Exercise 13.3

Answer 1E.

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, on an interval $[a, b]$, then

the arc length of C on the interval is $s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$.

$$= \int_a^b \|\mathbf{r}'(t)\|\,dt$$

We have $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Evaluate $x'(t)$.

$$\begin{aligned} x'(t) &= \frac{d}{dt}(t) \\ &= 1 \end{aligned}$$

Now, find $y'(t)$.

$$\begin{aligned} y'(t) &= \frac{d}{dt}(3\cos t) \\ &= -3\sin t \end{aligned}$$

Determine $z'(t)$.

$$\begin{aligned} z'(t) &= \frac{d}{dt}(3\sin t) \\ &= 3\cos t \end{aligned}$$

We get $\mathbf{r}'(t) = \mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

We know that $\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$.

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(1)^2 + (-3\sin t)^2 + (3\cos t)^2} \\ &= \sqrt{1 + 9\sin^2 t + 9\cos^2 t} \\ &= \sqrt{10} \end{aligned}$$

.

Substitute -5 for a, 5 for b, and $\sqrt{10}$ for $\|\mathbf{r}'(t)\|$.

$$s = \int_{-5}^{5} \sqrt{10}\, dt$$
$$= 10\sqrt{10}$$

The length of the curve is obtained as $\boxed{10\sqrt{10}}$.

Answer 2E.

(r)t = <2t, t2 , (1/3)t3>

(r)'t = <2, 2t, t2>

$|(r)'t|$ = √(2)2 + (2t)2 + (t2)2

= √4 + 4t2 + t4

= √ (t2+2)2

= t2 + 2

L = $\int_0^1 t^2 + 2\, \mathrm{d}\, t$

$$= \left[\frac{t^3}{3} + 2t\right]_0^1$$

= (1/3)(1)3 + 2(1) - (1/3)(0)3 + 2(0)

Answer 3E.

Consider the curve $r(t)=\left(\sqrt{2}t\right)\mathbf{i}+e^t\mathbf{j}+e^{-t}\mathbf{k}.$

Find the arc length of the given curve on the interval $0\le t\le 1.$

Here,

$x=\sqrt{2}t$ ...... (1)

$y=e^t$ ...... (2)

$z=e^{-t}$ ...... (3)

Differentiate (1) with respect to t, get

$$\frac{dx}{dt}=\frac{d}{dt}\left(\sqrt{2}t\right)$$
$$=\sqrt{2}$$

Differentiate (2) with respect to t, get

$$\frac{dy}{dt}=\frac{d}{dt}\left(e^t\right)$$
$$=e^t$$

Differentiate (3) with respect to t, get

$$\frac{dz}{dt}=\frac{d}{dt}\left(e^{-t}\right)$$
$$=-e^{-t}$$

The parameter t varies from 0 to 1.

Arc length of the curve on the interval $0\le t\le 1$ is,

$$L=\int_a^b\sqrt{\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2+\left[\frac{dz}{dt}\right]^2}\,dt$$
$$=\int_0^1\sqrt{\left(\sqrt{2}\right)^2+\left(e^t\right)^2+\left(-e^{-t}\right)^2}\,dt$$
$$=\int_0^1\sqrt{2+e^{2t}+e^{-2t}}\,dt$$
$$=\int_0^1\sqrt{\left(e^t+e^{-t}\right)^2}\,dt$$
$$=\int_0^1\left(e^t+e^{-t}\right)dt$$
$$=\left[e^t-e^{-t}\right]_0^1$$
$$=\left(e^1-e^{-1}\right)-\left(e^0-e^{-0}\right)$$
$$=\left(e-e^{-1}\right)-\left(1-1\right)$$
$$=e-e^{-1}$$

Hence, the required arc length of the given curve on the interval $0\le t\le 1$ is $\boxed{e-e^{-1}}$.

Answer 4E.

First find the derivative $\mathbf{r}'(t)$ of the vector function (1).

Differentiate each component of $\mathbf{r}(t)$ to find the derivative of the vector function. Then,

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d}{dt}\mathbf{r}(t)\\ &= \frac{d}{dt}(\cos t\mathbf{i} + \sin t\mathbf{j} + \ln\cos t\mathbf{k})\\ &= \frac{d}{dt}(\cos t)\mathbf{i} + \frac{d}{dt}(\sin t)\mathbf{j} + \frac{d}{dt}(\ln\cos t)\mathbf{k}\\ &= -\sin t\mathbf{i} + \cos t\mathbf{j} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{k}\end{aligned}$$

$$= -\sin t\mathbf{i} + \cos t\mathbf{j} - \tan t\mathbf{k}.$$

Now, find the value of $|\mathbf{r}'(t)|$:

$$|\mathbf{r}'(t)| = |-\sin t\mathbf{i} + \cos t\mathbf{j} - \tan t\mathbf{k}|$$

$= \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}$ Use $|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = \sqrt{a^2 + b^2 + c^2}$

$$\begin{aligned}&= \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}\\ &= \sqrt{(\sin^2 t + \cos^2 t) + \tan^2 t}\\ &= \sqrt{1 + \tan^2 t}\\ &= \sqrt{\sec^2 t}\\ &= \sec t.\end{aligned}$$

Now, use formula (2) and find the length of the curve (1) for $0 \le t \le \frac{\pi}{4}$ as follows:

$L = \int_a^b |\mathbf{r}'(t)|\,dt$ Write formula (2)

$L = \int_0^{\pi/4} \sec t\,dt$ Because $|\mathbf{r}'(t)| = \sec t$

$$\begin{aligned}&= \left[\ln|\sec t + \tan t|\right]_0^{\pi/4}\\ &= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|\\ &= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0|\\ &= \ln\left(\sqrt{2} + 1\right) - \ln 1\\ &= \ln\left(\sqrt{2} + 1\right) - 0\\ &= \ln\left(\sqrt{2} + 1\right).\end{aligned}$$

Therefore, the length of the curve (1) for $0 \le t \le \frac{\pi}{4}$ is $L = \boxed{\ln\left(\sqrt{2} + 1\right)}$.

Answer 5E.

$$\vec{r}(t)=\hat{i}+t^2\hat{j}+t^3\hat{k}$$

$$\vec{r}\,'(t)=2t\,\hat{j}+3t^2\hat{k}$$

$$\left|\vec{r}\,'(t)\right|=\sqrt{(2t)^2+\left(3t^2\right)^2}$$

$$=\sqrt{4t^2+9t^4}$$

$$=t\sqrt{9t^2+4}$$

The length of curve line is

$$L=\int_0^1 t\sqrt{9t^2+4}\;dt$$

Substitute $9t^2+4=u$

$$18t\,dt=du\Rightarrow t\,dt=\frac{du}{18}$$

When $t=0$, then $u=4$ and when $t=1$ then $u=13$

Therefore $L=\int_4^{13}(u)^{1/2}\frac{du}{18}$

$$=\left.\frac{1}{18}\frac{u^{3/2}}{3/2}\right]_4^{13}$$

$$=\frac{2}{18\times 3}\left[13^{3/2}-4^{3/2}\right]$$

$$=\frac{1}{27}\left[13^{3/2}-2^3\right]$$

$$=\boxed{\frac{1}{27}\left[13^{3/2}-8\right]}$$

Answer 6E.

$$\vec{r}(t)=12t\hat{i}+8t^{3/2}\hat{j}+3t^2\hat{k},\qquad 0\le t\le 1$$

$$\vec{r}\,'(t)=12\hat{i}+12t^{1/2}\hat{j}+6t\,\hat{k}$$

$$\left|\vec{r}\,'(t)\right|=\sqrt{12^2+(12)^2\left(t^{1/2}\right)^2+(6t)^2}$$

$$=\sqrt{144+144t+36t^2}$$

$$=6\sqrt{4+4t+t^2}$$

$$=6\sqrt{(t+2)^2}$$

$$=6(t+2)$$

The length of the curve is

$$L = \int_0^1 6(t+2)\,dt$$
$$= \left(6\frac{t^2}{2} + 12t\right)\Big]_0^1$$
$$= \left(3t^2 + 12t\right)\Big]_0^1$$
$$= (3+12) - (0)$$
$$= \boxed{15}$$

Answer 7E.

The objective is to find the length of the curve that has the vector equation $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle,\ 0 \le t \le 2.$

To find the arc length first differentiate $\mathbf{r}(t)$ with respect to t.

$$\mathbf{r}'(t) = \left\langle \frac{d}{dt}t^2, \frac{d}{dt}t^3, \frac{d}{dt}t^4 \right\rangle$$
$$= \langle 2t, 3t^2, 4t^3 \rangle$$

The formula for the arc length is $s(t) = \int_a^b |\mathbf{r}'(t)|\,dt$.

Here $a = 0$ and $b = 2$.

Determine $|\mathbf{r}'(t)|$ as:

$$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2}$$
$$= \sqrt{4t^2 + 9t^4 + 16t^6}$$

Substitute the values in the integral.

$$s(t) = \int_0^2 \left(\sqrt{4t^2 + 9t^4 + 16t^6}\right) dt$$
$$= \left[\frac{1}{4}t^2\sqrt{16t^4 + 9t^2 + 4} + \frac{175}{1024}\sinh^{-1}\left(\frac{1}{35}\sqrt{7}\left(32t^2 + 9\right)\right) + \frac{9}{128}\sqrt{16t^4 + 9t^2 + 4}\right]_0^2$$
$$= \frac{137}{128}\sqrt{296} + \frac{175}{1024}\sinh^{-1}\left(\frac{137}{35}\sqrt{7}\right) - \frac{175}{1024}\sinh^{-1}\left(\frac{9}{35}\sqrt{7}\right) - \frac{18}{128}$$
$$= \frac{137}{64}\sqrt{74} + \frac{175}{1024}\sinh^{-1}\left(\frac{137}{35}\sqrt{7}\right) - \frac{175}{1024}\sinh^{-1}\left(\frac{9}{35}\sqrt{7}\right) - \frac{9}{64}$$
$$\approx 18.6833$$

Therefore, the length of the curve is $\boxed{18.6833}$.

Answer 8E.

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, on an interval $[a, b]$, then the arc length of C on the interval is $s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt$.

We have $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Evaluate $x'(t)$.

$$\begin{aligned} x'(t) &= \frac{d}{dt}(t) \\ &= 1 \end{aligned}$$

Now, find $y'(t)$.

$$\begin{aligned} y'(t) &= \frac{d}{dt}\left(e^{-t}\right) \\ &= -e^{-t} \end{aligned}$$

Determine $z'(t)$.

$$\begin{aligned} z'(t) &= \frac{d}{dt}\left(te^{-t}\right) \\ &= e^{-t} - te^{-t} \end{aligned}$$

We get $\mathbf{r}'(t) = \mathbf{i} - e^{-t}\mathbf{j} + \left(e^{-t} - te^{-t}\right)\mathbf{k}$

We know that $\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$.

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(1)^2 + \left(-e^{-t}\right)^2 + \left(e^{-t} - te^{-t}\right)^2} \\ &= \sqrt{1 + 2e^{-2t} - 2te^{-2t} + t^2e^{-2t}} \end{aligned}$$

Substitute 1 for a, 3 for b, and $\sqrt{1 + 2e^{-2t} - 2te^{-2t} + t^2e^{-2t}}$ for $\|\mathbf{r}'(t)\|$.

$$\begin{aligned} s &= \int_1^3 \sqrt{1 + 2e^{-2t} - 2te^{-2t} + t^2e^{-2t}}\, dt \\ &= 2.0454 \end{aligned}$$

The length of the curve is obtained as $\boxed{2.0454}$.

Answer 9E.

Approximate L using a calculator.

r(t) = $0 \le t \le \frac{\pi}{4}$

$$r'(t) = (\cos t,\ -\sin t,\ \sec^2 t)$$

$$|r'(t)| = \sqrt{\cos^2 t + \sin^2 t + \sec^4 t} = \sqrt{1 + \sec^4 t}$$

$$L = \int_a^b |r'(t)|\, d\, t$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \sec^4 t}\ d\, t$$

Approximating on a calculator gives 1.2780

Answer 11E.

A curve C is the intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$.

The objective is to find the length of the curve *C* from origin to the point $(6,18,36)$.

The length *L* of the curve $\mathbf{r}(t)$ for $a \le t \le b$ is defined as,

$$L = \int_a^b |\mathbf{r}'(t)|\, dt \quad \text{......(1)}$$

First find the vector equations of the curve *C*.

The projection of the curve *C* onto the *xy*-plane is the curve,

$x^2 = 2y,\ z = 0$, or, $y = \frac{x^2}{2},\ z = 0$.

So, choose the parameter $x = t$. This implies that $y = \frac{t^2}{2}$.

Because, the curve *C* also lies on the surface $3z = xy$, then, $z = \frac{1}{3}xy$. This implies that,

$$\begin{aligned} z &= \frac{1}{3}xy \\ &= \frac{1}{3}t \cdot \frac{t^2}{2} \\ &= \frac{t^3}{6}. \end{aligned}$$

Hence, the parametric equations of the curve *C* are,

$x = t,\ y = \frac{t^2}{2},\ z = \frac{t^3}{6}$.

And, the corresponding vector equation of the curve *C* is,

$\mathbf{r}(t) = \left\langle t, \frac{t^2}{2}, \frac{t^3}{6} \right\rangle$. ....(2)

Now, find the derivative $\mathbf{r}'(t)$ of the vector function (2).

Differentiate each component of $\mathbf{r}(t)$ to find the derivative of the vector function. Then,

$$\begin{aligned} \mathbf{r}'(t) &= \frac{d}{dt}\mathbf{r}(t) \\ &= \frac{d}{dt}\left\langle t, \frac{t^2}{2}, \frac{t^3}{6} \right\rangle \\ &= \left\langle \frac{d}{dt}(t), \frac{d}{dt}\left(\frac{t^2}{2}\right), \frac{d}{dt}\left(\frac{t^3}{6}\right) \right\rangle \\ &= \left\langle 1, t, \frac{1}{2}t^2 \right\rangle. \end{aligned}$$

Now, find the value of $|\mathbf{r}'(t)|$:

$$|\mathbf{r}'(t)| = \left|\left\langle 1, t, \frac{1}{2}t^2 \right\rangle\right|$$

$$= \sqrt{1^2 + t^2 + \left(\frac{1}{2}t^2\right)^2}$$ Use $|\langle a,b,c\rangle| = \sqrt{a^2+b^2+c^2}$

$$= \sqrt{1 + t^2 + \frac{t^4}{4}}$$

$$= \sqrt{1 + 2\cdot\frac{t^2}{2} + \left(\frac{t^2}{2}\right)^2}$$

$$= \sqrt{\left(1 + \frac{t^2}{2}\right)^2}$$

$$= 1 + \frac{t^2}{2} \quad \text{.......(3)}$$

Find the parameter values *t* corresponding to the origin $(0,0,0)$ and $(6,18,36)$.

The parametric equations of the curve *C* are,

$$x = t,\ y = \frac{t^2}{2},\ z = \frac{t^3}{6}.$$

Substitute $x = 0,\ y = 0,\ z = 0$ into the parametric equations and solve for *t*:

$$0 = t, \quad 0 = \frac{t^2}{2}, \quad 0 = \frac{t^3}{6}.$$

Solve these equations for *t*, it implies that $t = 0$. Hence, the parameter value corresponding to $(0,0,0)$ is $t = 0$.

Substitute $x = 6,\ y = 18,\ z = 36$ into the parametric equations and solve for *t*:

$$6 = t, \quad 18 = \frac{t^2}{2}, \quad 36 = \frac{t^3}{6}$$

Solve these equations for *t*, it implies that $t = 6$. Hence, the parameter value corresponding to the point $(6,18,36)$ is $t = 6$.

Now, use formula (1) and find the length of the curve C from $(0,0,0)$ to $(6,18,36)$, that is, for $0 \le t \le 6$ as follows:

$L = \int_a^b |\mathbf{r}'(t)|\,dt$ Write formula (1)

$L = \int_0^6 \left(1+\frac{t^2}{2}\right)dt$ From (3); $|\mathbf{r}'(t)| = 1+\frac{t^2}{2}$

$= \left[t+\frac{t^3}{6}\right]_0^6$

$= \left(6+\frac{6^3}{6}\right)-\left(0+\frac{0^3}{6}\right)$

$= (6+36)-0$

$= 42$.

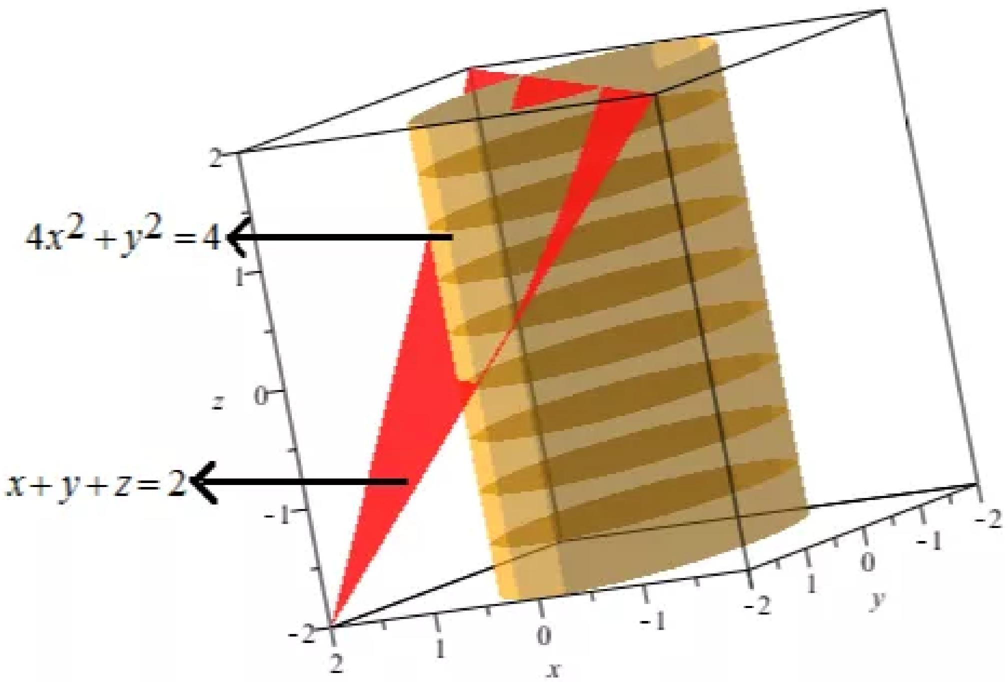
Therefore, the length of the curve C from origin to the point $(6,18,36)$ is $L = \boxed{42}$.

Answer 12E.

Consider the following length of the curve of intersection of the cylinder:

$4x^2 + y^2 = 4$

The plane is $x + y + z = 2$.



Find the parametric equations for the curve of intersection and substitute into the formula for arc length.

The first step is to find the parametric equations for the circular cylinder, which has the equation $4x^2 + y^2 = 4$.

Let $x = \cos t$, so the equation is as follows:

$$4x^2 + y^2 = 4$$
$$y^2 = 4 - 4x^2$$
$$y = \sqrt{4 - 4x^2}$$
$$y = 2\sqrt{1 - \cos^2 t} \quad \text{Since } x = \cos t$$
$$y = 2\sqrt{\sin^2 t}$$
$$y = 2\sin t$$

The curve of intersection with the other surface is the curve that satisfies the equations for both the surfaces.

Substitute $x = \cos t$ and $y = 2\sin t$ in the plane $x + y + z = 2$,

$$x + y + z = 2$$
$$\cos t + 2\sin t + z = 2$$
$$z = 2 - \cos t - 2\sin t$$

Since in order for a point to be on the curve of intersection it must satisfy all of the constraints, and have parametric equations for the curve of intersection:

$$x = \cos t$$
$$y = 2\sin t$$
$$z = 2 - \cos t - 2\sin t$$

Recollect the arc length equation:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Here $x = \cos t,\ y = 2\sin t,$ and $z = 2 - \cos t - 2\sin t$

Because all components depend only on $\cos t$ or $\sin t$.

The curve is periodic repeating itself every 2π. Therefore obtain limits of $t = 0$ and $t = 2\pi$ to traverse the curve of intersection once.

Now find the derivatives of the parametric equations:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(\cos t)\\ &= -\sin t\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}(2\sin t)\\ &= 2\cos t\end{aligned}$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{d}{dt}(2 - \cos t - 2\sin t)\\ &= \sin t - 2\cos t\end{aligned}$$

To find the range of *t* integrate at $t = 0$, the position being $(1,0,1)$. Return to this point after one cycle completes at $t = 2\pi$. Hence, equation is of the following form:

$$\begin{aligned}L &= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (2\cos t)^2 + (\sin t - 2\cos t)^2}\, dt\\ &= \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 t + \sin^2 t + 4\cos^2 t - 4\sin t\cos t}\, dt\\ L &= \int_0^{2\pi} \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\cos t}\, dt\end{aligned}$$

Use maple software to evaluate the above integral:

Maple Input:

```
int(sqrt(2*sin(t)^2+8*cos(t)^2-4*sin(t)*cos(t)), t = 0 .. 2*Pi);
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Maple Outputs:

$$\int_0^{2\pi} \sqrt{2\sin(t)^2 + 8\cos(t)^2 - 4\sin(t)\cos(t)}\, dt$$

$= 13.519$

Therefore, the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$ is $\boxed{13.519}$.

Answer 13E.

$$\vec{r}(t) = 2t\hat{i} + (1-3t)\hat{j} + (5+4t)\hat{k}$$

$$\vec{r}\,'(t) = 2\hat{i} + (-3)\hat{j} + 4\hat{k}$$

$$= 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\frac{ds}{dt} = \left|\vec{r}\,'(t)\right| = \sqrt{2^2 + (-3)^2 + 4^2}$$

$$= \sqrt{4+9+16}$$

$$= \sqrt{29}$$

And so $s = s(t) = \int_0^t \left|r\,'(u)\right| du$

$$= \int_0^t \sqrt{29}\, du$$

$$= \sqrt{29}\, u\Big]_0^t$$

$$= \sqrt{29}\, t$$

Therefore $t = \dfrac{s}{\sqrt{29}}$

The re parameterization of the curve is

$$\boxed{\vec{r}(t(s)) = 2\frac{s}{\sqrt{29}}\hat{i} + \left(1 - 3\frac{s}{\sqrt{29}}\right)\hat{j} + \left(5 + 4\frac{s}{\sqrt{29}}\right)\hat{k}}$$

Answer 14E.

$$\vec{r}(t) = e^{2t}\cos 2t\,\hat{i} + 2\hat{j} + e^{2t}\sin 2t\,\hat{k}$$

Then $\vec{r}\,'(t) = \left(2e^{2t}\cos 2t - 2e^{2t}\sin 2t\right)\hat{i} + \left(2e^{2t}\sin 2t + 2e^{2t}\cos 2t\right)\hat{k}$

Now $\dfrac{ds}{dt} = \left|\vec{r}\,'(t)\right|$

$$= \sqrt{\left[2e^{2t}\left(\cos 2t - \sin 2t\right)\right]^2 + \left[2e^{2t}\left(\sin 2t + \cos 2t\right)\right]^2}$$

$$= \sqrt{8e^{4t}}$$

$$= 2\sqrt{2}\, e^{2t}$$

Then $s = s(t) = \int_0^t |r'(t)| du$

$= \int_0^t 2\sqrt{2} e^{2u} du$

$= \frac{2\sqrt{2}}{2} \left[e^{2u} \right]_0^t$

$= \sqrt{2}\left(e^{2t} - 1\right)$

i.e. $e^{2t} = \frac{s}{\sqrt{2}} + 1$

i.e. $t = \frac{1}{2} \ln\left(\frac{s}{\sqrt{2}} + 1\right)$

Answer 15E.

Consider the parametric equations of the curve C:

$x = 3\sin t$ ...... (1)

$y = 4t$ ...... (2)

$z = 3\cos t$ ...... (3)

Substitute $t = 0$ in the given parametric equations, get

$x = 3\sin 0$
$= 3(0)$
$= 0$

$y = 4(0)$
$= 0$

$z = 3\cos 0$
$= 3(1)$
$= 3$

So, the point $(0,0,3)$ be any point on the curve C corresponding to $t = 0.$

Here, the initial point is $t = 0.$

Find the terminal point $t = b$ if the arc length is $L = 5$ units along the given curve.

Differentiate (1) with respect to t, get

$$\frac{dx}{dt} = 3\cos t$$

Differentiate (2) with respect to t, get

$$\frac{dy}{dt} = 4$$

Differentiate (3) with respect to t, get

$$\frac{dz}{dt} = -3\sin t.$$

The parameter t varies from 0 to b

Arc length of the curve is

$$L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt$$
$$= \int_0^b \sqrt{[3\cos t]^2 + [4]^2 + [-3\sin t]^2}\, dt$$
$$= \int_0^b \sqrt{9\cos^2 t + 16 + 9\sin^2 t}\, dt$$
$$= \int_0^b \sqrt{9\left(\cos^2 t + \sin^2 t\right) + 16}\, dt$$
$$= \int_0^b \sqrt{9(1) + 16}\, dt$$
$$= \int_0^b \sqrt{25}\, dt$$
$$= \int_0^b 5\, dt$$
$$= 5[t]_0^b$$
$$= 5[b - 0]$$
$$= 5b$$

Given that, arc length $L = 5.$

So,

$$L = 5b$$
$$5 = 5b$$
$$b = \frac{5}{5}$$
$$= 1$$

Therefore, the terminal point is $t(=b) = 1.$

Find the point on the curve C when $t = 1.$

Substitute $t = 1$ in the given parametric equations, get

$$x = 3\sin 1$$
$$y = 4(1)$$
$$= 4$$
$$z = 3\cos(1)$$

Therefore, after moving 5 units along the given curve in the positive direction from the initial point $(0,0,3)$, the position of the person on the curve is $\boxed{(3\sin(1), 4, 3\cos(1))}$.

Answer 16E.

The given vector equation is

$$\vec{r}(t)=\left(\frac{2}{t^2+1}-1\right)\hat{i}+\left(\frac{2t}{t^2+1}\right)\hat{j}$$

Then $$\vec{r}\,'(t)=\frac{-4t}{\left(t^2+1\right)^2}\hat{i}+\frac{\left(2-2t^2\right)}{\left(t^2+1\right)^2}\hat{j}$$

And $$\left|\vec{r}\,'(t)\right|=\sqrt{\frac{16t^2}{\left(t^2+1\right)^4}+\frac{4+4t^4-8t^2}{\left(t^2+1\right)^4}}$$

$$=\sqrt{\frac{4\left(t^2+1\right)^2}{\left(t^2+1\right)^4}}$$

$$=\frac{2}{t^2+1}$$

Now the point (1, 0) corresponds to parameter $t=0$

Then $$\frac{ds}{dt}=\left|\vec{r}\,'(t)\right|$$

And so $$s=s(t)$$

$$=\int_0^t\left|r'(u)\right|du$$

$$=\int_0^t\frac{2}{t_1^2+1}du$$

$$=2\left[\tan^{-1}u\right]_0^t$$

$$=2\left[\tan^{-1}t-\tan^{-1}0\right]$$

$$=2\tan^{-1}t$$

Therefore $t=\tan\left(\frac{s}{2}\right)$

And the required re parameterization is obtained by substituting for t:

$$\vec{r}(t(s))=\left(\frac{2}{\tan^2\left(\frac{s}{2}\right)+1}-1\right)\hat{i}+\left(\frac{2\tan\left(\frac{s}{2}\right)}{\tan^2\left(\frac{s}{2}\right)+1}\right)\hat{j}$$

$$=\left(\frac{2}{\sec^2\left(s/2\right)}-1\right)\hat{i}+\sin\left(2.\,s/2\right)\hat{j}$$

$$=\left(2\cos^2\left(\frac{s}{2}\right)-1\right)\hat{i}+\sin s\,\hat{j}$$

$$=\cos(s)\hat{i}+\sin s\,\hat{j}$$

Hence $\boxed{\vec{r}(t(s))=\cos s\,\hat{i}+\sin s\,\hat{j}}$

The curve is a circle with centre at origin

Answer 17E.

(a)

The vector function is $\mathbf{r}(t)=\langle t,3\cos t,3\sin t\rangle$.

Differentiate both sides with respect to t.

$$\mathbf{r}'(t)=\left\langle \frac{d}{dt}t,\frac{d}{dt}3\cos t,\frac{d}{dt}3\sin t\right\rangle$$
$$=\langle 1,-3\sin t,3\cos t\rangle$$
$$=\mathbf{i}-3\sin t\mathbf{j}+3\cos t\mathbf{k}$$

The unit normal is $\mathbf{T}(t)$ given by $\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

Also,

$$|\mathbf{r}'(t)| = \sqrt{(1)^2+(-3\sin t)^2+(3\cos t)^2}$$
$$= \sqrt{(1)^2+9\sin^2 t+9\cos^2 t}$$
$$= \sqrt{1+9}$$
$$= \sqrt{10}$$

Substitute these values in $\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

$$\mathbf{T}(t)=\frac{\mathbf{i}-3\sin t\mathbf{j}+3\cos t\mathbf{k}}{\sqrt{10}}$$

The unit tangent is $\mathbf{T}(t)=\left\langle \dfrac{1}{\sqrt{10}},\dfrac{-3\sin t}{\sqrt{10}},\dfrac{3\cos t}{\sqrt{10}}\right\rangle$.

The unit normal vector is $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.

Consider the expression, $\mathbf{T}(t) = \dfrac{\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}}{\sqrt{10}}$

Determine $\mathbf{T}'(t)$.

$$\begin{aligned}\mathbf{T}'(t) &= \frac{d}{dt}\left(\frac{\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}}{\sqrt{10}}\right)\\ &= \frac{-3\cos t\mathbf{j} - 3\sin t\mathbf{k}}{\sqrt{10}}\end{aligned}$$

Evaluate $|\mathbf{T}'(t)|$.

$$\begin{aligned}|\mathbf{T}'(t)| &= \sqrt{\left[\frac{-3\cos t}{\sqrt{10}}\right]^2 + \left[\frac{-3\sin t}{\sqrt{10}}\right]^2}\\ &= \sqrt{\frac{9\cos^2 t + 9\sin^2 t}{10}}\\ &= \frac{3}{\sqrt{10}}\end{aligned}$$

Substitute these values in $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}\\ &= \frac{\frac{-3\cos t\mathbf{j} - 3\sin t\mathbf{k}}{\sqrt{10}}}{\frac{3}{\sqrt{10}}}\\ &= -\cos t\mathbf{j} - \sin t\mathbf{k}\\ &= \boxed{\langle 0, -\cos t, -\sin t\rangle}\end{aligned}$$

(b)

The curvature of a curve is given by $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

$$\kappa(t) = \frac{\frac{3}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{10}$$

Therefore, the curvature is $\dfrac{3}{10}$.

Answer 18E.

$$\vec{r}(t) = < t^2,\ \sin t - t\cos t,\ \cos t + t\sin t >$$

$$\vec{r}\,'(t) = < 2t,\ \cos t - (\cos t - t\sin t),\ -\sin t + \sin t + t\cos t >$$
$$= < 2t,\ t\sin t,\ t\cos t >$$

$$|\vec{r}\,'(t)| = \left[(2t)^2 + (t\sin t)^2 + (t\cos t)^2\right]^{1/2}$$
$$= \left[4t^2 + t^2\sin^2 t + t^2\cos^2 t\right]^{1/2}$$
$$= \left[4t^2 + t^2\right]^{1/2}$$
$$= \left[5t^2\right]^{1/2}$$
$$= \sqrt{5}\,t$$

(A)

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$
$$= \frac{1}{\sqrt{5}\,t} < 2t,\ t\sin t,\ t\cos t >$$
$$= \boxed{< 2/\sqrt{5},\ \frac{1}{\sqrt{5}}\sin t,\ \frac{1}{\sqrt{5}}\cos t >}$$

$$\vec{T}\,'(t) = < 0,\ \frac{1}{\sqrt{5}}\cos t,\ \frac{-1}{\sqrt{5}}\sin t >$$

$$|\vec{T}\,'(t)| = \left[\left(\frac{1}{\sqrt{5}}\cos t\right)^2 + \left(-\frac{1}{\sqrt{5}}\sin t\right)^2\right]^{1/2}$$
$$= \left[\frac{1}{5}\cos^2 t + \frac{1}{5}\sin^2 t\right]^{1/2} = \frac{1}{\sqrt{5}}$$

$$\vec{N}(t)=\frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}=\frac{1}{1/\sqrt{5}}<0,\frac{1}{\sqrt{5}}\cos t,-\frac{1}{\sqrt{5}}\sin t>$$

$$=\boxed{<0,\cos t,-\sin t>}$$

(B)

The curvature is $k(t)=\frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$

$$=\frac{\frac{1}{\sqrt{5}}}{\sqrt{5}t}$$

$$=\boxed{\frac{1}{5t}}$$

Answer 19E.

$$\vec{r}(t)=<\sqrt{2}t,e^{t},e^{-t}>$$

$$\vec{r}'(t)=<\sqrt{2},e^{t},-e^{-t}>$$

$$\left|\vec{r}'(t)\right|=\sqrt{\left(\sqrt{2}\right)^{2}+\left(e^{t}\right)^{2}+\left(-e^{-t}\right)^{2}}$$

$$=\sqrt{2+e^{2t}+e^{-2t}}$$

$$=\sqrt{\frac{2e^{2t}+e^{4t}+1}{e^{2t}}}$$

$$=\sqrt{\frac{\left(e^{2t}+1\right)^{2}}{e^{2t}}}$$

$$=\frac{e^{2t}+1}{e^{t}}$$

(A)

$$\vec{T}(t)=\frac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}=\frac{e^{t}}{e^{2t}+1}<\sqrt{2},e^{t},-e^{-t}>$$

$$=\boxed{<\frac{\sqrt{2}e^{t}}{e^{2t}+1},\frac{e^{2t}}{e^{2t}+1},\frac{-1}{e^{2t}+1}>}$$

$$\vec{T}'(t)=<\frac{\left(e^{2t}+1\right).\sqrt{2}e^{t}-\sqrt{2}e^{t}\left(2e^{2t}\right)}{\left(e^{2t}+1\right)^{2}},\frac{\left(e^{2t}+1\right)2e^{2t}-e^{2t}\left(2e^{2t}\right)}{\left(e^{2t}+1\right)^{2}},\frac{2e^{2t}}{\left(e^{2t}+1\right)^{2}}>$$

$$=\frac{1}{\left(e^{2t}+1\right)^{2}}<\sqrt{2}\left(e^{3t}+e^{t}-2e^{3t}\right),2e^{4t}+2e^{2t}-2e^{4t},2e^{2t}>$$

$$=\frac{1}{\left(e^{2t}+1\right)^{2}}<\sqrt{2}\left(e^{t}-e^{3t}\right),2e^{2t},2e^{2t}>$$

$$\left|\vec{T}'(t)\right| = \frac{1}{\left(e^{2t}+1\right)^2}\sqrt{\left[\sqrt{2}\left(e^{t}-e^{3t}\right)\right]^2+\left(2e^{2t}\right)^2+\left(2e^{2t}\right)^2}$$

$$= \frac{1}{\left(e^{2t}+1\right)^2}\sqrt{2e^{2t}+2e^{6t}-4e^{4t}+4e^{4t}+4e^{4t}}$$

$$= \frac{1}{\left(e^{2t}+1\right)^2}\sqrt{2}\sqrt{e^{2t}}\sqrt{e^{4t}+2e^{2t}+1}$$

$$= \frac{\sqrt{2}\,e^{t}\left(e^{2t}+1\right)}{\left(e^{2t}+1\right)^2}$$

$$= \frac{\sqrt{2}\,e^{t}}{e^{2t}+1}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$$

$$= \boxed{\frac{1}{e^{2t}+1} < 1-e^{2t},\ \sqrt{2}\,e^{t},\ \sqrt{2}\,e^{t} >}$$

(B)

$$\text{Curvature } k(t) = \frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$$

$$= \frac{\sqrt{2}\,e^{t}\,.\,e^{t}}{\left(e^{2t}+1\right)^2}$$

$$= \boxed{\frac{\sqrt{2}\,e^{2t}}{\left(e^{2t}+1\right)^2}}$$

Answer 20E.

(a)

Consider the following position vector;

$$\mathbf{r}(t)=\left\langle t,\frac{1}{2}t^2,t^2\right\rangle$$

The objective is to find the unit tangent and unit normal vectors.

The unit tangent vector can be obtained by the following formula.

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Find the derivative of the vector.

$$\begin{aligned}\mathbf{r}'(t)&=\left\langle 1,\frac{1}{2}\cdot 2t,2t\right\rangle\\&=\langle 1,t,2t\rangle\end{aligned}$$

Find its magnitude.

$$\begin{aligned}|\mathbf{r}'(t)|&=\sqrt{(1)^2+(t)^2+(2t)^2}\\&=\sqrt{1+t^2+4t^2}\\&=\sqrt{1+5t^2}\end{aligned}$$

Now, find the unit tangent, using the formula, $\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Plug in to the unit tangent vector, to obtain $\mathbf{T}(t)$.

$$\begin{aligned}\mathbf{T}(t)&=\frac{\langle 1,t,2t\rangle}{\sqrt{1+5t^2}}\\&=\frac{1}{\sqrt{1+5t^2}}\langle 1,t,2t\rangle\end{aligned}$$

Therefore, the unit tangent vector for the vector is $\mathbf{T}(t)=\boxed{\frac{1}{\sqrt{1+5t^2}}\langle 1,t,2t\rangle}$.

The normal vector can be obtained by the following formula

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Find the derivative of the unit tangent, $\mathbf{T}(t)=\frac{1}{\sqrt{1+5t^2}}\langle 1,t,2t\rangle$.

$$\mathbf{T}'(t)=\left\langle -\frac{5t}{(1+5t^2)\sqrt{1+5t^2}},\frac{\sqrt{1+5t^2}-\frac{5t}{\sqrt{1+5t^2}}}{1+5t^2},\frac{2\sqrt{1+5t^2}-\frac{10t}{\sqrt{1+5t^2}}}{1+5t^2}\right\rangle$$

$$=\left\langle -\frac{5t}{(1+5t^2)\sqrt{1+5t^2}},\frac{1+5t^2-5t}{(1+5t^2)\sqrt{1+5t^2}},\frac{2+10t^2-10t}{(1+5t^2)\sqrt{1+5t^2}}\right\rangle$$

$$=\frac{1}{(1+5t^2)\sqrt{1+5t^2}}\langle -1,5t^2-5t+1,10t^2-10t+2\rangle$$

Find the magnitude of the vector.

$$|\mathbf{T}'(t)|=\frac{1}{(1+5t^2)\sqrt{1+5t^2}}\sqrt{1+(5t^2-5t+1)^2+4(5t^2-5t+1)^2}$$

$$=\frac{1}{(1+5t^2)\sqrt{1+5t^2}}\sqrt{1+5(5t^2-5t+1)^2}$$

Now, find the unit normal vector using the formula, $\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.

Plug in to the unit normal vector, to obtain $\mathbf{N}(t)$.

$$\mathbf{N}(t)=\frac{\frac{1}{(1+5t^2)\sqrt{1+5t^2}}\langle -1,5t^2-5t+1,10t^2-10t+2\rangle}{\frac{1}{(1+5t^2)\sqrt{1+5t^2}}\sqrt{1+5(5t^2-5t+1)^2}}$$

$$=\frac{1}{\sqrt{1+5(5t^2-5t+1)^2}}\langle -1,5t^2-5t+1,10t^2-10t+2\rangle$$

Therefore, the unit tangent vector for the vector $\mathbf{r}(t)=\left\langle t,\frac{1}{2}t^2,t^2\right\rangle$ is as follows.

$$\mathbf{N}(t)=\boxed{\frac{1}{\sqrt{1+5(5t^2-5t+1)^2}}\langle -1,5t^2-5t+1,10t^2-10t+2\rangle}.$$

b)

Find the curvature of the vector.

The curvature of the vector can be obtained the following formula.

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Plug in to the curvature of the given vector, to obtain κ.

$$\kappa = \frac{\frac{1}{(1+5t^2)\sqrt{1+5t^2}}\sqrt{1+5(5t^2-5t+1)^2}}{\sqrt{1+5t^2}}$$

$$= \frac{1}{(1+5t^2)^2}\sqrt{1+5(5t^2-5t+1)^2}$$

Therefore, the curvature of the given vector is $\boxed{\kappa = \frac{1}{(1+5t^2)^2}\sqrt{1+5(5t^2-5t+1)^2}}$.

Answer 21E.

Consider the vector function

$$\mathbf{r}(t) = t^3\mathbf{j} + t^2\mathbf{k} \quad \text{......} \quad (1)$$

Recall the theorem,

The curvature of the curve of the vector function $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad \text{......} \quad (2)$$

Compute the first and second derivative of vector function $\mathbf{r}(t)$.

Recall the theorem,

If $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$, where $f,g,$ and h are differentiable functions, then

$$\mathbf{r}'(t)=\langle f'(t),g'(t),h'(t)\rangle \text{ ...... } (3)$$

Using (3), the first derivative of the vector function $\mathbf{r}(t)$

$$\mathbf{r}'(t)=\frac{d}{dt}(t^3)\mathbf{j}+\frac{d}{dt}(t^2)\mathbf{k}$$
$$\mathbf{r}'(t)=3t^2\mathbf{j}+2t\mathbf{k}$$

Therefore,

$$\mathbf{r}'(t)=3t^2\mathbf{j}+2t\mathbf{k}$$

Using (3), the derivative of the vector function $\mathbf{r}'(t)$

$$\mathbf{r}''(t)=\frac{d}{dt}(3t^2)\mathbf{j}+\frac{d}{dt}(2t)\mathbf{k}$$
$$\mathbf{r}''(t)=6t\mathbf{j}+2\mathbf{k}$$

Therefore,

$$\mathbf{r}''(t)=6t\mathbf{j}+2\mathbf{k}$$

Compute, $|\mathbf{r}'(t)\times\mathbf{r}''(t)|$

First find the vector $\mathbf{r}'(t)\times\mathbf{r}''(t)$

$$\mathbf{r}'(t)\times\mathbf{r}''(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 3t^2 & 2t\\ 0 & 6t & 2\end{vmatrix}$$
$$=\mathbf{i}(6t^2-12t^2)-\mathbf{j}(0-0)+\mathbf{k}(0-0)$$
$$=-6t^2\mathbf{i}$$

Therefore,

$$\mathbf{r}'(t)\times\mathbf{r}''(t)=-6t^2\mathbf{i}$$

The magnitude of the vector function $\mathbf{r}'(t)\times\mathbf{r}''(t)$

$$\begin{aligned}\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right| &= \sqrt{\left(-6t^2\right)^2+0+0} \\ &= 6t^2\end{aligned}$$

Thus,

$$\boxed{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right| = 6t^2} \quad \cdots\cdots \quad (4)$$

The magnitude of the vector function $\mathbf{r}'(t)=3t^2\mathbf{j}+2t\mathbf{k}$

$$\begin{aligned}\left|\mathbf{r}'(t)\right| &= \left|3t^2\mathbf{j}+2t\mathbf{k}\right| \\ &= \sqrt{\left(3t^2\right)^2+\left(2t\right)^2} \\ &= \sqrt{9t^4+4t^2}\end{aligned}$$

Therefore

$$\boxed{\left|\mathbf{r}'(t)\right| = \sqrt{9t^4+4t^2}} \quad \cdots\cdots \quad (5)$$

Substitute the values of $\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|$ and $\left|\mathbf{r}'(t)\right|$ from (4) and (5) in equation (2)

$$\kappa(t) = \frac{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|}{\left|\mathbf{r}'(t)\right|^3}$$

$$\kappa(t) = \frac{6t^2}{\left(\sqrt{9t^4+4t^2}\right)^3}$$

$$\kappa(t) = \frac{6t^2}{\left(\sqrt{9t^4+4t^2}\right)^3}$$

Therefore the curvature of the curve of the vector function $\mathbf{r}(t)=t^3\mathbf{j}+t^2\mathbf{k}$ is

$$\boxed{\kappa(t) = \frac{6t^2}{\left(\sqrt{9t^4+4t^2}\right)^3}}$$

Answer 22E.

Consider the vector function

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k} \quad \ldots\ldots \quad (1)$$

Recall the theorem,

The curvature of the curve of the vector function $\mathbf{r}(t)$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad \ldots\ldots \quad (2)$$

Compute the first and second derivative of vector function $\mathbf{r}(t)$.

Recall the theorem,

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where $f, g,$ and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle \quad \ldots\ldots \quad (3)$$

Using (3), the first derivative of the vector function $\mathbf{r}(t)$

$$\mathbf{r}'(t) = \frac{d}{dt}(t)\mathbf{i} + \frac{d}{dt}(t^2)\mathbf{j} + \frac{d}{dt}(e^t)\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k}$$

Therefore,

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + e^t\mathbf{k}$$

Using (3), the derivative of the vector function $\mathbf{r}'(t)$

$$\mathbf{r}''(t) = \frac{d}{dt}(1)\mathbf{i} + \frac{d}{dt}(2t)\mathbf{j} + \frac{d}{dt}(e^t)\mathbf{k}$$

$$\mathbf{r}''(t) = 0\mathbf{i} + 2\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{r}''(t) = 2\mathbf{j} + e^t\mathbf{k}$$

Therefore,

$$\mathbf{r}''(t) = 2\mathbf{j} + e^t\mathbf{k}$$

Compute, $\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|$

First find the vector $\mathbf{r}'(t)\times\mathbf{r}''(t)$

$$\mathbf{r}'(t)\times\mathbf{r}''(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2t & e^t\\ 0 & 2 & e^t\end{vmatrix}$$

$$=\mathbf{i}\left(2e^t t-2e^t\right)-\mathbf{j}\left(e^t-0\right)+\mathbf{k}(2)$$

$$=2e^t\mathbf{i}(t-1)-e^t\mathbf{j}+2\mathbf{k}$$

Therefore,

$$\mathbf{r}'(t)\times\mathbf{r}''(t)=2e^t\mathbf{i}(t-1)-e^t\mathbf{j}+2\mathbf{k}$$

The magnitude of the vector function $\mathbf{r}'(t)\times\mathbf{r}''(t)$

$$\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|=\left|2e^t\mathbf{i}(t-1)-e^t\mathbf{j}+2\mathbf{k}\right|$$

$$=\sqrt{\left(2e^t(t-1)\right)^2+\left(-e^t\right)^2+(2)^2}$$

$$=\sqrt{4e^{2t}(t-1)^2+e^{2t}+4}$$

$$=\sqrt{4e^{2t}t^2-8e^{2t}t+5e^{2t}+4}$$

Thus,

$$\boxed{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|=\sqrt{4e^{2t}t^2-8e^{2t}t+5e^{2t}+4}} \quad \cdots\cdots \quad (4)$$

The magnitude of the vector function $\mathbf{r}'(t)=\mathbf{i}+2t\mathbf{j}+e^t\mathbf{k}$

$$\left|\mathbf{r}'(t)\right|=\left|\mathbf{i}+2t\mathbf{j}+e^t\mathbf{k}\right|$$

$$=\sqrt{(1)+(2t)^2+\left(e^t\right)^2}$$

$$=\sqrt{1+4t^2+e^{2t}}$$

Therefore

$$\boxed{\left|\mathbf{r}'(t)\right|=\sqrt{1+4t^2+e^{2t}}} \quad \cdots\cdots \quad (5)$$

Substitute the values of $|\mathbf{r}'(t)\times\mathbf{r}''(t)|$ and $|\mathbf{r}'(t)|$ from (4) and (5) in equation (2)

$$\kappa(t)=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\kappa(t)=\frac{\sqrt{4e^{2t}t^2-8e^{2t}t+5e^{2t}+4}}{\left(\sqrt{1+4t^2+e^{2t}}\right)^3}$$

Therefore the curvature of the curve of the vector function $\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j}+e^t\mathbf{k}$ is

$$\boxed{\kappa(t)=\frac{\sqrt{4e^{2t}t^2-8e^{2t}t+5e^{2t}+4}}{\left(\sqrt{1+4t^2+e^{2t}}\right)^3}}$$

Answer 23E.

$$\vec{r}(t)=3t\hat{i}+4\sin t\,\hat{j}+4\cos t\,\hat{k}$$

$$\vec{r}\,'(t)=3\hat{i}+4\cos t\,\hat{j}-4\sin t\,\hat{k}$$

$$\vec{r}\,''(t)=-4\sin t\,\hat{j}-4\cos t\,\hat{k}$$

$$\vec{r}\,'(t)\times\vec{r}\,''(t)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4\cos t & -4\sin t\\ 0 & -4\sin t & -4\cos t\end{vmatrix}$$

$$=\left(-16\cos^2 t-16\sin^2 t\right)\hat{i}+12\cos t\,\hat{j}-12\sin t\,\hat{k}$$

$$=-16\left(\cos^2 t+\sin^2 t\right)\hat{i}+12\cos t\,\hat{j}-12\sin t\,\hat{k}$$

$$=-16\hat{i}+12\cos t\,\hat{j}-12\sin t\,\hat{k}$$

$$\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right|=\sqrt{(-16)^2+(12\cos t)^2+(-12\sin t)^2}$$

$$=\sqrt{256+144\left(\cos^2 t+\sin^2 t\right)}$$

$$=\sqrt{400}=20$$

$$\left|\vec{r}\,'(t)\right|=\sqrt{3^2+(4\cos t)^2+(-4\sin t)^2}$$

$$=\sqrt{9+16\left(\cos^2 t+\sin^2 t\right)}$$

$$=\sqrt{9+16}$$

$$=\sqrt{25}=5$$

The curvature of the curve is

$$k(t) = \frac{\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right|}{\left|\vec{r}\,'(t)\right|^3}$$

$$= \frac{20}{(5)^3}$$

$$= \frac{4}{(5)^2}$$

$$= \boxed{\frac{4}{25}}$$

Answer 24E.

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then the curvature κ of C at t is given by $\kappa = \dfrac{\left\|\mathbf{r}'(t)\times\mathbf{r}''(t)\right\|}{\left\|\mathbf{r}'(t)\right\|^3}$.

We have $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Evaluate $x'(t)$, $y'(t)$, and $z'(t)$.

$$x'(t) = \frac{d}{dt}(t^2) = 2t \qquad y'(t) = \frac{d}{dt}(\ln t) = \frac{1}{t} \qquad z'(t) = \frac{d}{dt}(t\ln t) = 1+\ln t$$

We get $\mathbf{r}'(t) = 2t\mathbf{i} + \dfrac{1}{t}\mathbf{j} + (1+\ln t)\mathbf{k}$.

We know that $\left\|\mathbf{r}'(t)\right\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]}$.

$$\left\|\mathbf{r}'(t)\right\| = \sqrt{(2t)^2 + \left(\frac{1}{t}\right)^2 + (1+\ln t)^2}$$

$$= \sqrt{4t^2 + \frac{1}{t^2} + 1 + \ln t^2 + 2\ln t}$$

Now, determine $\mathbf{r}''(t)$.

$$\mathbf{r}''(t) = 2\mathbf{i} - \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}$$

Let us now evaluate $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & \frac{1}{t} & 1 + \ln t \\ 2 & -\frac{1}{t^2} & \frac{1}{t} \end{vmatrix}$$

$$= \left(\frac{2 + \ln t}{t^2}\right)\mathbf{i} + 2\ln t\mathbf{j} - \frac{4}{t}\mathbf{k}$$

We get $\left\| \mathbf{r}'(t) \times \mathbf{r}''(t) \right\|$ as $\sqrt{\left(\frac{2 + \ln t}{t^2}\right)^2 + (2\ln t)^2 + \left(-\frac{4}{t}\right)^2}$.

Substitute the known values in $\kappa = \frac{\left\| \mathbf{r}'(t) \times \mathbf{r}''(t) \right\|}{\left\| \mathbf{r}'(t) \right\|^3}$.

$$\kappa = \frac{\sqrt{\left(\frac{2 + \ln t}{t^2}\right)^2 + (2\ln t)^2 + \left(-\frac{4}{t}\right)^2}}{\left(\sqrt{4t^2 + \frac{1}{t^2} + 1 + \ln t^2 + 2\ln t}\right)^3}$$

From the given vector valued function and the point P, we get t as 1.

Replace t with 1 in $\kappa = \frac{\sqrt{\left(\frac{2 + \ln t}{t^2}\right)^2 + (2\ln t)^2 + \left(-\frac{4}{t}\right)^2}}{\left(\sqrt{4t^2 + \frac{1}{t^2} + 1 + \ln t^2 + 2\ln t}\right)^3}$.

$$\kappa = \frac{\sqrt{\left(\frac{2 + \ln t}{t^2}\right)^2 + (2\ln t)^2 + \left(-\frac{4}{t}\right)^2}}{\left(\sqrt{4(1)^2 + \frac{1}{1^2} + 1 + \ln(1)^2 + 2\ln(1)}\right)^3}$$

$$= \frac{\sqrt{20}}{6\sqrt{6}}$$

$$= \frac{\sqrt{5}}{3\sqrt{6}}$$

Therefore, we can say that the curvature of the plane curve is $\boxed{\frac{\sqrt{5}}{3\sqrt{6}}}$.

Answer 25E.

$$\vec{r}(t) = <t, t^2, t^3>$$

$$\vec{r}\,'(t) = <1, 2t, 3t^2>$$

$$\vec{r}\,''(t) = <0, 2, 6t>$$

$$\left|\vec{r}\,'(t)\right| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$= (12t^2 - 6t^2)\hat{i} - 6t\,\hat{j} + 2\hat{k}$$

$$= 6t^2\hat{i} - 6t\,\hat{j} + 2\hat{k}$$

$$\left|\vec{r}\,'(t) \times \vec{r}\,''(t)\right| = \sqrt{(6t^2)^2 + (-6t)^2 + 2^2}$$

$$= \sqrt{36t^4 + 36t^2 + 4}$$

$$= 2\sqrt{9t^4 + 9t^2 + 1}$$

The curvature the curve is

$$k(t) = \frac{\left|\vec{r}\,'(t) \times \vec{r}\,''(t)\right|}{\left|\vec{r}\,'(t)\right|^3}$$

$$= \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

The curvature at the point (1, 1, 1) is

$$k(1) = \frac{2\sqrt{9 + 9 + 1}}{(1 + 4 + 9)^{3/2}}$$

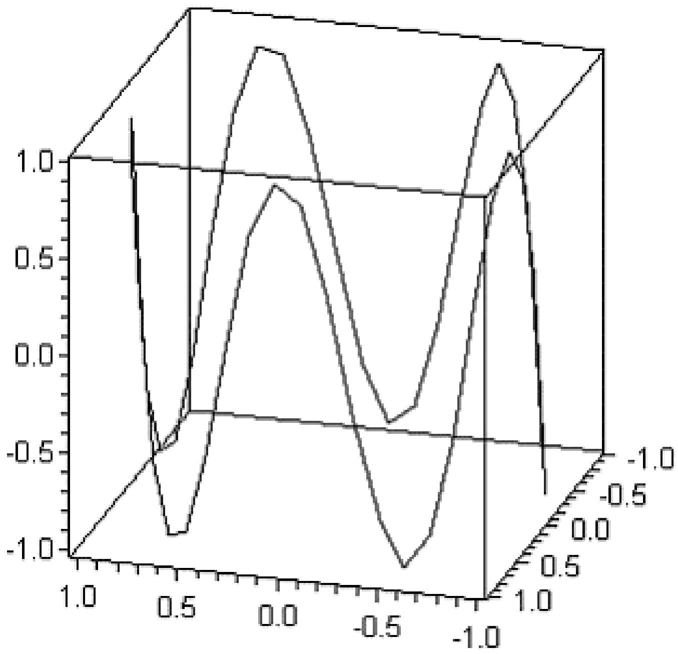
$$= \frac{2\sqrt{19}}{(14)^{3/2}}$$

$$= \frac{2\sqrt{19}}{14\sqrt{14}}$$

$$= \boxed{\frac{1}{7}\sqrt{\frac{19}{14}}}$$

Answer 26E.

Start by sketching the space curve of $\mathbf{r}(t)$.



If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ then the curvature $\kappa(t)$ of C at t is given by $\kappa(t) = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

Differentiating with respect to time t we have

$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Evaluate $x'(t), y'(t)$, and $z'(t)$.

$$\begin{aligned} x'(t) &= \frac{d}{dt}(\cos t) \\ &= -\sin t \end{aligned}$$

$$\begin{aligned} y'(t) &= \frac{d}{dt}(\sin t) \\ &= \cos t \end{aligned}$$

$$\begin{aligned} z'(t) &= \frac{d}{dt}(\sin 5t) \\ &= 5\cos 5t \end{aligned}$$

We get $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 5\cos 5t\mathbf{k}$.

We know that $\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]}$.

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{(-\cos t)^2 + (\sin t)^2 + (-5\cos 5t)^2} \\ &= \sqrt{\cos^2 t + \sin^2 t + 25\cos^2 5t} \\ &= \sqrt{1 + 25\cos^2 5t}\end{aligned}$$

Now, determine $\mathbf{r}''(t)$.

$$\mathbf{r}''(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} - 25\sin 5t\mathbf{k}$$

Let us now evaluate $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 5\cos 5t \\ -\cos t & -\sin t & -25\sin 5t \end{vmatrix} \\ &= (-25\sin 5t\cos t + 5\cos 5t\sin t)\mathbf{i} + (25\sin 5t\sin t + 5\cos 5t\cos t)\mathbf{j} + (\sin^2 t + \cos^2 t)\mathbf{k} \\ &= (-25\sin 5t\cos t + 5\cos 5t\sin t)\mathbf{i} + (25\sin 5t\sin t + 5\cos 5t\cos t)\mathbf{j} + \mathbf{k}\end{aligned}$$

Now $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|$ as

$$\sqrt{(-25\sin 5t\cos t + 5\cos 5t\sin t)^2 + (25\sin 5t\sin t + 5\cos 5t\cos t)^2 + 1}.$$

Substitute the known values in $\kappa = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$.

$$\kappa = \frac{\sqrt{(-25\sin 5t\cos t + 5\cos 5t\sin t)^2 + (25\sin 5t\sin t + 5\cos 5t\cos t)^2 + 1}}{\left(\sqrt{1 + 25\cos^2 5t}\right)^3}$$

From the given vector valued function and the point P, we get t as 0.
Plug in t with 0.

$$\begin{aligned}\kappa &= \frac{\sqrt{(-25\sin 5(0)\cos(0) + 5\cos 5(0)\sin(0))^2 + \begin{pmatrix} 25\sin 5(0)\sin(0) \\ + 5\cos 5(0)\cos(0) \end{pmatrix}^2 + 1}}{\left(\sqrt{1 + 25\cos^2 5(0)}\right)^3} \\ &= \frac{\sqrt{25 + 1}}{\left(\sqrt{1 + 25}\right)^3} \\ &= \frac{\sqrt{26}}{\left(\sqrt{26}\right)^3} \\ &= \frac{1}{26}\end{aligned}$$

Therefore, the curvature at point (1, 0, 0) is $\boxed{\dfrac{1}{26}}$

Answer 27E.

If C is a smooth curve given by the function $y = f(x)$, then the curvature κ of C at t is given by $\kappa = \dfrac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$.

We have $y' = 4x^3$ and $y'' = 12x^2$.

Substitute the known values in $\kappa = \dfrac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$.

$$K = \frac{12x^2}{\left[1+(4x^3)^2\right]^{3/2}}$$

$$= \frac{12x^2}{(1+16x^6)^{3/2}}$$

Therefore, we get the curvature as $\boxed{\dfrac{12x^2}{(1+16x^6)^{3/2}}}$.

Answer 28E.

Consider the function

$y = \tan x$ ...... (1)

Recall the theorem,

The curvature of the curve of the function $y = f(x)$ is

$$\kappa(x) = \frac{|f''(x)|}{\left(1+f'(x)^2\right)^{\frac{3}{2}}} \text{ ...... (2)}$$

Compute the first and second derivative of function $y = \tan x$.

The first derivative of the function $y = \tan x$

$$\frac{d}{dx}(y) = \frac{d}{dx}(\tan x)$$
$$= \sec^2 x$$

Therefore,

$$f'(x) = \sec^2 x$$

The derivative of the function $f'(x)$

$$f''(x) = \frac{d}{dt}\left(\sec^2 x\right)$$
$$f''(x) = 2\sec x\left(\sec x \tan x\right)$$
$$f''(x) = 2\sec^2 x \tan x$$

Therefore

$$f''(x) = 2\sec^2 x \tan x$$

Substitute the functions $f'(x) = \sec^2 x$ and $f''(x) = 2\sec^2 x \tan x$ in equation (2)

$$\kappa(x) = \frac{\left|f''(x)\right|}{\left(1 + f'(x)^2\right)^{\frac{3}{2}}}$$
$$= \frac{\left|2\sec^2 x \tan x\right|}{\left(1 + \left(\sec^2 x\right)^2\right)^{\frac{3}{2}}}$$
$$= \frac{\left|2\sec^2 x \tan x\right|}{\left(1 + \sec^4 x\right)^{\frac{3}{2}}}$$

Hence the curvature of the function $y = \tan x$

$$\boxed{\kappa(x) = \frac{\left|2\sec^2 x \tan x\right|}{\left(1 + \sec^4 x\right)^{\frac{3}{2}}}}$$

Answer 29E.

If C is a smooth curve given by the function $y = f(x)$, then the curvature κ of C at t is given by $\kappa = \dfrac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$.

We have $y' = e^x + xe^x$ and $y'' = 2e^x + xe^x$.

Substitute the known values in $\kappa = \dfrac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$.

$$K = \frac{2e^x + xe^x}{\left[1+\left(e^x + xe^x\right)^2\right]^{3/2}}$$

$$= \frac{2e^x + xe^x}{\left(1+e^{2x} + 2xe^{2x} + x^2e^{2x}\right)^{3/2}}$$

Therefore, we get the curvature as $\boxed{\dfrac{2e^x + xe^x}{\left(1+e^{2x} + 2xe^{2x} + x^2e^{2x}\right)^{3/2}}}$.

Answer 30E.

The equation of given curve is

$$y = \ln x$$

Then differentiating with respect to x

$$y' = \frac{dy}{dx}$$

$$= \frac{1}{x}$$

Again differentiating with respect to x

$$y'' = \frac{d^2y}{dx^2}$$

$$= \frac{-1}{x^2}$$

Now the radius of curvature is given by

$$k(x) = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$

i.e. $$k(x) = \frac{1/x^2}{\left(1+\frac{1}{x^2}\right)^{3/2}}$$

$$= \frac{1}{x^2\,\frac{\left(x^2+1\right)^{3/2}}{x^3}}$$

$$= \frac{x}{\left(1+x^2\right)^{3/2}}$$

Now $\dfrac{dk}{dx} = \dfrac{(1+x)^{3/2} - x\dfrac{3}{2}(1+x^2)^{1/2}.2x}{(1+x^2)^3}$

$$= \frac{(1+x^2)^{3/2} - 3x^2(1+x^2)^{1/2}}{(1+x^2)^3}$$

$$= \frac{(1+x^2) - 3x^2}{(1+x^2)^2}$$

$$= \frac{1-2x^2}{(1+x^2)^2}$$

And $\dfrac{d^2k}{dx^2} = \dfrac{(1+x^2)^2(-4x) - (1-2x^2)2(1+x^2).2x}{(1+x^2)^4}$

$$= \frac{-4x(1+x^2) - 4x(1-2x^2)}{(1+x^2)^3}$$

$$= \frac{-4x(2-x^2)}{(1+x^2)^3}$$

For the critical points put $\dfrac{dk}{dx} = 0$

i.e. $\dfrac{1-2x^2}{(1+x^2)^2} = 0$

i.e. $1-2x^2 = 0$

i.e. $x^2 = \dfrac{1}{2}$

i.e. $x = \pm\dfrac{1}{\sqrt{2}}$

When $x = \dfrac{1}{\sqrt{2}}$

$$\left(\frac{d^2k}{dx^2}\right)_{x=\frac{1}{\sqrt{2}}} = \frac{\frac{-4}{\sqrt{2}}\left(2-\frac{1}{2}\right)}{\left(1+\frac{1}{2}\right)^3} = \frac{-48}{27}$$

Since $\left(\dfrac{d^2k}{dx^2}\right)_{x=\frac{1}{\sqrt{2}}} < 0$

Then $x = \dfrac{1}{\sqrt{2}}$ is a point of maxima

When $x = \frac{-1}{\sqrt{2}}$

Then $\left(\frac{d^2k}{dx^2}\right)_{x=-\frac{1}{\sqrt{2}}} = \frac{48}{27} > 0$

Since $\left(\frac{d^2k}{dx^2}\right)_{x=\frac{-1}{\sqrt{2}}} > 0$

Then $x = -\frac{1}{\sqrt{2}}$ is a point of minima

Hence the radius of curvature of given curve is maximum when $x = \frac{1}{\sqrt{2}}$ and $y = \ln\left(\frac{1}{\sqrt{2}}\right)$

That is the radius of curvature is maximum at point

$$\boxed{\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\ln 2\right)}$$

Now

$$\lim_{x\to\infty} k(x) = \lim_{x\to\infty} \frac{x}{\left(1+x^2\right)^{3/2}}$$

$$= \lim_{x\to\infty} \frac{1/x^2}{\left(\frac{1}{x^2}+1\right)^{3/2}}$$

$$= \frac{1/\infty}{\left(\frac{1}{\infty}+1\right)^{3/2}}$$

$$= \frac{0}{0+0}$$

$$= \boxed{0}$$

ix

Answer 31E.

The equation of given curve is

$$y = e^x$$

Differentiating with respect to x

$$y' = \frac{dy}{dx}$$
$$= e^x$$

Again differentiating with respect to x

$$y'' = \frac{d^2y}{dx^2}$$
$$= e^x$$

The radius of curvature is given by

$$K(x) = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$

i.e. $$K(x) = \frac{e^x}{\left(1+e^{2x}\right)^{3/2}}$$

Now $$\frac{dK}{dx} = \frac{\left(1+e^{2x}\right)^{3/2} e^x - e^x \frac{3}{2}\left(1+e^{2x}\right)^{1/2} 2e^{2x}}{\left(1+e^{2x}\right)^3}$$

$$= \frac{e^x\left(1+e^{2x}\right)^{\frac{1}{2}}\left[\left(1+e^{2x}\right)-3e^{2x}\right]}{\left(1+e^{2x}\right)^3}$$

$$= \frac{e^x\left(1-2e^{2x}\right)}{\left(1+e^{2x}\right)^{\frac{5}{2}}}$$

And $$\frac{d^2K}{dx^2} = \frac{\left(1+e^{2x}\right)^{\frac{5}{2}} e^x\left(1-6e^{2x}\right) - e^x\left(1-2e^{2x}\right)\frac{5}{2}\left(1+e^{2x}\right)^{\frac{3}{2}} 2e^{2x}}{\left(1+e^{2x}\right)^5}$$

$$= \frac{e^x\left(1+e^{2x}\right)^{\frac{5}{2}}\left(1-6e^{2x}\right) - 5e^{3x}\left(1-2e^{2x}\right)\left(1+e^{2x}\right)^{\frac{3}{2}}}{\left(1+e^{2x}\right)^5}$$

For the critical points put $\frac{dK}{dx} = 0$

i.e. $\frac{e^x\left(1-2e^{2x}\right)}{\left(1+e^{2x}\right)^{\frac{5}{2}}} = 0$

i.e. $e^x\left(1-2e^{2x}\right) = 0$

i.e. $1-2e^{2x} = 0$

i.e. $e^{2x} = 1/2$

i.e. $2x = \ln\left(1/2\right)$

i.e. $x = \frac{1}{2}\ln(2)^{-1}$

i.e. $x = \frac{-1}{2}\ln 2$

When $x = \frac{-1}{2}\ln 2$

Then $$\frac{d^2K}{dx^2} = \frac{\frac{1}{\sqrt{2}}\left(1+\frac{1}{2}\right)^{\frac{5}{2}}\left(1-6\times\frac{1}{2}\right) - 5\times\frac{1}{2\sqrt{2}}\left(1-2\times\frac{1}{2}\right)\left(1+\frac{1}{2}\right)^{\frac{3}{2}}}{\left(1+\frac{1}{2}\right)^5}$$

$$= \frac{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{3}{2}\right)^{\frac{5}{2}}(-2) - \frac{5}{2\sqrt{2}}(0)}{\left(\frac{3}{2}\right)^5}$$

$$= -\left(\frac{2}{\sqrt{2}}\right)\left(\frac{2}{3}\right)^{\frac{5}{2}}$$

Since $\left(\frac{d^2K}{dx^2}\right)_{x=-\frac{1}{2}\ln 2} < 0$, then $x = \frac{-1}{2}\ln 2$ is a point of maxima

Hence the radius of curvature of given curve is maximum

at $x = -\frac{1}{2}\ln 2$ and $y = e^{-1/2}\ln 2$

i.e. $at \boxed{\left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)}$

$$\text{Now} \quad K(x) = \frac{e^x}{\left(1+e^{2x}\right)^{3/2}}$$

$$\text{As } x \to \infty \text{ then} \lim_{x\to\infty} K(x) = \lim_{x\to\infty} \frac{e^x}{\left(1+e^{2x}\right)^{3/2}}$$

$$= \lim_{x\to\infty} \frac{1}{\left(1+e^{2x}\right)^{3/2}}$$

$$= \lim_{x\to\infty} \frac{1}{\left(e^{-\frac{2}{3}x}+e^{\frac{4}{3}x}\right)^{3/2}}$$

$$= \frac{1}{0+\infty} = \frac{1}{\infty} = \boxed{0}$$

Answer 32E.

Consider the vertex equation for a parabola, $y = a(x-h)^2 + k$

Where h is the, $x-$component of vertex and k is the, $y-$ component of vertex.

But the vertex is at the origin that is $(0,0)$, so $h = 0, k = 0$.

Then, $y = a(x-h)^2 + k$

$$y = ax^2$$

Where a is some real number.

$$y = ax^2$$
$$= f(x)$$

Differentiating with respect to x.

$$f'(x) = 2ax$$

Again differentiating with respect to x.

$$f''(x) = 2a$$

So that, $f'(0) = 0$ and $f''(0) = 0$

Recollect that:

The curvature of the plane curve with equation $y = f(x)$ and x choose as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$.

$$\kappa(x) = \frac{|f''(x)|}{\left|1+\left(f'(x)\right)^2\right|^{\frac{3}{2}}}$$

Thus,

$$\kappa(0) = \frac{|f''(0)|}{\left|1+(f'(0))^2\right|^{\frac{3}{2}}}$$

$$= \frac{|2a|}{\left|1+0^2\right|^{\frac{3}{2}}}$$

$$= |2a|$$

The parabola that has curvature 4 at the origin.

To need a real number a such that $|2a| = 4$ that is $a = \pm 2$

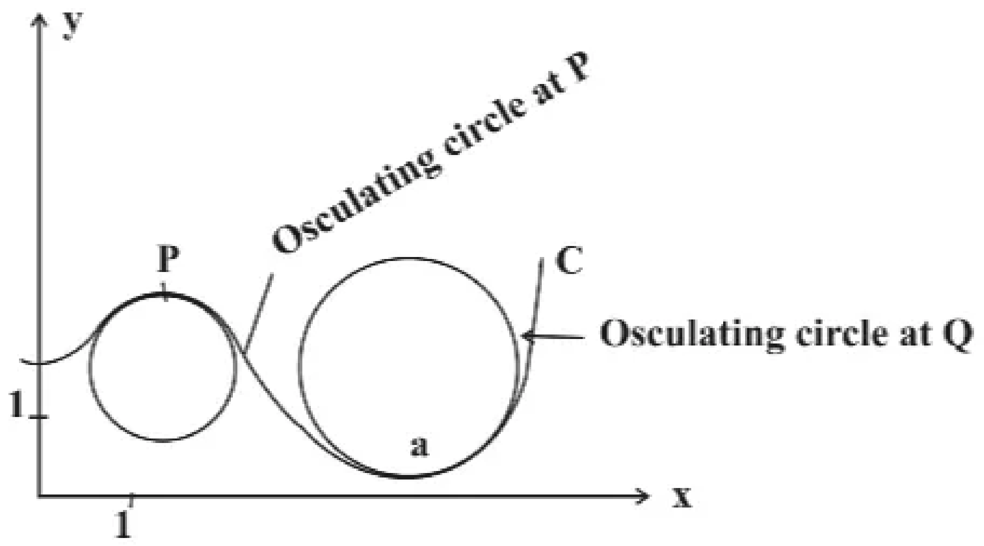
Therefore the equations, $y = 2x^2$ and $y = -2x^2$

Answer 33E.

(A)

The radius of curvature at a point is the rate of change of direction of curve at that point. Also the curvature is defined as the magnitude of the rate of change of the unit tangent vector with respect to the arc length. Since the arc length P is less than the arc length at Q then the rate of change of unit tangent vector with respect of the arc length is greater at P than at Q that is the curvature of curve C is grater at P.

(B)



Radius of osculating at P = 1/0.8 = 1.25
And radius of osculating circle at Q = 1.3
Then curvature at Q = 1/1.3
= 0.76

Answer 34E.

Consider the curve $y = x^4 - 2x^2$

Differentiating with respect to x

$$y' = 4x^3 - 4x$$

Again differentiating with respect to x

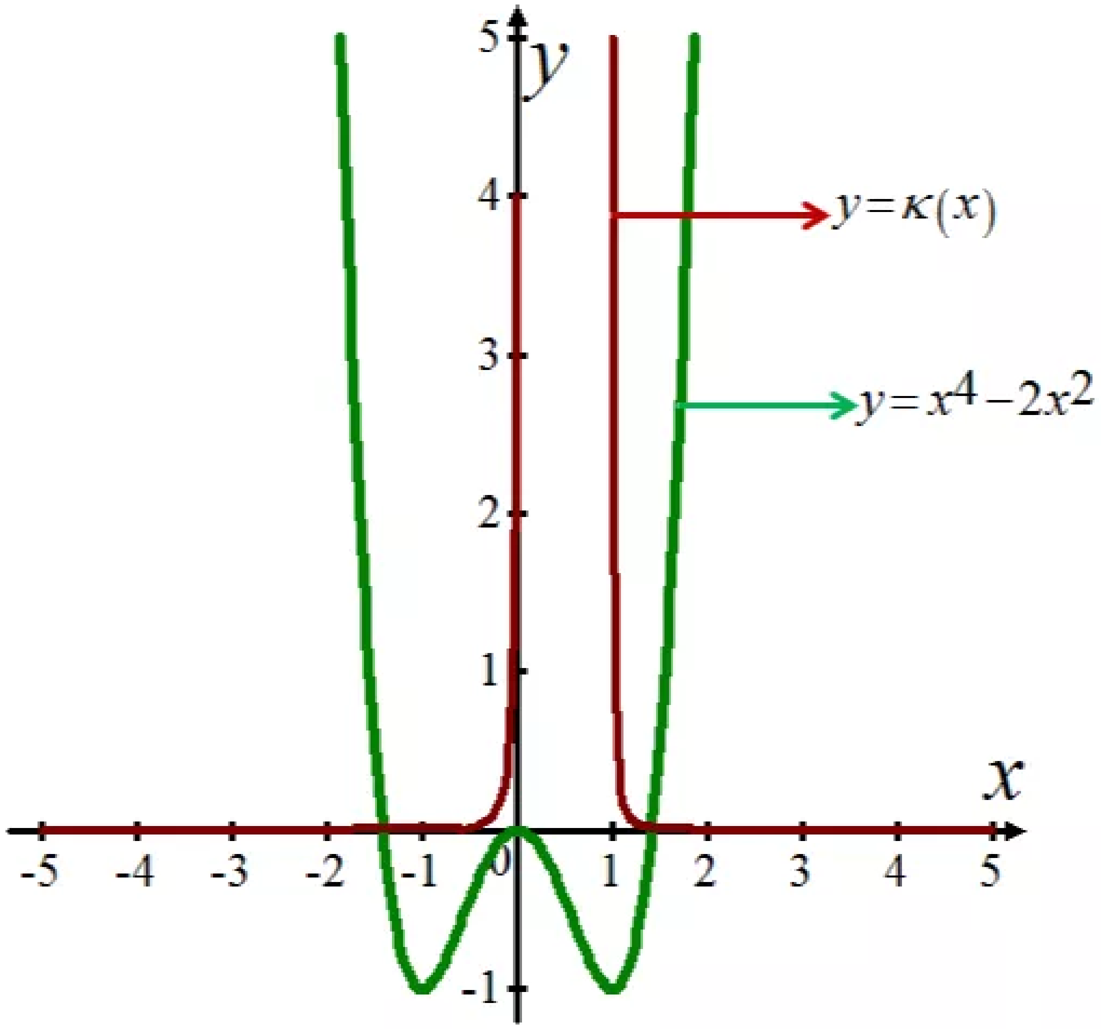
$$y'' = 12x^2 - 4$$

Recollect that: the curvature function $\kappa(x)$,

$$\begin{aligned}
\kappa(x) &= \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{\frac{3}{2}}} \\
&= \frac{|12x^2 - 4|}{\left[1+(4x^3-4x)^2\right]^{\frac{3}{2}}} \\
&= \frac{|12x^2 - 4|}{\left[1+16x^6+16x^2-32x\right]^{\frac{3}{2}}}
\end{aligned}$$

The diagram of the curve $y = x^4 - 2x^2$ and the curvature function

$$\kappa(x) = \frac{|12x^2 - 4|}{\left[1 + 16x^6 + 16x^2 - 32x\right]^{\frac{3}{2}}}$$



Answer 35E.

Consider the curve $y = x^{-2}$

Differentiating with respect to x

$$y' = (-2)x^{-3}$$

Again differentiating with respect to x

$$y'' = (6)x^{-4}$$

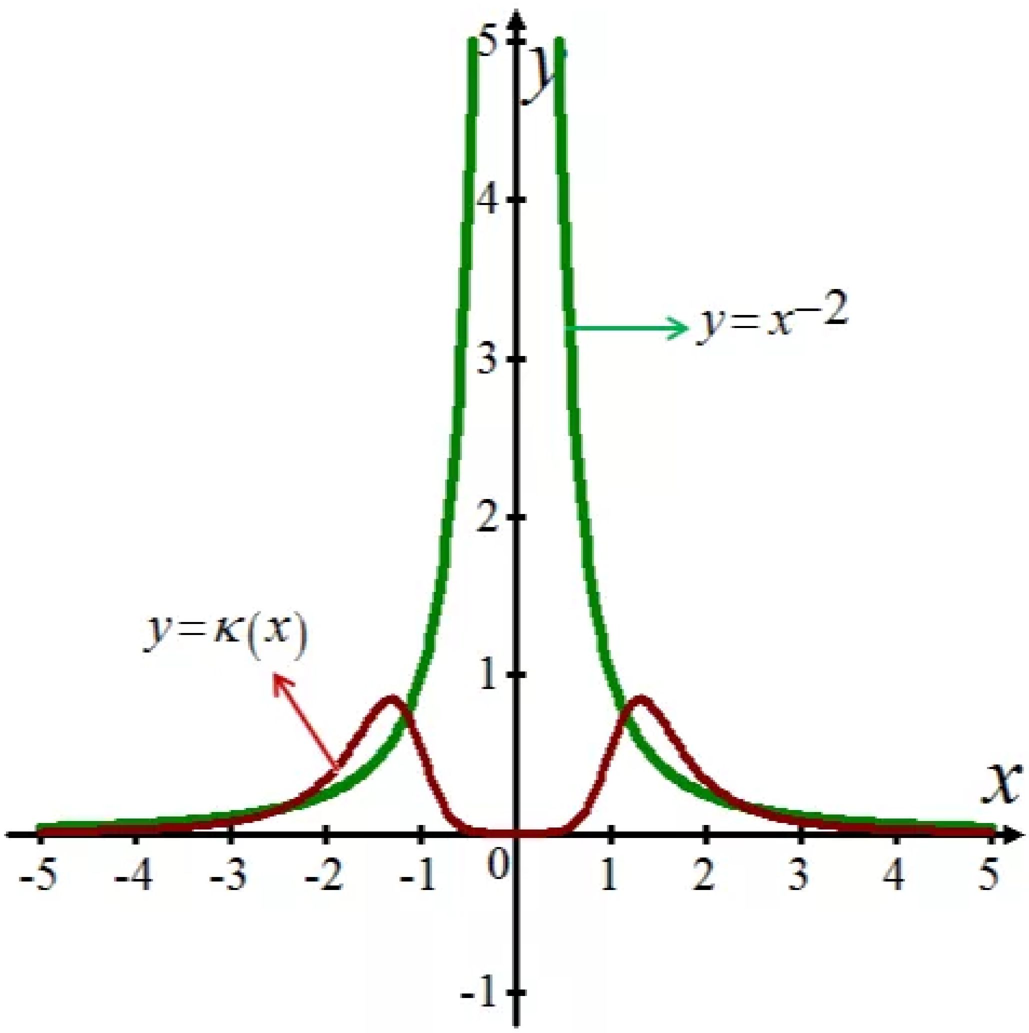
Recollect that:

$$\kappa(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{\frac{3}{2}}}$$

$$\kappa(x) = \frac{|(6)x^{-4}|}{\left[1+((-2)x^{-3})^2\right]^{\frac{3}{2}}}$$

$$= \frac{|6x^{-4}|}{\left[1+4x^{-6}\right]^{\frac{3}{2}}}$$

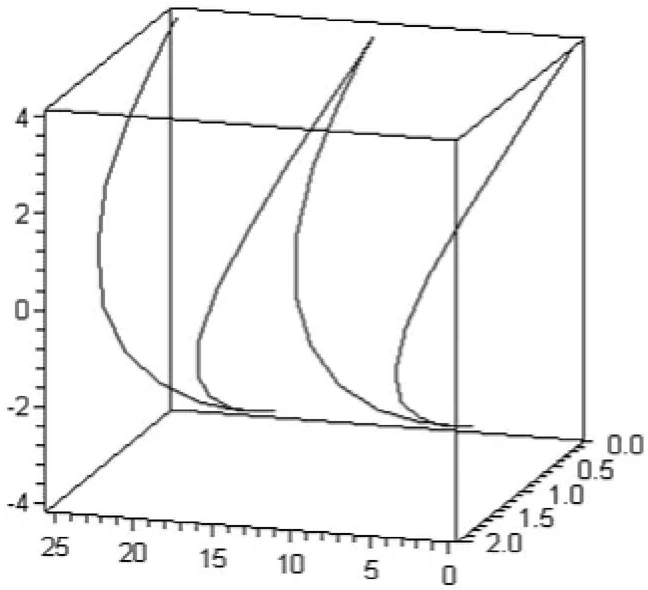
The diagram of the curve $y = x^{-2}$ and the curvature function $\kappa(x) = \dfrac{|6x^{-4}|}{\left[1+4x^{-6}\right]^{\frac{3}{2}}}$



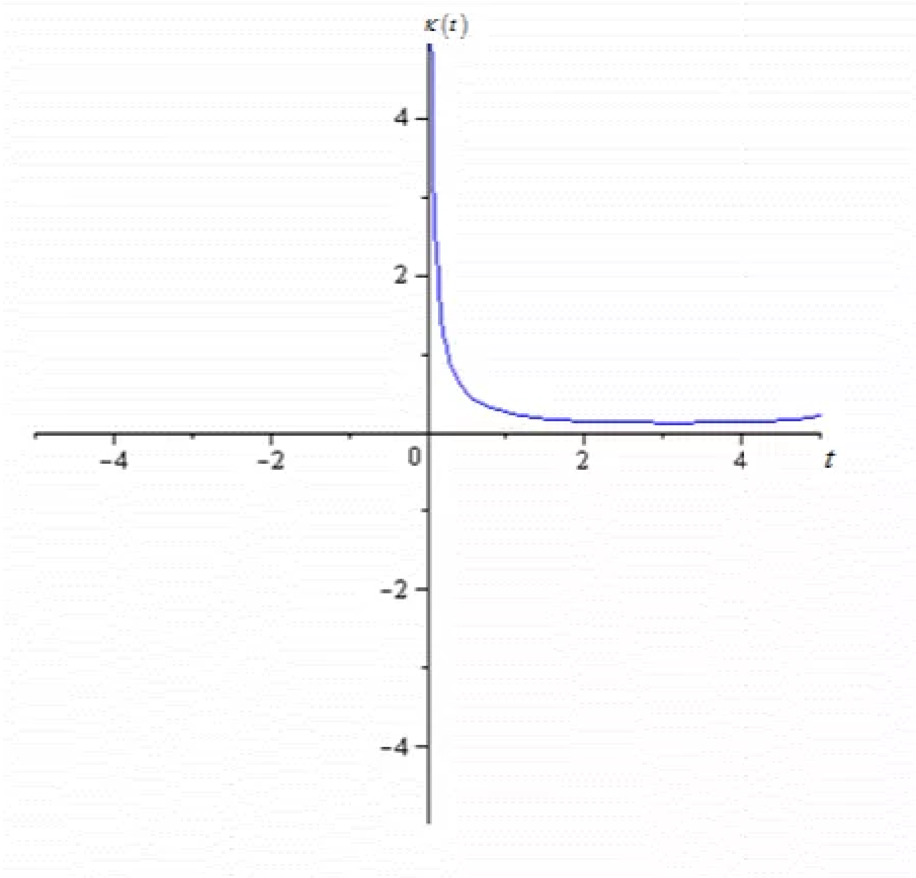
Answer 36E.

Given $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, 4\cos(t/2) \rangle$

The sketch of the space curve of $\mathbf{r}(t)$ is shown below.



Now, let us sketch the curvature function using a CAS.

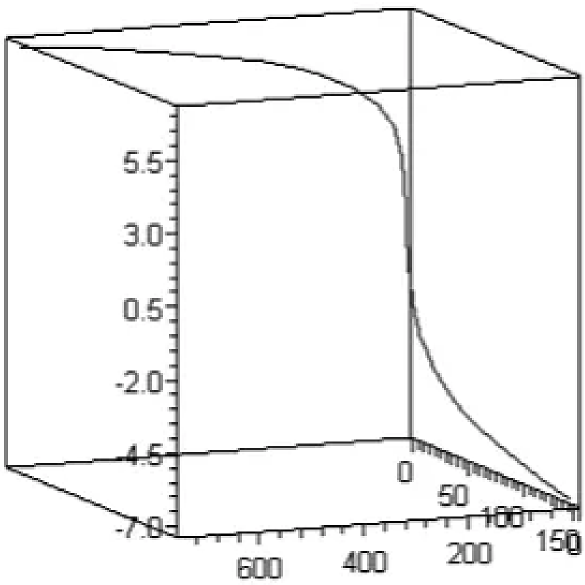


In the graph, we see that the curvature is maximum when t approaches zero. Thus, we can say that the graph has steep turns as t approaches zero. Also, as t approaches infinity the curvature tends to remain a constant. This means the curve changes its direction very slowly.

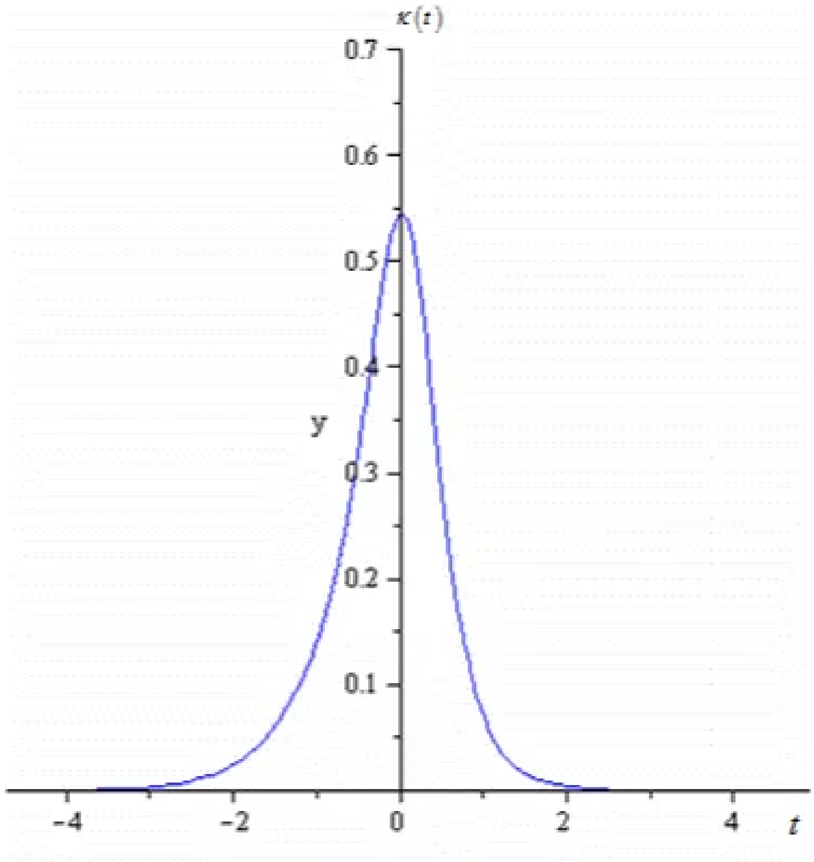
Answer 37E.

Given $\mathbf{r}(t)=\left\langle te^{t}, e^{-t}, \sqrt{2}t\right\rangle, \quad -5 \le t \le 5$

Start by sketching the space curve of $\mathbf{r}(t)$.



Now, let us sketch the curvature function using a CAS.



In the graph, we see that the curvature is maximum at $t=0$. Thus, we can say that the graph has steep turn as $t=0$. Also, as t approaches infinity the curvature tends to remain a constant. This means the curve changes its direction very slowly.

Answer 38E.

We know that if curvature is zero, then the curve is the straight line.
From the graphs it is clear that when the graph 'a' meets x axis, corresponding to it the graph 'b' is straight or nearly straight.
So, 'a' is the graph of $y = k(x)$ and 'b' is the graph of $y = f(x)$.

Answer 39E.

We know that if curvature is zero then the curve is straight line.
Now, from the given figure the graph 'b' appears to be straight at two places. But corresponding to it the graph 'a' does not intersect x axis. i.e. the curvature is not zero. Therefore 'a' cannot be the graph of $y = k(x)$.

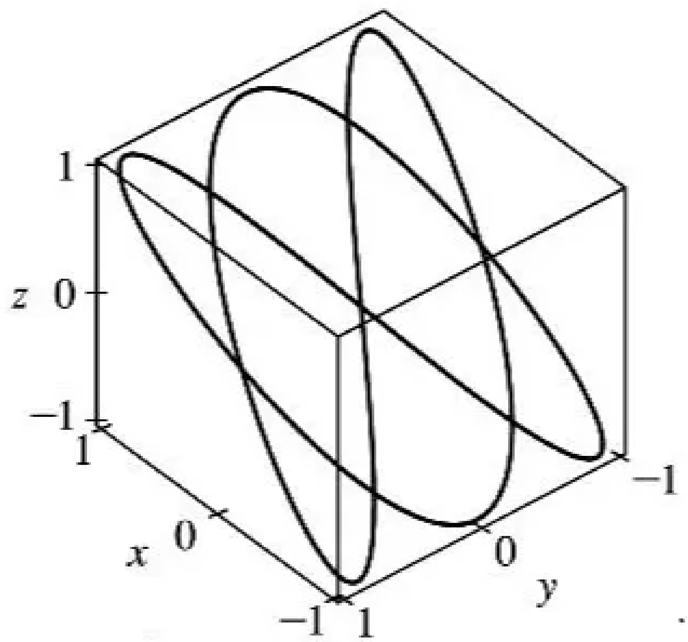
Hence, 'a' is the graph of curve $y = f(x)$

And 'b' is the graph of curve $y = k(x)$

Answer 40E.

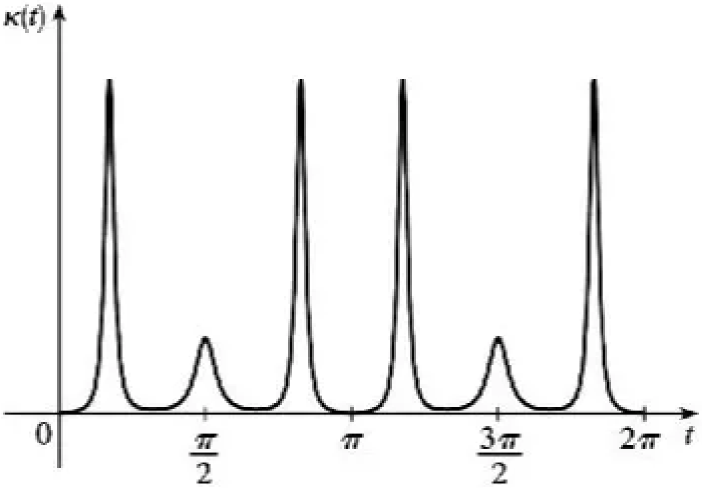
(a)

The complete curve is given by $0 \le t \le 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



(b)

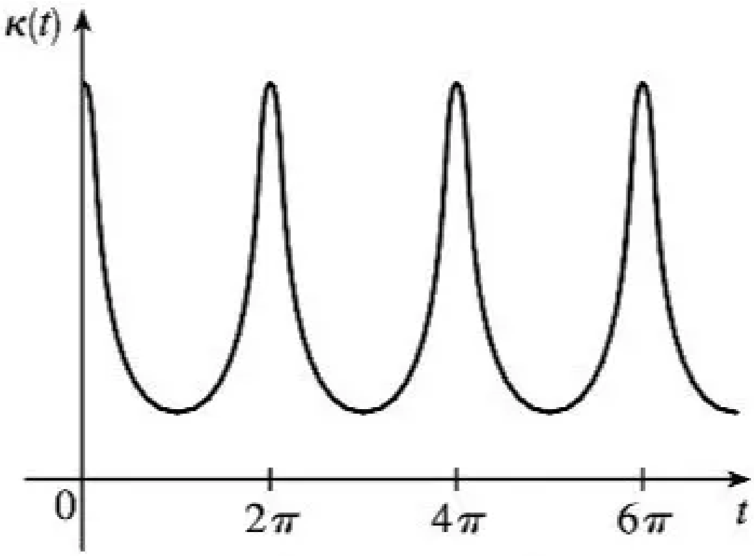
Using a CAS, we find (after simplifying) $\kappa(t) = \dfrac{3\sqrt{2}\sqrt{(5\sin t + \sin 5t)^2}}{(9\cos 6t + 2\cos 4t + 11)^{\frac{3}{2}}}$. (To compute cross products in Maple, use the Linalg package and the crossprod(a,b) command; in Mathematica, use Cross.) The graph shows 6 local (or absolute) maximum points for $0 \le t \le 2\pi$, as observed in part (a).



Answer 41E.

Using a CAS, we find (after simplifying) $\kappa(t) = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{\frac{3}{2}}}$.

(To compute cross products in Maple, use the Linalg package and the crossprod (a,b) command; in Mathematica, use Cross.) Curvature is largest at integer multiples of 2π.



Answer 42E.

The parametric curve is

$x = f(t)$, $y = g(t)$ ......(1)

Then the vector equation of curve is $\mathbf{r}(t) = \langle f(t), g(t), 0 \rangle$

The curvature of the curve is given by:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad \text{......(2)}$$

Now $\mathbf{r}'(t) = \langle f'(t), g'(t), 0 \rangle$

Then $|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$

And $\mathbf{r}''(t) = \langle f''(t), g''(t), 0 \rangle$

Then $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 0, 0, f'(t)g''(t) - g'(t)f''(t) \rangle$

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = f'(t)g''(t) - g'(t)f''(t)$

Using all these values in equation (2)

$$\kappa(t) = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{\left[(f'(t))^2 + (g'(t))^2\right]^{3/2}} \quad \text{......(3)}$$

Now from (1) $f(t) = x$, $g(t) = y$

Then $f'(t) = \frac{d}{dt} f(t) = \frac{d}{dt}(x) = \dot{x}$

$g'(t) = \frac{d}{dt} g(t) = \frac{d}{dt}(y) = \dot{y}$

And $f''(t) = \frac{d}{dt}\left(\dot{x}\right) = \ddot{x}$

$g''(t) = \frac{d}{dt}\left(\dot{y}\right) = \ddot{y}$

Then from equation (3)

$$\boxed{\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}}$$

Answer 43E.

Given $x = t^2$ and $y = t^3$

If C is a smooth curve given by $y = f(x)$, then the curvature κ of C at t is given by

$$\kappa(x) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\left[\dot{x}^2 + \dot{y}^2\right]^{3/2}}.$$

We have $x = t^2$ and $y = t^3$.

Differentiating we get

$\dot{x} = 2t$, $\ddot{x} = 2$, $\dot{y} = 3t^2$, and $\ddot{y} = 6t$.

Substituting the obtained values in the formula

$$\kappa(x) = \frac{|(2t)(6t) - (3t^2)(2)|}{\left[(2t)^2 + (3t^2)^2\right]^{3/2}}$$

$$= \frac{|12t^2 - 6t^2|}{\left[4t^2 + 9t^4\right]^{3/2}}$$

$$= \frac{6t^2}{\left[4t^2 + 9t^4\right]^{3/2}}$$

We get the curvature as $\boxed{\dfrac{6t^2}{\left[4t^2 + 9t^4\right]^{3/2}}}$.

Answer 44E.

Consider the plane parametric curve $x = a\cos \omega t, y = b\sin \omega t$

The curvature of the plane parametric curve $x = f(t), y = g(t)$ is,

$$\kappa = \frac{|\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}|}{\left[\dot{x}^2 + \dot{y}^2\right]^{\frac{3}{2}}} \quad \text{......(1)}$$

where the dots indicate derivatives with respect to t.

Take $x = a\cos\omega t$.

Differentiating with respect to 't' we get

$$\dot{x} = \frac{d}{dt}(a\cos\omega t)$$
$$= -a\sin\omega t\frac{d}{dt}(\omega t)$$
$$= -a\omega\sin\omega t$$

Again differentiating with respect to 't' we get

$$\ddot{x} = \frac{d}{dt}(-a\omega\sin\omega t)$$
$$= -a\omega\cos\omega t\frac{d}{dt}(\omega t)$$
$$= -a\omega^2\cos\omega t$$

Take $y = b\sin\omega t$.

Differentiating with respect to 't' we get

$$\dot{y} = \frac{d}{dt}(b\sin\omega t)$$
$$= b\cos\omega t\frac{d}{dt}(\omega t)$$
$$= b\omega\cos\omega t$$

Again differentiating with respect to 't' we get

$$\ddot{y} = \frac{d}{dt}(b\omega\cos\omega t)$$
$$= b\omega(-\sin\omega t)\frac{d}{dt}(\omega t)$$
$$= -b\omega^2\sin\omega t$$

Now

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = (-a\omega\sin\omega t)(-b\omega^2\sin\omega t) - (b\omega\cos\omega t)(-a\omega^2\cos\omega t)$$
$$= ab\omega^3\sin^2\omega t + ab\omega^3\cos^2\omega t$$
$$= ab\omega^3(\sin^2\omega t + \cos^2\omega t)$$
$$= ab\omega^3 \qquad \left[\because \sin^2 x + \cos^2 x = 1\right]$$

Further,

$$\dot{x}^2+\dot{y}^2=\left(-a\omega^2\sin\omega t\right)^2+\left(b\omega\cos\omega t\right)^2$$
$$=a^2\omega^2\sin^2\omega t+b^2\omega^2\cos^2\omega t$$
$$=\omega^2\left(a^2\sin^2\omega t+b^2\cos^2\omega t\right)$$

Therefore, from the curvature formula (1), we obtain

$$\kappa=\frac{\left|\dot{x}\,\ddot{y}-\dot{y}\,\ddot{x}\right|}{\left[\dot{x}^2+\dot{y}^2\right]^{\frac{3}{2}}}$$
$$=\frac{\left|ab\omega^3\right|}{\left[\omega^2\left(a^2\sin^2\omega t+b^2\cos^2\omega t\right)\right]^{\frac{3}{2}}}$$
$$=\frac{\left|ab\omega^3\right|}{\omega^3\left(a^2\sin^2\omega t+b^2\cos^2\omega t\right)^{\frac{3}{2}}}$$
$$=\frac{|ab|}{\left(a^2\sin^2\omega t+b^2\cos^2\omega t\right)^{\frac{3}{2}}}$$

Hence, the curvature of a given curve is $\boxed{\kappa=\frac{|ab|}{\left(a^2\sin^2\omega t+b^2\cos^2\omega t\right)^{\frac{3}{2}}}}$.

Answer 45E.

$x=e^t\cos t$

Differentiating with respect to t,

$$\dot{x}=\frac{dx}{dt}$$
$$=\frac{d}{dt}\left(e^t\cos t\right)$$
$$=e^t\left(-\sin t\right)+e^t\cos t$$
$$=e^t\left(-\sin t+\cos t\right)$$

Again differentiating with respect to t,

$$\ddot{x} = \frac{d\dot{x}}{dt}$$
$$= \frac{d}{dt}\left[e^t\left(-\sin t + \cos t\right)\right]$$
$$= e^t \frac{d}{dt}\left(-\sin t + \cos t\right) + \left(-\sin t + \cos t\right)\frac{d}{dt}e^t$$
$$= e^t\left(-\cos t - \sin t\right) + \left(-\sin t + \cos t\right)e^t$$
$$= e^t\left(-\cos t - \sin t - \sin t + \cos t\right)$$
$$= -2\sin t \cdot e^t$$

Differentiating y with respect to t,

$$\dot{y} = \frac{dy}{dt}$$
$$= \frac{d}{dt}\left(e^t \sin t\right)$$
$$= e^t \frac{d}{dt}\sin t + \sin t \frac{d}{dt}e^t$$
$$= e^t \cos t + \sin t \cdot e^t$$
$$= e^t\left(\cos t + \sin t\right)$$

Again differentiating with respect to t,

$$\ddot{y} = \frac{d}{dt}e^t\left(\cos t + \sin t\right)$$
$$= e^t \frac{d}{dt}\left(\cos t + \sin t\right) + \left(\cos t + \sin t\right)\frac{d}{dt}e^t$$
$$= e^t\left(-\sin t + \cos t\right) + \left(\cos t + \sin t\right)e^t$$
$$= e^t\left[-\sin t + \cos t + \cos t + \sin t\right]$$
$$= 2e^t \cos t$$

Now, $\dot{x}\ddot{y} - \dot{y}\ddot{x} = e^t\left(-\sin t + \cos t\right)2e^t \cos t - e^t\left(\cos t + \sin t\right)\left(-2e^t \sin t\right)$

$$= 2e^{2t}\left[-\sin t \cos t + \cos^2 t + \sin t \cos t + \sin^2 t\right]$$
$$= 2e^{2t}\left[\cos^2 t + \sin^2 t\right]$$
$$= 2e^{2t}$$

Also, $\dot{x}^2+\dot{y}^2=\left[e^t\left(-\sin t+\cos t\right)\right]^2+\left[e^t\left(\cos t+\sin t\right)\right]^2$

$$=e^{2t}\left(\sin^2 t+\cos^2 t-2\sin t\cos t\right)+e^{2t}\left(\cos^2 t+\sin^2 t+2\sin t\cos t\right)$$

$$=e^{2t}\left(1-2\sin t\cos t+1+2\sin t\cos t\right)$$

$$=2e^{2t}$$

Therefore the curvature,

$$k=\frac{\left|\dot{x}\ddot{y}-\dot{y}\ddot{x}\right|}{\left[\dot{x}^2+\dot{y}^2\right]^{3/2}}$$

$$=\frac{\left|2e^{2t}\right|}{\left(2e^{2t}\right)^{3/2}}$$

$$=\frac{2e^{2t}}{\left(2e^{2t}\right)^{3/2}}$$

$$=\frac{1}{\left(2e^{2t}\right)^{1/2}}$$

$$=\frac{1}{\sqrt{2}\,e^t}$$

Hence,

$$\boxed{\text{Curvature } k=\frac{1}{\sqrt{2}\,e^t}}$$

Answer 46E.

If C is a smooth curve given by $y=f(x)$, then the curvature κ of C at t is given by

$$\kappa(x)=\frac{\left|f''(x)\right|}{\left\{1+\left[f'(x)\right]^2\right\}^{3/2}}.$$

We have $f(x)=ce^{\alpha x}$.

Differentiating we have

$f'(x)=ce^{\alpha x}$ and $f''(x)=c^2e^{\alpha x}$.

Plug in the known values in the formula.

$$\kappa(x) = \frac{\left|c^2 e^{cx}\right|}{\left\{1+\left[ce^{cx}\right]^2\right\}^{3/2}}$$

$$= \frac{c^2 e^{cx}}{\left(1+c^2 e^{2cx}\right)^{3/2}}$$

We get the curvature as $\frac{c^2 e^{cx}}{\left(1+c^2 e^{2cx}\right)^{3/2}}$.

Replace x with 0.

$$\kappa(0) = \frac{c^2 e^{c(0)}}{\left(1+c^2 e^{2c(0)}\right)^{3/2}}$$

$$= \frac{c^2}{\left(1+c^2\right)^{3/2}}$$

In order to find the value of c which maximizes $\kappa(0)$ we have to set $\frac{d}{dc}\kappa(0) = 0$.

We have $\frac{d}{dc}\left[\frac{c^2}{\left(1+c^2\right)^{3/2}}\right] = -\frac{c\left(c^2-2\right)}{\left(1+c^2\right)^{5/2}}$.

Solve for c.

$$-\frac{c\left(c^2-2\right)}{\left(1+c^2\right)^{5/2}} = 0$$

$$c\left(c^2-2\right) = 0$$

$$c = 0, \pm\sqrt{2}$$

We note that $c=0$ minimizes $\kappa(0)$ and $c^2 = 2$ maximizes $\kappa(0)$.

The maximum value of $\kappa(0)$ is $\frac{2}{3\sqrt{3}}$.

Therefore $\boxed{c = \pm\sqrt{2}}$ maximizes $\kappa(0)$.

Answer 47E.

Given $\vec{r}(t)=\left\langle t^2,\frac{2}{3}t^3,t\right\rangle$

Differentiating with respect to t,

$$\vec{r}\,'(t)=\frac{d}{dt}\vec{r}(t)$$
$$=\frac{d}{dt}<t^2,\frac{2}{3}t^3,t>$$
$$=\left\langle 2t,2t^2,1\right\rangle$$
$$\left|\vec{r}\,'(t)\right|=\sqrt{(2t)^2+\left(2t^2\right)^2+(1)^2}$$
$$=\sqrt{4t^2+4t^4+1}$$
$$=\sqrt{\left(2t^2+1\right)^2}$$
$$=\left(2t^2+1\right)$$

Therefore,

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}$$
$$=\frac{\left\langle 2t,2t^2,1\right\rangle}{\left(2t^2+1\right)}$$

It is clear that at point $\left(1,\frac{2}{3},1\right)$, $\quad t=1$

Thus, $\quad \vec{T}(1)=\frac{\left\langle 2\times1,2\times1^2,1\right\rangle}{\left(2\times1^2+1\right)}$

$$=\left\langle\frac{2}{3},\frac{2}{3},\frac{1}{3}\right\rangle$$

Now $\bar{T}(t)=\dfrac{\langle 2t,2t^2,1\rangle}{(2t^2+1)}$

Differentiating with respect to t,

$$\bar{T}'(t)=\frac{(2t^2+1)\frac{d}{dt}\langle 2t,2t^2,1\rangle-\langle 2t,2t^2,1\rangle\frac{d}{dt}(2t^2+1)}{(2t^2+1)^2}$$

$$=\frac{(2t^2+1)\langle 2,4t,0\rangle-(2t,2t^2,1)(4t)}{(2t^2+1)^2}$$

$$=\frac{\langle 4t^2+2,8t^3+4t,0\rangle-\langle 8t^2,8t^3,4t\rangle}{(2t^2+1)^2}$$

$$=\frac{\langle 2-4t^2,4t,-4t\rangle}{(2t^2+1)^2}$$

Therefore,

$$\left|\dot{T}'(t)\right|=\sqrt{\left[\frac{2-4t^2}{(2t^2+1)^2}\right]^2+\left[\frac{4t}{(2t^2+1)^2}\right]^2+\left[\frac{-4t}{(2t^2+1)^2}\right]^2}$$

$$=\frac{1}{(2t^2+1)^2}\sqrt{(2-4t^2)^2+16t^2+16t^2}$$

$$=\frac{1}{(2t^2+1)^2}\sqrt{4+16t^4-16t^2+16t^2+16t^2}$$

$$=\frac{1}{(2t^2+1)^2}\sqrt{16t^4+16t^2+4}$$

$$=\frac{2}{(2t^2+1)^2}\sqrt{4t^4+4t^2+1}$$

$$=\frac{2}{(2t^2+1)^2}\cdot(2t^2+1)$$

$$=\frac{2}{(2t^2+1)}$$

Now, $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$

$$= \frac{\left\langle 2-4t^2, 4t, -4t \right\rangle}{\left(2t^2+1\right)^2 \cdot \dfrac{2}{\left(2t^2+1\right)}}$$

$$= \frac{\left\langle 1-2t^2, 2t, -2t \right\rangle}{\left(2t^2+1\right)}$$

At point $(1, 2/3, 1)$, $\quad t = 1$

Therefore, $N(t)$ at point $(1, 2/3, 1)$ is

$$\vec{N}(1) = \frac{\left\langle 1-2\times 1^2, 2\times 1, -2\times 1 \right\rangle}{\left(2\times 1^2+1\right)}$$

$$= \frac{\langle -1, 2, -2 \rangle}{3}$$

$$= \langle -1/3, 2/3, -2/3 \rangle$$

We know that $\bar{B}(t) = \bar{T}(t) \times \bar{N}(t)$

Therefore $\bar{B}(t)$ at point $(1, 2/3, 1)$

$$\vec{B}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{vmatrix}$$

$$= \frac{1}{9} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ -1 & 2 & -2 \end{vmatrix}$$

$$= \frac{1}{9} \left[\hat{i} \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} \right]$$

$$= \frac{1}{9}\left(-6\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$$= \langle -6/9, 3/9, 6/9 \rangle$$

$$= \langle -2/3, 1/3, 2/3 \rangle$$

Hence,

At point (1, 2/3, 1)
$\bar{T}(1) = \langle 2/3, 2/3, 1/3 \rangle$
$\vec{N}(1) = \langle -1/3, 2/3, -2/3 \rangle$
$\vec{B}(1) = \langle -2/3, 1/3, 2/3 \rangle$

Answer 48E.

Consider the following vector function:

$\mathbf{r}(t)=\langle\cos t,\sin t,\ln\cos t\rangle$. ......(1)

The objective is to find the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the point $(1,0,0)$.

If $\mathbf{r}(t)$ is a smooth curve, then the **unit tangent vector** $\mathbf{T}$, **unit normal vector N**, and **binormal vector B** are defined as;

(A) $\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$,

(B) $\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$,

(C) $\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$.

First, find the parameter values *t* corresponding to the point $(1,0,0)$.

The parametric equations of the vector function (1) are,

$x=\cos t,\ y=\sin t,\ z=\ln\cos t$.

Substitute $x=1,\ y=0,\ z=0$ into the parametric equations and solve for *t*:

$1=\cos t,\quad 0=\sin t,\quad 0=\ln\cos t$.

Solve these equations for *t*, it implies that $t=0$. Hence, the parameter value corresponding to the point $(1,0,0)$ is $t=0$.

Now, find the derivative $\mathbf{r}'(t)$ of the vector function (1).

Differentiate each component of $\mathbf{r}(t)$ to find the derivative of the vector function. Then,

$$\begin{aligned}\mathbf{r}'(t)&=\frac{d}{dt}\mathbf{r}(t)\\&=\frac{d}{dt}\langle\cos t,\sin t,\ln\cos t\rangle\\&=\left\langle\frac{d}{dt}(\cos t),\frac{d}{dt}(\sin t),\frac{d}{dt}(\ln\cos t)\right\rangle\end{aligned}$$

$$=\left\langle-\sin t,\cos t,-\frac{\sin t}{\cos t}\right\rangle$$

$=\langle-\sin t,\cos t,-\tan t\rangle$. ......(i)

Next, find the value of $|\mathbf{r}'(t)|$:

$$|\mathbf{r}'(t)| = |\langle -\sin t, \cos t, -\tan t\rangle|$$

$$= \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2} \quad \text{Use } |\langle a,b,c\rangle| = \sqrt{a^2+b^2+c^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{(\sin^2 t + \cos^2 t) + \tan^2 t}$$

$$= \sqrt{1 + \tan^2 t}$$

$$= \sqrt{\sec^2 t}$$

$$= \sec t \text{. ......(ii)}$$

Then, use formula (A) and find the **unit tangent vector T** as follows:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad \text{Write formula (A)}$$

$$\mathbf{T}(t) = \frac{\langle -\sin t, \cos t, -\tan t\rangle}{\sec t} \quad \text{From (i) and (ii)}$$

$$= \left\langle -\frac{\sin t}{\sec t}, \frac{\cos t}{\sec t}, -\frac{\tan t}{\sec t}\right\rangle$$

$$= \langle -\sin t\cos t, \cos^2 t, -\sin t\rangle \text{. ......(2)}$$

At the point $(1,0,0)$, that is, at $t=0$, the **unit tangent vector T** will be,

$$\mathbf{T}(0) = \langle -\sin 0\cos 0, \cos^2 0, -\sin 0\rangle$$

$$= \langle -0, 1^2, -0\rangle$$

$$= \langle 0,1,0\rangle \text{. ......(3)}$$

Therefore, the **unit tangent vector T** at the point $(1,0,0)$ is $\boxed{\mathbf{T}(0) = \langle 0,1,0\rangle}$.

Now, find the derivative $\mathbf{T}'(t)$ of the vector function (2).

$$\mathbf{T}'(t) = \frac{d}{dt}\mathbf{T}(t)$$

$$= \frac{d}{dt}\langle -\sin t\cos t, \cos^2 t, -\sin t\rangle$$

$$= \left\langle \frac{d}{dt}(-\sin t\cos t), \frac{d}{dt}(\cos^2 t), \frac{d}{dt}(-\sin t)\right\rangle$$

$$= \langle \sin^2 t - \cos^2 t, -2\sin t\cos t, -\cos t\rangle \text{.}$$

At the point $(1,0,0)$, that is, at $t=0$, the value of $\mathbf{T}'(t)$ will be,

$$\mathbf{T}'(0)=\langle \sin^2 0-\cos^2 0,-2\sin 0\cos 0,-\cos 0\rangle$$
$$=\langle 0-1,-2\cdot 0\cdot 1,-1\rangle$$
$$=\langle -1,0,-1\rangle \text{. ......(iii)}$$

Next, find the value of $|\mathbf{T}'(0)|$:

$$|\mathbf{T}'(0)|=|\langle -1,0,-1\rangle|$$
$$=\sqrt{(-1)^2+0^2+(-1)^2} \quad \text{Use } |\langle a,b,c\rangle|=\sqrt{a^2+b^2+c^2}$$
$$=\sqrt{1+0+1}$$
$$=\sqrt{2} \text{. ......(iv)}$$

Then, use formula (B) and find the **unit normal vector N** at the point $(1,0,0)$, that is, at $t=0$, as follows:

$$\mathbf{N}(0)=\frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} \quad \text{Write formula (B) at } t=0$$
$$\mathbf{N}(0)=\frac{\langle -1,0,-1\rangle}{\sqrt{2}} \quad \text{From (iii) and (iv)}$$
$$=\left\langle -\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\rangle \text{. ......(4)}$$

Therefore, the **unit normal vector N** at the point $(1,0,0)$ is $\boxed{\mathbf{N}(0)=\left\langle -\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\rangle}$.

Then, use formula (C) and find the **binormal vector B** at $(1,0,0)$, that is, at $t=0$, as follows:

$\mathbf{B}(0)=\mathbf{T}(0)\times\mathbf{N}(0)$ Write formula (C) at $t=0$

$\mathbf{B}(0)=\langle 0,1,0\rangle\times\left\langle -\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right\rangle$ From (3) and 4)

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 0\\ -1/\sqrt{2} & 0 & -1/\sqrt{2}\end{vmatrix}$$

$$=\left[(1)\left(-\frac{1}{\sqrt{2}}\right)-0\right]\mathbf{i}-[0-0]\mathbf{j}+\left[0-(1)\left(-\frac{1}{\sqrt{2}}\right)\right]\mathbf{k}$$

$$=-\frac{1}{\sqrt{2}}\mathbf{i}+0\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$$

$$=\left\langle -\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\rangle.$$

Therefore, the **binormal vector B** at the point $(1,0,0)$ is $\boxed{\mathbf{B}(0)=\left\langle -\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\rangle}$.

Answer 49E.

Consider the parametric curves

$x=2\sin 3t,\quad y=t,\quad z=2\cos 3t$

And the point $P(0,\pi,-2)$

So, the vector equation is

$\mathbf{r}(t)=\langle 2\sin 3t,\ t,\ 2\cos 3t\rangle$

Then

$\mathbf{r}'(t)=\langle 6\cos 3t,\ 1,\ -6\sin 3t\rangle$

Observe that at point $P(0,\pi,-2)$, we have $t=\pi$.

Note that, the equation of the plane is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

The normal plane at $P(0,\pi,-2)$ has normal vector $\mathbf{r}'(\pi)=\langle -6, 1, 0\rangle$, so an equation of the normal plane is

$$-6(x-0)+1(y-\pi)+0(z+2)=0$$

$$-6x+y-\pi=0$$

$$\boxed{y=6x+\pi}.$$

The osculating plane at P contains the vectors **T** and **N**, so its normal vector is

$$\mathbf{T}\times\mathbf{N}=\mathbf{B}$$

Now need to find the vector **B**.

Since, $\mathbf{r}'(t)=\langle 6\cos 3t, 1, -6\sin 3t\rangle$

Then

$$\begin{aligned}|\mathbf{r}(t)| &= \sqrt{(6\cos 3t)^2+1^2+(-6\sin 3t)^2}\\ &= \sqrt{36\cos^2 3t+1+36\sin^2 3t}\\ &= \sqrt{36(\cos^2 3t+\sin^2 3t)+1}\\ &= \sqrt{36+1}\\ &= \sqrt{37}\end{aligned}$$

Therefore,

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{1}{\sqrt{37}}\langle 6\cos 3t,\ 1,-6\sin 3t\rangle$$

Then,

$$\mathbf{T}'(t)=\frac{1}{\sqrt{37}}\langle -18\sin 3t,0,-18\cos 3t\rangle$$

$$\begin{aligned}|\mathbf{T}'(t)|&=\frac{1}{\sqrt{37}}\sqrt{(-18\sin 3t)^2+(-18\cos 3t)^2}\\&=\frac{1}{\sqrt{37}}\sqrt{18^2\sin^2 3t+18^2\cos^2 3t^2}\\&=\frac{1}{\sqrt{37}}\sqrt{18^2\left(\sin^2 3t+\cos^2 3t^2\right)}\\&=\frac{1}{\sqrt{37}}\sqrt{18^2(1)}\end{aligned}$$

$$=\frac{18}{\sqrt{37}}$$

Now

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

$$=\frac{1}{\left(\frac{18}{\sqrt{37}}\right)}\frac{1}{\sqrt{37}}\langle -18\sin 3t,0,-18\cos 3t\rangle$$

$$=\frac{1}{18}\cdot 18\langle -\sin 3t,0,-\cos 3t\rangle$$

$$=\langle -\sin 3t,0,-\cos 3t\rangle$$

Hence the binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{6}{\sqrt{37}}\cos 3t & \frac{1}{\sqrt{37}} & \frac{-6}{\sqrt{37}}\sin 3t \\ -\sin 3t & 0 & -\cos 3t \end{vmatrix}$$

$$= \frac{1}{\sqrt{37}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6\cos 3t & 1 & -6\sin 3t \\ -\sin 3t & 0 & -\cos 3t \end{vmatrix}$$

$$= \frac{1}{\sqrt{37}}\left[-\cos 3t\mathbf{i} + 6\mathbf{j} + \sin 3t\,\mathbf{k}\right]$$

And then $\mathbf{B}(\pi) = \frac{1}{\sqrt{37}}\langle 1, 6, 0\rangle$

So equation of osculating plane is

$$1(x-0) + 6(y-\pi) + 0(z+2) = 0$$
$$x + 6y - 6\pi = 0$$

$$\boxed{x + 6y = 6\pi}.$$

Answer 50E.

The vector equation of the curve is

$$\vec{r}(t) = <t, t^2, t^3>$$

The given point is (1, 1, 1)

The parameter corresponding to this point is t = 1

Now $\vec{r}\,'(t) = <1, 2t, 3t^2>$

The normal plane at given point has normal vector $\vec{r}\,'(1) = <1, 2, 3>$

Then the equation of normal plane is

$$1(x-1) + 2(y-1) + 3(z-1) = 0$$

i.e. $\boxed{x + 2y + 3z - 6 = 0}$

Now $\left|\vec{r}\,'(t)\right| = \sqrt{1 + 4t^2 + 9t^4}$

Then $\vec{T}(t) = \dfrac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}$

$$= \frac{1}{\sqrt{1+4t^2+9t^4}} <1, 2t, 3t^2>$$

At (1, 1, 1) i.e. $t = 1$

$$\vec{T}(t) = <\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}>$$

Now $\vec{T}'(t) = <\dfrac{-\left(4t+18t^3\right)}{\left(1+4t^2+9t^4\right)^{3/2}}, \dfrac{2-18t^4}{\left(1+4t^2+9t^4\right)^{3/2}}, \dfrac{12t^3+6t}{\left(1+4t^2+9t^4\right)^{3/2}}>$

At $t = 1$

$$\vec{T}'(t) = <\frac{-22}{(14)^{3/2}}, \frac{-16}{(14)^{3/2}}, \frac{18}{(14)^{3/2}}>$$

And $\left|\vec{T}'(t)\right|_{t=1} = \sqrt{\dfrac{484+256+324}{2744}}$

$$= \sqrt{\frac{1064}{2744}}$$

Now $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$

Then at $t = 1$, $\vec{N}(t) = \left[\dfrac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}\right]_{t=1}$

$$= <\frac{-22}{2\sqrt{266}}, \frac{-16}{2\sqrt{266}}, \frac{18}{2\sqrt{266}}>$$

$$= <\frac{-11}{\sqrt{266}}, \frac{-8}{\sqrt{266}}, \frac{9}{\sqrt{266}}>$$

Then $\left[\vec{B}(t)\right]_{t=1} = \left[\vec{T}(t)\times\vec{N}(t)\right]_{t=1}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{-11}{\sqrt{266}} & \frac{-8}{\sqrt{266}} & \frac{9}{\sqrt{266}} \end{vmatrix}$$

$$= \frac{1}{\sqrt{14}\sqrt{266}}\left(42\hat{i} - 42\hat{j} + 14\hat{k}\right)$$

i.e. $\vec{B}(1) = <\frac{42}{\sqrt{3724}}, \frac{-42}{\sqrt{3724}}, \frac{14}{\sqrt{3724}}>$

$$= <\frac{42}{14\sqrt{19}}, \frac{-42}{14\sqrt{19}}, \frac{14}{14\sqrt{19}}>$$

$$= <\frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}, \frac{1}{\sqrt{19}}>$$

The osculating plane at (1, 1, 1) contains vector $\vec{T}$ and $\vec{N}$, so its normal vector to

$$\vec{T}\times\vec{N} = \vec{B} = <\frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}, \frac{1}{\sqrt{19}}>$$

A simpler normal vector is $<3,-3,1>$

So an equation of osculating plane is:

$$3(x-1)-3(y-1)+1(z-1)=0$$

Or $3x-3y+z-3+3-1=0$

i.e. $\boxed{3x-3y+z-1=0}$

Answer 51E.

Consider the curve

$$9x^2 + 4y^2 = 36$$

Find the equations of the osculating circles

Rewrite the equation of the curve

$$4y^2 = 36 - 9x^2$$

$$y^2 = \frac{9}{4}(4 - x^2)$$

$$y = \pm\frac{3}{2}\sqrt{4 - x^2}$$

Find the first derivative

$$y' = \frac{3}{2} \cdot \frac{(-2x)}{2(4 - x^2)^{\frac{1}{2}}}$$

$$= \frac{-3x}{2(4 - x^2)^{1/2}}$$

$$y' = -\frac{3}{2}\left[x(4 - x^2)^{-\frac{1}{2}}\right]$$

Find the second derivative

$$y'' = -\frac{3}{2}\left[(4 - x^2)^{-\frac{1}{2}} + x \cdot -\frac{1}{2}(4 - x^2)^{-\frac{3}{2}}(-2x)\right]$$

$$= -\frac{3}{2}\left[(4 - x^2)^{-\frac{1}{2}} + x^2(4 - x^2)^{-\frac{3}{2}}\right]$$

$$= -\frac{3}{2}(4 - x^2)^{-\frac{3}{2}}\left[x^2 + (4 - x^2)\right]$$

$$= -\frac{3}{2}(4 - x^2)^{-\frac{3}{2}}(4)$$

$$= \frac{-6}{(4 - x^2)^{\frac{3}{2}}}$$

Find the radius of the curvature

The radius of curvature is given by the formula $K(x)=\frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$

$$k(x)=\frac{\left|\frac{-6}{\left(4-x^2\right)^{\frac{3}{2}}}\right|}{\left[1+\left(\frac{-3x}{2\left(4-x^2\right)^{1/2}}\right)^2\right]^{\frac{3}{2}}}$$

Radius of curvature at $(0,3)$ is

$$K(0)=\frac{\left|{-6}/{(4)^{3/2}}\right|}{[1+0]^{3/2}}$$

$$=\frac{6}{8}$$

$$=\frac{3}{4}$$

Then radius of osculating circle at $(0,3)$ is

$$\rho=\frac{1}{K(0)}$$

$$=\frac{4}{3}$$

Therefore the equation of the osculating circle at $(0,3)$ is

$$(x-x_1)^2+(y-y_1)^2=\rho^2$$

Where (x_1,y_1) is the centre

To find the center of the circle

Center lies in the *y*- axis, so center of the circle is of the form $(0,y)$

Radius=distance between center $(0,y)$ and $(0,3)$

$$3-y=\frac{4}{3}$$
$$y=\frac{5}{3}$$

Therefore, center is $(x_1,y_1)=\left(0,\ 5/3\right)$

Hence the required equation of the osculating circle is

$$\boxed{x^2+\left(y-\frac{5}{3}\right)^2=\frac{16}{9}}$$

Radius of curvature at $(2,0)$ is

$$K(x)=\frac{\dfrac{6}{(4-x^2)^{3/2}}}{\left[1+\dfrac{9x^2}{4(4-x^2)}\right]^{3/2}}$$

$$K(x)=\frac{6\Big/(4-x^2)^{3/2}}{\dfrac{\left[4(4-x^2)+9x^2\right]^{3/2}}{2^3(4-x^2)^{3/2}}}$$

$$K(x)=\frac{6\times 8}{\left[4(4-x^2)+9x^2\right]^{3/2}}$$

$$K(2)=\frac{6\times 8}{\left[4(4-2^2)+9(2)^2\right]^{3/2}}$$

$$K(2)=\frac{48}{(36)^{3/2}}$$

$$K(2)=\frac{48}{216}$$

$$K(2)=\frac{2}{9}$$

Then radius of osculating circle at $(2,0)$ is

$$\rho = \frac{1}{K(2)}$$

$$= \frac{9}{2}$$

Therefore the equation of the osculating circle at $(2,0)$ is

$$(x - x_2)^2 + (y - y_2)^2 = \rho^2$$

Where (x_2, y_2) is the centre

Find the center of the circle

Center lies in the *x*- axis, so center of the circle is of the form $(x,0)$

Radius=distance between center $(x,0)$ and $(2,0)$

$$2 - x = \frac{9}{2}$$

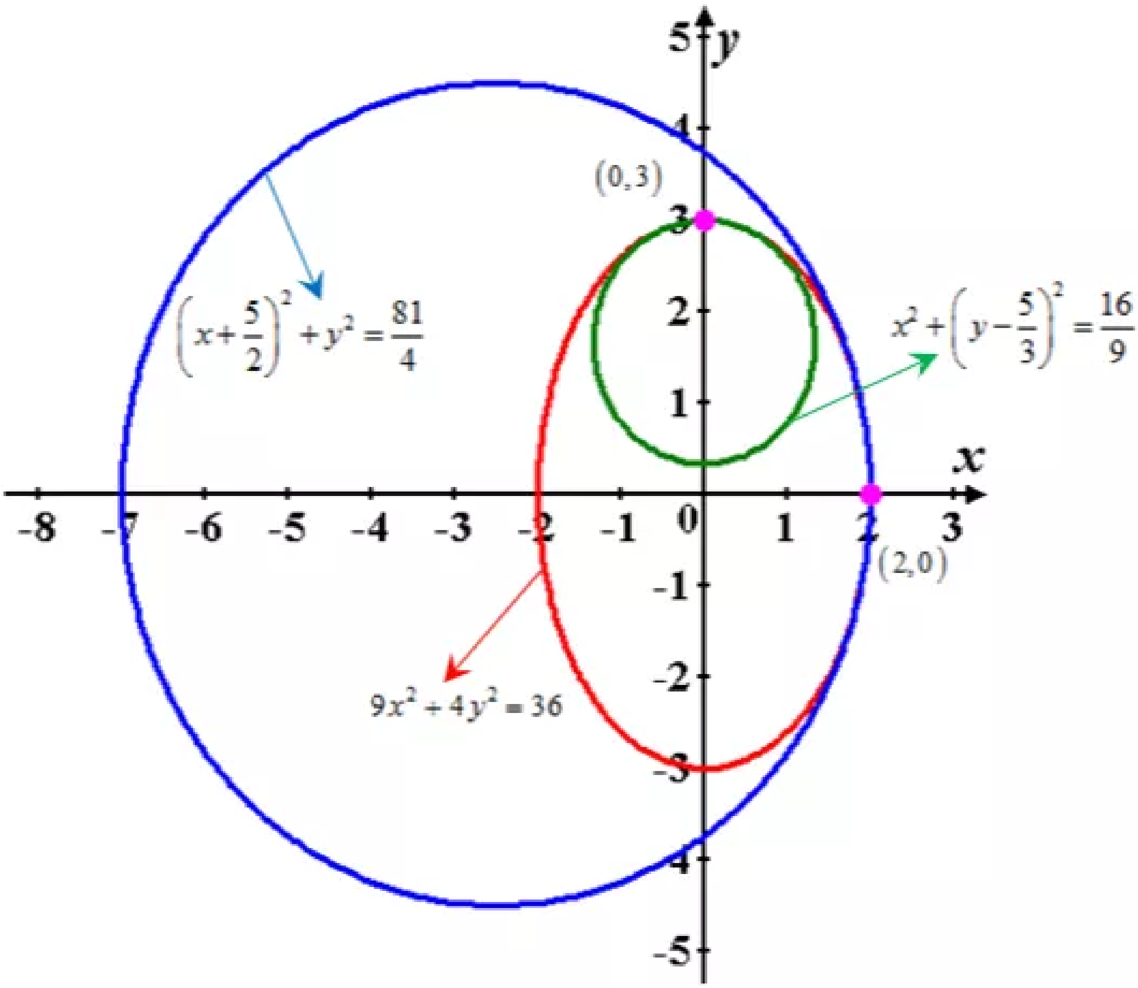
$$x = -\frac{5}{2}$$

Therefore, center is $(x_2, y_2) = \left(-\frac{5}{2}, 0\right)$

Hence the required equation of the osculating circle is

$$\boxed{\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}}$$

Graph of circles and ellipse



Answer 52E.

Consider the equation of a parabola $y=\frac{1}{2}x^2$.

The objective is to find equation of the osculating circle of the given parabola at the point $(0,0)$

.

Differentiate the curve $y=\frac{1}{2}x^2$ to obtain that,

$$\begin{aligned} y' &= \frac{1}{2}(2x) \\ &= x \end{aligned}$$

And,

$$\begin{aligned} y'' &= \frac{d}{dx}(x) \\ &= 1 \end{aligned}$$

Therefore, the radius of curvature is,

$$\begin{aligned} \kappa(x) &= \frac{|y''|}{\left[1+(y')^2\right]^{3/2}} \\ &= \frac{1}{\left[1+x^2\right]^{3/2}} \end{aligned}$$

At the point $(0,0)$, the radius of curvature is,

$$\begin{aligned} \kappa(0) &= \frac{1}{\left[1+0^2\right]^{3/2}} \\ &= 1 \end{aligned}$$

So the radius of curvature at the point $(0,0)$ is $\kappa(0)=1$.

The radius of the osculation circle at the origin is,

$$\rho = \frac{1}{\kappa(0)}$$
$$= \frac{1}{1}$$
$$= 1$$

Now write the vector equation of the given parabola.

Let $x = t$ then $y = \frac{1}{2}t^2$ and the point $(0,0)$ corresponds to the parameter value $t = 0.$

Therefore the vector equation to the given parabola is,

$$\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2 \right\rangle, \quad t = 0.$$

Then,

$$\mathbf{r}'(t) = \left\langle 1, \frac{1}{2}(2t) \right\rangle$$
$$= \langle 1, t \rangle$$

Find the tangent vector:

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$
$$= \frac{\langle 1, t \rangle}{\sqrt{1^2 + t^2}}$$
$$= \frac{\langle 1, t \rangle}{\sqrt{1 + t^2}}$$

And,

$$\mathbf{T}' = \frac{\left(\sqrt{1+t^2}\right)\langle 0,1 \rangle - \langle 1,t \rangle \left(\frac{2t}{2\sqrt{1+t^2}}\right)}{\left(\sqrt{1+t^2}\right)^2}$$
$$= \frac{(1+t^2)\langle 0,1 \rangle - \langle t,t^2 \rangle}{\left(\sqrt{1+t^2}\right)^2\left(\sqrt{1+t^2}\right)}$$
$$= \frac{\langle 0,1+t^2 \rangle - \langle t,t^2 \rangle}{\left(\sqrt{1+t^2}\right)^2\left(\sqrt{1+t^2}\right)}$$
$$= \frac{\langle -t,1 \rangle}{(1+t^2)\left(\sqrt{1+t^2}\right)}$$

The magnitude of the tangent vector is,

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{(1+t^2)\left(\sqrt{1+t^2}\right)}\sqrt{(-t)^2+1^2} \\ &= \frac{1}{(1+t^2)\left(\sqrt{1+t^2}\right)}\sqrt{t^2+1} \\ &= \frac{1}{1+t^2}\end{aligned}$$

Find the normal vector:

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \\ &= \frac{\langle -t,1\rangle}{(1+t^2)\left(\sqrt{1+t^2}\right)}\cdot(1+t^2) \\ &= \frac{\langle -t,1\rangle}{\left(\sqrt{1+t^2}\right)}\end{aligned}$$

$$\mathbf{N}(0) = \frac{\langle -0,1\rangle}{\left(\sqrt{1+0^2}\right)} = \langle 0,1\rangle$$

And the center of the circle will be $\rho = 1$ units from the point $\mathbf{r}(0) = (0,0)$ in the direction of $\mathbf{N}(0) = \langle 0,1\rangle$.

Center of the circle is,

$$\begin{aligned}C &= \mathbf{r}(0) + \rho\mathbf{N}(0) \\ &= \langle 0,0\rangle + 1\langle 0,1\rangle \\ &= (0,1)\end{aligned}$$

Thus, the equation of the osculating circle with center at $(a,b) = (0,1)$ and radius $\rho = r = 1$ is,

$$\begin{aligned}(x-a)^2 + (y-b)^2 &= r^2 \\ (x-0)^2 + (y-1)^2 &= 1^2 \\ x^2 + (y-1)^2 &= 1\end{aligned}$$

Therefore the required equation of the osculating circle of the given parabola at the point $(0,0)$ is $\boxed{x^2 + (y-1)^2 = 1}$.

Now need to find equation of the osculating circle of the given parabola at the point $\left(1,\frac{1}{2}\right)$.

At the point $\left(1,\frac{1}{2}\right)$, the radius of curvature is,

$$\kappa(1)=\frac{1}{\left[1+1^2\right]^{3/2}}$$
$$=\frac{1}{2^{\frac{3}{2}}}$$
$$=\frac{1}{2\sqrt{2}}$$

So the radius of curvature at the point $\left(1,\frac{1}{2}\right)$ is $\kappa(1)=\frac{1}{2\sqrt{2}}$.

The radius of the osculation circle at the origin is,

$$\rho=\frac{1}{\kappa(1)}$$
$$=\frac{1}{\frac{1}{2\sqrt{2}}}$$
$$=2\sqrt{2}$$

Write the vector equation of the given parabola.

Let $x=t$ then $y=\frac{1}{2}t^2$ and the point $\left(1,\frac{1}{2}\right)$ corresponds to the parameter value $t=1$.

Therefore the vector equation to the given parabola is,

$$\mathbf{r}(t)=\left\langle t,\frac{1}{2}t^2\right\rangle,\quad t=1.$$

Find the normal vector:

$$\mathbf{N}(t)=\frac{\langle -t,1\rangle}{\left(\sqrt{1+t^2}\right)}$$

$$\mathbf{N}(1)=\frac{\langle -1,1\rangle}{\left(\sqrt{1+1^2}\right)}$$

$$=\frac{\langle -1,1\rangle}{\sqrt{2}}$$

And the center of the circle will be $\rho=2\sqrt{2}$ units from the point $\mathbf{r}(1)=\left(1,\frac{1}{2}\right)$ in the direction of $\mathbf{N}(1)=\frac{\langle -1,1\rangle}{\sqrt{2}}$.

Center of the circle is,

$$C=\mathbf{r}(1)+\rho\mathbf{N}(1)$$

$$=\left\langle 1,\frac{1}{2}\right\rangle+2\sqrt{2}\frac{\langle -1,1\rangle}{\sqrt{2}}$$

$$=\left\langle 1,\frac{1}{2}\right\rangle+2\langle -1,1\rangle$$

$$=\left(1-2,\frac{1}{2}+2\right)$$

$$=\left(-1,\frac{5}{2}\right)$$

Thus, the equation of the osculating circle with center at $(a,b)=\left(-1,\frac{5}{2}\right)$ and radius $\rho=r=2\sqrt{2}$ is,

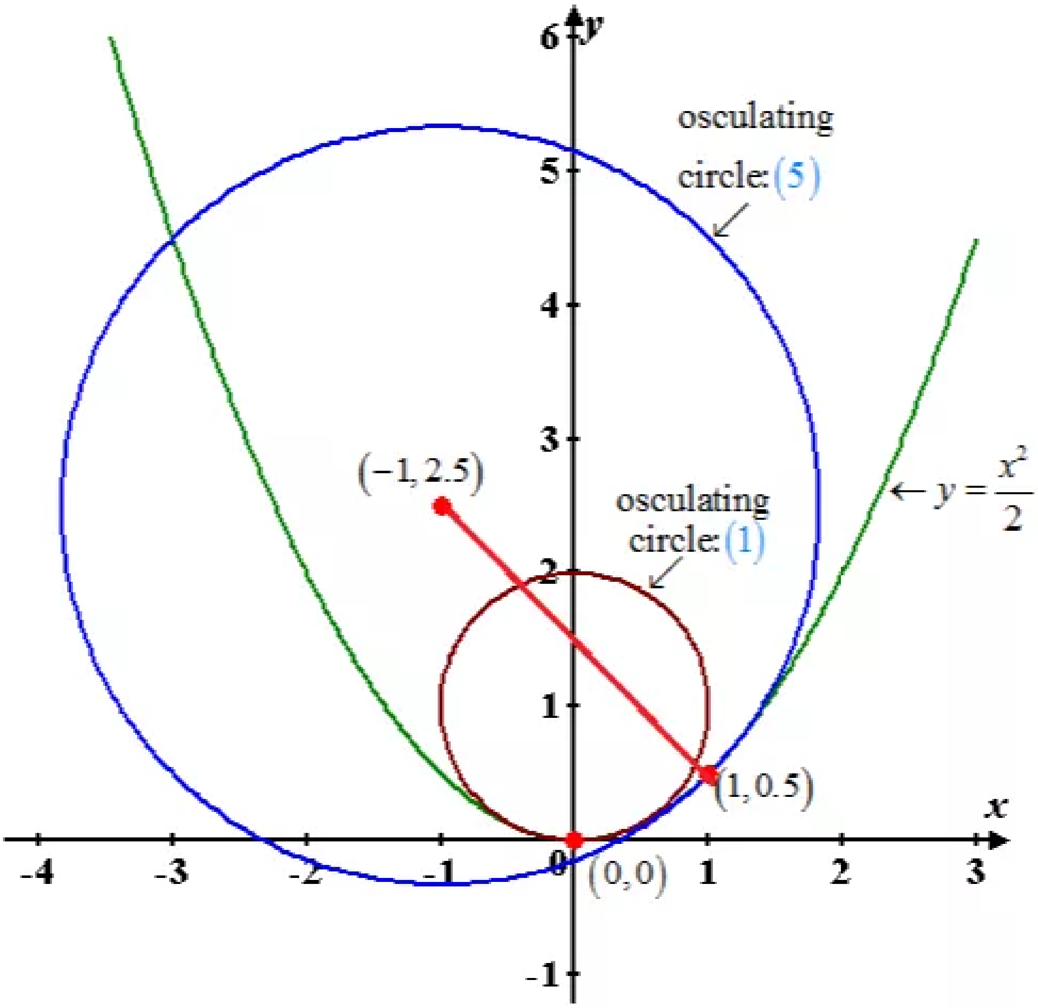
$$(x-a)^2+(y-b)^2=r^2$$

$$(x+1)^2+\left(y-\frac{5}{2}\right)^2=\left(2\sqrt{2}\right)^2$$

$$(x+1)^2+\left(y-\frac{5}{2}\right)^2=8$$

Therefore the required equation of the osculating circle of the given parabola at the point $\left(1,\frac{1}{2}\right)$ is $\boxed{(x+1)^2+\left(y-\frac{5}{2}\right)^2=8}$.

Graphs of the osculating circles and the parabola are shown below:



Answer 53E.

Consider the parametric equations of a plane $x = t^3, y = 3t, z = t^4$.

Then its vector equation form is $\vec{r}(t) = \langle t^3, 3t, t^4 \rangle$.

Differentiate $\vec{r}(t)$ with respect to t, get

$\vec{r}'(t) = \langle 3t^2, 3, 4t^3 \rangle$.

Let the parameter corresponding to the required point is t_1 then the normal plane at that point has the normal vector $\vec{r}'(t_1) = \langle 3t_1^2, 3, 4t_1^3 \rangle$.

Consider the plane $6x+6y-8z=1$.

Then the normals to the plane is $(6,6,-8)$.

Since the normal plane is parallel to the plane $6x+6y-8z=1$ (given) then the normals of both the planes must be parallel.

That is

$$\frac{3t_1^2}{6}=\frac{3}{6}=\frac{4t_1^3}{-8}$$
$$\frac{t_1^2}{2}=\frac{1}{2}=-\frac{t_1^3}{2}.$$

These relations give $t=-1$.

$$\vec{r}(t)=\langle t^3, 3t, t^4\rangle.$$

At $t=-1$,

$$\begin{aligned}\vec{r}(-1)&=\langle(-1)^3, 3(-1), (-1)^4\rangle\\&=\langle -1,-3,1\rangle\end{aligned}$$

Therefore the required point is $\boxed{(-1,-3,1)}$.

Answer 54E.

Consider the curve $x=t^3, y=3t, z=t^4$.

Then the vector function is $\mathbf{r}(t)=t^3\mathbf{i}+3t\mathbf{j}+t^4\mathbf{k}$.

Differentiate $\mathbf{r}(t)$ with respect to t, get

$$\mathbf{r}'(t)=(3t^2)\mathbf{i}+(3)\mathbf{j}+(4t^3)\mathbf{k}.$$

Then unit tangent vector is

$$\begin{aligned}\mathbf{T}(t)&=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\\&=\frac{(3t^2)\mathbf{i}+(3)\mathbf{j}+(4t^3)\mathbf{k}}{\sqrt{(3t^2)^2+3^2+(4t^3)^2}}\\&=\frac{(3t^2)\mathbf{i}+(3)\mathbf{j}+(4t^3)\mathbf{k}}{\sqrt{9t^4+9+16t^6}}\end{aligned}$$

Differentiate $\mathbf{T}(t)$ with respect to t, get

$$\mathbf{T}'(t)=\frac{\left(\sqrt{9t^4+9+16t^6}\right)\frac{d}{dt}\left(\left(3t^2\right)\mathbf{i}+(3)\mathbf{j}+\left(4t^3\right)\mathbf{k}\right)-\left(\left(3t^2\right)\mathbf{i}+(3)\mathbf{j}+\left(4t^3\right)\mathbf{k}\right)\frac{d}{dt}\left(\sqrt{9t^4+9+16t^6}\right)}{\left(\sqrt{9t^4+9+16t^6}\right)^2}$$

$$=\frac{\left(\sqrt{9t^4+9+16t^6}\right)\left((6t)\mathbf{i}+\left(12t^2\right)\mathbf{k}\right)-\left(\left(3t^2\right)\mathbf{i}+(3)\mathbf{j}+\left(4t^3\right)\mathbf{k}\right)\left(\frac{36t^3+96t^5}{2\sqrt{9t^4+9+16t^6}}\right)}{\left(9t^4+9+16t^6\right)}$$

$$=\frac{\left(9t^4+9+16t^6\right)\left((6t)\mathbf{i}+\left(12t^2\right)\mathbf{k}\right)-\left(\left(3t^2\right)\mathbf{i}+(3)\mathbf{j}+\left(4t^3\right)\mathbf{k}\right)\left(18t^3+48t^5\right)}{\left(9t^4+9+16t^6\right)^{\frac{3}{2}}}$$

$$=\frac{\left(-48t^7+54t\right)\mathbf{i}+\left(-54t^3-144t^5\right)\mathbf{j}+\left(36t^6+108t^2\right)\mathbf{k}}{\left(9t^4+9+16t^6\right)^{\frac{3}{2}}}$$

$$\left|\mathbf{T}'(t)\right|=\frac{\sqrt{\left(-48t^7+54t\right)^2+\left(-54t^3-144t^5\right)^2+\left(36t^6+108t^2\right)^2}}{\left(9t^4+9+16t^6\right)^{\frac{3}{2}}}.$$

The principal unit normal vector $\mathbf{N}(t)$ is

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{\left|\mathbf{T}'(t)\right|}$$

$$=\frac{\left(\left(-48t^7+54t\right)\mathbf{i}+\left(-54t^3-144t^5\right)\mathbf{j}+\left(36t^6+108t^2\right)\mathbf{k}\right)\left(9t^4+9+16t^6\right)^{\frac{3}{2}}}{\left(9t^4+9+16t^6\right)^{\frac{3}{2}}\sqrt{\left(-48t^7+54t\right)^2+\left(-54t^3-144t^5\right)^2+\left(36t^6+108t^2\right)^2}}$$

$$=\frac{\left(\left(-48t^7+54t\right)\mathbf{i}+\left(-54t^3-144t^5\right)\mathbf{j}+\left(36t^6+108t^2\right)\mathbf{k}\right)}{\sqrt{\left(-48t^7+54t\right)^2+\left(-54t^3-144t^5\right)^2+\left(36t^6+108t^2\right)^2}}$$

The binormal vector is

$$\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$$

$$=\frac{1}{\left(\sqrt{\left(-48t^{7}+54t\right)^{2}+\left(-54t^{3}-144t^{5}\right)^{2}+\left(36t^{6}+108t^{2}\right)^{2}}\right)\left(\sqrt{9t^{4}+9+16t^{6}}\right)}$$

$$\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3t^{2} & 3 & 4t^{3}\\ \left(-48t^{7}+54t\right) & \left(-54t^{3}-144t^{5}\right) & \left(36t^{6}+108t^{2}\right)\end{vmatrix}$$

$$=\frac{1}{\sqrt{36t^{2}\left(4t^{6}+36t^{2}+9\right)\left(16t^{6}+9t^{4}+9\right)^{2}}}$$

$$\cdot\begin{pmatrix}\mathbf{i}\left(3\left(36t^{6}+108t^{2}\right)-4t^{3}\left(-54t^{3}-144t^{5}\right)\right)\\ -\mathbf{j}\left(\left(3t^{2}\right)\left(36t^{6}+108t^{2}\right)-4t^{3}\left(-48t^{7}+54t\right)\right)\\ +\mathbf{k}\left(3t^{2}\left(-54t^{3}-144t^{5}\right)-3\left(-48t^{7}+54t\right)\right)\end{pmatrix}$$

$$=\frac{1}{6t\left(16t^{6}+9t^{4}+9\right)\sqrt{\left(4t^{6}+36t^{2}+9\right)}}$$

$$\cdot\begin{pmatrix}\mathbf{i}\left(576t^{8}+324t^{6}+324t^{2}\right)-\mathbf{j}\left(192t^{10}+108t^{8}+108t^{4}\right)+\\ +\mathbf{k}\left(-288t^{7}-162t^{5}-162t\right)\end{pmatrix}$$

Continuation to the above

$$=\frac{\left(36t\left(16t^6+9t^4+9\right)\mathbf{i}-12t^3\left(16t^6+9t^4+9\right)\mathbf{j}-18\left(16t^6+9t^4+9\right)\mathbf{k}\right)}{6\left(16t^6+9t^4+9\right)\sqrt{\left(4t^6+36t^2+9\right)}}$$

$$=\frac{\left(16t^6+9t^4+9\right)\left(6t\mathbf{i}-2t^3\mathbf{j}-3\mathbf{k}\right)}{\left(16t^6+9t^4+9\right)\sqrt{\left(4t^6+36t^2+9\right)}}$$

$$=\frac{\left(6t\mathbf{i}-2t^3\mathbf{j}-3\mathbf{k}\right)}{\sqrt{\left(4t^6+36t^2+9\right)}}$$

$$=\frac{1}{\sqrt{\left(4t^6+36t^2+9\right)}}\left\langle 6t,-2t^3,-3\right\rangle$$

A simpler normal vector is $\left\langle 6t,-2t^3,-3\right\rangle$.

This is also normal vector for the osculating plane.

The normal vector for the plane $x+y+z=1$ is $\langle 1,1,1\rangle$.

Since the plane $x+y+z=1$ is parallel to the osculating plane, so the two normal vectors are proportional to each other.

Therefore,

$$\frac{6t}{1}=\frac{-2t^3}{1}=\frac{-3}{1}$$
$$6t=-2t^3=-3$$

There is no t value exists to satisfy this relation.

Therefore conclude that there is no point on the curve exists where the osculating plane is parallel to the plane $x+y+z=1$.

Another method:

Use CAS to solve this problem.

Maple keystrokes:

```
with(VectorCalculus);

r := `<,>`(t^3, 3*t, t^4);

drdt := diff(r, t)

dsdt := Norm(drdt, 2);

T := drdt/dsdt;

dTdt := diff(T, t);

dSdt := Norm(dTdt, 2);

N := dTdt/dSdt;

B := simplify(CrossProduct(T, N));

simplify(Vector(3, {(1) = 6*t^2/((t^2*(4*t^6+36*t^2+9)/(16*t^6+9*t^4+9)^2)^(1/2)*
(16*t^6+9*t^4+9)), (2) = -2*t^4/((16*t^6+9*t^4+9)*(t^2*
(4*t^6+36*t^2+9)/(16*t^6+9*t^4+9)^2)^(1/2)), (3) = -3*t/((t^2*
(4*t^6+36*t^2+9)/(16*t^6+9*t^4+9)^2)^(1/2)*(16*t^6+9*t^4+9))}, attributes = [coords =
cartesian]), 'assume = positive');
```

Maple Result:

Maple Result:

with(*VectorCalculus*);

> *r* := `<,>`(*t*^3, 3**t*, *t*^4);

$$r := \left(t^3\right)e_x + 3\,t e_y + \left(t^4\right)e_z$$

> *drdt* := *diff*(*r*, *t*)

$$drdt := 3\,t^2 e_x + 3 e_y + 4\,t^3 e_z$$

> *dsdt* := *Norm*(*drdt*, 2);

$$dsdt := \sqrt{16\,t^6 + 9\,t^4 + 9}$$

> *T* := *drdt* / *dsdt*;

$$T := \frac{3\,t^2}{\sqrt{16\,t^6 + 9\,t^4 + 9}} e_x + \frac{3}{\sqrt{16\,t^6 + 9\,t^4 + 9}} e_y + \frac{4\,t^3}{\sqrt{16\,t^6 + 9\,t^4 + 9}} e_z$$

> *dTdt* := *diff*(*T*, *t*);

$$dTdt := \left(-\frac{3}{2}\,\frac{t^2\left(96\,t^5 + 36\,t^3\right)}{\left(16\,t^6 + 9\,t^4 + 9\right)^{3/2}} + \frac{6\,t}{\sqrt{16\,t^6 + 9\,t^4 + 9}}\right)e_x - \frac{3}{2}\,\frac{96\,t^5 + 36\,t^3}{\left(16\,t^6 + 9\,t^4 + 9\right)^{3/2}} e_y + \left(-\frac{2\,t^3\left(96\,t^5 + 36\,t^3\right)}{\left(16\,t^6 + 9\,t^4 + 9\right)^{3/2}} + \frac{12\,t^2}{\sqrt{16\,t^6 + 9\,t^4 + 9}}\right)e_z$$

> *dSdt* := *Norm*(*dTdt*, 2);

$$dSdt := 6\sqrt{\frac{t^2\left(4\,t^6 + 36\,t^2 + 9\right)}{\left(16\,t^6 + 9\,t^4 + 9\right)^2}}$$

> $N := dTdt/dSdt;$

$$N := \frac{1}{6}\,\frac{-\frac{3}{2}\,\frac{t^2\left(96\,t^5+36\,t^3\right)}{\left(16\,t^6+9\,t^4+9\right)^{3/2}}+\frac{6\,t}{\sqrt{16\,t^6+9\,t^4+9}}}{\sqrt{\frac{t^2\left(4\,t^6+36\,t^2+9\right)}{\left(16\,t^6+9\,t^4+9\right)^2}}}e_x - \frac{1}{4}\,\frac{96\,t^5+36\,t^3}{\sqrt{\frac{t^2\left(4\,t^6+36\,t^2+9\right)}{\left(16\,t^6+9\,t^4+9\right)^2}}\left(16\,t^6+9\,t^4+9\right)^{3/2}}e_y + \frac{1}{6}\,\frac{-\frac{2\,t^3\left(96\,t^5+36\,t^3\right)}{\left(16\,t^6+9\,t^4+9\right)^{3/2}}+\frac{12\,t^2}{\sqrt{16\,t^6+9\,t^4+9}}}{\sqrt{\frac{t^2\left(4\,t^6+36\,t^2+9\right)}{\left(16\,t^6+9\,t^4+9\right)^2}}}e_z$$

> $B := simplify(CrossProduct(T,N));$

$$B := \frac{6\,t^2}{\sqrt{\frac{t^2\left(4\,t^6+36\,t^2+9\right)}{\left(16\,t^6+9\,t^4+9\right)^2}}\left(16\,t^6+9\,t^4+9\right)}e_x - \frac{2\,t^4}{\left(16\,t^6+9\,t^4+9\right)\sqrt{\frac{t^2\left(4\,t^6+36\,t^2+9\right)}{\left(16\,t^6+9\,t^4+9\right)^2}}}e_y - \frac{3\,t}{\sqrt{\frac{t^2\left(4\,t^6+36\,t^2+9\right)}{\left(16\,t^6+9\,t^4+9\right)^2}}\left(16\,t^6+9\,t^4+9\right)}e_z$$

> $simplify((9), 'assume = positive')$

$$\frac{6t}{\sqrt{4t^6+36t^2+9}}e_x - \frac{2t^3}{\sqrt{4t^6+36t^2+9}}e_y - \frac{3}{\sqrt{4t^6+36t^2+9}}e_z$$

A simpler normal vector is $\langle 6t, -2t^3, -3 \rangle$.

This is also normal vector for the osculating plane.

The normal vector for the plane $x+y+z=1$ is $\langle 1,1,1 \rangle$.

Since the plane $x+y+z=1$ is parallel to the osculating plane, so the two normal vectors are proportional to each other.

Therefore,

$$\frac{6t}{1}=\frac{-2t^3}{1}=\frac{-3}{1}$$
$$6t=-2t^3=-3$$

There is no t value exists to satisfy this relation.

Therefore conclude that there is no point on the curve exists where the osculating plane is parallel to the plane $x+y+z=1$.

Answer 55E.

Given $x=y^2$ and $z=x^2$ at (1, 1, 1).
Let $y=t$.
Then, we get $x=t^2, z=t^4$.
Also, we have at (1, 1, 1) we have $t=1$ and
$$\mathbf{r}(t)=t^2\mathbf{i}+t\mathbf{j}+t^4\mathbf{k}$$

Differentiating we have
$$\mathbf{r}'(t) = 2t\mathbf{i}+\mathbf{j}+4t^3\mathbf{k}$$
Substitute 1 for t and simplify.
$$\begin{aligned}\mathbf{r}'(1) &= 2(1)\mathbf{i}+\mathbf{j}+4(1)^3\mathbf{k}\\ &= 2\mathbf{i}+\mathbf{j}+4\mathbf{k}\end{aligned}$$

Now $\left|\mathbf{r}'(t)\right|$.

$$\left|\mathbf{r}'(t)\right| = \sqrt{(2t)^2 + (1)^2 + \left(4t^3\right)}$$
$$= \sqrt{4t^2 + 1 + 16t^6}$$

Plug in t with 1 in $\left|\mathbf{r}'(t)\right| = \sqrt{4t^2 + 1 + 16t^6}$ and simplifying we get

$$\left|\mathbf{r}'(1)\right| = \sqrt{4(1)^2 + 1 + 16(1)^6}$$
$$= \sqrt{4 + 1 + 16}$$
$$= \sqrt{21}$$

We know that $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|}$.

Then, $\mathbf{T}(t) = \dfrac{2t\mathbf{i} + \mathbf{j} + 4t^3\mathbf{k}}{\sqrt{4t^2 + 1 + 16t^6}}$.

Plugging in the known values in the equation we get

$$\mathbf{T}(1) = \frac{2(1)\mathbf{i} + \mathbf{j} + 4(1)^3\mathbf{k}}{\sqrt{4(1)^2 + 1 + 16(1)^6}}$$

$$\mathbf{T}(1) = \frac{2\mathbf{i} + \mathbf{j} + 4\mathbf{k}}{\sqrt{21}}$$

We have $\mathbf{T}'(t)$ as $\dfrac{\left(4t^2 + 1 + 16t^6\right)\left(2\mathbf{i} + 12t^2\mathbf{k}\right) - \left(4t + 48t^5\right)\left(2t\mathbf{i} + \mathbf{j} + 4t^3\mathbf{k}\right)}{\left(4t^2 + 1 + 16t^6\right)^{3/2}}$.

Then, $\mathbf{T}'(1) = \dfrac{-62\mathbf{i} - 52\mathbf{j} + 44\mathbf{k}}{21\sqrt{21}}$ and

$$\left\|\mathbf{T}'(t)\right\| = \frac{\sqrt{62^2 + 52^2 + 44^2}}{21\sqrt{21}}.$$

Let C be a smooth curve represented by $\mathbf{r}$ on an interval I. if $\mathbf{T}'(t) \neq 0$, then the principal unit normal vector at t is defined as $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\left\|\mathbf{T}'(t)\right\|}$.

We thus get $\mathbf{N}(1) = \dfrac{\mathbf{T}'(1)}{\left\|\mathbf{T}'(1)\right\|}$.

Plugging in the known values in $\mathbf{N}(1) = \dfrac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|}$.

$$\mathbf{N}(1) = \dfrac{\dfrac{-62\mathbf{i} - 52\mathbf{j} + 44\mathbf{k}}{21\sqrt{21}}}{\dfrac{\sqrt{62^2 + 52^2 + 44^2}}{21\sqrt{21}}}$$

$$= \dfrac{-62\mathbf{i} - 52\mathbf{j} + 44\mathbf{k}}{\sqrt{62^2 + 52^2 + 44^2}}$$

Now, the normal plane at $P(1,1,1)$ has normal vector $\mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and so its equation is $2x + y + 4z - 7 = 0$

The osculating plane at $P(1, 1, 1)$ contains the vectors $\mathbf{T}(1)$ and $\mathbf{N}(1)$. Thus, the normal vector is given by $\mathbf{T}(1) \times \mathbf{N}(1) = \mathbf{B}(1)$.

$$\mathbf{B}(1) = \dfrac{1}{\sqrt{62^2 + 52^2 + 44^2}\,\sqrt{21}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 4 \\ -62 & 52 & 44 \end{vmatrix}$$

$$= \dfrac{42}{422}(6\mathbf{i} - 8\mathbf{j} - \mathbf{k})$$

A simple vector parallel to $\mathbf{B}(1)$ is $6\mathbf{i} - 8\mathbf{j} - \mathbf{k}$ and so the equation of the osculating plane is $\boxed{6x - 8y - z + 3 = 0}$

Answer 56E.

Given $\mathbf{r}(t) = (t + 2)\mathbf{i} + (1 - t)\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

Finding $\mathbf{r}'(t)$.

$\mathbf{r}'(t) = \mathbf{i} - \mathbf{j} + t\mathbf{k}$

Now $|\mathbf{r}'(t)|$.

$$|\mathbf{r}'(t)| = \sqrt{(1)^2 + (-1)^2 + (t)^2}$$

$$= \sqrt{2 + t^2}$$

We know that $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

On substituting the known values, we get $\mathbf{T}(t) = \dfrac{\mathbf{i} - \mathbf{j} + t\mathbf{k}}{\sqrt{2 + t^2}}$.

We have $\mathbf{T}'(t)$ as $\dfrac{-t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}}{(t^2 + 2)^{3/2}}$ and $|\mathbf{T}'(t)| = \dfrac{\sqrt{2}}{t^2 + 2}$.

It is known that the unit normal vector at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

$$\mathbf{N}(t) = \frac{\dfrac{-t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}}{(t^2 + 2)^{3/2}}}{\dfrac{\sqrt{2}}{t^2 + 2}}$$

$$= \frac{-t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}}{\sqrt{2}\left(\sqrt{t^2 + 2}\right)}$$

The binormal vector $\mathbf{B}(t)$ is given by

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

$$\mathbf{B}(t) = \frac{1}{\sqrt{2}(t^2 + 2)} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & t \\ -t & t & 2 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}(t^2 + 2)} \left[(-2 - t^2)\mathbf{i} - (2 + t^2)\mathbf{j} + \mathbf{k}(t - t)\right]$$

$$= \frac{1}{\sqrt{2}(t^2 + 2)} \left[(-2 - t^2)\mathbf{i} - (2 + t^2)\mathbf{j}\right]$$

$$= \frac{-\mathbf{i} - \mathbf{j}}{\sqrt{2}}$$

We note that $\mathbf{B}(t)$ is independent of t.

Therefore, the osculating plane at every point on the curve is the same.

Answer 57E.

By definition of $\vec{N}$ we know

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$$

i.e. $\vec{T}'(t) = \vec{N}(t)\left|\vec{T}'(t)\right|$

i.e. $\frac{d}{dt}\left(\vec{T}(t)\right) = \vec{N}(t)\left|\vec{T}'(t)\right|$

i.e. $\frac{d\vec{T}}{ds}\frac{ds}{dt} = \vec{N}(t)\left|\vec{T}'(t)\right|$

i.e. $\frac{d\vec{T}}{ds}\left|\vec{r}'(t)\right| = \vec{N}(t)\left|\vec{T}'(t)\right|$

(As $\left|\vec{r}'(t)\right| = {}^{ds}/_{dt}$)

i.e. $\frac{d\vec{T}}{ds} = \vec{N}(t)\frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$

But $\frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|} = K$

Hence $\frac{d\vec{T}}{ds} = \vec{N}K$

Hence proved

Answer 58E.

Since ϕ is the angle between $\vec{T}$ and $\hat{i}$

Then,

$$\vec{T} = \left|\vec{T}\right|\cos\phi\hat{i} + \left|\vec{T}\right|\sin\phi\hat{j}$$

As, $\left|\vec{T}\right| = 1$ then

$$\vec{T} = \cos\phi\hat{i} + \sin\phi\hat{j}$$

Differentiating $\vec{T}$ with respect to ϕ

$$\frac{d\vec{T}}{d\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

We know $\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{d\phi}\frac{d\phi}{ds}$

Then $\frac{d\vec{T}}{ds} = \left(-\sin\phi\hat{i} + \cos\phi\hat{j}\right)\frac{d\phi}{ds}$

Thus, $\left|\frac{d\vec{T}}{ds}\right| = \left|-\sin\phi\hat{i} + \cos\phi\hat{j}\right|\left|\frac{d\phi}{ds}\right|$

$= \left|\frac{d\phi}{ds}\right|$ ----- (1) $\left\{\text{As}\left|-\sin\phi\hat{i} + \cos\phi\hat{j}\right| = 1\right.$

By the definition of curvature,

$k = \left|\frac{d\vec{T}}{ds}\right|$ ----- (2)

Therefore, from (1) and (2)

$k = \frac{d\phi}{ds}$

Answer 59E.

(A) Since $\vec{B}$ is a unit vector.

Therefore, $\vec{B}\cdot\vec{B} = 1$

Differentiating with respect to s.

$$\frac{d}{ds}\left(\vec{B}\cdot\vec{B}\right) = \frac{d}{ds}(1)$$

$$\Rightarrow \frac{d\vec{B}}{ds}\cdot\vec{B} + \vec{B}\cdot\frac{d\vec{B}}{ds} = 0$$

$$\Rightarrow \frac{d\vec{B}}{ds}\cdot\vec{B} + \frac{d\vec{B}}{ds}\cdot\vec{B} = 0 \qquad \left[\text{since} \quad a\cdot b = b\cdot a\right]$$

$$\Rightarrow 2\frac{d\vec{B}}{ds}\cdot\vec{B} = 0$$

$$\Rightarrow \frac{d\vec{B}}{ds}\cdot\vec{B} = 0$$

Now, two vectors are perpendicular if their dot product is zero.

Therefore,

$\frac{d\vec{B}}{ds}$ Is perpendicular to $\vec{B}$

(B) Unit binormal $\vec{B}$ is given by

$$\vec{B} = \vec{T} \times \vec{N}$$

Obviously $\vec{B}$ is normal to the osculating plane and $\vec{B}$ is a unit vector perpendicular to both $\vec{T}$ and $\vec{N}$

Therefore $\quad \vec{T} \cdot \vec{B} = 0 \quad$ and $\quad \vec{N} \cdot \vec{B} = 0$

We have $\vec{T} \cdot \vec{B} = 0$

Differentiating with respect to s,

$$\frac{d}{ds}\left(\vec{T} \cdot \vec{B}\right) = \frac{d}{ds}(0)$$

$$\Rightarrow \quad \frac{d\vec{T}}{ds} \cdot \vec{B} + \vec{T} \cdot \frac{d\vec{B}}{ds} = 0$$

$$\Rightarrow \quad k\vec{N} \cdot \vec{B} + \vec{T} \cdot \frac{d\vec{B}}{ds} = 0 \qquad \left[\text{since } \frac{d\vec{T}}{ds} = k\vec{N}\right]$$

$$\Rightarrow \quad 0 + \vec{T} \cdot \frac{d\vec{B}}{ds} = 0 \qquad \left[\text{since } \vec{N} \cdot \vec{B} = 0\right]$$

$$\Rightarrow \quad \vec{T} \cdot \frac{d\vec{B}}{dS} = 0$$

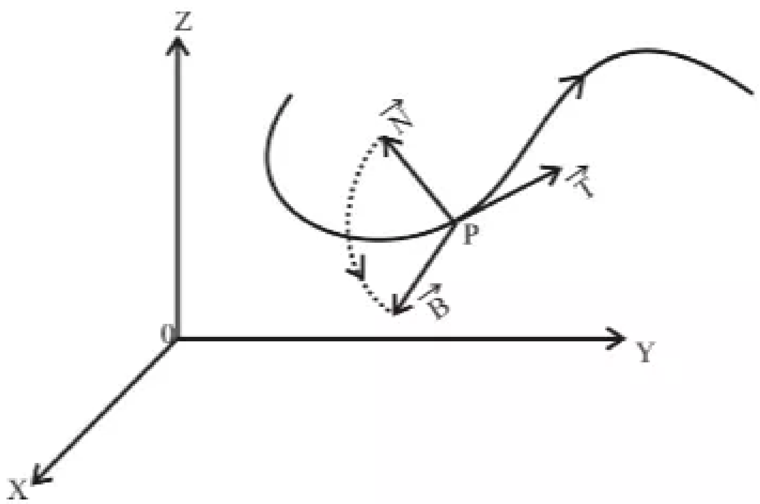
If the dot product of two vectors is zero then the vectors are perpendicular.

Therefore, $\frac{d\vec{B}}{ds}$ is perpendicular to $\vec{T}$

Hence,

$$\boxed{\frac{d\vec{B}}{ds} \text{ is perpendicular to } \vec{T}}$$

(C)



We know that the triplet $\vec{T}, \vec{N}$ and $\vec{B}$ constitute an orthonormal set of vectors.

From part (a) $\frac{d\vec{B}}{ds}$ is perpendicular to $\vec{B}$

From part (b) $\frac{d\vec{B}}{ds}$ is perpendicular to $\vec{T}$

Now, $\frac{d\vec{B}}{ds}$ is perpendicular to both $\vec{B}$ and $\vec{T}$.

So, $\frac{d\vec{B}}{ds}$ will be parallel to $\vec{B}\times\vec{T}$

i.e. $\frac{d\vec{B}}{ds}$ will be parallel to $\vec{N}$

Thus, we have $\frac{d\vec{B}}{ds}$ = scalar multiple of $\vec{N}$

Or, $\frac{d\vec{B}}{ds} = -\tau(s)\vec{N}$

Where $\tau(s)$ is called the torsion of the curve.

Now, here minus sign has the following meaning:-

When $\tau > 0$, $\frac{d\vec{B}}{ds}$ has the direction of $-\vec{N}$

Then, as P moves along the curve in a positive direction, $\vec{B}$ revolves about $\vec{T}$ in the same sense as a right handed screw advancing in direction of $\vec{T}$

Hence,

$$\boxed{\frac{d\vec{B}}{ds} = -\tau(s)\vec{N}}$$

(D) We know that for a plane curve $\vec{T}$ and $\vec{N}$ always lie on a fixed plane while $\vec{B}$ is a unit normal to that plane; so there will be no change in $\vec{B}$ with respect to arc length s.

Hence $\frac{d\vec{B}}{ds} = 0$ at all points where $\vec{N}$ is defined $(k \neq 0)$

$$\text{Now,}\quad \frac{d\vec{B}}{ds} = -\tau(s)\vec{N}$$

$$\Rightarrow \quad 0 = -\tau(s)\vec{N}$$

Since, $\vec{N}$ is a unit normal vector

Therefore, $\tau(s) = 0$

Hence,

For a plane curve the torsion $\tau(s) = 0$

Answer 60E.

$$\text{We know} \quad \vec{N} = \vec{B} \times \vec{T}$$

$$\text{Then} \quad \frac{d\vec{N}}{dS} = \frac{d}{dS}\left(\vec{B} \times \vec{T}\right)$$

$$= \frac{d\vec{B}}{ds} \times \vec{T} + \vec{B} \times \frac{d\vec{T}}{ds}$$

$$= -\tau\vec{N} \times \vec{T} + \vec{B} \times K\vec{N}$$

$$\left(\text{Since } \frac{d\vec{T}}{ds} = K\vec{N} \text{ and } \frac{d\vec{B}}{ds} = \tau\vec{N}\right)$$

$$= -\tau\left(\vec{N} \times \vec{T}\right) + K\left(\vec{B} \times \vec{N}\right)$$

$$\text{But} \quad \vec{B} \times \vec{N} = \vec{B} \times \left(\vec{B} \times \vec{T}\right) \quad \left(\text{Since } \vec{N} = \vec{B} \times \vec{T}\right)$$

$$= \left(\vec{B} \times \vec{T}\right)\vec{B} - \left(\vec{B} \times \vec{B}\right)\vec{T}$$

$$= (0)\vec{B} - \left|\vec{B}\right|^2 \vec{T} \left(\text{As } \vec{B} \;\; 1_{ar} \; \vec{T}\right)$$

$$= -\vec{T} \quad \left(\text{As } \left|\vec{B}\right| = 1\right)$$

$$\text{Hence} \quad \frac{d\vec{N}}{ds} = -\tau\left(\vec{N} \times \vec{T}\right) + K\left(-\vec{T}\right)$$

$$= \tau\left(\vec{T} \times \vec{N}\right) - K\vec{T}$$

$$= \tau\vec{B} - K\vec{T}$$

Answer 61E.

(A)

Since $\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$ and $|\vec{r}\,'| = \frac{ds}{dt}$

Then $\vec{r}\,' = |\vec{r}\,'|\vec{T} = \frac{ds}{dt}\vec{T}$ ------------------ (1)

And thus $\vec{r}\,'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}\,'$

$= s''\vec{T} + s'\vec{T}\,'$

$= s''\vec{T} + s'\vec{N}|\vec{T}\,'(t)|$

(As $\vec{N} = \vec{T}(t) / |\vec{T}\,'(t)|$)

$= s''\vec{T} + s'\vec{N}\frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}|\vec{r}\,'(t)|$

$= s''\vec{T} + s'\vec{N}K\frac{ds}{dt}$

$= s''\vec{T}K\vec{N}(s')^2$

i.e. $\vec{r}\,'' = s''\vec{T} + k\,\vec{N}(s')^2$ ------------------ (2)

(B)

Taking cross product of equations (1) and (2)

$\vec{r}\times\vec{r}\,'' = (s'\vec{T})\times(s''\vec{T} + \vec{K}(s')^2\vec{N})$

$= (s'\vec{T})\times(s''\vec{T}) + (s'\vec{T})\times\left[K(s')^2\vec{N}\right]$

$= s's''(\vec{T}\times\vec{T}) + K(s')^3\,\vec{T}\times\vec{N}$

$= s's''(0) + K(s')^3\vec{B}$

i.e. $\vec{r}\times\vec{r}\,'' = K(s')^3\vec{B}$ --------------- (3)

(C)

Differentiating equation (2) with respect to t

$$\vec{r}''' = s'''\vec{T} + s''\vec{T}' + K\,2s's''\,\vec{N} + K(s')^2\,\vec{N}' + K'(s')^2\,\vec{N}$$

$$= s'''\vec{T} + s''\vec{N}\left|\vec{T}'(t)\right| + 2K\,s's''\,\vec{N} + K(s')^2\,\vec{N}' + K'(s')^2\,\vec{N}$$

(Using definition of N)

$$= s'''\vec{T} + s''\vec{N}s'K + 2Ks's''\vec{N} + k(s')^2\,\vec{N}' + K'(s')^2\,\vec{N} \qquad \text{------------ (4)}$$

(Because $K = \dfrac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|} = \dfrac{\left|\vec{T}'(t)\right|}{ds/dt}$)

Now consider $\vec{N} = \vec{B}\times\vec{T}$

Then $\quad \vec{N}' = \vec{B}'\times\vec{T} + \vec{B}\times\vec{T}'$

$$= -\vec{T}\times\vec{B}' + \vec{B}\times\left(s's''K\vec{N}\right)$$

$$= -\vec{T}\times\frac{d\vec{B}}{dt} + \left(\vec{B}\times\vec{N}\right)(s's''K)$$

$$= -\vec{T}\times\left(\frac{d\vec{B}}{ds}\right)\frac{ds}{dt} - \vec{T}(s's''K)$$

$$= -\vec{T}\times\left(-\tau\vec{N}\right)\frac{ds}{dt} - (s's''K)\vec{T}$$

$$= \tau s'\left(\vec{T}\times\vec{N}\right) - (ss''K)\vec{T}$$

$$= \tau s'B - (ss''K)T$$

Using this result in equation (4)

$$\vec{r}''' = s'''\vec{T} + s''s'\vec{N}K + 2Ks's''N + K(s')^2\left(\tau s'\vec{B} - ss''K\vec{T}\right) + K'(s')^2\,\vec{N}$$

$$= \left[s''' - k^2(s')^3\right]\vec{T} + \left[3K.s's'' + K'(s')^2\right]\vec{N} + K\tau(s')^3\,\vec{B} \qquad \text{----------- (5)}$$

(D)

Consider $\left(\vec{r}\,'\times\vec{r}\,''\right)\vec{r}\,'''$

$$=\left[K(s')^3\vec{B}\right].K\tau(s')^3\vec{B}$$

Using (3) and (5)

(As $\vec{B}.\vec{T}=\vec{B}.\vec{N}=0$)

Then $\left(\vec{r}\,'\times\vec{r}\,''\right).\vec{r}\,'''=K^2(s')^3\,\tau$

But $K^2(s')^3=\left|\vec{r}\,'\times\vec{r}\,''\right|^2$

(From (3), $\left|\vec{r}\,'\times\vec{r}\,''\right|^2=K^2(s')^6\,\vec{B}\times\vec{B}=K^2(s')^6$)

Thus $\tau=\dfrac{\left[\vec{r}\,'\times\vec{r}\,''\right]\vec{r}\,'''}{\left|\vec{r}\,'\times\vec{r}\,''\right|^2}$

Answer 62E.

$$\vec{r}(t)=<a\cos t, a\sin t, bt>$$

Then $\vec{r}\,'(t)=<-a\sin t,\ a\cos t, b>$

And $\left|\vec{r}\,'(t)\right|=\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}$

$$=\sqrt{a^2+b^2}$$

Also $\vec{T}(t)=\dfrac{\vec{r}\,'(t)}{\left|\vec{r}\,'(t)\right|}$

$$=<-\frac{a}{\sqrt{a^2+b^2}}\sin t,\ \frac{a}{\sqrt{a^2+b^2}}\cos t, \frac{b}{\sqrt{a^2+b^2}}>$$

Then $\vec{T}\,'(t)=<\dfrac{-a}{\sqrt{a^2+b^2}}\cos t,\ \dfrac{-a}{\sqrt{a^2+b^2}}\sin t,\ 0>$

And thus $\left|\vec{T}\,'(t)\right|=\sqrt{\dfrac{a^2}{a^2+b^2}\left(\sin^2 t+\cos^2 t\right)}$

$$=\frac{a}{\sqrt{a^2+b^2}}$$

As we know curvature $K=\frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$

$$=\frac{a}{\sqrt{a^2+b^2}}\Big/\sqrt{a^2+b^2}$$

i.e. $K=\frac{a}{a^2+b^2}=\text{constant}$ (As a, b are constant)

Now $\vec{r}''=<-a\cos t,-a\sin t,0>$

And $\vec{r}'''=<a\sin t,-a\cos t,0>$

Then $\left(\vec{r}'\times\vec{r}''\right).\vec{r}'''$

$$=\begin{vmatrix} a\sin t & -a\cos t & 0 \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$=a\sin t\left(ab\sin t\right)+a\cos t\left(ab\cos t\right)$$

$$=a^2b\sin^2 t+a^2b\cos^2 t$$

$$=a^2b$$

And $\vec{r}'\times\vec{r}''=<ab\sin t,ab\cos t,a^2>$

Then $\left|\vec{r}'\times\vec{r}''\right|=\sqrt{a^2b^2\sin^2 t+a^2b^2\cos^2 t+a^4}$

$$=\sqrt{a^2+b^2+a^4}$$

$$=\sqrt{a^2\left(b^2+a^2\right)}$$

$$=a\sqrt{a^2+b^2}$$

As torsion $\tau=\frac{\left(\vec{r}'\times\vec{r}''\right).\vec{r}'''}{\left|\vec{r}'\times\vec{r}''\right|^2}$

$$=\frac{a^2b}{a^2\left(a^2+b^2\right)}$$

$$=\frac{b}{a^2+b^2}$$

$$=\text{constant}$$

Answer 63E.

$$\vec{r}(t) = <t, \frac{1}{2}t^2, \frac{1}{3}t^3>$$

Then $\vec{r}'(t) = <1, t, t^2>$

$\vec{r}''(t) = <0, 1, 2t>$

$\vec{r}'''(t) = <0, 0, 2>$

Then $\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & t & t^2 \\ 0 & 1 & 2t \end{vmatrix}$

$= \hat{i}(t^2) - \hat{j}(2t) + \hat{k}(1)$

$= <t^2, -2t, 1>$

And $\left|\vec{r}' \times \vec{r}''\right| = \sqrt{t^4 + 4t^2 + 1}$

Also $\left(\vec{r}' \times \vec{r}''\right) . \vec{r}''' = <t^2, -2t, 1> . <0, 0, 2>$

$= 2$

As torsion $(\tau) = \dfrac{\left(\vec{r}' \times \vec{r}''\right) . \vec{r}'''}{\left|\vec{r}' \times \vec{r}''\right|^2}$

i.e. $\boxed{\tau = \dfrac{2}{\left(t^4 + 4t^2 + 1\right)}}$

Answer 64E.

Consider the parametric equations of the curve $x = \sin ht, y = \cos ht, z = t$.

Then the corresponding vector equation is $\mathbf{r}(t) = \langle \sin ht, \cos ht, t \rangle$.

Need to find the curvature of the given curve at the point $(0,1,0)$.

To find the curvature, use the formula $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

The point $(0,1,0)$ corresponds to parameter $t = 0$.

So the curvature of the curve at $t = 0$ is $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3}$. ...... (1)

$\mathbf{r}(t) = \langle \sin ht, \cos ht, t \rangle$. ...... (2)

Differentiate (2) with respect to *t*, get

$\mathbf{r}'(t) = \langle \cos ht,\ \sin ht,\ 1 \rangle$. ...... (3)

At $t = 0$,

$$\begin{aligned} \mathbf{r}'(0) &= \langle \cos h(0),\ \sin h(0),\ 1 \rangle \\ &= \langle 1,\ 0,\ 1 \rangle \end{aligned}$$

Find the magnitude of $\mathbf{r}'(0)$:

$$\begin{aligned} |\mathbf{r}'(0)| &= \sqrt{(1)^2 + (0)^2 + 1^2} \\ &= \sqrt{1+0+1} \\ &= \sqrt{2} \end{aligned} \qquad \left[\begin{aligned} &\text{If } \mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &\text{then } |\mathbf{u}| = \sqrt{x^2 + y^2 + z^2} \end{aligned} \right]$$

Differentiate (3) with respect to t, get

$$\mathbf{r}''(t) = \langle \sinh t, \cosh t, 0 \rangle. \quad \text{......} \ (4)$$

At $t = 0$,

$$\mathbf{r}''(t) = \langle \sinh t, \cosh t, 0 \rangle.$$

$$\mathbf{r}''(0) = \langle \sinh(0), \cosh(0), 0 \rangle \qquad [\text{Substitute } t = 0]$$

$$= \langle 0, 1, 0 \rangle$$

Find $\mathbf{r}'(0) \times \mathbf{r}''(0)$:

$$\mathbf{r}'(0) \times \mathbf{r}''(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \mathbf{i}(0-1) - \mathbf{j}(0-0) + \mathbf{k}(1-0)$$

$$= -\mathbf{i} - 0 \cdot \mathbf{j} + \mathbf{k}$$

$$|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-1)^2 + (0)^2 + 1^2} \qquad \left[\begin{array}{l} \text{If } \mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \text{then } |\mathbf{u}| = \sqrt{x^2 + y^2 + z^2} \end{array}\right]$$

$$= \sqrt{1 + 0 + 1}$$

$$= \sqrt{2}$$

Substitute known values in equation (1), get the curvature as

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3}$$

$$= \frac{\sqrt{2}}{(\sqrt{2})^3}$$

$$= \frac{\sqrt{2}}{2\sqrt{2}}$$

$$= \frac{1}{2}$$

Therefore the curvature of the given curve at the point $(0,1,0)$ is $\kappa(0) = \boxed{\frac{1}{2}}$.

Now find the torsion of the given curve $\mathbf{r}(t) = \langle \sin ht, \cos ht, t \rangle$.

Use the formula torsion $\tau = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$. ...... (5)

Differentiate (4) with respect to *t*, get

$\mathbf{r}'''(t) = \langle \cos ht, \sin ht, 0 \rangle$.

At $t = 0$,

$$\begin{aligned} \mathbf{r}'''(0) &= \langle \cos h(0), \sin h(0), 0 \rangle \\ &= \langle 1, 0, 0 \rangle \end{aligned}$$

Find the dot product of $\mathbf{r}'(0) \times \mathbf{r}''(0)$ and $\mathbf{r}'''(0)$.

$$\begin{aligned} (\mathbf{r}'(0) \times \mathbf{r}''(0)) \cdot \mathbf{r}'''(0) &= \langle -1, 0, 1 \rangle \cdot \langle 1, 0, 0 \rangle \qquad [\text{Since } \mathbf{r}'(0) \times \mathbf{r}''(0) = -\mathbf{i} - 0 \cdot \mathbf{j} + \mathbf{k}] \\ &= (-1)(1) + (0)(0) + (1)(0) \\ &= -1 \end{aligned}$$

And $|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{2}$.

Substitute known values in equation (5), get the torsion as

$$\begin{aligned} \tau &= \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2} \\ &= \frac{-1}{(\sqrt{2})^2} \\ &= \frac{-1}{2} \end{aligned}$$

Therefore the torsion of the given curve at the point $(0,1,0)$ is $\tau = \boxed{\frac{-1}{2}}$.

Answer 65E.

Consider the diagram of double helix,

The above helix looks like a cylinder.

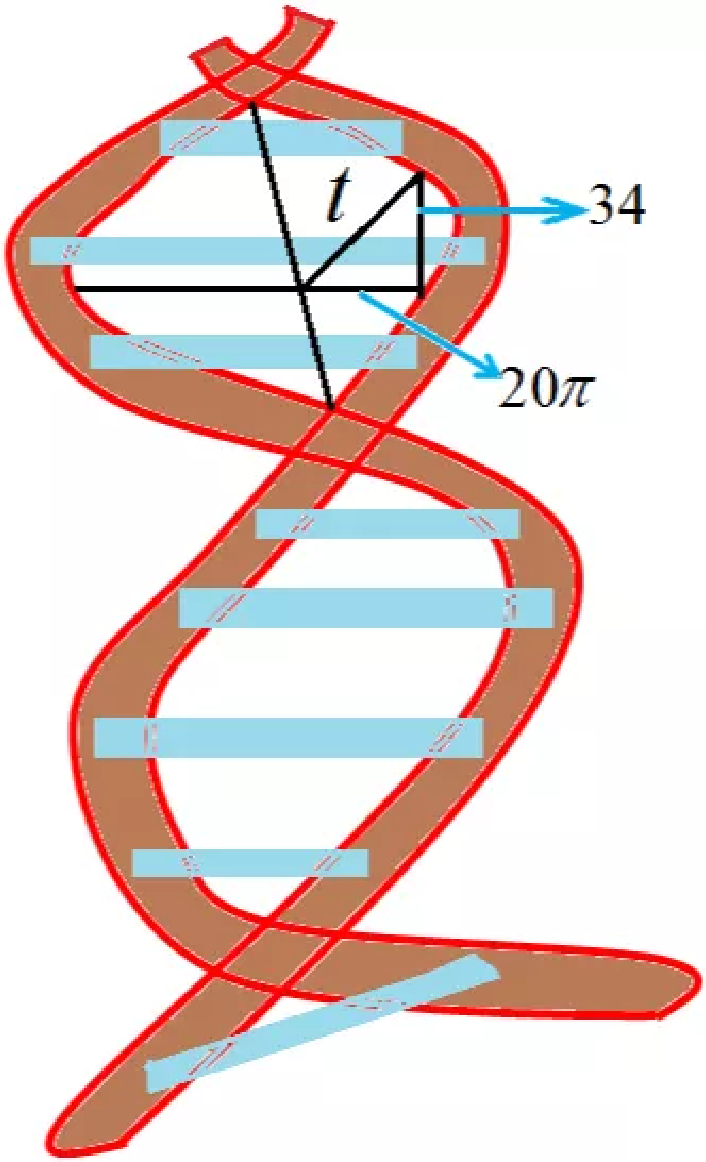
The circumference of helix is $2\pi(\text{radius})$

But the radius of each helix is about 10 angstroms.

$$\begin{aligned}2\pi(\text{radius}) &= 2\pi(10)\\ &= 20\pi\end{aligned}$$

Let l be the length of the curve through one turn.

The length l is the hypotenuse of a right triangle and lengths 20π, 34


t
34
20π

By the right triangle,

$$t^2 = (20\pi)^2 + (34)^2$$

$$t^2 = 400\pi^2 + 1156$$
$$= 3947.84176 + 1156$$
$$= 5103.84176$$

$$t = \sqrt{5103.84176}$$

$$= 71.441177$$

But there are about 2.9×10^8 complete turns.

The total length,

$$= (2.9 \times 10^8) \times t$$

$$= (2.9 \times 10^8) \times 71.441177$$

$$= 2.9 \times 71.441177 \times 10^8$$

$$= 207.179413 \times 10^8$$

$$= 207 \times 10^{10}\ \overset{0}{\text{A}} \quad \text{Since } 1\overset{0}{\text{A}} = 10^{-8} cm$$

$$= \boxed{2\text{ m}}$$

Answer 6E.

(A) The given function F(x) is defined as

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Since F(x) is continuous at x = 0 and at x = 1.

Therefore, $P(0) = \lim_{x \to 0^-} F(x) = 0$

And $P(1) = \lim_{x \to 1^+} F(x) = 1$

Thus, we have $P(0) = 0, \qquad P(1) = 1$

Now, $F'(x)=\begin{cases}0 & \text{if } x\le 0\\ P'(x) & \text{if } 0<x<1\\ 0 & \text{if } x\ge 1\end{cases}$

Since $F'(x)$ is continuous at x = 0 and x = 1.

Therefore,

$$P'(0)=\lim_{x\to 0^-} F'(x)=0$$

$$P'(1)=\lim_{x\to 1^+} F'(x)=0$$

Thus, we have $P'(0)=0$, $P'(1)=0$

The curvature of the curve y = F(x) at point (x, F(x)) is given as:

$$k=\frac{F''(x)}{\left[1+\left(F'(x)\right)^2\right]^{3/2}}$$

Now, $F''(x)=\begin{cases}0 & \text{if } x\le 0\\ P''(x) & \text{if } 0<x<1\\ 0 & \text{if } x\ge 1\end{cases}$

Since $F''(x)$ is continuous at x = 0 and at x = 1.

Therefore,

$$P''(0)=\lim_{x\to 0^-} F''(x)=0$$

$$P''(1)=\lim_{x\to 1^+} F''(x)=0$$

Thus, we have $P''(0)=0$, $P''(1)=0$

Let $P(x)=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5$ ----- (1)

Be a polynomial of degree 5.

Differentiating P(x) with respect to x,

$$P'(x)=B+2Cx+3Dx^2+4Ex^3+5Fx^4 \quad \text{------ (2)}$$

Again differentiating with respect to x,

$$P''(x)=2C+6Dx+12Ex^2+20Fx^3 \quad \text{---- (3)}$$

Since $P(0)=0$

Therefore, from (1)

$$0=A+B\times 0+C\times 0^2+D\times 0^3+E\times 0^4+F\times 0^5$$

$\Rightarrow \quad A=0$

Also, $P'(0)=0$

Therefore, from (2)

$$0=B+2C\times 0+3D\times 0^2+4E\times 0^3+5F\times 0^4$$

$$0=B$$

Also, $P''(0)=0$

Therefore, from (3) we have,

$$0=2C+6D\times 0+12E\times 0^2+20F\times 0^3$$

$\Rightarrow \quad 0=2C$

$\Rightarrow \quad C=0$

Now $P(1)=1$

Therefore, from (1) we have

$$1=A+B\times 1+C\times 1^2+D\times 1^3+E\times 1^4+F\times 1^5$$

$\Rightarrow \quad D+E+F=1$ ----- (4)[Since $A=B=C=0$]

Also, $P'(1)=0$

$\Rightarrow \quad B+2C\times 1+3D\times 1^2+4E\times 1^3+5F\times 1^4=0$

$\Rightarrow \quad 3D+4E+5F=0$ ----- (5) (Since A = B = C = 0)

And $P''(1)=0$

$\Rightarrow \quad 2C+6D\times 1+12E\times 1^2+20F\times 1^3=0$

$\Rightarrow \quad 6D+12E+20F=0$

$\Rightarrow \quad 3D+6E+10F=0$ ---- (6)

Solving equation (5) and (6) we get,

$$\frac{D}{\begin{vmatrix}4 & 5\\ 6 & 10\end{vmatrix}}=\frac{E}{\begin{vmatrix}5 & 3\\ 10 & 3\end{vmatrix}}=\frac{F}{\begin{vmatrix}3 & 4\\ 3 & 6\end{vmatrix}}$$

$$\Rightarrow \quad \frac{D}{40-30}=\frac{E}{15-30}=\frac{F}{18-12}=m\,(\text{say})$$

$$\Rightarrow \quad \frac{D}{10}=\frac{E}{-15}=\frac{F}{6}=m$$

$\Rightarrow \quad D = 10m, \quad E = -15m, \quad F = 6m$

Putting these values of D,E ,F in equation (4) we get,

$$D + E + F = 1$$

$$\Rightarrow \quad 10m - 15m + 6m = 1$$

$$\Rightarrow \quad m = 1$$

Therefore, $D = 10m = 10 \times 1 = 10$

$E = -15m = -15 \times 1 = -15$

$F = 6m = 6 \times 1 = 6$

Putting values of A,B,C,D,E,F in equation (1) we get polynomial of degree 5 as.

$$P(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$$

$$= 0 + 0 \times x + 0 \times x^2 + 10x^3 - 15x^4 + 6x^5$$

$$= 10x^3 - 15x^4 + 6x^5$$

Hence,

$$\boxed{P(x) = 10x^3 - 15x^4 + 6x^5}$$