MATHEMATICS

Time allowed : 3 hours   **Maximum Marks: 100**

General Instructions:

1. All questions are compulsory.

2. This question paper contains 29 questions.

3. Questions 1 – 4 in Section A are very short-answer type questions carrying 1 mark each.

4. Questions 5 – 12 in Section B are short-answer type questions carrying 2 marks each.

5. Questions 13 – 23 in Section C are long-answer I type questions carrying 4 marks each.

6. Questions 24 – 29 in Section D are long-answer II type questions carrying 6 marks each.

Section A

1. If A and B are invertible matrices of order 3, $|A| = 2$ and $|(AB)^{-1}| = -\dfrac{1}{6}$. Find |B|.

2. Differentiate $\sin^2(x^2)$ w.r.t x^2.

3. Write the order of the differential equation:

$$\log\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^3 + x$$

4. Find the acute angle which the line with direction cosines $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{6}}, n$ makes with

    positive direction of z-axis.

                                    OR

    Find the direction cosines of the line: $\dfrac{x-1}{2} = -y = \dfrac{z+1}{2}$

Section B

5. Let $A = Z \times Z$ and * be a binary operation on A defined by

    $(a, b) * (c, d) = (ad + bc, bd)$.

    Find the identity element for * in the set A.

6. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = 5A + kI$.

7. Find : $\int \dfrac{(x^2 + \sin^2 x)\sec^2 x}{1+x^2}\,dx$

8. Find: $\int \dfrac{e^x(x-3)}{(x-1)^3}\,dx$

OR

Find: $\int \dfrac{(x^4 - x)^{\frac{1}{4}}}{x^5}\,dx$

9. Form the differential equation of all circles which touch the x-axis at the origin.

10. Find the area of the parallelogram whose diagonals are represented by the vectors

$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

OR

Find the angle between the vectors

$\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

11. If A and B are two independent events, prove that A′ and B are also independent.

12. One bag contains 3 red and 5 black balls. Another bag contains 6 red and 4 black balls. A ball is transferred from first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is red.

OR

If P(A) = 0.6, P(B) = 0.5 and P(A|B) = 0.3, then find P(A ∪·B).

Section C

13. Prove that the function f:[0, ∞) → R given by $f(x) = 9x^2 + 6x - 5$ is not invertible. Modify the codomain of the function f to make it invertible, and hence find f^{-1}.

OR

Check whether the relation R in the set **R** of real numbers, defined by

R = {(a, b) : 1 + ab > 0}, is reflexive, symmetric or transitive.

14. Find the value of : $\sin\left(2\tan^{-1}\dfrac{1}{4}\right) + \cos(\tan^{-1} 2\sqrt{2})$

15. Using properties of determinants, prove that:

$$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)\,(a^2+b^2+c^2)$$

16. If $y = x^{\sin x} + \sin(x^x)$, find $\dfrac{dy}{dx}$

OR

If $y = \log(1+2t^2+t^4)$, $x = \tan^{-1} t$, find $\dfrac{d^2y}{dx^2}$

17. If $y = \cos(m\cos^{-1} x)$

Show that: $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2 y = 0$

18. Find the equations of the normal to the curve $y = 4x^3 - 3x + 5$ which are perpendicular to the line $9x - y + 5 = 0$.

19. Find: $\displaystyle\int \dfrac{x^4+1}{x(x^2+1)^2}\,dx$

20. Evaluate: $\displaystyle\int_{-1}^{1} \dfrac{x+|x|+1}{x^2+2|x|+1}\,dx$

21. Find the particular solution of the following differential equation.

$$\cos y\,dx + (1+2e^{-x})\sin y\,dy = 0;\; y(0) = \dfrac{\pi}{4}$$

OR

Find the general solution of the differential equation:

$$\dfrac{dx}{dy} = \dfrac{y\tan y - x\tan y - xy}{y\tan y}$$

22. If $\vec{p} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{q} = \hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude $5\sqrt{3}$ units perpendicular to the vector $\vec{q}$ and coplanar with vectors $\vec{p}$ and $\vec{q}$.

23. Find the vector equation of the line joining (1, 2, 3) and (−3, 4, 3) and show that it is perpendicular to the z-axis.

Section D

24. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & -3 \\ 2 & 0 & -1 \end{bmatrix}$, find A^{-1}.

Hence, solve the system of equations:

$$3x + 3y + 2z = 1$$

$$x + 2y = 4$$

$$2x - 3y - z = 5$$

OR

Find the inverse of the following matrix using elementary transformations.

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$$

25. A cuboidal shaped godown with square base is to be constructed. Three times as much cost per square meter is incurred for constructing the roof as compared to the walls. Find the dimensions of the godown if it is to enclose a given volume and minimize the cost of constructing the roof and the walls.

26. Find the area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and $x - $ axis.

OR

Find the area of the region.

$\{(x, y) : x^2 + y^2 \le 8, \ x^2 \le 2y\}$

27. Find the equation of the plane through the line $\dfrac{x-1}{3} = \dfrac{y-4}{2} = \dfrac{z-4}{-2}$ and parallel to the line $\dfrac{x+1}{2} = \dfrac{1-y}{4} = \dfrac{z+2}{1}$. Hence, find the shortest distance between the lines.

OR

Show that the line of intersection of the planes $x + 2y + 3z = 8$ and $2x + 3y + 4z = 11$ is coplanar with the line $\dfrac{x+1}{1} = \dfrac{y+1}{2} = \dfrac{z+1}{3}$. Also find the equation of the plane containing them.

28. A manufacturer makes two types of toys A and B. Three machine are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Types of Toys	Machines		
	I	II	III
A	20	10	10
B	10	20	30

The machines I, II and III are available for a maximum of 3 hours, 2 hours and 2 hours 30 minutes respectively. The profit on each toy of type A is ₹ 50 and that of type B is ₹ 60. Formulate the above problem as a L.P.P and solve it graphically to maximize profit.

29. The members of a consulting firm rent cars from three rental agencies:
50% from agency X, 30% from agency Y and 20% from agency Z.
From past experience it is known that 9% of the cars from agency X need a service and tuning before renting, 12% of the cars from agency Y need a service and tuning before renting and 10% of the cars from agency Z need a service and tuning before renting. If the rental car delivered to the firm needs service and tuning, find the probability that agency Z is not to be blamed.

Marking Scheme

Section A

1. $\dfrac{1}{|\,AB\,|} = -\dfrac{1}{6} \Rightarrow \dfrac{1}{|\,A\,||\,B\,|} = -\dfrac{1}{6} \Rightarrow |\,B\,| = -3$. [1]

2. $2\sin(x^2)\cos(x^2)$ or $\sin(2x^2)$ [1]

3. 2 [1]

4. $l^2 + m^2 + n^2 = 1 \Rightarrow \left(\dfrac{1}{\sqrt{3}}\right)^2 + \left(\dfrac{1}{\sqrt{6}}\right)^2 + n^2 = 1 \Rightarrow \cos\gamma = \dfrac{1}{\sqrt{2}} \Rightarrow \gamma = 45° \text{ or } \dfrac{\pi}{4}$ [1]

OR

Direction ratios of the given line are 2, –1, 2. [1/2]

Hence, direction cosines of the line are:

$\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{2}{3}$ or $\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{-2}{3}$ [1/2]

Section B

5. An element $(e, f) \in Z \times Z$ bethe identity element, if

$(a, b) * (e, f) = (a, b) = (e, f) * (a, b) \ \forall \ (a, b) \in Z \times Z$ [1/2]

i.e., if, $(af + be, bf) = (a, b) = (eb + fa, fb)$

i.e., if, $af + be = a = eb + fa$ and $bf = b = fb$ …(1)

i.e., if, $f = 1, e = 0$ …(2) [1]

Hence, (0, 1) is the identity element. [1/2]

6. $A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$ [1/2]

$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}, \quad kI = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ [1/2]

$A^2 - 5A = kI$

$$\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \Rightarrow k = -7$$ [1]

7. $I = \int \left\{ \dfrac{(1+x^2)+(\sin^2 x - 1)}{1+x^2} \right\} \sec^2 x\, dx$ [1/2]

$\int \left[1 - \dfrac{\cos^2 x}{1+x^2} \right] \sec^2 x\, dx$

$\int \left[\sec^2 x - \dfrac{1}{1+x^2} \right] dx$ [1/2]

$= \tan x - \tan^{-1} x + c$ [1]

8. $\int \dfrac{e^x (x-3)}{(x-1)^3}\, dx$

$= \int e^x \left[\dfrac{(x-1)-2}{(x-1)^3} \right] dx$ [1/2]

$= \int e^x \left[\dfrac{1}{(x-1)^2} + \left(\dfrac{-2}{(x-1)^3} \right) \right]$ $\left(\because \int e^x [f(x)+f'(x)]dx = e^x f(x) + c \right)$ [1/2]

$= \dfrac{e^x}{(x-1)^2} + C$ [1]

OR

$I = \int \dfrac{(x^4 - x)^{\frac{1}{4}}}{x^5}\, dx$

$= \int \left[x^4 \left(1 - \dfrac{x}{x^4} \right) \right]^{\frac{1}{4}} \dfrac{1}{x^5}\, dx$ $\left[\dfrac{1}{2} \right]$

$= \int x \left(1 - \dfrac{1}{x^3} \right)^{\frac{1}{4}} \dfrac{1}{x^5}\, dx$

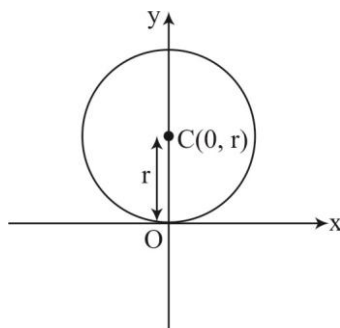$= \int \left(1 - \dfrac{1}{x^3} \right)^{\frac{1}{4}} \dfrac{1}{x^4}\, dx$

Put $1 - \dfrac{1}{x^3} = t$ $\left[\dfrac{1}{2} \right]$

So that, $\dfrac{3}{x^4}\,dx = dt$

$$= \dfrac{1}{3}\int t^{\frac{1}{4}}\,dt = \dfrac{1}{3}\cdot\dfrac{4}{5}\,t^{\frac{5}{4}} = \dfrac{4}{15}\left(1-\dfrac{1}{x^3}\right)^{\frac{5}{4}}+C$$ [1]

9. $(x-0)^2+(y-r)^2 = r^2$ [1/2]

$\Rightarrow \qquad\qquad x^2+y^2 = 2ry \qquad\qquad\qquad …(1)$

Differentiating both sides w.r.t. x, we get

$$2x+2yy' = 2ry'$$

$\Rightarrow \qquad\qquad r = \dfrac{x+yy'}{y'} \qquad\qquad\qquad …(2)$ [1]

Substituting r from (2) in (1), we get

$$(x^2+y^2)y' = 2y(x+yy')$$ [1/2]

10. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 2 & -1 & 2 \end{vmatrix} = -2\hat{i}+4\hat{j}+4\hat{k}$ [1]

$|\vec{a}\times\vec{b}| = \sqrt{4+16+16} = 6$ [1/2]

Area of the parallelogram $= \dfrac{|\vec{a}\times\vec{b}|}{2} = 3\text{sq units.}$ [1/2]

OR

The angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$ [1/2]

i.e., $\cos \theta = \dfrac{(\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}-\hat{j}+\hat{k})}{\sqrt{(1)^2+(1)^2+(-1)^2}\cdot\sqrt{(1)^2+(-1)^2+(1)^2}}$

i.e., $\cos \theta = \dfrac{1-1-1}{\sqrt{3}.\sqrt{3}}$ [1]

i.e., $\cos \theta = -\dfrac{1}{3}$

$\Rightarrow \theta = \cos^{-1}\left(-\dfrac{1}{3}\right)$ [1/2]

11. $P(A' \cap B) = P(B) - P(A \cap B)$ [1/2]

$\qquad\qquad = P(B) - P(A) \cdot P(B)[\because \text{ A and B are independentevents}]$

$\qquad\qquad = (1 - P(A)) \, P(B)$

$\qquad\qquad = P(A') \, P(B)$ [1]

Since, $P(A' \cap B) = P(A')P(B)$

Therefore A′ and B are independent events.   [1/2]

12. P (Red transferred and red drawn or black transferred and red drawn)

$\qquad\qquad = \dfrac{3}{8}\times\dfrac{7}{11}+\dfrac{5}{8}\times\dfrac{6}{11}$ [1]

$\qquad\qquad = \dfrac{51}{88}$ [1]

<div align="center">OR</div>

$P(A|B) = 0.3$

$\Rightarrow \dfrac{P(A \cap B)}{P(B)} = 0.3$

$\Rightarrow P(A \cap B) = 0.5 \times 0.3 = 0.15$ [1]

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) = 0.6 + 0.5 - 0.15$

$\Rightarrow P(A \cup B) = 0.95$ [1]

Section C

13. Let $y \in R$, then for any x, $f(x) = y$ if $y = 9x^2 + 6x - 5$

$\Rightarrow \qquad y = (3x)^2 + 2(3x)(1) + (1)^2 - 5 - (1)^2$

$\Rightarrow \qquad y = (3x + 1)^2 - 6$

$\Rightarrow \qquad 3x + 1 = \pm\sqrt{y + 6}$

$\Rightarrow \qquad x = \dfrac{\pm\sqrt{y + 6} - 1}{3}$

$\qquad x = \dfrac{\sqrt{y + 6} - 1}{3} \left[\because \dfrac{-\sqrt{y + 6} - 1}{3} \notin [0, \infty) \text{ for any value of } y \right]$

Now, for $y = -6 \in R$, $x = \dfrac{-1}{3} \notin [0, \infty)$

Hence, f(x) is not onto

$\Rightarrow$f(x) is not invertible. [1]

Since, $x \geq 0$, therefore $\dfrac{\sqrt{y + 6} - 1}{3} \geq 0$

i.e., $\qquad \sqrt{y + 6} \geq 1$

i.e., $\qquad y + 6 \geq 1$

$\qquad y \geq -5$

Redefining, $f : [0, \infty) \rightarrow [-5, \infty)$ [1]

makesf(x) $= 9x^2 + 6x - 5$ an onto function

Let $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$

$\Rightarrow (3x_1 + 1)^2 = (3x_2 + 1)^2$

$\Rightarrow [3(x_1 + x_2) + 2][3(x_1 - x_2)] = 0$

$\Rightarrow x_1 = x_2$ (as $3(x_1 + x_2) + 2 > 0$)

$\therefore f(x)$ is one-one [1]

Thus, f(x) is bijective, hence f is invertible

and $f^{-1} : [-5, \infty) \rightarrow [0, \infty)$

$f^{-1}(y) = \dfrac{\sqrt{y + 6} - 1}{3}$ [1]

OR

Reflexive:

R is reflexive, as $1 + a.a = 1 + a^2 > 0 \Rightarrow (a, a) \in R \; \forall \, a \in \mathbf{R}$ [1]

Symmetric:

If $(a, b) \in R$

then, $1 + ab > 0$

$\Rightarrow \qquad 1 + ba > 0$

$\Rightarrow \qquad (b, a) \in R$

Hence, R is symmetric. [1]

Transitive:

Let $a = -8, b = -1, c = \dfrac{1}{2}$

Since, $\qquad 1 + ab = 1 + (-8)(-1) = 9 > 0 \quad \therefore \; (a, b) \in R$

also, $\qquad 1 + bc = 1 + (-1)\left(\dfrac{1}{2}\right) = \dfrac{1}{2} > 0 \;\; \therefore \; (b, c) \in R$

But, $\qquad 1 + ac = 1 + (-8)\left(\dfrac{1}{2}\right) = -3 < 0$

Hence, R is not transitive. [2]

14. $\sin\left(2\tan^{-1}\dfrac{1}{4}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$

Lets evaluate, $\sin\left(2\tan^{-1}\dfrac{1}{4}\right)$

Put $\qquad \tan^{-1}\dfrac{1}{4} = \theta$

$\Rightarrow \qquad \tan\theta = \dfrac{1}{4}$

Now, $\qquad \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{2 \times \dfrac{1}{4}}{1 + \left(\dfrac{1}{4}\right)^2} = \dfrac{8}{17}$ $\qquad \left[1\dfrac{1}{2}\right]$

To evaluate $\cos(\tan^{-1}2\sqrt{2})$, put $\tan^{-1}2\sqrt{2} = \phi$

$$\Rightarrow \qquad \tan\phi = 2\sqrt{2}$$

$$\Rightarrow \qquad \cos\phi = \frac{1}{3} \qquad\qquad \left[1\frac{1}{2}\right]$$

$$\sin\left(2\tan^{-1}\frac{1}{4}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right) = \frac{8}{17} + \frac{1}{3} = \frac{41}{51} \qquad\qquad [1]$$

15.

$$\text{LHS} \qquad = \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} \; C_1 \to aC_1$$

$$= \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ca & b & c-a \\ a^2-ab & b+a & c \end{vmatrix} \; C_1 \to C_1 + bC_2 + cC_3 \qquad\qquad [1]$$

$$= \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix} \; \text{Taking } (a^2+b^2+c^2) \text{ common from } C_1$$

$$[1]$$

$$= \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix} \; \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix} \qquad\qquad \left[\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{(a^2+b^2+c^2)}{a}(-bc+a^2+ac+ba+bc)$$

$$= (a^2+b^2+c^2)(a+b+c) = \text{R.H.S} \qquad\qquad [1]$$

16. Let $u = x^{\sin x}$

$\Rightarrow \ln u = \sin x \ln x$

$\Rightarrow \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{\sin x}{x} + \ln x \cos x$

$\Rightarrow \dfrac{du}{dx} = x^{\sin x}\left[\dfrac{\sin x}{x} + \cos x \ln x\right]$ ...(1) $\left[1\dfrac{1}{2}\right]$

Let $v = \sin(x^x)$

Put $t = x^x$

     $\log t = x \ln x$

     $\dfrac{1}{t}\dfrac{dt}{dx} = x\dfrac{1}{x} + \ln x\, 1$

     $\dfrac{dt}{dx} = x^x(1+\ln x)$ ...(2)

$\dfrac{dv}{dx} = \cos(x^x) \times \dfrac{dt}{dx}$ (where $t = x^x$)

$\dfrac{dv}{dx} = \cos(x^x)\, x^x(1+\ln x)$ ...(3)   (using (2)) $\left[1\dfrac{1}{2}\right]$

Since, $y = u + v$

Therefore, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

$\dfrac{dy}{dx} = x^{\sin x}\left[\dfrac{\sin x}{x} + \cos x \ln x\right] + x^x(1+\ln x)\cos(x^x)$ [1]

<center>OR</center>

$y = \log(1+2t^2+t^4)$

$y = \log(1+t^2)^2$

$y = 2\log(1+t^2)$

$\dfrac{dy}{dt} = \dfrac{2(2t)}{1+t^2} = \dfrac{4t}{1+t^2}$ $\left[1\dfrac{1}{2}\right]$

$x = \tan^{-1} t$

$$\frac{dx}{dt} = \frac{1}{1+t^2}$$ $\left[\frac{1}{2}\right]$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dfrac{4t}{1+t^2}}{\dfrac{1}{1+t^2}} = 4t$$ [1]

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(4t) \times \frac{dt}{dx}$$

$$= 4 \times (1+t^2) = 4(1+t^2)$$ [1]

17. $y = \cos(m\cos^{-1}x)$

$\Rightarrow$ $\quad \dfrac{dy}{dx} = -\sin(m\cos^{-1}x) \times \dfrac{-m}{\sqrt{1-x^2}}$ [1]

$\Rightarrow$ $\quad \sqrt{1-x^2}\,\dfrac{dy}{dx} = m\sin(m\cos^{-1}x)$

Squaring both sides, we get

$\Rightarrow$ $\quad (1-x^2)\left(\dfrac{dy}{dx}\right)^2 = m^2\sin^2(m\cos^{-1}x)$ [1]

$\Rightarrow$ $\quad (1-x^2)\left(\dfrac{dy}{dx}\right)^2 = m^2(1-y^2)$ $\left[\frac{1}{2}\right]$

Differentiating again,

$(1-x^2)2\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) + \left(\dfrac{dy}{dx}\right)^2(-2x) = m^2\left(-2y\dfrac{dy}{dx}\right)$ [1]

$\Rightarrow$ $\quad (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2y = 0$ $\left[\frac{1}{2}\right]$

18. The given curve is

$y = 4x^3 - 3x + 5$

Let the required normal be at (x_1, y_1)

Slope of the tangent $= \dfrac{dy}{dx} = 12x^2 - 3$

m_1 = slope of the normal = $\dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1,\, x_2)}} = \dfrac{-1}{12x_1^2 - 3}$ [1]

m_2 = Slope of the line = 9

Since normal is perpendicular to the line.

Therefore, $m_1 \cdot m_2 = -1$

$$\dfrac{-1}{12x_1^2 - 3} \times 9 = -1$$

$\Rightarrow$ $12x_1^2 - 3 = 9$

$\Rightarrow$ $x_1 = \pm\, 1$

Hence, the points are (1, 6) and (−1, 4)   [1]

Equations of normals are:

$$y - 6 = -\dfrac{1}{9}(x - 1)\ \text{i.e., } x + 9y = 55$$ [1]

and $y - 4 = -\dfrac{1}{9}(x + 1)\ \text{i.e., } x + 9y = 35$ [1]

19. $I = \displaystyle\int \dfrac{x^4 + 1}{x(x^2 + 1)^2}\, dx$

$= \displaystyle\int \dfrac{(x^4 + 1)x}{x^2(x^2 + 1)^2}\, dx$ $\left[\dfrac{1}{2}\right]$

Put $x^2 = t$

So that $2x\, dx = dt$

$$I = \dfrac{1}{2}\int \dfrac{(t^2 + 1)\, dt}{t(t + 1)^2}$$ $\left[\dfrac{1}{2}\right]$

Now, $\dfrac{t^2 + 1}{t(t + 1)^2} = \dfrac{A}{t} + \dfrac{B}{t + 1} + \dfrac{C}{(t + 1)^2}$

$$t^2 + 1 = A(t + 1)^2 + Bt(t + 1) + Ct \qquad \text{...(1)}$$

On comparing the coefficients of like terms in (1), we get

$A = 1,\ B = 0,\ C = -2$ $\left[1\dfrac{1}{2}\right]$

$$I = \int \left[\frac{1}{t} - \frac{2}{(t+1)^2} \right] dt$$

$$= \log|t| - \frac{2(t+1)^{-1}}{-1} + C \qquad [1]$$

$$= 2\log|x| + \frac{2}{x^2+1} + C \qquad \left[\frac{1}{2}\right]$$

20. $I = \int_{-1}^{1} \frac{x+|x|+1}{x^2+2|x|+1} dx$

$$= \int_{-1}^{1} \frac{x}{x^2+2|x|+1} dx + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1} dx \qquad [1]$$

$$= I_1 + I_2 (\text{say}) \quad ....(1)$$

Now, $\qquad I_1 = \int_{-1}^{1} \frac{x}{x^2+2|x|+1} dx$

Let $\qquad f(x) = \frac{x}{x^2+2|x|+1}$

$$f(-x) = \frac{-x}{(-x)^2+2|-x|+1} = \frac{-x}{x^2+2|x|+1} = -f(x)$$

∵ f(x) is odd function.

Hence, $\qquad I_1 = 0...(2) \qquad [1]$

Also, $\qquad I_2 = \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1} dx$

Let $\qquad g(x) = \frac{|x|+1}{x^2+2|x|+1} \Rightarrow g(-x) = \frac{|-x|+1}{(-x)^2+2|-x|+1}$

$$g(-x) = \frac{|x|+1}{x^2+2|x|+1} = g(x)$$

∵ g(x) is even function $\qquad \left[\frac{1}{2}\right]$

∴ $\qquad I_2 = 2\int_{0}^{1} \frac{x+1}{x^2+2x+1} = 2\int_{0}^{1} \frac{1}{x+1} dx$

$$= 2\big[\ln|x+1|\big]_0^1 = 2\big[\ln 2 - \ln 1\big]$$

$$I_2 = 2\ln 2 \qquad\qquad \text{...(3)} \qquad\qquad [1]$$

From (1), (2) and (3), we get

$$I = 2\ln 2 \qquad\qquad\qquad \left[\dfrac{1}{2}\right]$$

21. $\cos y\, dx + (1 + 2e^{-x})\sin y\, dy = 0$

$\Rightarrow \qquad \displaystyle\int \dfrac{dx}{1+2e^{-x}} = \int \dfrac{-\sin y}{\cos y}\, dy \qquad\qquad [1]$

$\Rightarrow \qquad \displaystyle\int \dfrac{e^x}{2+e^x}\, dx = \int \dfrac{-\sin y}{\cos y}\, dy$

$\Rightarrow \qquad \ln(e^x + 2) = \ln|\cos y| + \ln C$

$\Rightarrow \qquad \ln(e^x + 2) = \ln|\cos y|\, C$

$\Rightarrow \qquad e^x + 2 = C\cos y \qquad\qquad \text{...(1)} \qquad \left[1\dfrac{1}{2}\right]$

$\Rightarrow e^x + 2 = \pm C\cos y \Rightarrow e^x + 2 = k\cos y$

Substituting $x = 0$, $y = \dfrac{\pi}{4}$ in (1), we get

$$1 + 2 = k\cos\dfrac{\pi}{4}$$

$\Rightarrow \qquad k = 3\sqrt{2} \qquad\qquad\qquad [1]$

$\therefore \qquad e^x + 2 = 3\sqrt{2}\,\cos y$ is the particular solution. $\qquad \left[\dfrac{1}{2}\right]$

OR

$\Rightarrow \qquad \dfrac{dx}{dy} = \dfrac{y\tan y - x\tan y - xy}{y\tan y}$

$$\dfrac{dx}{dy} + \left(\dfrac{1}{y} + \dfrac{1}{\tan y}\right)x = 1 \qquad\qquad [1]$$

$$\text{I.F} = e^{\int\left(\frac{1}{y}+\cot y\right)dy} = e^{\ln y + \ln\sin y}$$

$$\text{I.F} = e^{\ln(y\sin y)} = y\sin y \qquad\qquad [1]$$

Solution of the D.E. is:

$$x \times \text{I.F} = \int (Q \times \text{I.F})dy$$

$$\Rightarrow \qquad xy\sin y = \int y\sin\,dy \qquad\qquad [1]$$

$$\Rightarrow \qquad xy\sin y \ = \ y(-\cos y) - \int(-\cos y)dy$$

$$\Rightarrow \qquad xy\sin y = -y\cos y + \sin y + C$$

$$\Rightarrow \qquad x = \frac{\sin y - y\cos y + C}{y\sin y} \qquad\qquad [1]$$

22. Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ be the required vector.

Since, $\vec{r} \perp \vec{q}$

therefore, $\quad 1a - 2b + 1c = 0 \qquad\qquad \dots(1) \qquad [1]$

Also, $\vec{p}, \vec{q}$ and $\vec{r}$ are coplanar.

$\therefore [\vec{p}, \vec{q}\ \vec{r}] = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ a & b & c \end{vmatrix} = 0 \Rightarrow a - c = 0 \qquad\qquad \dots(2) \qquad [1]$$

Solving equation (1) and (2)

$$\frac{a}{2-0} = \frac{b}{1+1} = \frac{c}{0+2}$$

$$\Rightarrow \qquad \frac{a}{2} = \frac{b}{2} = \frac{c}{2}$$

i.e., $\qquad \dfrac{a}{1} = \dfrac{b}{1} = \dfrac{c}{1}$

$$\therefore \qquad \vec{r} \ = \ 1\hat{i} + 1\hat{j} + 1\hat{k} \qquad\qquad [1]$$

$$\left| \vec{r} \right| = \sqrt{3}$$

$\therefore$ Unit vector $\hat{r} = \dfrac{\vec{r}}{|r|} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

$\therefore$ Required vector $= 5\sqrt{3}\,\hat{r} = 5(\hat{i}+\hat{j}+\hat{k}) \qquad\qquad [1]$

23. Vector equation of the line passing through

(1, 2, 3) and (–3, 4, 3) is

$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ where

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \text{ and } \vec{b} = -3\hat{i} + 4\hat{j} + 3\hat{k}$$

$\Rightarrow \qquad \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(-4\hat{i} + 2\hat{j})$ ...(1)　　　　　　　　[2]

Equation of z-axis is

$$\vec{r} = \mu\hat{k} \qquad \qquad ...(2) \qquad \qquad [1]$$

Since $(-4\hat{i} + 2\hat{j}) \cdot \hat{k} = 0$

$\therefore$ line (1) is $\perp$ to z-axis.　　　　　　　　　　　　　　　　[1]

Section D

24. $A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & -3 \\ 2 & 0 & -1 \end{bmatrix}$

$|A| = 3(-2) - 1(3) + 2(-4) = -6 - 3 - 8 = -17 \neq 0$　　　　　[1]

$\therefore A^{-1}$ exists.　　　　　　　　　　　　　　　　　　　　　[1/2]

Cofactor matrix of $A = \begin{bmatrix} -2 & -3 & -4 \\ 1 & -7 & 2 \\ -7 & 15 & 3 \end{bmatrix}$ 　　　　[2]

$A^{-1} = \dfrac{1}{|A|} \text{Adj } A = \dfrac{1}{-17} \begin{bmatrix} -2 & 1 & -7 \\ -3 & -7 & 15 \\ -4 & 2 & 3 \end{bmatrix}$ 　　　$\left[\dfrac{1}{2}\right]$

Now for given system of equations.

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$(A^t)X = B$$

$\Rightarrow \qquad X = (A^t)^{-1}B$

$\Rightarrow \qquad X = (A^{-1})^t B \qquad \left[\because (A^t)^{-1} = (A^{-1})^t \right]$ $\qquad \left[\dfrac{1}{2} \right]$

$$X = \frac{1}{-17} \begin{bmatrix} -2 & -3 & -4 \\ 1 & -7 & 2 \\ -7 & 15 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$X = \frac{1}{-17} \begin{bmatrix} -34 \\ -17 \\ 68 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

$x = 2, \ y = 1, \ z = -4$ $\qquad \left[1\dfrac{1}{2} \right]$

<p style="text-align:center">OR</p>

<p style="text-align:center">$A = IA$</p> $\qquad \left[\dfrac{1}{2} \right]$

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_1 \to R_1 + R_3$$ $\qquad \left[\dfrac{1}{2} \right]$

$$\begin{bmatrix} -1 & 1 & 6 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_1 \to (-1) R_1$$ $\qquad \left[\dfrac{1}{2} \right]$

$$\begin{bmatrix} 1 & -1 & -6 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \begin{matrix} R_2 \to R_2 + 5R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$ $\qquad [1]$

$$\begin{bmatrix} 1 & -1 & -6 \\ 0 & -2 & -29 \\ 0 & -1 & -15 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ -5 & 1 & -5 \\ -3 & 0 & -2 \end{bmatrix} A \qquad R_2 \leftrightarrow R_3$$ $\qquad \left[\dfrac{1}{2} \right]$

$$\begin{bmatrix} 1 & -1 & -6 \\ 0 & -1 & -15 \\ 0 & -2 & -29 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ -3 & 0 & -2 \\ -5 & 1 & -5 \end{bmatrix} A \qquad R_2 \to (-1)R_2$$ $\qquad \left[\dfrac{1}{2} \right]$

$$\begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 15 \\ 0 & -2 & -29 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 3 & 0 & 2 \\ -5 & 1 & -5 \end{bmatrix} A$$

$$R_1 \to R_1 + R_2$$
$$R_3 \to R_3 + 2R_2$$

[1]

$$\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 15 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} A$$

$$R_1 \to R_1 - 9R_3$$
$$R_2 \to R_2 - 15R_3$$

[1]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix} A$$

Hence, $A^{-1} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$

$\left[\dfrac{1}{2}\right]$

25. Let the length and breadth of the base $=x$.

Also let the height of the godown $= y$.

Let C be the cost of constructing the godown and V be the given volume.

$\left[\dfrac{1}{2}\right]$

Since cost is proportional to the area, therefore

$$C = k[3x^2 + 4xy], \text{ where } k > 0 \text{ is constant of proportionality} \quad \ldots(1)$$ $\left[1\dfrac{1}{2}\right]$

$$x^2 y = V(\text{constant}) \ldots(2)$$ $\left[\dfrac{1}{2}\right]$

$$y = \frac{V}{x^2} \ldots(3)$$

Substituting value of y from equation (3), in equation (1), we get

$$C = k\left[3x^2 + 4x\left(\frac{V}{x^2}\right)\right] = k\left[3x^2 + \frac{4V}{x}\right]$$

$$\frac{dC}{dx} = k\left[6x - \frac{4V}{x^2}\right] \qquad \ldots(1)$$ [1]

For maximum or minimum value of S

$$\frac{dC}{dx} = 0$$

$\Rightarrow \qquad 6x - \dfrac{4V}{x^2} = 0$

$\Rightarrow \qquad x = \left(\dfrac{2V}{3}\right)^{\frac{1}{3}}$ [1]

when, $x = \left(\dfrac{2V}{3}\right)^{1/3}$, $\dfrac{d^2C}{dx^2} = 6 + \dfrac{8V}{x^3} = 18 > 0$ [1]

$\therefore$ C is minimum when $x = \left(\dfrac{2V}{3}\right)^{1/3}$ and $y = \dfrac{(18V)^{1/3}}{2}$ $\left[\dfrac{1}{2}\right]$

26. The given curves are

$$y = \sqrt{x} \qquad\qquad \ldots(1)$$

$$2y + 3 = x \qquad\qquad \ldots(2)$$

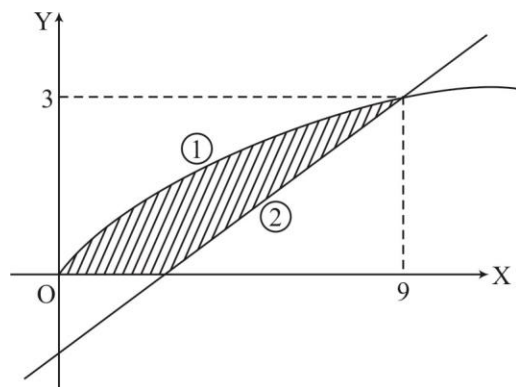Solving equation (1) and (2), we get

$$2y + 3 = y^2$$

$\Rightarrow \qquad y = -1, 3$

$\Rightarrow \qquad y = 3 (\text{as } y > 0)$

Substituting value of y = 3 in (2) we get

x = 2(3) + 3 = 9

i.e., (1) and (2) intersects at (9, 3) [1]

$\left[1\dfrac{1}{2}\right]$

Required Area $= \displaystyle\int_0^3 (2y+3)dy - \int_0^3 y^2 dy$ $\left[1\dfrac{1}{2}\right]$

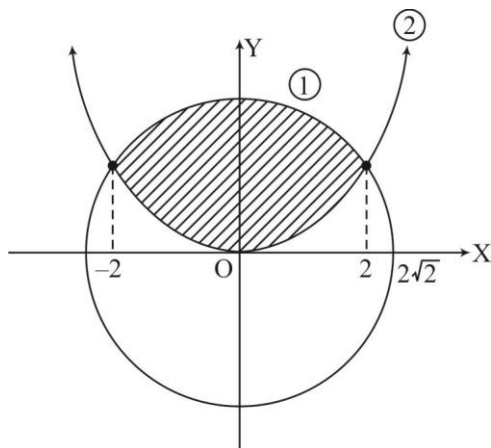$$= \left[y^2 + 3y - \frac{y^3}{3} \right]_0^3 \qquad\qquad [1]$$

$$= 9 + 9 - 9$$

$$= 9 \text{ sq units.} \qquad\qquad [1]$$

OR

The given curves are

$$x^2 + y^2 = 8 \qquad\qquad \ldots(1)$$

$$x^2 = 2y \qquad\qquad \ldots(2)$$

$$\left[1\frac{1}{2} \right]$$

Solving (1) and (2)

$$8 - y^2 = 2y \Rightarrow y = 2, -4 \Rightarrow y = 2 (\text{as } y > 0)$$

Substituting $y = 2$ in (2) we get $x^2 = 4 \Rightarrow x = -2$ or 2 $\qquad\qquad [1]$

Required Area $= \displaystyle\int_{-2}^{2} \sqrt{8 - x^2}\, dx - \int_{-2}^{2} \frac{x^2}{2}\, dx \qquad\qquad [1]$

$$= 2\left[\int_0^2 \sqrt{(2\sqrt{2})^2 - x^2}\, dx - \int_0^2 \frac{x^2}{2}\, dx \right]$$

$$= 2\left[\frac{x}{2}\sqrt{8 - x^2} + \frac{8}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}} \right) \right]_0^2 - \frac{1}{3}\left[x^3 \right]_0^2 \qquad\qquad \left[1\frac{1}{2} \right]$$

$$= 2\left[2 + 4\left(\frac{\pi}{4}\right) - 0\right] - \frac{1}{3}[8 - 0]$$

$$= 4 + 2\pi - \frac{8}{3}$$

$$= \left(2\pi + \frac{4}{3}\right) \text{sq. units} \qquad\qquad [1]$$

27. The two given lines are

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2} \qquad\qquad \ldots(1)$$

and $$\frac{x+1}{2} = \frac{y-1}{-4} = \frac{z+2}{1} \qquad\qquad \ldots(2)$$

Let a, b, c be the D.R's of the normal to the plane containing the line (1). Therefore, equation of plane is

$$a(x-1) + b(y-4) + c(z-4) = 0 \qquad \ldots(3) \qquad\qquad [1]$$

$$3a + 2b - 2c = 0 \ldots(4) \quad (\because \text{ Required plane contains line (1)})$$

$$2a - 4b + 1c = 0 \ldots(5) \ (\because \text{ line (2) is parallel to the required plane}) \quad \left[\frac{1}{2} + \frac{1}{2}\right]$$

$$\Rightarrow \qquad \frac{a}{2-8} = \frac{b}{-4-3} = \frac{c}{-12-4}$$

$$\Rightarrow \qquad \frac{a}{6} = \frac{b}{7} = \frac{c}{16} = \lambda$$

Putting, $a = 6\lambda, b = 7\lambda, c = 16\lambda$ in (3), we get $\qquad\qquad [1]$

$$\Rightarrow \qquad 6\lambda(x-1) + 7\lambda(y-4) + 16\lambda(z-4) = 0$$

$$\Rightarrow \qquad 6x + 7y + 16z - 98 = 0, \text{ which is the required equation of the plane} \qquad [1]$$

Since line (2) is parallel to required plane

$\therefore$ SD between two lines = Perpendicular distance of the point (–1, 1 –2) from the plane.

i.e., $$\text{SD} = \left|\frac{6(-1) + 7(1) + 16(-2) - 98}{\sqrt{6^2 + 7^2 + 16^2}}\right|$$

$$\text{SD} = \frac{129}{\sqrt{341}} \text{ units} \qquad\qquad [2]$$

OR

The given line $\dfrac{x+1}{1} = \dfrac{y+1}{2} = \dfrac{z+1}{3}$ …(1) is coplanar with the line determined by the planes

$x+2y+3z-8=0$ …(2) and $2x+3y+4z-11=0$ …(3), if we are able to show there exists a

plane passing through intersection of planes (2) and (3) containing the line (1).

Equation of the plane passing through the intersection of planes (2) and (3) is

$(x+2y+3z-8)+k(2x+3y+4z-11)=0$ ……(4) [2]

We find, value of k for which the plane given by (4) passes through the point $(-1, -1, -1)$ lying

on line (1).

Substituting the coordinates of the point $(-1, -1, -1)$ in (4), we get

$(-1-2-3-8) + k(-2-3-4-11) = 0$

$-14 - 20k = 0$

$k = -\dfrac{7}{10}$

Putting, $k = -\dfrac{7}{10}$ in (4) we get

$(x+2y+3z-8) - \dfrac{7}{10}(2x+3y+4z-11) = 0$

$\Rightarrow 4x+y-2z+3=0$ …(5) [2]

Now we find value of $a_1a_2 + b_1b_2 + c_1c_2$, where a_1, b_1, c_1 and D. Ratios of the line (1) and

a_2, b_2, c_2 and D. Ratios of the normal to the plane (5)

$a_1a_2 + b_1b_2 + c_1c_2 = 1(4) + 2(1) + 3(-2)$

i.e., $a_1a_2 + b_1b_2 + c_1c_2 = 0$ [1]

which implies line (1) lies in plane (5)

Hence the two lines are coplanar and the equation of the plane containing them is

$4x+y-2z+3=0$ [1]

28. Let the manufacturer make x and y quantity of toy A and toy B respectively.

$$\text{Max } P = 50x + 60y \qquad\qquad\qquad [1]$$

Subject to

$$20x + 10y \le 180 \qquad\qquad …(1)$$

$$10x + 20y \leq 120 \qquad \qquad \dots(2) \qquad \qquad \left[1\frac{1}{2}\right]$$

$$10x + 30y \leq 150 \qquad \qquad \dots(3)$$

$$x, y \geq 0$$

Corner Points	P = 50x + 60y
O (0, 0)	0
A (0, 5)	300
B (6, 3)	480
C (8, 2)	520 (Max)
D (9, 0)	450

[1]
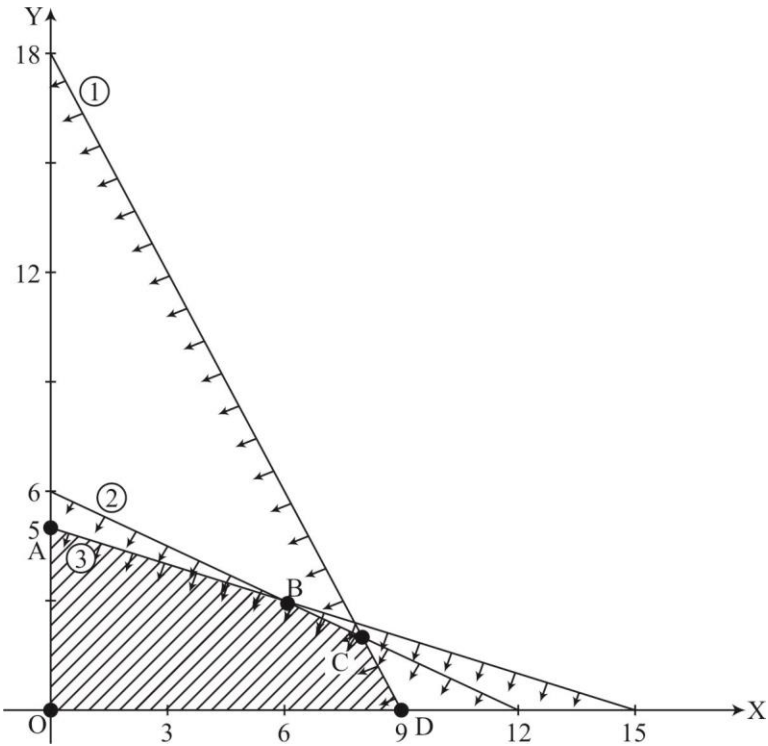
[2]

Hence, Max Profit is ₹ 520, at x = 8 and y = 2 $\qquad \left[\frac{1}{2}\right]$

29. Let A be the event thatcar delivered to firm needs service and tuning. Also let E_1, E_2 and

E_3 be the events ofcar being rented from agencies X, Y and Z respectively. $\qquad \left[\frac{1}{2}\right]$

$$P(E_1) = \frac{50}{100} \qquad P(E_2) = \frac{30}{100} \qquad P(E_3) = \frac{20}{100} \qquad\qquad [1]$$

$$P(A \mid E_1) = \frac{9}{100} \qquad P(A \mid E_2) = \frac{12}{100} \qquad P(A \mid E_3) = \frac{10}{100} \qquad \left[1\frac{1}{2}\right]$$

$$P(E_3 \mid A) = \frac{P(E_3)P(A \mid E_3)}{P(E_1)\,P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)} \qquad [1]$$

$$\Rightarrow \qquad P(E_3 \mid A) = \frac{\dfrac{20}{100} \times \dfrac{10}{100}}{\dfrac{50}{100} \times \dfrac{9}{100} + \dfrac{30}{100} \times \dfrac{12}{100} + \dfrac{20}{100} \times \dfrac{10}{100}} = \frac{20}{101} \qquad [1]$$

$$P(E'_3 \mid A) = 1 - P(E_3 \mid A) = 1 - \frac{20}{101} = \frac{81}{101} \qquad [1]$$