S. is the position vector of a point trisecting *A* an *C*, and R.H.S. that of *B* and *D*.

**13. b.** Vector in the direction of angular bisector of $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}+\vec{b}}{2}$

Unit vector in this direction is $\dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$

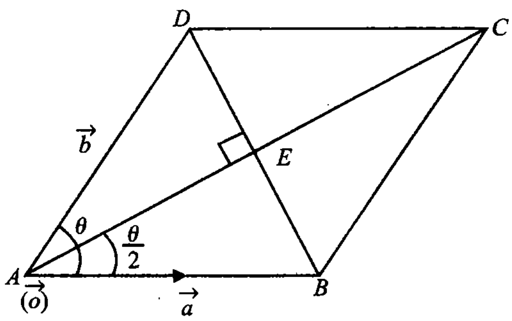



**Fig. 1.54**

From the figure, position vector of $E$ is $\dfrac{\vec{a}+\vec{b}}{2}$

Now in triangle $AEB$, $AE = AB\cos\dfrac{\theta}{2}$

$$\Rightarrow \quad \left|\frac{\vec{a}+\vec{b}}{2}\right| = \cos\frac{\theta}{2}$$

Hence unit vector along the bisector is $\dfrac{\vec{a}+\vec{b}}{2\cos\frac{\theta}{2}}$

**14.** **b.** $|a|+|b|+|c| = \sqrt{a^2+b^2+c^2} \Leftrightarrow 2\,|ab|+2\,|bc|+2\,|ca|=0$

$\Leftrightarrow ab = bc = ca = 0 \Leftrightarrow$ any two of $a$, $b$ and $c$ are zero

**15.** **c.** $\vec{\alpha} = \vec{a}+\vec{b}+\vec{c} = 6\hat{i}+12\hat{j}$

Let $\vec{\alpha} = x\vec{a}+y\vec{b} \Rightarrow 6x+2y=6$

and $-3x-6y=12$

$\therefore x=2, y=-3$

$\therefore \vec{\alpha} = 2\vec{a}-3\vec{b}$

**16.** **c.** Given $\vec{\alpha}+\vec{\beta}+\vec{\gamma} = a\vec{\delta}$ (i)

$\vec{\beta}+\vec{\gamma}+\vec{\delta} = b\vec{\alpha}$ (ii)

From (i), $\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta} = (a+1)\,\vec{\delta}$ (iii)

From (ii), $\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta} = (b+1)\,\vec{\alpha}$ (iv)

From (iii) and (iv),

$(a+1)\,\vec{\delta} = (b+1)\,\vec{\alpha}$ (v)

Since $\vec{\alpha}$ is not parallel to $\vec{\delta}$,
From (v), $a + 1 = 0$ and $b + 1 = 0$

From (iii), $\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = 0$

**17.** **a.** Let the origin of reference be the circumcentre of the triangle.

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OT} = \vec{t}$

Then $|\vec{a}| = |\vec{b}| = |\vec{c}| = R$ (circumradius)

Again $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OH}$

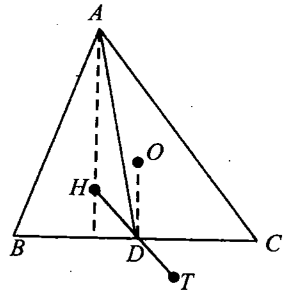



**Fig. 1.55**

Therefore, the P.V. of $H$ is $\vec{a} + \vec{b} + \vec{c}$. Since $D$ is the midpoint of $HT$, we have

$$\frac{\vec{a} + \vec{b} + \vec{c} + \vec{t}}{2} = \frac{\vec{b} + \vec{c}}{2} \Rightarrow \vec{t} = -\vec{a}$$

$\therefore \overrightarrow{AT} = -2\vec{a} \Rightarrow \overrightarrow{AT} = |-2\vec{a}| = 2|\vec{a}| = 2R$. But $BC = 2R \sin A = R$, therefore $AT = 2BC$

**18.** **c.** Given $a_1 \vec{r}_1 + a_2 \vec{r}_2 + \cdots + a_n \vec{r}_n = 0$

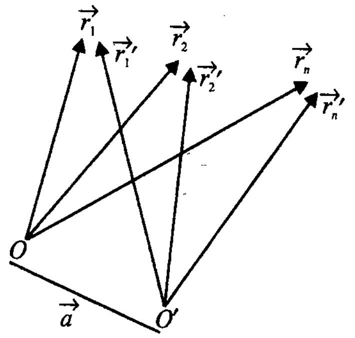



**Fig. 1.56**

Now $\vec{a}+\vec{r_1'}=\vec{r_1}$ and so on

Hence $a_1\,(\vec{a}+\vec{r_1'})+a_2\,(\vec{a}+\vec{r_2'})+\cdots+a_n\,(\vec{a}+\vec{r_n'})=0$

$a_1\vec{r_1'}+a_2\vec{r_2'}+\cdots+a_n\vec{r_n'}+\vec{a}(a_1+a_2+\cdots+a_n)=0$

Hence $a_1\vec{r_1'}+a_2\vec{r_2'}+\cdots+a_n\vec{r_n'}=0$ if $a_1+a_2+\cdots+a_n=0.$

**19. d.** $\vec{r_1}+2\vec{r_2}=(p\vec{a}+q\vec{b}+\vec{c})+2(\vec{a}+p\vec{b}+q\vec{c})=(p+2)\vec{a}+(q+2p)\vec{b}+(1+2q)\vec{c}$

$2\vec{r_1}+\vec{r_2}=(2p+1)\,\vec{a}+(2q+p)\,\vec{b}+(2+q)\,\vec{c}$

$$\frac{p+2}{2p+1}=\frac{q+2p}{2q+p}=\frac{1+2q}{2+q}=\frac{p+q+2p+2q+3}{p+q+2p+2q+3}=1$$

$\Rightarrow p=1$ and $q=1$

**20. d.** $\sqrt{3}\tan\theta+1=0$ and $\sqrt{3}\sec\theta-2=0$

$\Rightarrow \theta=\dfrac{11\pi}{6}$

$\Rightarrow \theta=2n\pi+\dfrac{11\pi}{6},\ n\in Z$

**21. d.** $\vec{c}-\vec{b}=\alpha\vec{d}$ and $\vec{P}=\overrightarrow{AC}+\overrightarrow{BD}=\mu\,\overrightarrow{AD}$

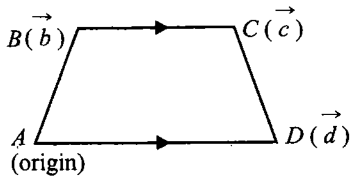



**Fig. 1.57**

Hence $\vec{p}=\vec{c}+\vec{d}-\vec{b}=\mu\,\vec{d}$ (using $\vec{c}-\vec{b}=\alpha\vec{d}$)

or $\alpha+1=\mu$

**22. b.** Note that $\vec{a}+\vec{b}=\vec{c}$

**23. a.** $\hat{a}=\dfrac{-4\hat{i}+3\hat{k}}{5}$; $\hat{b}=\dfrac{14\hat{i}+2\hat{j}-5\hat{k}}{15}$

A vector $\vec{V}$ bisecting the angle between $\vec{a}$ and $\vec{b}$ is $\vec{V}=\hat{a}+\hat{b}$

$$=\frac{-12\hat{i}+9\hat{k}+14\hat{i}+2\hat{j}-5\hat{k}}{15}$$

$$=\frac{2\hat{i}+2\hat{j}+4\hat{k}}{15}$$

Required vector $\vec{d} = \sqrt{6}\,\hat{V} = \hat{i} + \hat{j} + 2\hat{k}$

**24.** **d.** We must have $\lambda(\hat{i} - 3\hat{j} + 5\hat{k}) = \hat{a} + \frac{2\hat{k} + 2\hat{j} - \hat{i}}{3}$. Therefore,

$3\hat{a} = 3\lambda(\hat{i} - 3\hat{j} + 5\hat{k}) - (2\hat{k} + 2\hat{j} - \hat{i}) = \hat{i}(3\lambda + 1) - \hat{j}(2 + 9\lambda) + \hat{k}(15\lambda - 2)$

$\Rightarrow 3|\hat{a}| = \sqrt{(3\lambda+1)^2 + (2+9\lambda)^2 + (15\lambda-2)^2}$

$\Rightarrow 9 = (3\lambda + 1)^2 + (2 + 9\lambda)^2 + (15\lambda - 2)^2 \Rightarrow 315\lambda^2 - 18\lambda = 0 \Rightarrow \lambda = 0, \frac{2}{35}$

If $\lambda = 0$, $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$ (not acceptable)

For $\lambda = \frac{2}{35}$, $\vec{a} = \frac{41}{105}\hat{i} - \frac{88}{105}\hat{j} - \frac{40}{105}\hat{k}$

**25.** **c.** Suppose the bisector of angle $A$ meets $BC$ at $D$. Then $AD$ divides $BC$ in the ratio $AB : AC$.

So, P.V. of $D = \frac{|\overrightarrow{AB}|(2\hat{i} + 5\hat{j} + 7\hat{k}) + |\overrightarrow{AC}|\ (2\hat{i} + 3\hat{j} + 4\hat{k})}{|\overrightarrow{AB}| + |\overrightarrow{AC}|}$

But $\overrightarrow{AB} = -2\hat{i} - 4\hat{j} - 4\hat{k}$ and $\overrightarrow{AC} = -2\hat{i} - 2\hat{j} - \hat{k}$

$\Rightarrow |\overrightarrow{AB}| = 6$ and $|\overrightarrow{AC}| = 3$

$\therefore$ P.V. of $D = \frac{6(2\hat{i} + 5\hat{j} + 7\hat{k}) + 3(2\hat{i} + 3\hat{j} + 4\hat{k})}{6+3}$

$= \frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$

**26.** **b.** The point that divides $5\hat{i}$ and $5\hat{j}$ in the ratio of $k : 1$ is $\frac{(5\hat{j})k + (5\hat{i})1}{k+1}$

$\therefore \vec{b} = \frac{5\hat{i} + 5k\hat{j}}{k+1}$

Also, $|\vec{b}| \le \sqrt{37}$


**Fig. 1.58**

$\Rightarrow \frac{1}{k+1}\sqrt{25 + 25k^2} \le \sqrt{37}$

$\Rightarrow 5\sqrt{1+k^2} \le \sqrt{37}(k+1)$

Squaring both sides

$25(1+k^2) \le 37(k^2+2k+1)$

or $6k^2+37k+6 \ge 0 \Rightarrow (6k+1)(k+6) \ge 0$

$k \in (-\infty, -6] \cup \left[-\frac{1}{6}, \infty\right)$

**27.** **b.** Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{OP} = \frac{1}{3}\vec{a}, \overrightarrow{OQ} = \frac{1}{2}\vec{b}, \overrightarrow{OR} = \frac{1}{3}\vec{c}$.

Since $P, Q, R$ and $S$ are coplanar, then

$\overrightarrow{PS} = \alpha\,\overrightarrow{PQ} + \beta\,\overrightarrow{PR}$ ($\overrightarrow{PS}$ can be written as a linear combination of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$)

$= \alpha(\overrightarrow{OQ} - \overrightarrow{OP}) + \beta(\overrightarrow{OR} - \overrightarrow{OP})$

i.e., $\overrightarrow{OS} - \overrightarrow{OP} = -(\alpha+\beta)\frac{\vec{a}}{3} + \frac{\alpha}{2}\vec{b} + \frac{\beta}{3}\vec{c}$

$\Rightarrow \overrightarrow{OS} = (1-\alpha-\beta)\frac{\vec{a}}{3} + \frac{\alpha}{2}\vec{b} + \frac{\beta}{3}\vec{c}$ (i)

Given $\overrightarrow{OS} = \lambda\,\overrightarrow{AB} = \lambda(\vec{b} - \vec{a})$ (ii)

From (i) and (ii), $\beta = 0, \frac{1-\alpha}{3} = -\lambda$ and $\frac{\alpha}{2} = \lambda$

$\Rightarrow 2\lambda = 1 + 3\lambda$

$\Rightarrow \lambda = -1$

**28.** **a.** Let the incentre be at the origin and be $A(\vec{p}), B(\vec{q})$ and $C(\vec{r})$. Then

$\overrightarrow{IA} = \vec{p}, \overrightarrow{IB} = \vec{q}$ and $\overrightarrow{IC} = \vec{r}$.

Incentre $I$ is $\frac{a\vec{p} + b\vec{q} + c\vec{r}}{a+b+c}$, where $p = BC, q = AC$ and $r = AB$

Incentre is at the origin. Therefore,

$\frac{a\vec{p} + b\vec{q} + c\vec{r}}{a+b+c} = \vec{0}$, or $a\vec{p} + b\vec{q} + c\vec{r} = \vec{0}$

$\Rightarrow a\,\overrightarrow{IA} + b\,\overrightarrow{IB} + c\,\overrightarrow{IC} = \vec{0}$

**29. a.**

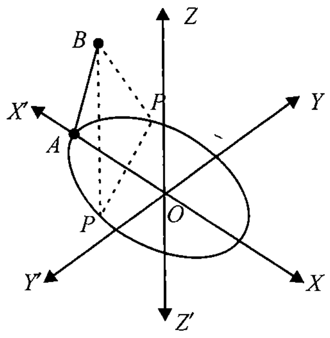



**Fig. 1.59**

Point $P$ lies on $x^2 + 3y^2 = 3$ (i)

Now from the diagram, according to the given conditions, $AP = AB$

or $(x + \sqrt{3})^2 + (y - 0)^2 = 4$ or $(x + \sqrt{3})^2 + y^2 = 4$ (ii)

Solving (i) and (ii), we get $x = 0$ and $y = \pm 1$

Hence point $P$ has position vector $\pm \hat{j}$

**30. b.**

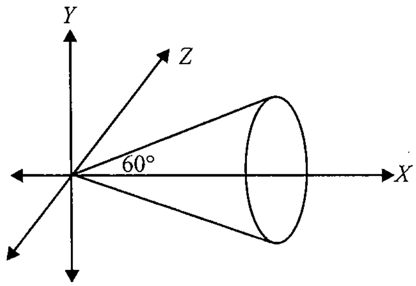



**Fig. 1.60**

From the diagram, it is obvious that locus is a cone concentric with the positive $x$-axis having vertex at the origin and the slant height equal to the magnitude of the vector.

**31. c.** Since $\vec{x}, \vec{y}$ and $\vec{x} \times \vec{y}$ are linearly independent,

$20a - 15b = 15b - 12c = 12c - 20a = 0$

$\Rightarrow \frac{a}{3} = \frac{b}{4} = \frac{c}{5}$

$\Rightarrow c^2 = a^2 + b^2$

Hence, $\Delta ABC$ is right angled.

**32.** **a.** The position vector of any point at $t$ is

$$\vec{r} = (2+t^2)\hat{i} + (4t-5)\hat{j} + (2t^2-6)\hat{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$$

$$\Rightarrow \left.\frac{d\vec{r}}{dt}\right|_{t=2} = 4\hat{i} + 4\hat{j} + 2\hat{k} \text{ and } \left|\frac{d\vec{r}}{dt}\right|_{t=2} = \sqrt{16+16+4} = 4$$

Hence, the required unit tangent vector at $t = 2$ is $\frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$.

**33.** **a.** As $\vec{x}$, $\vec{y}$ and $\vec{x} \times \vec{y}$ are non-collinear vectors, vectors are linearly independent.

$\Rightarrow a - b = 0 = b - c = c - a$

$\Rightarrow a = b = c$

Therefore, the triangle is equilateral.

**34.** **c.**

## Multiple Correct Answers Type

**1.** **a.,b.,c.,d.**

$x\hat{i} + (x+1)\hat{j} + (x+2)\hat{k}$, $(x+3)\hat{i} + (x+4)\hat{j} + (x+5)\hat{k}$ and $(x+6)\hat{i} + (x+7)\hat{j} + (x+8)\hat{k}$ are coplanar

We have determinant of their coefficients as $\begin{vmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$$\begin{vmatrix} x & 1 & 2 \\ x+3 & 1 & 2 \\ x+6 & 1 & 2 \end{vmatrix} = 0$$

Hence $x \in R$

**2.** **a., d.** Let $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Then the diagonals of the parallelogram are $\vec{p} = \vec{a} + \vec{b}$ and $\vec{q} = \vec{b} - \vec{a}$,

i.e., $\vec{p} = 3\hat{i} + 6\hat{j} - 2\hat{k}$, $\vec{q} = -\hat{i} - 2\hat{j} + 8\hat{k}$

So, unit vectors along the diagonals are $\frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$ and $\frac{1}{\sqrt{69}}(-\hat{i} - 2\hat{j} + 8\hat{k})$.

**3.** **b., c.** We have, $\vec{a} = 2p\hat{i} + \hat{j}$

On rotation, let $\vec{b}$ be the vector with components $(p + 1)$ and 1 so that $\vec{b} = (p+1)\hat{i} + \hat{j}$.

Now, $|\vec{a}| = |\vec{b}| \Rightarrow a^2 = b^2$

$\Rightarrow 4p^2 + 1 = (p + 1)^2 + 1$

$\Rightarrow 4p^2 = (p + 1)^2$

$\Rightarrow 2p = \pm (p + 1)$

$\Rightarrow 3p = -1$ or $p = 1$

$\therefore p = -1/3$ or $p = 1$

**4. a.,b.,d.**

Points $A(\hat{i} + \hat{j})$, $B(\hat{i} - \hat{j})$ and $C(p\hat{i} + q\hat{j} + r\hat{k})$ are collinear

Now $\vec{AB} = -2\hat{j}$ and $\vec{BC} = (p-1)\hat{i} + (q-1)\hat{j} + r\hat{k}$

Vectors $\vec{AB}$ and $\vec{BC}$ must be collinear

$\Rightarrow \quad p = 1, r = 0$ and $q \neq 1$

**5. a., b., c.**

For coplanar vectors, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & \mu \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0$

$\Rightarrow (2\lambda - 1)\lambda = 0 \Rightarrow \lambda = 0, \dfrac{1}{2}$

**6. b., c.**

Let $\vec{R}$ be the resultant.

Then $\vec{R} = \vec{F_1} + \vec{F_2} + \vec{F_3} = (p+1)\hat{i} + 4\hat{j}$

Given, $|\vec{R}| = 5$. Therefore,

$(p + 1)^2 + 16 = 25$

$\Rightarrow p + 1 = \pm 3$

$\therefore \; p = 2, -4$

**7. a., b., d.**

$$\vec{a} = \left[\pm\left(\hat{i} - \hat{j}\right) \pm \left(\hat{j} + \hat{k}\right)\right]$$

$$= \pm\left(\hat{i} + \hat{k}\right), \pm\left(\hat{i} - 2\hat{j} - \hat{k}\right)$$

**8. b., c.** Let $\vec{\alpha} = \hat{i} + x\hat{j} + 3\hat{k}$, $\vec{\beta} = 4\hat{i} + (4x-2)\hat{j} + 2\hat{k}$

Given, $2|\vec{\alpha}| = |\vec{\beta}|$

$\Rightarrow 2\sqrt{10 + x^2} = \sqrt{20 + 4(2x-1)^2}$

$\Rightarrow 10 + x^2 = 5 + (4x^2 - 4x + 1)$

$\Rightarrow 3x^2 - 4x - 4 = 0$

$\Rightarrow x = 2, -\dfrac{2}{3}$

**9. c., d.**

Let $\vec{a}, \vec{b}$ and $\vec{c}$ lie in the $x$-$y$ plane.

Let $\vec{a} = \hat{i}, \vec{b} = -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{i}$ and $\vec{c} = -\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}$. Therefore,

$$|\vec{p}+\vec{q}+\vec{r}| = |\lambda\vec{a}+\mu\vec{b}+\nu\vec{c}|$$

$$= \left|\lambda\hat{i} + \mu\left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) + \nu\left(-\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right)\right|$$

$$= \left|\left(\lambda - \frac{\mu}{2} - \frac{\nu}{2}\right)\hat{i} + \frac{\sqrt{3}}{2}(\mu - \nu)\hat{j}\right|$$

$$= \sqrt{\left(\lambda - \frac{\mu}{2} - \frac{\nu}{2}\right)^2 + \frac{3}{4}(\mu - \nu)^2}$$

$$= \sqrt{\lambda^2 + \mu^2 + \nu^2 - \lambda\mu - \lambda\nu - \mu\nu}$$

$$= \frac{1}{\sqrt{2}}\sqrt{(\lambda-\mu)^2 + (\mu-\nu)^2 + (\nu-\lambda)^2}$$

$$\geq \frac{1}{\sqrt{2}}\sqrt{1+1+4} = \sqrt{3}$$

$\Rightarrow |\vec{p}+\vec{q}+\vec{r}|$ can take a value equal to $\sqrt{3}$ and 2.

**10. b., d.** Since $\vec{a}$ and $\vec{b}$ are equally inclined to $\vec{c}$, $\vec{c}$ must be of the form $t\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$.

Now $\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{a} + \frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\vec{b} = \frac{|\vec{a}|\,|\vec{b}|}{|\vec{a}|+|\vec{b}|}\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$

Also, $\frac{|\vec{b}|}{2|\vec{a}|+|\vec{b}|}\vec{a} + \frac{|\vec{a}|}{2|\vec{a}|+|\vec{b}|}\vec{b} = \frac{|\vec{a}|\,|\vec{b}|}{2|\vec{a}|+|\vec{b}|}\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$

Other two vectors cannot be written in the form $t\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$.

**11. a., c., d.**

$\overrightarrow{OA} = -4\hat{i} + 3\hat{k}$; $\overrightarrow{OB} = 14\hat{i} + 2\hat{j} - 5\hat{k}$

$\hat{a} = \frac{-4\hat{i} + 3\hat{k}}{5}$; $\hat{b} = \frac{14\hat{i} + 2\hat{j} - 5\hat{k}}{15}$

$\vec{r} = \frac{\lambda}{15}[-12\hat{i} + 9\hat{j} + 14\hat{i} + 2\hat{j} - 5\hat{k}]$

$$\vec{r} = \frac{\lambda}{15}[2\hat{i} + 2\hat{j} + 4\hat{k}]$$

$$\vec{r} = \frac{2\lambda}{15}[\hat{i} + \hat{j} + 2\hat{k}]$$

**12. a., b., d.**

$$(\lambda - 1)\ (\vec{a}_1 - \vec{a}_2) + \mu\ (\vec{a}_2 + \vec{a}_3) + \gamma\ (\vec{a}_3 + \vec{a}_4 - 2\vec{a}_2) + \vec{a}_3 + \delta\,\vec{a}_4 = \vec{0}$$

i.e., $(\lambda - 1)\ \vec{a}_1 + (1 - \lambda + \mu - 2\gamma)\vec{a}_2 + (\mu + \gamma + 1)\ \vec{a}_3 + (\gamma + \delta)\ \vec{a}_4 = \vec{0}$

Since $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{a}_4$ are linearly independent

$\lambda - 1 = 0,\ 1 - \lambda + \mu - 2\gamma = 0,\ \mu + \gamma + 1 = 0$ and $\gamma + \delta = 0$

i.e., $\lambda = 1, \mu = 2\gamma, \mu + \gamma + 1 = 0, \gamma + \delta = 0$

i.e., $\lambda = 1,\ \mu = -\frac{2}{3},\ \gamma = -\frac{1}{3},\ \delta = \frac{1}{3}$

**13. a., c.** We have, $\overrightarrow{AB} = -\hat{i} - \hat{j} - 4\hat{k}$, $\overrightarrow{BC} = -3\hat{i} + 3\hat{j}$ and $\overrightarrow{CA} = 4\hat{i} - 2\hat{j} + 4\hat{k}$. Therefore,

$|\overrightarrow{AB}| = |\overrightarrow{BC}| = 3\sqrt{2}$ and $|\overrightarrow{CA}| = 6$

Clearly, $|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 = |\overrightarrow{AC}|^2$

Hence, the triangle is right-angled isosceles triangle.

## Reasoning Type

**1. a.**

$\sqrt{(p+2)^2 + 1} = \sqrt{p^2 + 1}$

$\Rightarrow p^2 + 4 + 4p + 1 = p^2 + 1$

$\Rightarrow 4p = -4$

$\Rightarrow p = -1$

Hence a is the correct option.

**2. a.** $2\vec{a} + 3\vec{b} - 5\vec{c} = 0$

$\Rightarrow 3\,(\vec{b} - \vec{a}) = 5\,(\vec{c} - \vec{a}) \Rightarrow \overrightarrow{AB} = \frac{5}{3}\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}$ and $\overrightarrow{AC}$ must be parallel since there is a common point $A$. The points $A$, $B$ and $C$ must be collinear.

**3. d.** We know that the unit vector along bisector of unit vectors $\vec{u}$ and $\vec{v}$ is $\dfrac{\vec{u} + \vec{v}}{2\cos\frac{\theta}{2}}$, where $\theta$ is the angle between vectors $\vec{u}$ and $\vec{v}$.

Hence Statement 1 is false, however Statement 2 is true.

4. **b.** Obviously, Statement 1 is true.

$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2\cos^2 \alpha - 1 + 2\cos^2 \beta - 1 + 2\cos^2 \gamma - 1$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 = 2 - 3 = -1$

Hence, Statement 2 is true but does not explain Statement 1 as it is result derived using the result in the statement.

5. **b.**

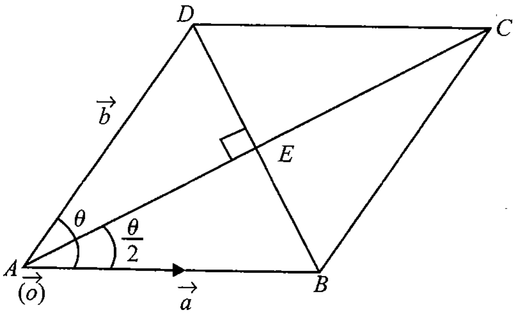



**Fig. 1.61**

We know that vector in the direction of angular bisector of unit vectors $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}+\vec{b}}{2\cos\frac{\theta}{2}}$

where $\vec{a} = \overrightarrow{AB} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$ and $\vec{b} = \overrightarrow{AD} = l_2\hat{i} + m_2\hat{j} + n_3\hat{k}$

Thus unit vector along the bisector is $\dfrac{l_1 + l_2}{\cos\frac{\theta}{2}}\hat{i} + \dfrac{m_1 + m_2}{\cos\frac{\theta}{2}}\hat{j} + \dfrac{n_1 + n_2}{\cos\frac{\theta}{2}}\hat{k}$

Hence Statement 1 is true.

Also, in triangle ABD, by cosine rule

$$\cos\theta = \frac{AB^2 + AD^2 - BD^2}{2AB \cdot AD}$$

$$\Rightarrow \quad \cos\theta = \frac{1 + 1 - |(l_1 - l_2)\hat{i} + (m_1 - m_2)\hat{j} + (n_1 - n_2)\hat{k}|^2}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{2 - [(l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2]}{2}$$

$$= \frac{2 - [2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2}$$

$$= l_1l_2 + m_1m_2 + n_1n_2$$

Hence, Statement 2 is true but does not explain Statement 1.

6. **c.** In $\Delta ABC$, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = -\overrightarrow{CA} \Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{O}$

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ is the triangle law of addition.

Hence Statement 1 is true and Statement 2 is false.

**7.** a.

$$\frac{3}{2}=\frac{p}{3}=\frac{3}{q} \Rightarrow p=\frac{9}{2} \text{ and } q=2$$

Thus, both the statements are true and Statement 2 is the correct explanation for Statement 1.

**8.** **a.** $\vec{a}+\vec{b} = \vec{a}-\vec{b}$ are the diagonals of a parallelogram whose sides are $\vec{a}$ and $\vec{b}$.

$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$\Rightarrow$ Diagonals of the parallelogram have the same length.

$\Rightarrow$ The parallelogram is a rectangle $\Rightarrow \vec{a} \perp \vec{b}$

**9.** **a.** Given vectors are non-coplanar. Hence the answer is (a)

**10.** a. $3\vec{a}-2\vec{b}+5\vec{c}-6\vec{d}=(2\vec{a}-2\vec{b})+(-5\vec{a}+5\vec{c})+(6\vec{a}-6\vec{d})$

$$=-\,2\overrightarrow{AB}+5\overrightarrow{AC}-6\overrightarrow{AD}=\vec{0}$$

Therefore, $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are linearly dependent. Hence by Statement 2, Statement 1is true.

**11.** **a.** We have adjacent sides of triangle $|\vec{a}| = 3$, $|\vec{b}| = 4$.

The length of the diagonal is $|\vec{a}+\vec{b}| = 5$.

Since it satisfies the Pythagoras theorem, $\vec{a} \perp \vec{b}$.

Hence the parallelogram is a rectangle.

Hence length of the other diagonal is $\left|\vec{a}-\vec{b}\right|=5$

## Linked Comprehension Type

**For Problems 1–3**

**1. c., 2. b., 3. c.**

**Sol.**

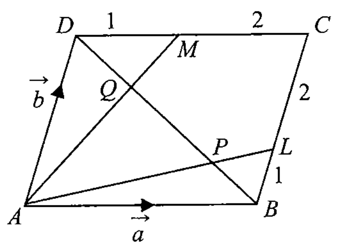



**Fig. 1.62**

$\overrightarrow{BL}=\frac{1}{3}\vec{b}$

$\therefore\ \overrightarrow{AL}=\vec{a}+\frac{1}{3}\vec{b}$

Let $\overrightarrow{AP} = \lambda \overrightarrow{AL}$ and $P$ divides $DB$ in the ratio $\mu : 1-\mu$

Then $\overrightarrow{AP} = \lambda \vec{a} + \frac{\lambda}{3}\vec{b}$ (i)

Also $\overrightarrow{AP} = \mu \vec{a} + (1-\mu)\vec{b}$ (ii)

From (i) and (ii), $\lambda \vec{a} + \frac{\lambda}{3}\vec{b} = \mu \vec{a} + (1-\mu)\vec{b}$

$\therefore \lambda = \mu$ and $\frac{\lambda}{3} = 1 - \mu$

$\therefore \lambda = \frac{3}{4}$

Hence, $P$ divides $AL$ in the ratio 3 : 1 and $P$ divides $DB$ in the ratio 3 : 1.
Similarly $Q$ divides $DB$ in the ratio 1 : 3.

Thus $DQ = \frac{1}{4}DB$ and $PB = \frac{1}{4}DB$

$\therefore PQ = \frac{1}{2}DB$, i.e., $PQ : DB = 1 : 2$

**For Problems 4–5**

**4. c., 5. b.**

**Sol.**

Let the position vectors of $A$, $B$, $C$ and $D$ be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, respectively.
Then, $OA : CB = 2 : 1$

$\Rightarrow \overrightarrow{OA} = 2\overrightarrow{CB}$

$\Rightarrow \vec{a} = 2(\vec{b} - \vec{c})$ (i)

and $OD : AB = 1 : 3$

$3\overrightarrow{OD} = \overrightarrow{AB}$

$\Rightarrow 3\vec{d} = (\vec{b} - \vec{a}) = \vec{b} - 2(\vec{b} - \vec{c})$ (using (i))

$\Rightarrow 3\vec{d} = -\vec{b} + 2\vec{c}$ (ii)

Let $OX : XC = \lambda : 1$ and $AX : XD = \mu : 1$

Now, $X$ divides $OC$ in the ratio $\lambda : 1$. Therefore,

P.V. of $X = \frac{\lambda \vec{c}}{\lambda + 1}$ (iii)

$X$ also divides $AD$ in the ratio $\mu : 1$

P.V. of $X = \frac{\mu \vec{d} + \vec{a}}{\mu + 1}$ (iv)

From (iii) and (iv), we get

$$\frac{\lambda \vec{c}}{\lambda+1} = \frac{\mu \vec{d} + \vec{a}}{\mu+1}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c} = \left(\frac{\mu}{\mu+1}\right)\vec{d} + \left(\frac{1}{\mu+1}\right)\vec{a}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c} = \left(\frac{\mu}{\mu+1}\right)\left(\frac{-\vec{b}+2\vec{c}}{3}\right) + \left(\frac{1}{\mu+1}\right)2\left(\vec{b}-\vec{c}\right) \quad \text{(using (i) and (ii))}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c} = \left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b} + \left(\frac{2\mu}{3(\mu+1)} - \frac{2}{\mu+1}\right)\vec{c}$$

$$\Rightarrow \left(\frac{\lambda}{\lambda+1}\right)\vec{c} = \left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b} + \left(\frac{2\mu-6}{3(\mu+1)}\right)\vec{c}$$

$$\Rightarrow \left(\frac{6-\mu}{3(\mu+1)}\right)\vec{b} + \left(\frac{2\mu-6}{3(\mu+1)} - \frac{\lambda}{\lambda+1}\right)\vec{c} = \vec{0}$$

$$\Rightarrow \frac{6-\mu}{3(\mu+1)} = 0 \text{ and } \frac{2\mu-6}{3(\mu+1)} - \frac{\lambda}{\lambda+1} = 0 \text{ (as } \vec{b} \text{ and } \vec{c} \text{ are non-collinear)}$$

$$\Rightarrow \mu = 6,\ \lambda = \frac{2}{5}$$

Hence $OX : XC = 2 : 5$

**For Problems 6–7**

**6. c., 7. d.**

**Sol.**

Consider the regular hexagon $ABCDEF$ with centre at $O$ (origin).

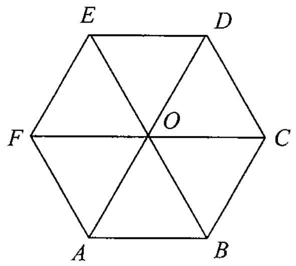



**Fig. 1.63**

$$\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC} = 2\overrightarrow{AO} + 2\overrightarrow{OB} + 2\overrightarrow{OC}$$
$$= 2(\overrightarrow{AO} + \overrightarrow{OB}) + 2\overrightarrow{OC}$$

$$= 2\overrightarrow{AB} + 2\overrightarrow{AB} \qquad (\because\ \overrightarrow{OC} = \overrightarrow{AB})$$

$$= 4\overrightarrow{AB}$$

$$\vec{R} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$$

$$= \overrightarrow{ED} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{CD} \quad (\because\ \overrightarrow{AB} = \overrightarrow{ED} \text{ and } \overrightarrow{AF} = \overrightarrow{CD})$$

$$= (\overrightarrow{AC} + \overrightarrow{CD}) + (\overrightarrow{AE} + \overrightarrow{ED}) + \overrightarrow{AD}$$

$$= \overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{AD} = 3\overrightarrow{AD} = 6\overrightarrow{AO}$$

## Matrix-Match Type

**1. a → p, r, s; b → q, r, s; c → p, r; d → r, s**

**2. a → q, r; b → p, r; c → q, s; d → p**

**Sol.**

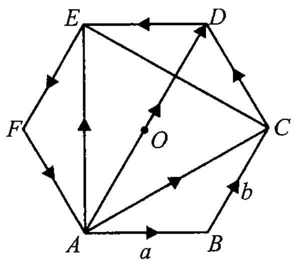



**Fig. 1.64**

$$\overrightarrow{AB} = \vec{a},\ \overrightarrow{BC} = \vec{b}$$

$$\therefore\ \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{a} \qquad \text{(i)}$$

$$\overrightarrow{AD} = 2\overrightarrow{BC} = 2\vec{b} \qquad \text{(ii)}$$

(because $AD$ is parallel to $BC$ and twice its length).

$$\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC} = 2\vec{b} - (\vec{a} + \vec{b})$$

$$= \vec{b} - \vec{a}$$

$$\overrightarrow{FA} = -\overrightarrow{CD} = \vec{a} - \vec{b} \qquad \text{(iii)}$$

$$\overrightarrow{DE} = -\overrightarrow{AB} = -\vec{a} \qquad \text{(iv)}$$

$\overrightarrow{EF} = -\overrightarrow{BC} = -\vec{b}$ (v)

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE} = 2\vec{b} - \vec{a}$ (vi)

$\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DE} = \vec{b} - \vec{a} - \vec{a} = \vec{b} - 2\vec{a}$ (vii)

## Integer Answer Type

**1.** **(2)** L.H.S. $= \vec{d} - \vec{a} + \vec{d} - \vec{b} + \vec{h} - \vec{c} + 3(\vec{g} - \vec{h})$

$$= 2\vec{d} - (\vec{a} + \vec{b} + \vec{c}) + 3\frac{(\vec{a} + \vec{b} + \vec{c})}{3} - 2\vec{h}$$

$$= 2\vec{d} - 2\vec{h} = 2(\vec{d} - \vec{h}) = 2\overrightarrow{HD}$$

$\Rightarrow \quad \lambda = 2$

**2.** **(6)** Let $\vec{R}$ be the resultant

Then $\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (p+1)\hat{i} + 4\hat{j}$

Given, $|\vec{R}| = 5$, therefore $R^2 = 25$

$\because \quad (p + 1)^2 + 16 = 25 \Rightarrow p + 1 = \pm 3$

$\therefore \quad p = 2, -4$

**3.** **(3)** Given, $\vec{a} + \vec{b} = \vec{c}$

Now vector $\vec{c}$ is along the diagonal of the parallelogram which has adjacent side vectors $\vec{a}$ and $\vec{b}$.

Since $\vec{c}$ is also a unit vector, triangle formed by vectors $\vec{a}$ and $\vec{b}$ is an equilateral triangle.

Then, area of triangle is $\frac{\sqrt{3}}{4}$

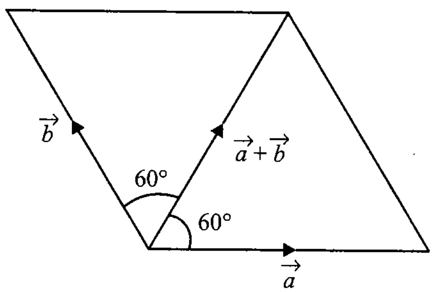



**Fig. 1.65**

**4.** **(2)** Let $\vec{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ be the adjacent sides of the parallelogram.

Now angle between $\vec{a}$ and $\vec{b}$ is acute,

$\Rightarrow \quad |\vec{a} + \vec{b}| > |\vec{a} - \vec{b}|$

$\Rightarrow \quad \left|3x\hat{i} + (x-3)\hat{j} - 2\hat{k}\right|^2 > \left|-x\hat{i} - (x+3)\hat{j}\right|^2$

$\Rightarrow \quad 9x^2 + (x-3)^2 + 4 > x^2 + (x+3)^2$

$\Rightarrow \quad 8x^2 - 12x + 4 > 0$

$\Rightarrow \quad 2x^2 - 3x + 1 > 0$

$\Rightarrow \quad (2x-1)(x-1) > 0$

$\Rightarrow \quad x < 1/2 \quad \text{or} \quad x > 1$

Hence the least positive integral value is 2

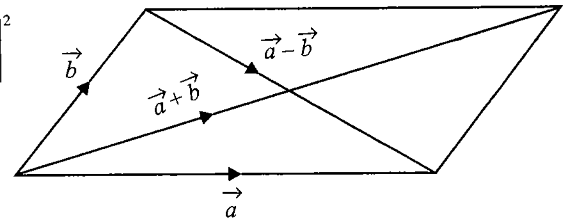



**Fig. 1.66**

**5.** **(7)** Vectors along the sides are $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$

Clearly the vector along the longer diagonal is $\vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} + 2\hat{k}$

Hence length of the longer diagonal is $|\vec{a} + \vec{b}| = |3\hat{i} + 6\hat{j} + 2\hat{k}| = 7$

**6.** **(9)** Vector $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k}, \vec{c} = \lambda\hat{i} + \hat{j} + 2\hat{k}$ are coplanar

$$\Rightarrow \quad \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ \lambda & 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda - 3 + 2(-5) = 0$

$\Rightarrow \quad \lambda = 13$

## Archives

## Subjective Type

**1.** $(\hat{i} + \hat{j} + 3\hat{k})x + (3\hat{i} - 3\hat{j} + \hat{k})y + (-4\hat{i} + 5\hat{j})z = \lambda(x\hat{i} + y\hat{j} + z\hat{k})$

Comparing coefficient of $\hat{i}$, $x + 3y - 4z = \lambda x$

$\Rightarrow (1 - \lambda)x + 3y - 4z = 0$ (i)

Comparing coefficient of $\hat{j}$, $x - 3y + 5z = \lambda y$

$\Rightarrow x - (3 + \lambda)y + 5z = 0$ (ii)

Comparing coefficient of $\hat{k}$, $3x + y + 0z = \lambda z$

$3x + y - \lambda z = 0$ (iii)

All the above three equations are satisfied for $x$, $y$ and $z$ not all zero if

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[3\lambda + \lambda^2 - 5] - 3[-\lambda - 15] - 4[1 + 9 + 3\lambda] = 0$

$\Rightarrow \lambda^3 + 2\lambda^2 + \lambda = 0$

$\Rightarrow \lambda(\lambda + 1)^2 = 0$

$\Rightarrow \lambda = 0, -1$

**2.** Since vector $\vec{A}$ has components $A_1$, $A_2$ and $A_3$, in the coordinate system $OXYZ$,

$\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$

When given system is rotated through $\pi/2$, the new $x$-axis is along the old $y$-axis and the new $y$-axis is along the old negative $x$-axis; $z$ remains same as before.

Hence the components of $A$ in the new system are $A_2$, $-A_1$ and $A_3$.

Therefore, $\vec{A}$ becomes $A_2\hat{i} - A_1\hat{j} + A_3\hat{k}$.

**3.** Given that P.V.'s of points $A$, $B$, $C$ and $D$ are $3\hat{i} - 2\hat{j} - \hat{k}$, $2\hat{i} + 3\hat{j} - 4\hat{k}$, $-\hat{i} + \hat{j} + 2\hat{k}$ and $4\hat{i} + 5\hat{j} + \lambda\hat{k}$, respectively.

Given that $A$, $B$, $C$ and $D$ lie in a plane if $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar. Therefore,

$$\begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & 1+\lambda \end{vmatrix} = 0$$

$\Rightarrow -1(3 + 3\lambda - 21) - 5(-4 - 4\lambda - 3) - 3(-28 - 3) = 0$

$\Rightarrow -3\lambda + 18 + 20\lambda + 35 + 93 = 0$

$\Rightarrow 17\lambda = -146$

$\Rightarrow \lambda = -\dfrac{146}{17}$

**4.** $OACB$ is a parallelogram with $O$ as origin. Let with respect to $O$, position vectors of $A$ and $B$ be $\vec{a}$ and $\vec{b}$, respectively. Then P.V. of $C$ is $\vec{a} + \vec{b}$.

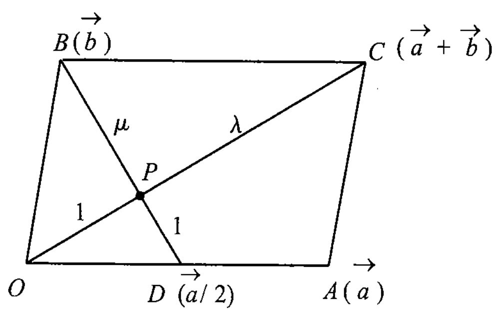



**Fig. 1.67**

Also $D$ is the midpoint of $OA$; therefore, the position vector of $D$ is $\vec{a}/2$.
$CO$ and $BD$ intersect each other at $P$.
Let $P$ divide $CO$ in the ratio $\lambda : 1$ and $BD$ in the ratio $\mu : 1$. Then by section theorem, position vector of point $P$ dividing $CO$ in ratio $\lambda : 1$ is

$$\frac{\lambda \times 0 + 1 \times (\vec{a}+\vec{b})}{\lambda+1} = \frac{\vec{a}+\vec{b}}{\lambda+1} \quad \text{(i)}$$

and position vector of point $P$ dividing $BD$ in the ratio $\mu : 1$ is

$$\frac{\mu\left(\frac{\vec{a}}{2}\right)+1(\vec{b})}{\mu+1} = \frac{\mu\vec{a}+2\vec{b}}{2(\mu+1)} \quad \text{(ii)}$$

As (i) and (ii) represent the position vector of the same point, hence

$$\frac{\vec{a}+\vec{b}}{\lambda+1} = \frac{\mu\vec{a}+2\vec{b}}{2(\mu+1)}$$

Equating the coefficients of $\vec{a}$ and $\vec{b}$, we get

$$\frac{1}{\lambda+1} = \frac{\mu}{2(\mu+1)} \quad \text{(iii)}$$

$$\frac{1}{\lambda+1} = \frac{1}{\mu+1} \quad \text{(iv)}$$

From (iv) we get $\lambda = \mu \Rightarrow P$ divides $CO$ and $BD$ in the same ratio.
Putting $\lambda = \mu$ in Eq. (iii), we get $\mu = 2$
Thus the required ratio is 2 : 1.

**5.** Let the vertices of the triangle be $A\ (\vec{0}), B(\vec{b})$ and $C(\vec{c})$.

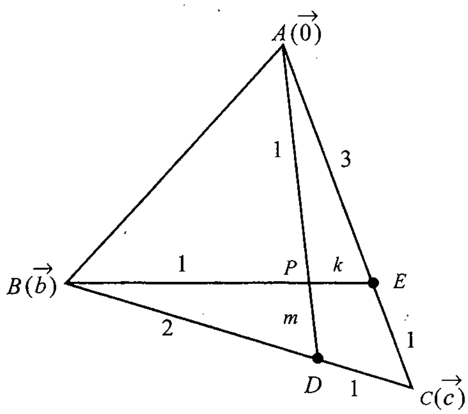



**Fig. 1.68**

Given that $D$ divides $BC$ in the ratio 2 : 1.

Therefore, position vector of $D$ is $\frac{\vec{b}+2\vec{c}}{3}$.

$E$ divides $AC$ in the ratio 3 : 1.

Therefore, position vector of $E$ is $\frac{\vec{0}+3\vec{c}}{4}=\frac{3\vec{c}}{4}$.

Let point of intersection $P$ of $AD$ and $BE$ divide $BE$ in the ratio $1:k$ and $AD$ in the ratio $1:m$. Then position vectors of $P$ in these two cases are $\frac{k\vec{b}+1(3\vec{c}/4)}{k+1}$ and $\frac{m\vec{0}+m\ \ ((\vec{b}+2\vec{c})/3)}{m+1}$, respectively.

Equating the position vectors of $P$ in these two cases, we get

$$\frac{k\vec{b}}{k+1}+\frac{3\vec{c}}{4(k+1)}=\frac{m\vec{b}}{3(m+1)}+\frac{2m\vec{c}}{3(m+1)}$$

$$\Rightarrow \frac{k}{k+1}=\frac{m}{3(m+1)} \text{ and } \frac{3}{4(k+1)}=\frac{2m}{3(m+1)}$$

Dividing, we have $\frac{4k}{3}=\frac{1}{2}\Rightarrow k=\frac{3}{8}$

Required ratio is 8 : 3.

**6.** Let the P.V.s of the points $A$, $B$, $C$ and $D$ be $\vec{O}$, $B(\vec{b})$, $D(\vec{d})$ and $C(\vec{d}+t\vec{b})$

For any point $\vec{r}$ on $\overrightarrow{AC}$ and $\overrightarrow{BD}$, $\vec{r}=\lambda\ (\vec{d}+t\vec{b})$ and $\vec{r}=(1-\mu)\ \vec{b}+\mu\vec{d}$, respectively.

For the point of intersection, say $T$, compare the coefficients.

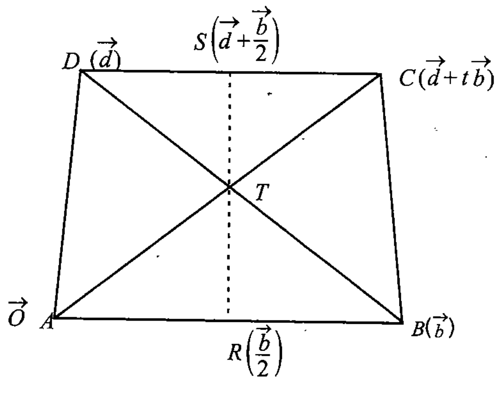



**Fig. 1.69**

$\lambda=\mu$, $t\lambda=1-\mu=1-\lambda$ or $(t+1)\lambda=1$

$\therefore \lambda = \frac{1}{t+1} = \mu$

Therefore, $\vec{r}$ ( position vector of $T$) is $\frac{\vec{d} + t\vec{b}}{t+1}$. (i)

Let $R$ and $S$ be the midpoints of the parallel sides $AB$ and $DC$; then $R$ is $\frac{b}{2}$ and $S$ is $d + t\frac{b}{2}$.
Let $T$ divide $SR$ in the ratio $m:1$.

Position vector of $T$ is $\frac{m\frac{\vec{b}}{2} + \vec{d} + t\frac{\vec{b}}{2}}{m+1}$, which is equivalent to $\frac{\vec{d} + t\vec{b}}{t+1}$.

Comparing coefficients of $\vec{b}$ and $\vec{d}$, $\frac{1}{m+1} = \frac{1}{t+1}$ and $\frac{m+t}{2(m+1)} = \frac{t}{t+1}$.

From the first relation, $m = t$, which satisfies the second relation. Hence proved.

**7.** Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of $A$, $B$ and $C$, respectively.

Let $AD$, $BE$ and $CF$ be the bisectors of $\angle A$, $\angle B$ and $\angle C$, respectively.

$a$, $b$ and $c$ are the lengths of sides $BC$, $CA$ and $AB$, respectively.

Now $AD$ divides $BC$ in the ratio $BD : DC = AB : AC = c : b$.

Hence, the position vector of $D$ is $\vec{d} = \frac{b\vec{b} + c\vec{c}}{b+c}$.

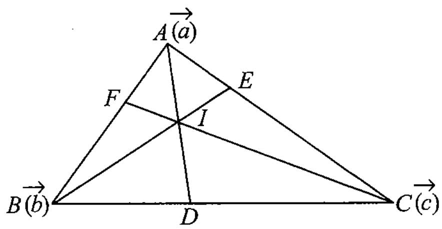


**Fig. 1.70**

Let $I$ be the point of intersection of $BE$ and $AD$.

Then in $\Delta ABC$, $BI$ is bisector of $\angle B$. Therefore,
$DI : IA = BD : BA$

But $\frac{BD}{DC} = \frac{c}{b} \Rightarrow \frac{BD}{BD+DC} = \frac{c}{c+b}$

$\Rightarrow \frac{BD}{BC} = \frac{c}{c+b}$

$\Rightarrow BD = \frac{ac}{b+c}$

$\therefore \; DI : IA = \frac{ac}{b+c} : c = a : (b+c)$

$\therefore$ P.V. of $I = \frac{a\vec{a} + \vec{d}(b+c)}{a+b+c}$

$$= \frac{a\vec{a} + \left(\frac{b\vec{b} + c\vec{c}}{b+c}\right)(b+c)}{a+b+c} = \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}$$

As P.V. of $I$ is symmetrical in $\vec{a}$, $\vec{b}$, $\vec{c}$ and $a$, $b$, $c$, it must lie on $CF$ as well.

**8.** $\vec{A}(t)$ is parallel to $\vec{B}(t)$ for some $t \in [0, 1]$ if and only if $\frac{f_1(t)}{g_1(t)} = \frac{f_2(t)}{g_2(t)}$ for some $t \in [0, 1]$

or $f_1(t) \cdot g_2(t) = f_2(t) g_1(t)$ for some $t \in [0, 1]$

Let $h(t) = f_1(t) \cdot g_2(t) - f_2(t) \cdot g_1(t)$

$h(0) = f_1(0) \cdot g_2(0) - f_2(0) \cdot g_1(0)$

$= 2 \times 2 - 3 \times 3 = -5 < 0$

$h(1) = f_1(1) \cdot g_2(1) - f_2(1) \cdot g_1(1)$

$= 6 \times 6 - 2 \times 2 = 32 > 0$

Since $h$ is a continuous function, and $h(0) \cdot h(1) < 0$, there are some $t \in [0, 1]$ for which $h(t) = 0$, i.e., $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for this $t$.

**9.** With $O$ as origin let $\vec{a}$ and $\vec{b}$ be the position vectors of $A$ and $B$, respectively.

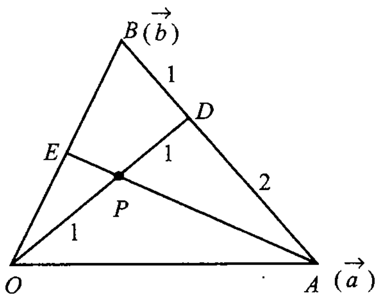



**Fig. 1.71**

Then the position vector of $E$, the midpoint of $OB$, is $\vec{b}/2$.

Again since $AD : DB = 2 : 1$, the position vector of $D$ is

$$\frac{1 \cdot \vec{a} + 2\vec{b}}{1+2} = \frac{\vec{a} + 2\vec{b}}{3}$$

Let $\frac{OP}{OD} = \frac{1}{\lambda}$

$\Rightarrow$ P.V. of $P$ is $\frac{\vec{a} + 2\vec{b}}{3(\lambda + 1)}$

Let $\frac{AP}{PE} = \frac{1}{\mu}$

$\Rightarrow$ P.V. of $P$ is $\frac{\mu \vec{a} + \frac{\vec{b}}{2}}{\mu + 1}$

Comparing P.V. of $P$, we have

$$\frac{1}{3(\lambda + 1)} = \frac{\mu}{\mu + 1} \text{ and } \frac{2}{3(\lambda + 1)} = \frac{1}{2(\mu + 1)}$$

Dividing $\mu = \frac{1}{4} \Rightarrow \lambda = \frac{2}{3}$

$$\Rightarrow \frac{OP}{PA} = \frac{3}{2}$$

## Objective Type

***Fill in the blanks***

**1.** Given that $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

Operating $C_2 \leftrightarrow C_3$ and then $C_1 \leftrightarrow C_2$ in first determinant

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$(1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

either $1 + abc = 0$ or $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

Also given that vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non-coplanar.

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

So we must have $1 + abc = 0$

$abc = -1$

**2.** Given that the vectors $\vec{u} = a\hat{i} + \hat{j} + \hat{k}$, $\vec{v} = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{w} = \hat{i} + \hat{j} + c\hat{k}$, where $a, b, c \neq 1$ are coplanar. Therefore,

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Operating $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$\begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix} = 0$$

Expanding

$c(a-1)(b-1)+(1-b)(1-c)-(1-c)(a-1)=0$

$$\frac{c}{1-c}+\frac{1}{1-a}+\frac{1}{1-b}=0$$

$$\frac{c}{1-c}+1+\frac{1}{1-a}+\frac{1}{1-b}=1$$

$$\frac{1}{1-c}+\frac{1}{1-a}+\frac{1}{1-b}=1$$

***True or false***

1. Let position vectors of points $A$, $B$ and $C$ be $\vec{a}+\vec{b}, \vec{a}-\vec{b}$ and $\vec{a}+k\vec{b}$, respectively.

   Then $\overrightarrow{AB}=(\vec{a}-\vec{b})-(\vec{a}+\vec{b})=-2\vec{b}$

   Similarly, $\overrightarrow{BC}=(\vec{a}+k\vec{b})-(\vec{a}-\vec{b})=(k+1)\vec{b}$

   Clearly $\overrightarrow{AB} \parallel \overrightarrow{BC} \quad \forall\, k \in R$

   $\Rightarrow A, B$ and $C$ are collinear $\quad \forall\, k \in R$

   Therefore, the statement is true.

***Multiple choice questions with one correct answer***

1. **a.** Three points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ are collinear if $\overline{AB} \parallel \overline{AC}$

   $\overrightarrow{AB}=-20\hat{i}-11\hat{j}; \overrightarrow{AC}=(a-60)\hat{i}-55\hat{j}$

   $$\Rightarrow \overrightarrow{AB} \parallel \overrightarrow{AC} \Rightarrow \frac{a-60}{-20}=\frac{-55}{-11} \Rightarrow a=-40$$

2. **b.** $a$, $b$ and $c$ are distinct negative numbers and vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar

   $$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix}=0$$

   $\Rightarrow ac+c^2-ab-ac=0$

   $\Rightarrow c^2=ab$

   $\Rightarrow a, c, b$ are in G.P.

   So $c$ is the G.M. of $a$ and $b$.

3. **c.** $\vec{a}=\hat{i}-\hat{k}$

   $\vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$

   $\vec{c}=y\vec{i}+x\vec{j}+(1+x-y)\vec{k}$

$$= \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

$= (1 + x - y - x + x^2) - 1(x^2 - y)$

$= 1$

**4. b.** Let the given position vectors be of points $A$, $B$ and $C$, respectively. Then

$$|\overrightarrow{AB}| = \sqrt{(\beta-\alpha)^2+(\gamma-\beta)^2+(\alpha-\gamma)^2}$$

$$|\overrightarrow{BC}| = \sqrt{(\gamma-\beta)^2+(\alpha-\gamma)^2+(\alpha-\beta)^2}$$

$$|\overrightarrow{CA}| = \sqrt{(\alpha-\gamma)^2+(\beta-\alpha)^2+(\gamma-\beta)^2}$$

$\because |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$

Hence, $\Delta ABC$ is an equilateral triangle.

**5. c.** We know that three vectors are coplanar if their scalar triple product is zero.

$$\Rightarrow \begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0 \quad R_1 \to R_1 + R_2 + R_3$$

$$\Rightarrow \begin{vmatrix} 2-\lambda^2 & 2-\lambda^2 & 2-\lambda^2 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(1+\lambda^2) & 0 \\ 0 & 0 & -(1+\lambda^2) \end{vmatrix} = 0 \qquad (R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$$

$\Rightarrow (2 - \lambda^2)(1 + \lambda^2)^2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$

Hence two real solutions.

**6. d.** Given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent,

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0$$

$\Rightarrow 1 - \beta = 0$

$\Rightarrow \beta = 1$

Also given that $|\vec{c}| = \sqrt{3} \Rightarrow 1 + \alpha^2 + \beta^2 = 3$

Substituting the value of $\beta$, we get $\alpha^2 = 1$

$\Rightarrow \alpha = \pm 1$