Subject Code: 35 (NS)

MATHEMATICS

(English Version)

Instructions:

1. **The question paper has five parts namely A, B, C, D, and E. Answer all the parts.**
2. **Use the Graph Sheet for the question on Linear Programming Problem on Part –E.**

PART –A

Answer all the ten questions: $10\times1=10$

1. **Find** $\int \cos ecx\left(\cos ecx+\cot x\right)dx$.

Sol.

Simplify the expression

$$\int \cos ecx\left(\cos ecx+\cot x\right)dx$$
$$\int \cos ec^2 xdx+\int \cos ecx.\cot xdx=-\cot x-\cos ecx+c$$

2. **Find a value of x if** $\begin{vmatrix} x & 2 \\ 18 & \text{x} \end{vmatrix}=\begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

Sol.

Simplify the expression,

X=6 or x=-6

$$x^2-36=36-36$$
$$x^2-36=0\Rightarrow x^2=36$$
$$x=\pm6$$

3. **If** $y=a^{1/2\log x,\cos x}$, $find\ \frac{dy}{dx}$

Sol.

Simplify the expression,

$$y=a^{\log(\cos x)^{\frac{1}{2}}}$$
$$\left(\cos x\right)^{1/2}=\sqrt{\cos x}$$
$$\frac{dy}{dx}=\frac{1}{2\sqrt{\cos x}}\left(-\sin x\right).=\frac{-\sin x}{2\sqrt{\cos x}}$$

4. Find the value of $\cos\left(\sec^{-1} x+\cos ec^{-1}x\right),|x|\geq 1$**.**

Sol.

Simplify the expression,

$$\cos\left(\frac{\pi}{2}\right)=0\sec^{-1} x+\cos ec^{-1}x=\frac{\pi}{2}$$

5. If vector $\overrightarrow{AB}=2\hat{i}-\hat{j}+\hat{k}$ **and** $\overrightarrow{OB}=3\hat{i}-4\hat{j}+4\hat{k}$ **find the position vector** $\overrightarrow{OA}$.

Sol.

Simplify the expression,

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$
$$\overrightarrow{OA}=\overrightarrow{OB}-\overrightarrow{AB}=\left(3\hat{i}-4\hat{j}+4\hat{k}\right)-\left(2\hat{i}-\hat{j}+\hat{k}\right)$$
$$=\hat{i}-3\hat{j}+3\hat{k}$$

6. Find the distance of the point (-6,0,0) from the plane 2x-3y+6z=2

Sol.

Simplify the expression,

$$d=\left|\frac{2(-6)-3(0)+6(0)-2}{\sqrt{4+9+36}}\right|=\left|\frac{-12-2}{\sqrt{49}}\right|$$
$$=\left|\frac{-14}{7}\right|=|-2|=2$$

7. If $\begin{vmatrix} x+2 & y-3 \\ 0 & 4 \end{vmatrix}$ **is a scalar matrix. Find x and y.**

Sol.

$$x+2=4\Rightarrow x=2$$
$$y-3=0\Rightarrow y=3$$

8. If P(A) =0.8 and P(B/A) =0.4, then find $P\left(A\cap B\right)$

Sol.

Simplify the expression,

$$P\left(\frac{B}{A}\right)=\frac{P\left(A\cap B\right)}{P\left(A\right)}=P\left(A\cap B\right)=P\left(\frac{B}{A}\right)\times P\left(A\right)$$

9. An operation * on Z^+(the set of all non-negative integers) is defined

$a \in b = a - b, \forall a, b \in Z^+$. Is $\in$ a binary operation on Z^+

Sol.

Simplify the expression,

$2*3 = 2 - 3 = -1 \notin Z^+$

* is not a binary operation on Z^+

10. Define feasible region in linear programming problem.

Sol.

The region containing the set of points satisfying all the constraints of an LPP is called as feasible region.

PART-B

Answer any ten questions: $10 \times 2 = 20$

11. Write the simplest form of $\tan^{-1}\left[\dfrac{3\cos x - 4\sin x}{4\cos x + 3\sin x}\right], If\ \dfrac{3}{4}\tan x > -1$

Sol.

$$\tan^{-1}\left[\frac{4\cos\left(\frac{3}{4} - \tan x\right)}{4\cos\left(1 + \frac{3}{4}\tan x\right)}\right] = \tan^{-1}\left[\frac{\frac{3}{4} - \tan x}{1 + \frac{3}{4}\tan x}\right]$$

$$= \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}(\tan x) = \tan^{-1}\left(\frac{3}{4}\right) - x$$

12. Using determinants show that points A(a,b+c),B(b,c+a) and C (c,a+b) are collinear.

Sol.

Simplify the expression,

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix}$$

$$= (a+b+c)(0) = 0$$

13. If functions $f : R \to R$ and $g : R \to R$ are given by $f(x) = |x|$ and $g(x) = |x|$ where [x] is greatest integer function find $f0g\left(-\frac{1}{2}\right) and\ g0f\left(-\frac{1}{2}\right)$.

Sol.

Simplify the expression,

$$fog\left(-\frac{1}{2}\right) = f\left[g\left(-\frac{1}{2}\right)\right] = f(-1)$$
$$= f(-1) = |-1| = 1$$
$$gof\left(-\frac{1}{2}\right) = g\left[f\left(-\frac{1}{2}\right)\right] = g\left[\left|-\frac{1}{2}\right|\right]$$
$$= g\left[\frac{1}{2}\right] = \left[\frac{1}{2}\right] = 0$$

14. Prove that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x, \frac{1}{\sqrt{2}} \le x \le 1$.

Sol.

Prove this expression,

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\cos\theta \times \sin\theta\right)$$
$$= \sin^{-1}\left(\sin 2\theta\right)$$
$$= 2\theta$$
$$= 2\cos^{-1} x$$

Hence, proved

15. Find $\frac{dy}{dx}$ **if** $y = \sec^{-1}\left[\frac{1}{2x^2 - 1}\right], 0 < x < \frac{1}{\sqrt{2}}$

Sol.

$$\text{suppose } \cos^{-1}x = \theta, \quad x = \cos\theta$$

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$
$$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)$$
$$y = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$
$$y = \sec^{-1}(\sec 2\theta)$$
$$y = 2\theta \Rightarrow (\sec 2\theta)$$
$$y = 2\theta \Rightarrow y = 2\cos^{-1}x$$

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \left(2\cos^{-1}x\right) = \frac{-2}{\sqrt{1-x^2}}$$

16. If $x^y = a^x$, **prove that** $\frac{dy}{dx} = \frac{x\log_e a - y}{x\log_e x}$.

Sol.

$$x^y = a^x$$
$$\log x^y = \log a^x$$
$$y\log x = x\log a$$

Differentiate both side w.r.t x, we get

$$y\frac{1}{x} + \log x\frac{dy}{dx} = (1)\log a$$
$$\log x\frac{dy}{dx} = \log a - \frac{y}{x}$$
$$\log x\frac{dy}{dx} = \frac{x\log a - y}{x}$$
$$\frac{dy}{dx} = \frac{x\log_e a - y}{x\log_e x}$$

17. Find $\int \frac{1}{\sin x \cos^3 x} dx$.

Sol.

Simplify the expression,

$$\int \frac{dx}{\sin x \cos^3 x} = \int \frac{dx}{\frac{\sin x}{\cos x}\cos^4 x} = \int \frac{\sec^4 x dx}{\tan x}$$

$$\int \frac{\sec^2 x \sec^2 x dx}{\tan x} = \int \frac{\left(1+\tan^2 x\right)\sec^2 x dx}{\tan x}$$

$$\tan x = t$$

$$\sec^2 x dx = dt$$

$$= \int \frac{\left(1+t^2\right)dt}{t} = \left(\int \frac{1}{t} + t\right)dt$$

$$= \log|t| + \frac{1}{t^2} + c = \log|\tan x| + \frac{(\tan x)^2}{2} + c$$

18. Using differentials find the approximate value of $(25)^{1/3}$.

Sol.

Simplify the expression,

Let $f(x) = \sqrt[3]{x} = x^{1/3}$

$$f(x) = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3}$$

$$f(x+\Delta x) = f(x) + \Delta x f(x)$$

$$(x+\Delta x)^{1/3} = x^{1/3} + \Delta x.\frac{1}{3}x^{-2/3}$$

$$x = 27 \ \Delta \text{x} = -2$$

$$(27-2)^{1/3} = (27)^{1/3} + (-2)\frac{1}{3}(27)^{-2/3}$$

$$(25)^{1/3} = 3 - 2.\frac{1}{3}\left(3^3\right)^{-2/3} = 3 - 2.\frac{1}{3}.\frac{1}{9}$$

$$= 3 - \frac{2}{27} = \frac{79}{27} = 2.92$$

19. Evaluate : $\int_{0}^{\pi}\left(\sin^2\left(\frac{1}{2}\right)-\cos^2\left(\frac{x}{2}\right)\right)dx$

Sol.

Simplify the expression,

We know that,

$$\cos^2 x-\sin^2 x=\cos 2x$$

$$\sin^2\frac{x}{2}-\cos^2\frac{x}{2}=-\left[\cos 2\frac{x}{2}\right]$$

$$\int_0^{\pi}-\cos xdx=\int_0^{\pi}\cos xdx=$$

$$=-\left[\sin x\right]_0^{\pi}$$

$$=-\left[\sin\pi-\sin 0\right]=-\left(0-0\right)=0$$

20. If $\left|\vec{a}+\vec{b}\right|=\left|\vec{a}-\vec{b}\right|$, **Prove that** $\vec{a}$ and $\vec{b}$ **are perpendicular.**

Sol.

Simplify the expression,

$$\left|\vec{a}+\vec{b}\right|=\left|\vec{a}-\vec{b}\right|$$

$$\left|\vec{a}+\vec{b}\right|^2=\left|\vec{a}-\vec{b}\right|^2$$

$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)=\left(\vec{a}-\vec{b}\right).\left(\vec{a}-\vec{b}\right)$$

$$\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{b}+\vec{b}.\vec{b}=\vec{a}.\vec{a}+\vec{b}.\vec{b}-\vec{a}.\vec{b}-\vec{a}.\vec{b}$$

$$2\vec{a}.\vec{b}=-2\vec{a}.\vec{b}$$

$$4\vec{a}.\vec{b}=0\Rightarrow\vec{a}.\vec{b}=0$$

$$\Rightarrow\vec{a}\perp\vec{b}$$

21. Find order degree (if defined) of the differential equation $\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)$

Sol.

Given the equation $\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)$

Order =4

Degree = not defined.

22. Find angle between the vector the vectors $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ **and** $\vec{b}=\hat{i}+\hat{j}+\hat{k}$.

Sol.

Simplify the expression,

$$\vec{a}.\vec{b}=(1)(1)+(1)(1)+(-1)(1)$$
$$1+1-1=1$$
$$\vec{a}=\sqrt{1+1+1}=\sqrt{3}$$
$$\vec{b}=\sqrt{1+1+1}=\sqrt{3}$$
$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$
$$=\frac{1}{\sqrt{3}\sqrt{3}}=\frac{1}{3}$$
$$\theta=\cos^{-1}\left(\frac{1}{3}\right)$$

23. The random variable X has a probability distribution P(X) of the following form where k is some number.

$$P(X)=\begin{cases} k & \text{if x=0} \\ 2\text{k} & \text{if x=1} \\ 3k & \text{if x=2} \\ 0 & \text{otherwise} \end{cases}$$

Sol.

$$\sum_{i=1}^{n} Pi = 1$$

$$k + 2k + 3k + 0 = 1$$

$$6k = 1$$

$$k = \frac{1}{6}$$

$$P(x \le 2) = k + 2k + 3k = 6k$$

$$= 6 \times \frac{1}{6} = 1$$

24. Find the Cartesian equation of the line parallel to y-axis and passing through the point (1, 1, 1).

Sol.

Direction ratios of y-axis are 0, 1, 0 the line passes through (1, 1, 1)……

The equation of the line is $\frac{x-1}{0} = \frac{y-1}{1} = \frac{z-1}{0}$

PART – C

Answer any ten questions: $10 \times 3 = 30$

25. Show $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{4}{3} = \frac{\pi}{2}$.

Sol.

Simplify the expression,

We know that,

$$\tan^{-1}(a+b) = \frac{a+b}{1-ab}$$

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\left[\frac{\frac{2}{11}+\frac{4}{3}}{1-\frac{2}{11}.\frac{4}{3}}\right]$$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\left[\frac{\frac{6+44}{33}}{\frac{33-8}{33}}\right]=\tan^{-1}\frac{1}{2}+\tan^{-1}\left[\frac{50}{25}\right]$$

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}[2]=\tan^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{2}=\frac{\pi}{2}$$

26. **By using elementary transformations, find the inverse of the matrix** $A=\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$.

Sol. Simplify the expression,

$$A=IA$$

$$\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}A$$

$$R_2\to R_2-2R_1$$

$$\begin{pmatrix}1 & -1\\ 0 & 5\end{pmatrix}=\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}A$$

$$R_2\to\frac{R_2}{5}$$

$$\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{2}{5} & \frac{1}{5}\end{pmatrix}$$

$$R_1\to R_1+R_2$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{3}{5} & \frac{1}{5}\\ -\frac{2}{5} & \frac{1}{5}\end{pmatrix}A$$

$$A^{-1}\begin{pmatrix}\frac{3}{5} & \frac{1}{5}\\ -\frac{2}{5} & \frac{1}{5}\end{pmatrix}$$

27. Show that the relation R in the set $A = \{x : x \in z, 0 \le x \le 12\}$ given by $R = \{(a,b) : |a-b|$ is multiple of 4} is an equivalence relation.

Sol.

$A\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$

$|a-a| = 0$ divisible by 4 $\Rightarrow (a,a) \in R \forall a \in A \Rightarrow R$ is reflexive.

$(a,a) \in R \Rightarrow |a-b|$ is divisible by 4 $\Rightarrow |-(b-a)| is$ symmetric.

Suppose,

$(a,b) and$ $(b,c) \in R \Rightarrow |a-b|$ and $|b-c|$ are divisble by 4.

$\Rightarrow$ a $- b$ and b $- c$ are divisible by 4 $\Rightarrow (a-b)+(b-c)$ is divisible by 4

$\Rightarrow$ a $- c$ is divisible by 4 $\Rightarrow |a-c| is$ divisible by 4 $\Rightarrow (a,c) \Rightarrow R$

$\Rightarrow R$ is transitive.

Hence, R is an equivalence relation.

28. Verify mean value Theorem if $f(x) = x^3 - 5x^2 - 3x$ in the interval [1,3].

Sol.

Simplify the expression,

$f(x)$ is a polynomial in x

It continous in [1,3] and differential in $(1,3)$

There exits a C $\in$ (1,3) such that,

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$= \frac{-27-(-7)}{3-1} = 3C^2 - 10C - 3$$

$$= 3C^2 - 3C - 7C + 7 = 0$$

$$= 3C(C-1) - 7(C-1) = 0$$

$$(C-1)(3C-7) = 0$$

$$C = 1, C = \frac{7}{3} \in (1,3)$$

29. If $x = a\cos^3\theta$ and y=$a\sin^3\theta$, **prove that** $\frac{dy}{dx} = \sqrt[3]{\frac{y}{x}}$.

Sol.

$$\frac{dx}{d\theta} = a.3\cos^2\theta\left(-\sin\theta\right)$$
$$\frac{dy}{d\theta} = a.3\sin^2\theta\cos\theta$$
$$\frac{dy}{d\theta} = \frac{dy}{d\theta} / \frac{dx}{d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$
$$\frac{y}{x} = \frac{a\sin^3\theta}{a\cos^3\theta} = \tan^3\theta$$
$$\tan\theta = \left(\frac{y}{x}\right)^{1/3}$$
$$\tan\theta = \sqrt[3]{\frac{y}{x}}$$

30. Box-I contains 2 golds coins, while another Box-II contains 1 gold and 1 silver coin. A person chooses a box at random and takes out a coin. If the coin is a gold , what is the probability that the other coin in the box is also of gold?

Sol:

$Box - I \rightarrow$ 2 gold coins and

Box-II $\rightarrow$ gold and 1 silver coin

Let E_1 be the event of the selecting box I

E_2 be the event of selecting box II

$$P\left(E_1\right) = \frac{1}{2}$$
$$P\left(E_2\right) = \frac{1}{2}$$

Let A be the event of selecting a gold coin

$$P\left(\frac{A}{E_1}\right)=\frac{2}{2}=1$$

$$P\left(\frac{A}{E_2}\right)=\frac{1}{2}$$

$$P\left(\frac{E_1}{A}\right)=\frac{P\left(\frac{A}{E_1}\right)P(E_1)}{P\left(\frac{A}{E_1}\right)P(E_1)+P\left(\frac{A}{E_2}\right)P(E_2)}$$

$$=\frac{(1)\left(\frac{1}{2}\right)}{(1)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$$

$$=\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{4}}=\frac{\frac{1}{2}}{\frac{3}{4}}$$

$$=\frac{1}{2}\times\frac{4}{3}$$

$$=\frac{2}{3}$$

31. Find $\int\frac{x}{(x-1)(x-2)}dx$.

Sol:

$$\frac{x}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$$

$$\Rightarrow x=A(x-1)+B(x-2)$$

For x=1

$$\Rightarrow 2=A(1-2)$$

$$\Rightarrow 1=A(-1)$$

$$\Rightarrow A=-1$$

For x=2

$$\Rightarrow 2=\text{A}(2-2)+B(2-1)$$

$$\Rightarrow B=2$$

$$\therefore \int \frac{x}{(x-1)(x-2)} dx$$
$$= \int \frac{-1}{x-1} dx + \int \frac{2}{x-2} dx$$
$$= -\log|x-1| + 2\log|x-2| + C$$
$$= -\log|x-1| + \log|x-2|^2 + C$$
$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

32. Integrate: $\int \frac{2x}{(x^2+1)(x^2+2)}$

Sol

Simplify the expression,

$$x^2 = t$$
$$2xdx = dt$$

$$\int \frac{2x}{(t+1)(t+2)}$$
$$\int \frac{2x}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$
$$1 = A(t+1) + B(t+2)$$
$$t = -1 \Rightarrow 1 = A(1)$$
$$t = -2 \Rightarrow 1 = B(-2+1)$$
$$1 = B(-1) \Rightarrow B = -1$$
$$\int \frac{dt}{(t+1)(t+2)} = \int \left(\frac{1}{t+1} + \frac{-1}{t+2}\right) dt$$
$$= \log|t+1| - \log|t+2| + c$$
$$= \log|x^2+1| - \log|x^2+2| + c$$
$$= \log\frac{|x^2+1|}{|x^2+2|} + c$$

33. Find the number whose product is 100 and whose sum is minimum.

Sol.

Let the number be x and $\frac{100}{x}$

Let,

$$f(x) = x + \frac{100}{x}$$

$$f'(x) = 1 - \frac{100}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \frac{100}{x^2} = 0$$

$$1 = \frac{100}{x^2}$$

$$x^2 = 100$$

$$x = 10$$

$$f'(x) = \frac{200}{x^3} > 0$$

At x= 10, $\frac{100}{x} = 10$ the numbers are 10, 10.

34. Find the area lying between the curve $y^2 = 4x$ and the line $y = 2x$.

Sol.

The given curve is $y^2 = 4x$ …………………….. (i)

And the given line is $y = 2x$ …………………….. (ii)

The two curve meet where

$$(2x)^2 = 4x \Rightarrow 4x(x-1) = 0 \Rightarrow x = 0,1$$

When $x = 0, y = 2 \times 0 = 0$ and when $x = 1, y = 2 \times 1 = 2$.

Equation (i) and (ii) meet at the points (0, 0) and (1, 2).

Required area $= \int_0^1 \sqrt{4}xdx - \int_0^1 2xdx$

$$= 2\int_0^1 \sqrt{x}dx - \int_0^1 2xdx$$

$$= \left[2\frac{x^{3/2}}{\frac{3}{2}} - x^2 \right]_0^1 = \frac{4}{3}.1^{3/2} - 1^2 - 0 = \frac{1}{3} sq.unit.$$

35. For any three vector $\vec{a},\vec{b}$ and $\vec{c}$, prove that vector $\vec{a}-\vec{b},\vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ are coplanar.

Sol.

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$$
$$\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$$
$$\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$$
$$\vec{a}-\vec{b}=(a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}+(a_3-b_3)\hat{k}$$
$$\vec{b}-\vec{c}=(b_1-c_1)\hat{i}+(b_2-c_2)\hat{j}+(b_3-c_3)\hat{k}$$
$$\vec{c}-\vec{a}=(c_1-a_1)\hat{i}+(c_2-a_2)\hat{j}+(c_3-a_3)\hat{k}$$

$$\left[\vec{a}-\vec{b},\vec{b}-\vec{c},\vec{c}-\vec{a}\right]=\begin{vmatrix} a_1-b_1 & a_2-b_2 & b_3-c_3 \\ b_1-c_1 & b_2-c_2 & b_3-c_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \end{vmatrix}$$

$$R_1=R_1+R_2+R_3$$

$$\begin{vmatrix} 0 & 0 & 0 \\ b_1-c_1 & b_2-c_2 & b_3-c_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \end{vmatrix}=0$$

Since, $\vec{a}-\vec{b},\vec{b}-\vec{c},\vec{c}-\vec{a}$ are coplanar.

36. **Find the distance between the liens $\vec{l_1}$ and $\vec{l_2}$ given by**

$$\vec{l_1}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right) \text{ and } \mathrm{l}_2=3\hat{i}+3\hat{j}-5\hat{k}+\mu\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$

Sol.

Simplify the expression,

$$\overrightarrow{a_1}=\hat{i}+2\hat{j}-4\hat{k}$$
$$\overrightarrow{a_2}=3\hat{i}+3\hat{j}-5\hat{k}$$
$$\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$$

Thus, the distance between the lines is given by

$$d=\left|\frac{\vec{b}\times\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)}{\left|\vec{b}\right|}\right|=\left|\frac{\begin{vmatrix}\hat{i} & \hat{\text{j}} & \hat{\text{k}}\\ 2 & 3 & 6\\ 2 & 1 & -1\end{vmatrix}}{\sqrt{4+9+36}}\right|$$

$$=\frac{\left|-9\hat{i}+14\hat{\text{j}}\ -4\hat{\text{k}}\right|}{\sqrt{49}}=\frac{\sqrt{293}}{\sqrt{49}}$$

37. Find the sine of the angle between the vectors $\hat{i}+2\hat{j}+2\hat{k}$ and $3\hat{i}+2\hat{j}+6\hat{k}$.

Sol.

Suppose,

$\vec{a}=\hat{i}+2\hat{j}+2\hat{k}$

$\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{\text{j}} & \hat{\text{k}}\\ 1 & 2 & 2\\ 3 & 3 & 6\end{vmatrix}$$

$=\hat{i}\left(12-4\right)-\hat{\text{j}}\ \left(6-6\right)+\ \hat{\text{k}}\left(2-6\right)=8\hat{i}-4\hat{\text{k}}$

$=\left|\vec{a}\times\vec{b}\right|=\sqrt{64+16}=\sqrt{80}$

$\left|\vec{a}\right|=\sqrt{1+4+4}=\sqrt{9}=3$

$\left|\vec{b}\right|=\sqrt{9+4+36}=\sqrt{49}=7$

$$\sin\theta=\frac{\left|\vec{a}\times\vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|}=\frac{\sqrt{80}}{\left(3\right)\left(7\right)}=\frac{\sqrt{80}}{21}=\frac{4\sqrt{5}}{21}$$

38. Find the equation of the curve passing though the point (1,1), given that the slope of the tangent to the curve at points is $\frac{x}{y}$

Sol.

Simplify the expression,

$$\left(x_1, y_1\right) = \left(1,1\right)$$
$$\frac{dy}{dx} = \frac{x}{y}$$
$$ydy = xdx$$
$$\int ydy = \int xdx$$
$$\frac{y^2}{2} = \frac{x^2}{2} + c$$
$$x = 1, y = 1$$
$$\frac{1}{2} = \frac{1}{2} + c$$
$$C = 0$$
$$\frac{y^2}{2} = \frac{x^2}{2} \text{ or } y^2 = x^2$$

PART-D

Answer any six question **(6x5=30)**

39. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ and B=$\left[1, 3-6\right]$, **verify that** $\left(AB\right)' = B'A'$

Sol.

$$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \left[1, 3, -6\right]$$

$$\begin{bmatrix} (-2)(1) & (-2)(3) & (-2)(-6) \\ (4)(1) & (4)(3) & (4)(-6) \\ (5)(1) & (5)(3) & (5)(-6) \end{bmatrix}$$

$$AB=\begin{bmatrix}-2 & -6 & 12\\ 4 & 12 & -24\\ 5 & 15 & -30\end{bmatrix}$$

$$\left(AB\right)'=\begin{bmatrix}-2 & 4 & 5\\ -6 & 12 & 15\\ 12 & -24 & -30\end{bmatrix}......................(i)$$

$$A'=\left[-2,4,5\right],B'=\begin{bmatrix}1\\ 3\\ -6\end{bmatrix}$$

$$B'A'=\begin{bmatrix}1\\ 3\\ -6\end{bmatrix}\begin{bmatrix}-2 & 4 & 5\end{bmatrix}$$

$$=\begin{bmatrix}(1)(-2) & (1)(4) & (1)(5)\\ (4)(1) & (3)(4) & (3)(5)\\ (5)(1) & (-6)(4) & (6)(5)\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 4 & 5\\ -6 & 12 & 15\\ 5 & 24 & -30\end{bmatrix}................(ii)$$

$eq(i)$ and (ii)

(AB)'=B'A'

$$\text{A}=\begin{bmatrix}2 & 3 & 5\\ -6 & 2 & -4\\ 1 & 1 & -2\end{bmatrix},X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\text{ and B}=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$$

$$|A|=-1$$

$$AdjA=\begin{bmatrix}0 & 2 & 1\\ -1 & -9 & -5\\ 5 & 23 & -13\end{bmatrix}^T=\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 5 & -5 & 13\end{bmatrix}$$

$$X=A^{-1}B=\frac{1}{|A|}\left(adjA\right)B$$

$x=1,y=2,$ and z=3

40. Solve the system of linear equation by matrix method:
$2x-3y+5z=11, 3x+2y-4z=-5, x+y-2z=-3$

Sol.

$A^{-1} exits$ $\qquad\qquad$ $|A|=-1\neq 0$

$$adjA=\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}AdjA$$

$$=\frac{1}{-1}\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}$$

$$X=A^{-1}B$$

$$=\frac{1}{-1}\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix} \therefore x=1, y=2, z=3$$

41. Let $f:N\to R$ **be defined by f(x) =** $f(x)=4x^2+12x+15$. **Show that** $f:N\to S$ **where S is the range of function F, is invertible. Also find the inverse of f.**

Sol.

$$f(x_1)=f(x_2)$$

$$4x_1^2+12x_1+15=4x_2^2+12x_2+15$$

$$4\left(x_1^2-x_2^2\right)+12(x_1-x_2)=0$$

$$4(x_1-x_2)(x_1+x_2)+12(x_1-x_2)$$

$$(x_1-x_2)\left[4(x_1+x_2)+12\right]=0$$

$$(x_1-x_2)=0$$

$$x_1=x_2$$

f is one-one

For the function $f: N \rightarrow S$, condomain of f=S= Range of f (given)

f is onto $\therefore$ f is bijective $\therefore f^{-1}$ exits.

Suppose y be an arbitrary element of S. Then $y = 4x^2 + 12x + 15$ for some x in N

$$y = (2x)^2 + 2.2x.3 + 3^2 + 6 = (2x+3)^2 + 6$$

$$(2x+3)^2 = y - 6$$

$$2x = \sqrt{y-6} - 3$$

$$x = \frac{\sqrt{y-6} - 3}{2}$$

Define a function $g: S \rightarrow N$ *by* $g(y) = \frac{\sqrt{y-6} - 3}{2}$

$$gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g(2x+3)^2 + 6)$$

$$= \frac{\left(\sqrt{(2x+3)^2 + 6 - 6}\right) - 3}{2} = \frac{2x+3-3}{2} = x$$

$$fog(y) = f(g(y)) = f\left(\frac{\sqrt{y-6} - 3}{2}\right)$$

$$= 4\left(\frac{y-6+9-6\sqrt{y-6}}{4}\right) + 6\left(\sqrt{y-6} - 3\right) + 15$$

$$= y + 3 - 6\sqrt{y-6} + 6\sqrt{y-6} - 18 + 15$$

$$= y - 15 + 15 = y$$

$fog = I_s$ and gof $= I_N$

$f^{-1} = g$

42. If length x of a rectangle is decreasing at the rate of 3cm/minute and the width y is increasing at the rate of 2cm/minute, when x=10cm and y=6cm, find the rates of change of (i) the perimeter, (ii) the area of the rectangle.

Sol.

Simplify the expression,

$$\frac{dx}{dt} = -3cm/\min, \frac{dy}{dt} = +2cm/\min$$

$$x = 10cm, y = 6cm$$

$$p = 2(x + y)$$

$$\frac{dp}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2(-3 + 2) = 2(-1) = -2cm/\min$$

$$A = xy$$

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$= 10(2) + 6(-3)$$

$$20 - 18 = 2sq.cm/\min$$

43. If $y = (\sin^{-1} x)^2$ **Show that** $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$.

Sol.

Prove the expression,

$$y = (\sin^{-1} x)^2$$

$$y_1 = \sin^{-1} x.\frac{1}{\sqrt{1 - x^2}}$$

$$\sqrt{1 - x^2}\, y_1 = 2\sin^{-1} x$$

$$\sqrt{1 - x^2}\, y_2 + y_1 \frac{1}{2\sqrt{1 - x^2}}(-2x) = 2\frac{1}{\sqrt{1 - x^2}}$$

multiply by $\sqrt{1 - x^2}$ on both the sides .

$$(1 - x^2)y_1 - xy_1 - 2 = 0$$

$$(1 - x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$$

44. Find the integral of $\frac{1}{x^2+a^2}$ w.r.t x and hence find $\int\frac{1}{3+2x+x^2}$.

Sol. Substitute,

$$x = a\tan\theta; \frac{x}{a} = \tan\theta = \tan^{-1}\frac{x}{a}$$

$$dx = a\sec^2\theta d\theta$$

$$\int\frac{1}{x^2+a^2}dx = \int\frac{1}{a^2\tan^2\theta+a^2}a\sec^2\theta d\theta$$

$$= \frac{1}{a^2}\int\frac{a\sec^2\theta d\theta}{(\tan^2\theta+1)}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$\int\frac{1}{x^2+2x+3}dx = \int\frac{1}{(x+1)^2+(\sqrt{2})^2}dx$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

45. Using intergration find the area of region bounded by the triangle whose verticals are (1, 0), (2, 2) and (3, 1).

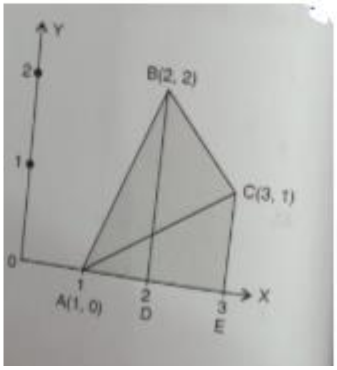
Sol.

Area of $\Delta ABC = Area$ of $\Delta ABD + Area$ of TripeziumBDEC $-$ $Area$ of ΔAEC

Equation of AB y=2(x $-$1)

Equation of BC y=4 $-$ x

Equation of CA y= $\frac{1}{2}(x-1)$



Area of $\Delta ABC = \int_1^2 2(x-1)dx + \int_1^2 (4-x)dx - \int_1^3 \frac{(x-1)}{2}dx$

$$2\left[\frac{x^2}{2}-x\right]_1^2 + \left[4x-\frac{x^2}{2}\right]_1^2 - \frac{1}{2}\left[\frac{x^2}{2}-x\right]_1^3$$

$$= \left[\left[\frac{(2)^2}{2}-2\right]-\left[\frac{1}{2}-1\right]\right]+\left[\left\{4\times3-\frac{3^2}{2}\right\}-\left\{4\times2-\frac{2^2}{2}\right\}-\frac{1}{2}\left\{\frac{3^2}{2}-3\right\}-\left\{\frac{1}{2}-1\right\}\right]$$

$$= \frac{3}{2} sq.unit$$

46. Derive the equation of a plane perpendicular to a given vector and passing through a given both in vector and Cartesian form.

Sol.

Suppose a plane pass through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$

Let $\vec{r}$ be the position vector of any point (x, y, z) in the plane.

The point P lies in the plane if and only if $\overrightarrow{AP}$ is perpendicular to $\vec{N}$

$\overrightarrow{AP}.\vec{N} = 0$

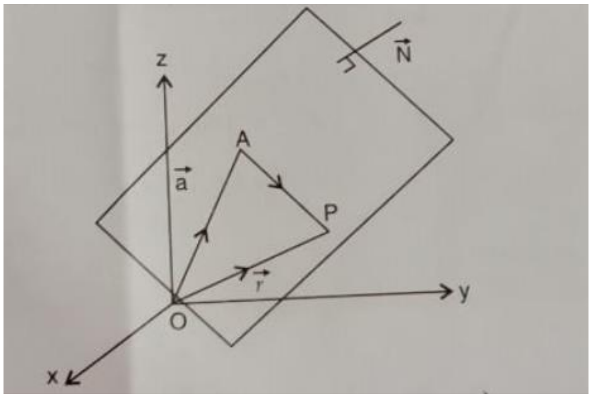
$\overrightarrow{AP} = (\vec{r} - \vec{a})$

Therefore,

$(\vec{r} - \vec{a}).\vec{N} = 0$

This is the vector equation of the plane

Cartesian form



Suppose the given point A be $\left(x_1, y_1, z_1\right),$ P be (x,y,z) and direction ratios of $\vec{N}$ and A, B and C. Then

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

And, then

$$\left(\vec{r} - \vec{a}\right).\vec{N} = 0$$

So,

$$\left(x - x_1\right)\hat{i} + \left(y - y_1\right)\hat{j} + \left(z - z_1\right)\hat{k} = 0$$
$$A\left(x - x_1\right) + B\left(y - y_1\right) + C\left(y - y_1\right) = 0$$

47. The probability that a student is not a swimmer is $\frac{1}{5}$. Find the probability that out of 5 student (i) at least four are swimmer and (ii) at most three are swimmer.

Sol.

Simplify the probability expression,

P= success (a student is a swimmer)

q= failure (a student is not a swimmer)

$$q = \frac{1}{5}$$
$$p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$$

Suppose x denotes the no, of swimmers clearly, x has a binomial distribution with

$n=5$ and p=$\frac{4}{5}$

$p(x=r)={}^{n}C_{r}q^{n-r}.p^{r}$

(i) At least four are swimmers

$$P(X\geq 4)=P(X=4)+P(X=5)$$

$$={}^{5}C_{4}\left(\frac{1}{5}\right)^{5-4}\left(\frac{4}{5}\right)^{4}+{}^{5}C_{5}\left(\frac{1}{5}\right)^{5-5}\left(\frac{4}{5}\right)^{5}$$

$$=5\times\left(\frac{1}{5}\right)^{1}\left(\frac{4}{5}\right)^{1}+1\left(\frac{4}{5}\right)^{5}$$

$$=\left(\frac{4}{5}\right)^{4}+\left(\frac{4}{5}\right)^{5}=\left(\frac{4}{5}\right)^{4}\left[1+\frac{4}{5}\right]$$

$$=\left(\frac{4}{5}\right)^{4}\left(\frac{9}{5}\right)$$

(ii)

$$P(X\leq 3)=P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

$$={}^{5}C_{0}\left(\frac{1}{5}\right)^{5}.\left(\frac{4}{5}\right)^{0}+{}^{5}C_{1}\left(\frac{1}{5}\right)^{4}\left(\frac{4}{5}\right)^{2}+{}^{5}C_{3}\left(\frac{1}{5}\right)^{2}\left(\frac{4}{5}\right)^{2}$$

$$=\left(\frac{1}{5}\right)^{5}+5\times\left(\frac{1}{5}\right)^{4}\times\frac{4}{5}+10\left(\frac{1}{5}\right)^{3}+10\left(\frac{1}{5}\right)^{2}\left(\frac{4}{5}\right)^{3}=$$

$$=\left(\frac{1}{5}\right)^{5}\left[1+5\times 4+10\times 16+10\times 64\right]=\left(\frac{1}{5}\right)^{5}[821]$$

$$=\frac{821}{(5)^{5}}$$

48. Solve the differential equation $ydx+\left(x-ye^{y}\right)dy=0$.

Sol.

Simplify the differential expression,

$$ydx+\left(x-ye^{y}\right)dy=0$$
$$ydx=\left(ye^{y}-x\right)dy$$
$$\frac{dx}{dy}+\frac{1}{y}x=e^{y}$$

Comparing above with $\frac{dy}{dx}+Px=Q.P=\frac{1}{p}$ and Q=e^{y}

$$e^{\int Pdy}=e^{\int\frac{1}{y}dy}=e^{\log y}=y$$
$$x\left(I.F\right)=\int Q\left(I.F\right)dy+c$$

I.F = $xy=\int e^{y}ydy+c$

$$xy=ye^{y}-\int e^{y}dy+c$$
$$xy=ye^{y}-e^{y}+c$$

49. Answer any one question:

(a) $Z=5x+10y$

Subject to the constraints

$x+2y\le 120$

$x+y\ge 60$

$x-2y\ge 0$ and $x\ge 0$ graphically method.

(b) Find the value of k, if

$$f\left(x\right)=\frac{1-\cos 2x}{1-\cos x},x\ne 0$$
$$=k \qquad x=0$$

is continous at x=0

(a)

Sol:

Draw the graph of the line, $x+2y=120$

x	**0**	**120**
y	**60**	**0**

Substitute (0,0) in the inequality $x+2y\le 120$ we have

$$0+2\times 0\le 120\Rightarrow 0\ge 120$$

So, the half plane is away from the origin

Draw the graph of the line $x-2y=0$.

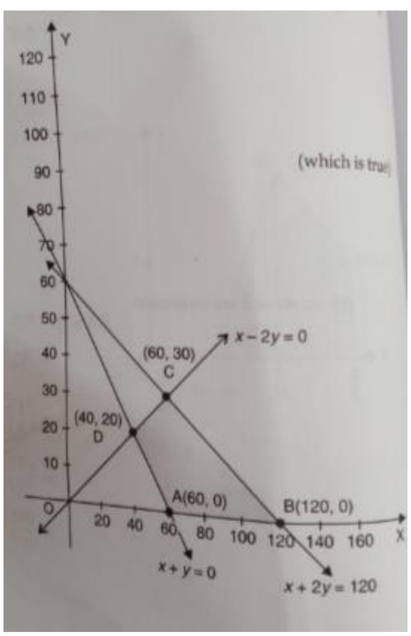
x	**0**	**60**
y	**60**	**0**

Subsitute (0,0) in the inequality $x+y\ge 60$, We have $0+0\le 60\Rightarrow 0\ge 60$

x	**0**	**10**
y	**0**	**5**

Substitute (5,0) in the inequality $x-2y\ge 0$

$5-2\times 0\ge 0\Rightarrow 5\ge 0$



So, the feasible region lies in the first quadrant.
Feasible region is ABCDA
On solving equation $x-2y=0$
And $x+2y=120$, We get C$\left(60,30\right)$
The corner points of the feasible region are A (60,0) B (120,0) ,C (60,30) and D(40,20).

Corner point	**Z=5x+10y**
A(60,0)	$300 \rightarrow Minimum$
B(120,0)	$600 \rightarrow Maximum$
C(60,30)	$600 \rightarrow \max imum$
D(40,20)	400

The minimum value of Z is 300 at (60,0) and the maximum value of Z is 600 at all points on the line segment joining the points (120,0) and (60,30) (120,0),and (60,30).

(b)

Sol:

$$f\left(x\right)=\frac{1-\cos 2x}{1-\cos x}=\lim_{x\to 0}\frac{1-\cos x}{1-\cos x}=\lim_{x\to 0}\frac{2\sin^2 x}{2\sin^2\left(\frac{x}{2}\right)}$$

$$=\lim_{x\to 0}\left[\frac{\sin x}{\sin\left(\frac{x}{2}\right)}\right]^2=\lim_{x\to 0}\left[\frac{2\sin\left(\frac{x}{2}\right)\cos\frac{x}{2}}{\sin\left(\frac{x}{2}\right)}\right]^2$$

$$\lim_{x\to 0}\left(2\cos\left(\frac{x}{2}\right)\right)^2=\left(2\cos\frac{0}{2}\right)^2=\left[2\times\cos 0\right]^2$$

$$=\left(2\times 1\right)^2=4$$

$$k=4$$

50. Prove that $\int_0^{2a} f(x)dx = 2f(x)\,dx,\ f(2a-x) = f(x)$
$= 0,\ f(2a-x) = -f(x)$

And hence, evaluate $\int_o^{2\pi} \cos^5 x dx$

Sol:

$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \ ...(i)$

Consider

$\int_a^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt)$

$= -\int_a^0 f(2a-t)\,dt$

$= \int_0^a f(2a-x)\,dx$

$\therefore Eq(i) becomes$

$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_a^{2a} f(2a-x)\,dx \ ...(ii)$

$Now, case(i): f(2a-x) = f(x)$ then eq(ii) becomes

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$

$\int_0^{2a} f(x)dx = 2\int_0^a f(x)\,dx$

$case(ii)$ If f $(2a-x) = -f(x)$ then eq(ii) becomes

$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_a^{2a} -f(x)\,dx$

$= \int_0^a f(x)\,dx - \int_0^a f(x)\,dx$

$\Rightarrow \int_0^{2a} f(x)\,dx = 0$

$$\therefore \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{iff } (2a-x) = f(x) \\ 0 & ,\ \text{if f } (2a-x) = -f(x) \end{cases},$$

And,

$\int_{O}^{2\pi} \cos^5 xdx$

$= 2\int_0^{\pi} \cos^5 xdx$

$I = 2\int_0^{\pi} \cos^5 (\pi - x) dx$

$I = 2\int_0^{\pi} -\cos^5 xdx$

$= -2\int_0^{\pi} \cos^5 xdx$

$I = -I$

$2I = 0$

$I = 0$

(b)

Prove that.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Sol.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$ Operating $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

We get,

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Taking common (a-b) from C_1 and (b-c) from C_1 and C_2, we get

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Now, expanding along R_3 We get,

$$= (a-b)(b-c)\left[1\times\left(b^2+bc+c^2\right)-1\times\left(a^2+ab+b^2\right)\right]$$

$$= (a-b)(b-c)\left[b^2+bc+c^2-a^2-ab-b^2\right]$$

$$= (a-b)(b-c)\left[b(c-a)+(c-a)(a+c)\right]$$

$$= (a-b)(b-c)(c-a)(a+b+c) = RHS$$