Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. The length of a rectangle is decreasing at the rate of 5 cm/min. and the width *y* is increasing at the rate of 4 cm/min. When *x* = 8 cm and *y* = 6 cm, find the change of (*a*) the perimeter (*b*) area of the rectangle.

Ans.

Given, $\frac{dx}{dt} = -5\,\text{cm}/\,\text{min}, \frac{dy}{dt} = 4\,\text{cm}/\,\text{min}$

Let x = length, and y = breadth

Perimeter of rectangle $P = 2(x + y)$

∴ Rate of change of P is

$$\frac{dP}{dt} = 2.\frac{dx}{dt} + 2\frac{dy}{dt}$$

$$\Rightarrow \frac{dP}{dt} = 2(-5) + 2(4) = -2$$

∴ Perimeter is decreasing at 2 m/s

If A be the area of rectangle then

$A = x \,.\, y$

Differentiating w.r.t. 't', we get

$$\frac{dA}{dt} = x.\frac{dy}{dt} + y.\frac{dx}{dt}$$

$= x \times 4 + y \times (-5)$

$= 4x - 5y$

$$\therefore \left.\frac{dA}{dt}\right]_{\substack{x=8 \\ y=6}} = 4 \times 8 - 5 \times 6$$

$= 32 - 30$

$= 2 \text{ cm}^2/\text{min.}$

Q.2. Find the intervals in which the function is

$$f(\mathbf{x}) = \frac{\mathbf{3}}{\mathbf{2}}x^4 - 4x^3 - 45x^2 + 51$$

(a) strictly increasing

(b) strictly decreasing.

Ans.

Here, $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$

$\Rightarrow \quad f'(x) = 6x^3 - 12x^2 - 90x$

$\Rightarrow \quad f'(x) = 6x(x^2 - 2x - 15) = 6x(x + 3)(x - 5)$

Now for critical point $f'(x) = 0$

$6x(x + 3)(x - 5) = 0$

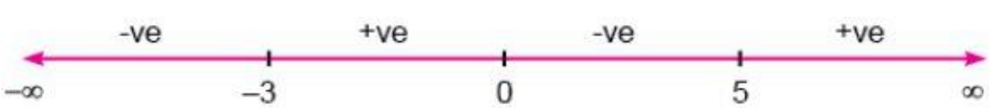
$x = 0, -3, 5$

i.e., $-3, 0, 5$ are critical points which divides domain R of given function into four disjoint sub intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 5)$, $(5, \infty)$.

For $(-\infty, -3)$

$f'(x) = +\text{ve} \times (-\text{ve}) \times (-\text{ve}) \times (-\text{ve}) = -\text{ve}$

i.e., $f(x)$ is decreasing in $(-\infty, -3)$



For $(-3, 0)$

$f'(x) = +\text{ve} \times (-\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = +\text{ve}$

i.e., $f(x)$ is increasing in $(-3, 0)$

For $(0, 5)$

$f'(x) = +\text{ve} \times (+\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = -\text{ve}$

i.e., $f(\text{x})$ is decreasing in $(0, 5)$

For $(5, \infty)$

$f'(x) = +\text{ve} \times (+\text{ve}) \times (+\text{ve}) \times (+\text{ve}) = +\text{ve}$

i.e., $f(x)$ is increasing in $(5, \infty)$

Hence $f(x)$ is

(a) strictly increasing in $(-3, 0) \cup (5, \infty)$

(b) strictly decreasing in $(-\infty, -3) \cup (0, 5)$

Q.3. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing.

Ans.

Given function is

$$f(x) = \sin 3x - \cos 3x$$
$$f'(x) = 3\cos 3x + 3\sin 3x$$

For critical points of function $f(x)$

$f'(x) = 0$

$\Rightarrow 3\cos 3x + 3\sin 3x = 0$

$\Rightarrow \cos 3x + \sin 3x = 0$

$$\Rightarrow \sin 3x = -\cos 3x$$

$$\Rightarrow \frac{\sin 3x}{\cos 3x} = -1$$

$$\Rightarrow \tan 3x = -\tan\frac{\pi}{4}$$

$$\Rightarrow \tan 3x = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$\Rightarrow \tan 3x = \tan\frac{3\pi}{4}$$

$$\Rightarrow 3x = n\pi + \frac{3\pi}{4}$$

where $n = 0, \pm 1, \pm 2, ...$

Putting $n = 0, \pm 1, \pm 2, ...$ we get

$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12} \in (0, \pi)$

Hence required possible intervals are

$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right) \left(\frac{11\pi}{12}, \pi\right)$

For $\left(0, \frac{\pi}{4}\right), f'(x) = +\text{Ve}$

For $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right), f'(x) = -\text{Ve}$

Hence, given function $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ and strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$.

Q.4. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Ans.

Given, curve $ay^2 = x^3$

On differentiating, we get

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\Rightarrow \quad \frac{dy}{dx} \text{ at } (am^2, am^3) = \frac{3\times a^2m^4}{2a\times am^3} = \frac{3m}{2}$$

$$\therefore \text{ Slope of normal} = -\frac{1}{\text{slope of tangent}} = -\frac{1}{\frac{3m}{2}} = -\frac{2}{3m}$$

Equation of normal at the point (am^2, am^3) is given by

$$\frac{y-am^3}{x-am^2} = -\frac{2}{3m}$$

$$\Rightarrow \quad 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow 2x + 3my - am^2(2 + 3m^2) = 0$$

Hence, equation of normal is $2x + 3my - am^2(2 + 3m^2) = 0$

Q.5. Find the approximate value of *f* (3.02), upto 2 places of decimal, where *f* (*x*) = $3x^2 + 5x + 3$.

Ans.

Here, $f(x) = 3x^2 + 5x + 3$

Let $x = 3$ and $dx = 0.02$

$\therefore x + dx = 3.02$

By definition, the approximate value of $f(x)$ is

$$f'(x) = \frac{f(x+dx) - f(x)}{dx}$$
$$\Rightarrow \quad f'(3) = \frac{f(3+0.02) - f(3)}{0.02}$$

[Putting $x = 3$ and $dx = 0.02$]

$$\Rightarrow \begin{array}{l} f'(x) = \frac{f(x+dx) - f(x)}{dx} \\ \Rightarrow \quad f'(3) = \frac{f(3+0.02) - f(3)}{0.02} \end{array} \quad \ldots(i)$$

Now, $\quad f(x) = 3x^2 + 5x + 3$

$\Rightarrow f'(x) = 6x + 5$

$\Rightarrow f'(3) = 23$

Also $\quad f(3) = 3 \times 3^2 + 5 \times 3 + 3 = 27 + 15 + 3 = 45$

Putting in (i), we get

$$23 = \frac{f(3.02) - 45}{0.02}$$

$\Rightarrow f(3.02) = 23 \times 0.02 + 45 = 45.46$

Q.6. Show that $y = \log(1 + x) - \frac{2x}{2+x}, x >-$ **is an increasing function of *x* throughout its domain.**

Here, $f(x) = \log\,(1 + x) - \frac{2x}{2+x}$

[where $y = f\,(x)$]

$$\Rightarrow\ f'(x) = \frac{1}{1+x} - 2\left[\frac{(2+x).1\ -\ x}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \frac{2(2+x\ -\ x)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(x-1)\,(x+2)^2}$$

$$= \frac{x^2}{(x+1)\,(x+2)^2}$$

For $f(x)$ being increasing function

$f'(x) > 0$

$$\Rightarrow \quad \frac{x^2}{(x+1)\,(x+2)^2} 0$$

$$\Rightarrow \quad \frac{1}{x+1} \cdot \frac{x^2}{(x+2)^2} 0$$

$$\Rightarrow \quad \frac{1}{x+1} 0 \qquad \left[\frac{x^2}{(x+2)^2} 0\right]$$

$\Rightarrow\ x + 1 > 0 \qquad$ or $\qquad x > -1$

i.e., $f(x) = y = \log\,(1 + x) - \frac{2x}{2+x}$ is increasing function in its domain $x > -1$ *i.e.,* $(-1, \infty)$.

Q.7. Find the equation of tangent to the curve $x = \sin\,3t, y = \cos\,2t$ at $t = \frac{\pi}{4}$.

Ans.

Here, $y = \cos 2t \quad \therefore \quad \frac{dy}{dt} = -2\sin 2t$

Also, $x = \sin 3t \quad \therefore \quad \frac{dx}{dt} = 3\cos 3t$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{3\cos 3t}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{-2\sin\frac{\pi}{2}}{3\cos\frac{3\pi}{4}}$$

$$= \frac{-2\times 1}{3\times\left(-\frac{1}{\sqrt{2}}\right)} = \frac{2\sqrt{2}}{3}$$

If $t = \frac{\pi}{4}$ then

$$x = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}},$$
$$y = \cos\frac{2\pi}{4} = \cos\frac{\pi}{2} = 0$$

Therefore, equation of tangent at $t = \frac{\pi}{4}$ *i.e.*, at $\left(\frac{1}{\sqrt{2}}, 0\right)$ is given by

$$y - 0 = \frac{dy}{dx}\left(x - \frac{1}{\sqrt{2}}\right)$$
$$\Rightarrow \quad y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad 3y = 2\sqrt{2x} - 2$$

Q.8. Using differential, find the approximate value of $f(2.01)$, where $f(x) = 4x^3 + 5x^2 + 2$.

Ans.

Let $x = 2$, $\Delta x = 0.01$ where $f(x) = 4x^3 + 5x^2 + 2$

$\Rightarrow \quad x + \Delta x = 2 + 0.01 = 2.01$

By definition, approximate value of $f(x)$ is

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\left[\because f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$\Rightarrow f'(2) = \frac{f(2+0.01) - f(2)}{0.01} \qquad \ldots(i)$$

$\because \quad f(x) = 4x^3 + 5x^2 + 2$

$\Rightarrow \quad f'(x) = 12x^2 + 10x$

$\Rightarrow \quad f'(2) = 48 + 20 = 68$

Also, $f(2) = 4 \times 2^3 + 5 \times 2^2 + 2$

$= 32 + 20 + 2 = 54$

Putting the values of $f'(2)$ and $f(2)$ in (i) we get

$$68 = \frac{f(2.01) - 54}{0.01}$$

$\Rightarrow f(2.01) = 68 \times 0.01 + 54 = 54.68$

Q.9. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the *y*-coordinate of the point.

Ans.

Let $P(x_1, y_1)$ be the required point on the curve

$y = x^3$ ...(i)

$\frac{dy}{dx} = 3x^2$

$\Rightarrow \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 3x_1{}^2$

$\Rightarrow$ Slope of tangent at $(x_1, y_1) = 3x_1^2$

According to the question,

$3x_1^2 = y_1$ ...(ii)

Also (x_1, y_1) lies on (i)

$\Rightarrow y_1 = x_1^3$ ...(iii)

From (ii) and (iii), we get

$3x_1^2 = x_1^3$

$\Rightarrow x_1^3 - 3x_1^2 = 0$

$\Rightarrow x_1^2 (x_1 - 3) = 0$

$\Rightarrow x_1 = 0$ or $x_1 = 3$

$\Rightarrow y_1 = 0$ or $y_1 = 27$

Hence, required points are $(0, 0)$ and $(3, 27)$.

Q.10. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm?

Ans.

Let 'A' be the area and 'a' be the side of an equilateral triangle.

$$A = \frac{\sqrt{3}}{4} a^2$$

Differentiating with respect to t we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2a \,.\, \frac{da}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2a \times 2$$
$$\left[\text{Given } \frac{da}{dt} = 2 \text{ cm/sec}\right]$$

$$\frac{dA}{dt} = \sqrt{3a}$$

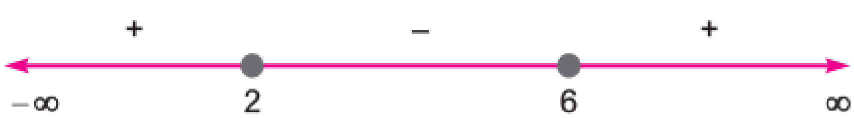
$$\Rightarrow \quad \left.\frac{dA}{dt}\right]_{a=20\,\text{cm}} = 20\sqrt{3}\,\text{sq cm/s}$$

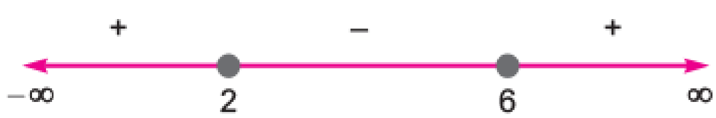
Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. Find the intervals in which the function $f(x) = x^3 - 12x^2 + 36x + 17$ is (*a*) increasing, (*b*) decreasing.

Ans.





We have, $f(x) = x^3 - 12x^2 + 36x + 17$

$\Rightarrow f(x) = 3x^2 - 24x + 36 = 3\ (x - 6)\ (x - 2)$

a. For $f(x)$ to be increasing, we must have

$f'(x) > 0$

$\Rightarrow 3\ (x - 6)\ (x - 2) > 0$

$\Rightarrow x < 2$ or $x > 6$

$\Rightarrow x \in (-\infty, 2) \cup (6, \infty)$

So, $f(x)$ is increasing on $(-\infty, 2) \cup (6, \infty)$

b. For $f(x)$ to be decreasing, we must have

$f'(x) < 0$

$\Rightarrow 3(x - 2)(x - 6) < 0$

$\Rightarrow 2 < x < 6$

So, $f(x)$ is decreasing on $(2, 6)$.

Q.2. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line 4*x* – 2*y* + 5 = 0.

Ans.

Given, curve $y = \sqrt{3x - 2} \quad \Rightarrow \quad y^2 = 3x - 2$ ...(i)

To get the equation of tangent to the curve $y^2 = 3x - 2$, which is a parabola, first we have to find the coordinates of point from where tangent line passes.

Let the coordinates of the point on parabola be (a, b), then this coordinate will satisfy the equation (i).

Therefore, from (i), we have $b^2 = 3a - 2$ ...(ii)

Now differentiating the equation (i) with respect to x, we get

$$2y\frac{\mathrm{dy}}{\mathrm{dx}} = 3$$
$$\Rightarrow \qquad \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right) = \frac{3}{2y}$$

Now slope of the required tangent line

$$m_1 = \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{3}{2b}$$

As it is given that required tangent line is parallel to the given line $4x - 2y + 5 = 0$, so, the slopes of lines are equal.

Therefore,

$$\frac{3}{2b} = \frac{-4}{-2}$$
$$\Rightarrow \; 4b = 3$$
$$\Rightarrow \quad b = \frac{3}{4}$$

Substituting this value $b = \frac{3}{4}$ in equation (ii), we get

$$\left(\frac{3}{4}\right)^2 + 2 = 3a$$
$$\Rightarrow \qquad 3a = \frac{41}{16} \qquad \Rightarrow \qquad a = \frac{41}{48}$$

Now the coordinates of the point on tangent are $\frac{41}{48}, \frac{3}{4}$ and slope is 2.

Hence, equation of tangent is obtained by $y - b = m(x - a)$

$$\Rightarrow \qquad y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$
$$\Rightarrow \qquad \frac{4y - 3}{4} = \frac{2(48x - 41)}{48}$$
$$\Rightarrow \qquad (4y - 3) = \frac{48x - 41}{6}$$
$$\Rightarrow \qquad 24y - 18 = 48x - 41$$

$\Rightarrow 48x - 24y = 41 - 18$ $\qquad \Rightarrow 48x - 24y - 23 = 0$ is the required equation of tangent.

Q.3. The fuel cost for running a train is proportional to the square of the speed generated in km per hour. If the fuel costs ₹ 48 per hour at speed 16 km per hour and the fixed charges amount to ₹ 1200 per hour then find the most economical speed of train, when total distance covered by train is *S* km.

Ans.

Let F be the fuel cost per hour and v the speed in km/h

From question,

$F \propto v^2 \quad \Rightarrow F = Kv^2 \quad \ldots(i)$

Given, $F =$ ₹ 48/hour, when $v = 16$ km/hour

$\Rightarrow 48 = K.16^2 \quad \Rightarrow \quad K = \frac{3}{16}$

Now (i) becomes, $F = \frac{3v^2}{16}$

If t is the time taken by train in covering given distance S km and C the total cost for running the train then

$$\Rightarrow C = 1200\, t + \frac{3v^2}{16} t \qquad \left[\because t = \frac{S}{v}\right]$$

$$\Rightarrow C = 1200 \times \frac{S}{v} + \frac{3v^2}{16} \times \frac{S}{v}$$

$$\Rightarrow \frac{dC}{dv} = -1200 \times \frac{S}{v^2} + \frac{3S}{16}$$

For maximum or minimum value of $C, \frac{dC}{dv} = 0$

$$\Rightarrow \quad -1200 \times \frac{S}{v^2} + \frac{3S}{16} = 0$$

$$\Rightarrow \quad 1200 \times \frac{S}{v^2} = \frac{3S}{16}$$

$$\Rightarrow \quad v^2 = \frac{1200 \times 16}{3}$$

$$\Rightarrow \quad v^2 = 6400 \quad \Rightarrow \quad v = 80 \text{ km / hour}$$

Also,

$$\Rightarrow \quad \frac{d^2C}{dv^2} = 2400 \times \frac{S}{v^3}$$

$$\Rightarrow \quad \left(\frac{d^2C}{dv^2}\right)_{v=80} > 0$$

Hence, C is minimum, when v = 50 km/hour.