JEE ADVANCE - 2016 (Paper 2)

37. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If Q = $[q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals

(A) 52 (B) 103 (C) 201 (D) 205

Sol. : $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{bmatrix}$. So, $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ A^n is a zero matrix, $\forall n \geq 3$.

$P^{50} = (1 + A)^{50} = I + 50A + \frac{50 \times 49}{2} A^2$

$\therefore \quad Q + I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 50 \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{bmatrix} + 25 \times 49 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{bmatrix}$

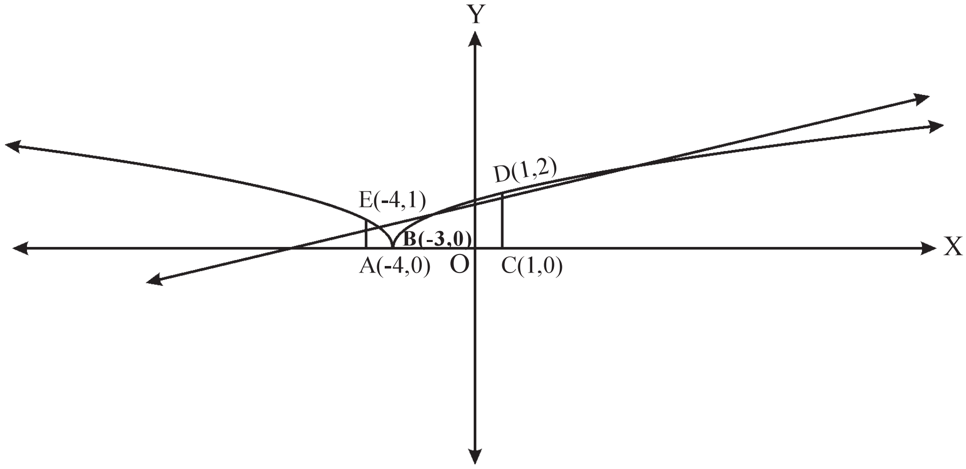
$\therefore \left(\frac{q_{31}+q_{32}}{q_{21}}\right) = \frac{16(50+25\times 49)+50\times 4}{50\times 4}$

$= \frac{16\times 51+8}{8} = 102 + 1 = 103$ **Ans. (B)**

38. Area of region $\left\{(x, y) \in R^2 \mid y \geq \sqrt{|x+3|}, 5y \leq x+9 \leq 15\right\}$ is equal to

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

Sol. :



Solving for points of intersection $y^2 = |x+3| = \left(\frac{x+9}{5}\right)^2$

$\therefore$ $x^2 + 18x + 81 = 25x + 75$ or $-25x - 75$

$x^2 - 7x + 6 = 0$ or $x^2 + 43x + 156 = 0$

$\therefore$ $x = 6$ or 1. So, $y = 3$ or 2

$\therefore$ D(1, 2)

or $(x + 39)(x + 4) = 0$

$\therefore$ $x = -39$ or -4. So, $y = -6$ or 1

$\therefore$ E(–4, 1)

The required area = Area of trapezium DCAE – Area EAB under parabola – Area DBC under parabola

$$= \frac{1}{2}(1+2)5 - \int_{-4}^{-3} \sqrt{-x-3}\,dx - \int_{-3}^{1} \sqrt{x+3}\,dx$$

$$= \frac{15}{2} + \left[\frac{2(-x-3)^{\frac{3}{2}}}{3}\right]_{-4}^{-3} - \left[\frac{2(x+3)^{\frac{3}{2}}}{3}\right]_{-3}^{1}$$

$$= \frac{15}{2} - \left(\frac{2}{3}\right) - \frac{2}{3}(8) = \frac{15}{2} - 6 = \frac{3}{2}$$

Ans. (C)

39. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to

(A) $3 - \sqrt{3}$ (B) $2(3 - \sqrt{3})$ (C) $2(\sqrt{3} - 1)$ (D) $2(2 + \sqrt{3})$

Sol. :

$$T_k = 2\left(\frac{\sin\left(\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)-\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right)\right)}{\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)\sin\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right)}\right)$$

$$= 2\left(\cot\left(\frac{\pi}{4}+(k-1)\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)\right)$$

$\therefore$ The sum $= 2\left(\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4}+\frac{13\pi}{6}\right)\right)$

$$= 2\left(\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right)$$

$$= 2\left(1 - \cot\frac{5\pi}{12}\right)$$

$$= 2\left(1-\left(2-\sqrt{3}\right)\right) = 2\left(\sqrt{3}-1\right)$$

Ans. (C)

40. Let $b_i > 1$ for $i = 1, 2, \ldots, 101$. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \ldots + b_{51}$ and $s = a_1 + a_2 + \ldots + a_{51}$, then

(A) $s > t$ and $a_{101} > b_{101}$ (B) $s > t$ and $a_{101} < b_{101}$

(C) $s < t$ and $a_{101} > b_{101}$ (D) $s < t$ and $a_{101} < b_{101}$

Sol. : $a_1 = b_1 = a,\ t = b_1 + b_2 + \ldots + b_{51},\ s = a_1 + a_2 + \ldots + a_{51}$

$t = a\frac{(2^{51}-1)}{2-1} = a(2^{51} - 1)$ (G.P. with ratio 2)

$s = \frac{51}{2}(2a + (n-1)d) = \frac{51}{2}(2a + 50d)$

We know $a_{51} = b_{51}$. So, $a + 50d = a(2^{50})$

$\therefore\ 50d = a(2^{50} - 1)$. So, $s = a \cdot \frac{51}{2}(2^{50} + 1) = a\left(51 \cdot 2^{49} + \frac{51}{2}\right)$

$\therefore\ s = a\left(4 \cdot 2^{49} + 47 \cdot 2^{49} + \frac{51}{2}\right) = a\left(2^{51} - 1 + 47 \cdot 2^{49} + \frac{53}{2}\right)$

$\therefore\ s - t = a\left(47 \cdot 2^{49} + \frac{53}{2}\right)$

$\therefore\ s > t$

Also, $a_{101} = a + 100d = a + 2a(2^{50} - 1) = a(2^{51} - 1)$

$b_{101} = a \cdot 2^{100}$

$\therefore\ b_{101} > a_{101}$ **Ans. (B)**

41. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x}\,dx$ is equal to

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

Sol. : $I = \int_0^{\frac{\pi}{2}} \left(\frac{x^2 \cos x}{1+e^x} + \frac{x^2 \cos x}{1+e^{-x}}\right) dx$

$= \int_0^{\frac{\pi}{2}} \frac{x^2 \cos x + x^2 e^x \cos x}{1+e^x}\,dx$

$= \int_0^{\frac{\pi}{2}} x^2 \cos x\,dx$

$$= [x^2 \sin x]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} 2x \sin x \, dx$$

$$= \frac{\pi^2}{4} - 2\left[\left[(x(-\cos x))\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} -\cos x \, dx\right]$$

$$= \frac{\pi^2}{4} - 2\,[-(0-0) + (\sin x)]_0^{\pi/2}$$

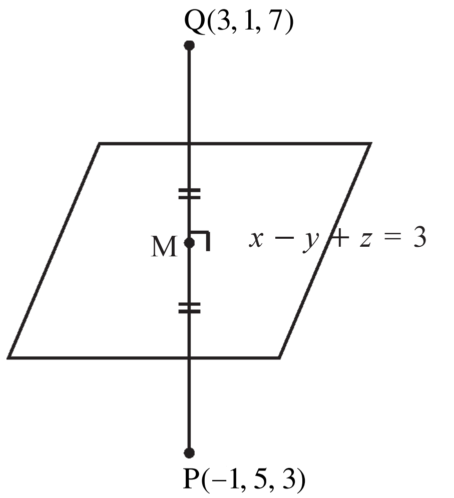
$$= \frac{\pi^2}{4} - 2$$

Ans. (A)

42. Let P be the image of the point (3, 1,7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is

(A) $x + y - 3z = 0$ (B) $3x + z = 0$ (C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

Sol. : Mirror image of (3, 1, 7)



$$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-7}{1} = \frac{-2(3-1+7-3)}{3} = -4.$$

So, P = (–1, 5, 3)

The equation of the plane passing through the line and (−1, 5, 3) has normal

$$\vec{n} = (1, 2, 1) \times (\vec{b} - \vec{a})$$

$$= (1, 2, 1) \times (-1, 5, 3) = (1, -4, 7)$$

∴ The equation is $(x - 0) - 4(y - 0) + 7(z - 0) = 0$.

∴ $x - 4y + 7z = 0$

Ans. (C)

SECTION 2

43. Let $a, b \in$ R and f : R → R be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 - x|)$. Then f is

(A) differentiable at $x = 0$ if $a = 0$ and $b = 1$

(B) differentiable at $x = 1$ if $a = 1$ and $b = 0$

(C) not differentiable at $x = 0$ if $a = 1$ and $b = 0$

(D) not differentiable at $x = 1$ if $a = 1$ and $b = 1$

Sol. : $f(x) = a\cos(x^3 - x) + bx\sin(x(x^2 - 1))$ ($\cos$ is even)

It is a differentiable function, $\forall x \in$ R

Ans. (A), (B)

44. Let $f(x) = \lim_{n\to\infty}\left(\dfrac{n^n(x+n)\left(x+\frac{n}{2}\right)\ldots\left(x+\frac{n}{n}\right)}{n!\,(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\ldots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all $x > 0$. Then

(A) $f\left(\frac{1}{2}\right) \geq f(1)$ (B) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$ (C) $f'(2) \leq 0$ (D) $\dfrac{f'(3)}{f(3)} \geq \dfrac{f'(2)}{f(2)}$

Sol. : $f(x) = \lim_{n\to\infty}\left(\dfrac{n^{2n}\left(\frac{x}{n}+1\right)\left(\frac{x}{n}+\frac{1}{2}\right)\ldots\left(\frac{x}{n}+\frac{1}{n}\right)}{n!\,n^{2n}\left(\frac{x^2}{n^2}+1\right)\left(\frac{x^2}{n^2}+\frac{1}{2^2}\right)\ldots\left(\frac{x^2}{n^2}+\frac{1}{n^2}\right)}\right)^{\frac{x}{n}}$

$\therefore \quad \log f(x) = \lim_{n\to\infty}\dfrac{x}{n}\left\{\Sigma\log\left(\dfrac{x}{n}+\dfrac{1}{r}\right) - \Sigma\log\left(\dfrac{rx^2}{n^2}+\dfrac{1}{r}\right)\right\}$

$= \lim_{n\to\infty}\dfrac{x}{n}\,\Sigma\left\{\log\left(1+\dfrac{rx}{n}\right) - \log\left(1+\dfrac{r^2x^2}{n^2}\right)\right\}$

$= x\int_0^1 \log(1+xy)dy - x\int_0^1 \log(1+x^2y^2)dy$

Let $xy = t$

$\log f(x) = \int_0^x \log(1+t)dt - \int_0^x \log(1+t^2)dt$

$\therefore \quad \dfrac{f'(x)}{f(x)} = \log\left(\dfrac{1+x}{1+x^2}\right)$

$\dfrac{f'(2)}{f(2)} = \log\dfrac{3}{5} < 0$ and $\dfrac{f'(3)}{f(3)} = \log\dfrac{2}{5} < 0$

Also, $f'(2) < 0$ as $f(2) > 0$

$\dfrac{f'(2)}{f(2)} = \log\dfrac{3}{5} < 0$ and $\dfrac{f'(3)}{f(3)} = \log\dfrac{2}{5} < 0$. So, $\dfrac{f'(2)}{f(2)} > \dfrac{f'(3)}{f(3)}$

Also in (0, 1), $\dfrac{f'(x)}{f(x)} > 0$ and in $(1, \infty)$, $\dfrac{f'(x)}{f(x)} < 0$

$\therefore \quad f$ is $\uparrow$ in (0, 1) and $\downarrow$ in $(1, \infty)$.

$\therefore \quad f(1) \geq f\left(\frac{1}{2}\right)$ and $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$ **Ans. (B), (C)**

45. Let $f: \mathrm{R} \to (0, \infty)$ and $g: \mathrm{R} \to \mathrm{R}$ be twice differentiable functions such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x\to 2}\dfrac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

(A) f has a local minimum at $x = 2$

(B) f has a local maximum at $x = 2$

(C) $f''(2) > f(2)$

(D) $f(x) - f''(x) = 0$ for at least one $x \in R$

Sol. : $\lim_{x \to 2} \dfrac{f(x)g(x)}{f'(x)g'(x)} = 1$ ($\frac{0}{0}$ form)

$\therefore \quad \lim_{x \to 2} \dfrac{f'(x)g(x) + g'(x)f(x)}{f''(x)g'(x) + f'(x)g''(x)} = 1$

$\therefore \quad \dfrac{g'(2)f(2)}{f''(2)g'(2)} = 1$

$\therefore \quad f''(2) = f(2)$. Also, $f(2) \in (0, \infty)$

Since $f(2) > 0, f''(2) > 0$

$\therefore \quad f$ has a local minimum at $x = 2$. **Ans. (A), (D)**

46. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in R^3 and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector $\vec{v}$ in R^3 such that $|\hat{u} \times \hat{v}| = 1$ and $\hat{w} \cdot (\hat{u} \times \hat{v}) = 1$. Which of the following statement(s) is(are) correct ?

(A) There is exactly one choice for such $\vec{v}$

(B) There are in infinitely many choices for such $\vec{v}$

(C) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$

(D) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$

Sol. : $\hat{w} \cdot (\hat{u} \times \hat{v}) = 1$

$\therefore \quad |\hat{w}||\hat{u} \times \hat{v}| \cos\alpha = 1$. So, $\cos\alpha = 1$ where $\alpha = (\hat{w}, \hat{u} \times \hat{v})$. $|\hat{w}| = 1, |\hat{u} \times \hat{v}| = 1$.

So, $\hat{w}$ is parallel to $\hat{u} \times \hat{v}$

$\therefore \quad \hat{w} = k(\hat{u} \times \hat{v})$. So, $\hat{w} \cdot \hat{u} = 0, \hat{w} \cdot \hat{v} = 0$

$\therefore \quad \hat{w} \perp \hat{u}$ and $\hat{w} \perp \hat{v}$

As it is given there exists a vector $\vec{v}$, $\hat{w}$ must be perpendicular to $\hat{u}$

Infinitely many such $\vec{v}$ exists.

If $\hat{u} = u_1\hat{i} + u_2\hat{j}$

$\hat{u} \cdot \hat{w} = 0 \Rightarrow (u_1 + u_2) = 0$

$\Rightarrow |u_1| = |u_2|$

if $u = u_1\hat{i} + u_3\hat{k}$, $\vec{u} \cdot \vec{w} = 0$

$u_1 + 2u_3 = 0$

$\therefore |u_1| = 2|u_3|$ **Ans. (B), (C)**

47. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then

(A) $SP = 2\sqrt{5}$

(B) $SQ : QP = (\sqrt{5} + 1) : 2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$

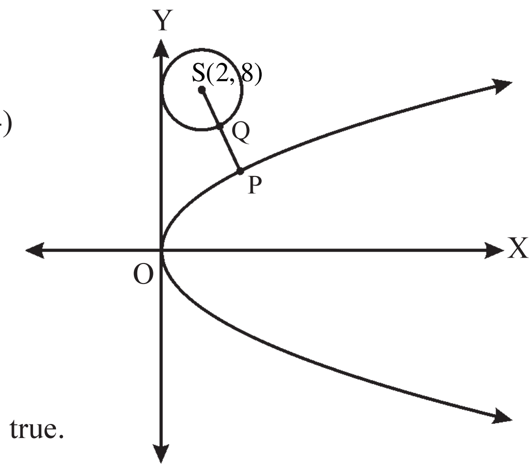
Sol. : The point at the shortest distance lies along their common normal.

S (centre of the circle) = (2, 8) $\quad$ $P(t^2, 2t)$

The slope of normal = $\frac{2t-8}{t^2-2} = -t$ (for parabola)

$\therefore \quad t^3 = 8$

$\therefore \quad t = 2$ $\quad$ P(4, 4)



The equation of the normal at P,

$y + tx = 2t + t^3$

$y + 2x = 4 + 8$

$\therefore \quad \frac{2x}{12} + \frac{y}{12} = 1$

$\therefore$ The X-intercept of the normal at P is 6. $\quad$ (C) is true.

$SP = \sqrt{4+16} = 2\sqrt{5}$. So, (A) is true.

The slope of the normal at Q is $-t = -2$.

The slope of the tangent at Q is $\frac{1}{2}$.

(D) is true.

SQ = radius = $\sqrt{4+64-64} = 2$

$QP = SP - SQ = 2\sqrt{5} - 2$

$\therefore \quad \frac{SQ}{QP} = \frac{2}{2(\sqrt{5}-1)} = \frac{\sqrt{5}+1}{4}$. (B) is not true. **Ans. (A), (C), (D)**

48. Let $a, b \in$ R and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in C \,\middle|\, z = \frac{1}{a+ibt}, t \in R, t \neq 0\right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in$ S, then (x, y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) the X-axis for $a \neq 0, b = 0$

(D) the Y-axis for $a = 0, b \neq 0$

Sol. : $x + iy = \frac{1}{a+ibt} = \frac{a-ibt}{a^2+b^2t^2}$. So, $x = \frac{a}{a^2+b^2t^2}$, $y = \frac{-bt}{a^2+b^2t^2}$

$\therefore \quad \frac{y}{x} = -\frac{bt}{a}$. So, $t = \frac{-ay}{bx}$. Let $a \neq 0$. So, $x \neq 0$.

Thus, $a^2 + b^2t^2 = \frac{a}{x} \Rightarrow a^2 + b^2 \frac{a^2y^2}{b^2x^2} = \frac{a}{x}$

$\therefore \quad a^2(x^2 + y^2) = ax$

$\therefore \quad x^2 + y^2 - \frac{x}{a} = 0.$

$\therefore$ This is a circle with centre $\left(\frac{1}{2a}, 0\right)$ and radius $\frac{1}{2a}$, if $a > 0$. So, (A) is true.

If $a = 0$, $x = 0$, $y = \frac{-1}{bt}$. This is Y-axis, if $b \neq 0$. (Infact Y-axis – {(0, 0)})

If $b = 0$, then $y = 0$, $x = \frac{1}{a}$. $a \neq 0$

This is X-axis. (Infact X-axis – {(0, 0)}) **Ans. (A), (C), (D)**

49. Let $a, \lambda, \mu \in$ R. Consider the system of linear equations

$ax + 2y = \lambda$

$3x - 2y = \mu$

Which of the following statement(s) is(are) correct ?

(A) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ

(B) If $a \neq -3$, then the system has a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$

(D) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$

Sol. : System has unique solution for $\frac{a}{3} \neq \frac{2}{-2}$. i.e. $a \neq -3$. So, (B) is true.

System has infinitely many solutions for $\frac{a}{3} = \frac{2}{-2} = \frac{\lambda}{\mu}$. i.e. $a = -3$ and $\lambda + \mu = 0$.

$\therefore$ (C) is true.

and no solution for $\frac{a}{3} = \frac{2}{-2} \neq \frac{\lambda}{\mu}$. i.e. $a = -3$, $\lambda + \mu \neq 0$.

$\therefore$ (D) is true. **Ans. (B), (C), (D)**

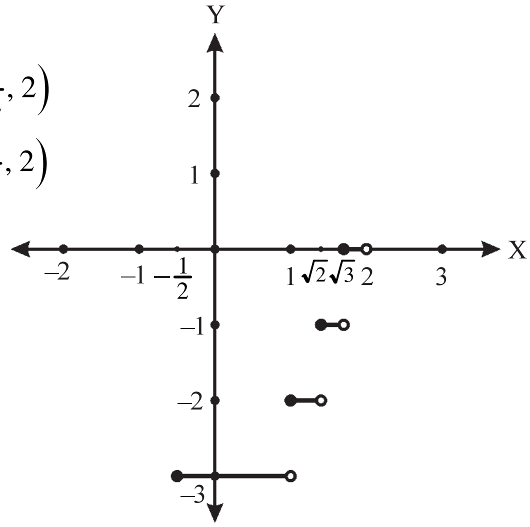
50. Let $f : \left[-\frac{1}{2}, 2\right] \to$ R and $g : \left[-\frac{1}{2}, 2\right] \to$ R be functions defined by $f(x) = [x^2 - 3]$ and g$(x) = |x| f(x) + |4x - 7| f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in$ R. Then,

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$

(C) g is not differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$

(D) g is not differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$



Sol. : $f(x) = [x^2 - 3] = [x^2] - 3 = \begin{cases} -3 & -\frac{1}{2} \le x < 1 \\ -2 & 1 \le x < \sqrt{2} \\ -1 & \sqrt{2} \le x < \sqrt{3} \\ 0 & \sqrt{3} \le x < 2 \\ 1 & x = 2 \end{cases}$

$g(x) = |x| f(x) + |4x - 7| f(x)$

$$= (|x| + |4x - 7|)[x^2 - 3] = \begin{cases} (-x - 4x + 7)(-3) & -\frac{1}{2} \le x < 0 \\ (x - (4x - 7))(-3) & 0 \le x < 1 \\ (x - (4x - 7))(-2) & 1 \le x < \sqrt{2} \\ (x - (4x - 7))(-1) & \sqrt{2} \le x < \sqrt{3} \\ (x - (4x - 7))(0) & \sqrt{3} \le x < \frac{7}{4} \\ x + (4x - 7)(0) & \frac{7}{4} \le x < 2 \\ (x + (4x - 7)(1) & x = 2 \end{cases}$$

$$= \begin{cases} 15x - 21 & -\frac{1}{2} \le x < 0 \\ 9x - 21 & 0 \le x < 1 \\ 6x - 14 & 1 \le x < \sqrt{2} \\ 3x - 7 & \sqrt{2} \le x < \sqrt{3} \\ 0 & \sqrt{3} \le x < 2 \\ 3 & x = 2 \end{cases}$$

Clearly f is not continuous at exactly 4 points in $\left[-\frac{1}{2}, 2\right]$ and g is not differentiable at 4 points in $\left(-\frac{1}{2}, 2\right)$. Hence, answer is (B), (C).

f is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}, 2$

$g(x) = f(x)\ [|x| + |4x - 7|]$

$f(x)$ is non-differentiable at $x = 1, \sqrt{2}, \sqrt{3}$

and $|x| + |4x - 7|$ is non differentiable at $x = 0, \frac{7}{4}$

But $f(x) = 0.\ \forall \in [\sqrt{3}, 2)$

Hence $g(x)$ is non-differentiable $x = 0, 1, \sqrt{2}, \sqrt{3}$ **Ans. (B), (C)**

SECTION 3

Paragraph 1

Football terms T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total number of points scored by teams T_1 and T_2, respectively, after two games.

51. P(X > Y) is

(A) $\frac{1}{4}$ (B) $\frac{5}{12}$ (C) $\frac{1}{2}$ (D) $\frac{7}{12}$

Sol. : P(X > Y) = P(T_1 wins both) + P(T_1 wins either of the matches and other is draw)

$= \frac{1}{2} \times \frac{1}{2} + 2 \times \frac{1}{2} \times \frac{1}{6} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ **Ans. (B)**

52. P(X = Y) is

(A) $\frac{11}{36}$ (B) $\frac{1}{3}$ (C) $\frac{13}{36}$ (D) $\frac{1}{2}$

Sol. : P(X = Y) = P(T_1 and T_2 win alternately) + P(Both matches are drawn)

$= 2 \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{6} \times \frac{1}{6} = \frac{1}{3} + \frac{1}{36} = \frac{13}{36}$ **Ans. (C)**

Paragraph 2

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$, for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

53. The orthocentre of the triangle F_1MN is ....

(A) $\left(-\frac{9}{10}, 0\right)$ (B) $\left(\frac{2}{3}, 0\right)$ (C) $\left(\frac{9}{10}, 0\right)$ (D) $\left(\frac{2}{3}, \sqrt{6}\right)$

Sol. : $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{8}{9}} = \frac{1}{3}$

$F_1 = (-1, 0)$ $x_1 < 0$

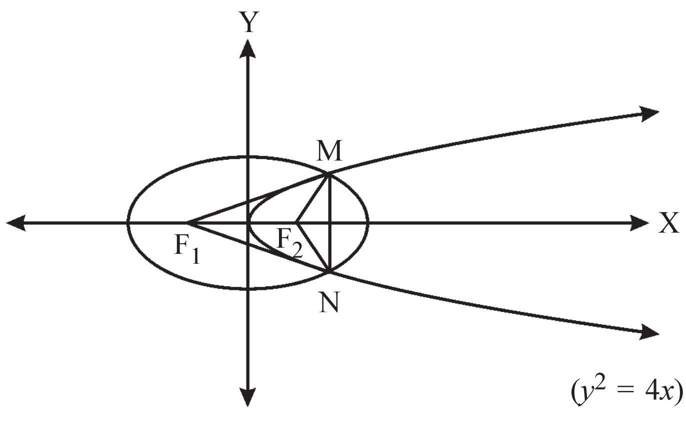
$F_2 = (1, 0)$ $x_2 > 0$

Parabola is $y^2 = 4x$.

For intersection of ellipse and parabola,

$\frac{x^2}{9} + \frac{4x}{8} = 1$

$\therefore$ $(x + 6)(2x - 3) = 0$



$\therefore \quad x = \frac{3}{2}$ as $x \neq -6$ for M

$\therefore \quad M = \left(\frac{3}{2}, \sqrt{6}\right), N = \left(\frac{3}{2}, -\sqrt{6}\right)$

The altitude from M is $y - \sqrt{6} = \frac{5}{2\sqrt{6}}\left(x - \frac{3}{2}\right)$ as slope of $\overrightarrow{F_1N} = \frac{-\sqrt{6}-0}{\frac{3}{2}+1} = \frac{-2\sqrt{6}}{5}$

One altitude is $y = 0$. ($\overleftrightarrow{MN} \perp$ X-axis)

$\therefore$ The orthocentre is $\left(-\frac{9}{10}, 0\right)$.

$\left(0 - \sqrt{6} = \frac{5}{2\sqrt{6}}\left(x - \frac{3}{2}\right) \text{ gives } x = -\frac{9}{10}\right)$ **Ans. (A)**

54. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the X-axis at Q, then the ratio of area of the triangle MQR to the area of the quadrilateral MF_1NF_2 is

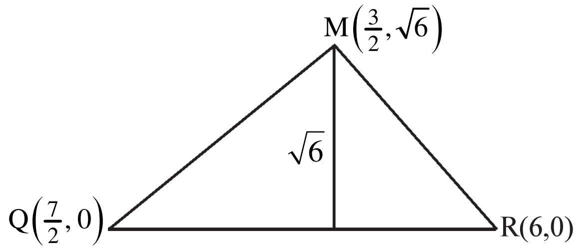
(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3

Sol. : The equation of the tangents at M and N are $\frac{x}{6} \pm \frac{y\sqrt{6}}{8} = 1$

They intersect in R(6, 0).

The equation of the normal to the parabola $(y - \sqrt{6}) = -\frac{\sqrt{6}}{2}\left(x - \frac{3}{2}\right)$ $\left(\text{Slope is } -\frac{y_1}{2a}\right)$

Q is $\left(\frac{7}{2}, 0\right)$.



Area of $\Delta MQR = \frac{1}{2} \times \sqrt{6} \times \frac{5}{2} = \frac{5\sqrt{6}}{4}$

Area of $MF_1NF_2 = 2\sqrt{6}$

Ratio $= \frac{5}{8}$

$(MF_1NF_2 = 2MF_1F_2 = 2 \times 1 \times \sqrt{6} = 2\sqrt{6})$ **Ans. (C)**

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