DAY FOURTEEN

Maxima and Minima

Learning & Revision for the Day

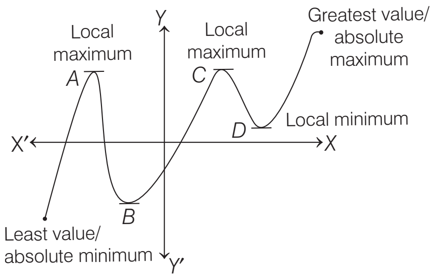
- Maxima and Minima of a Function
- Concept of Global Maximum/Minimum

Maxima and Minima of a Function

A function $f(x)$ is said to attain a **maximum** at $x = a$, if there exists a neighbourhood $(a - \delta, a + \delta)$, $x \neq a$ i.e. $f(x) < f(a)$, $\forall\ x \in (a - \delta, a + \delta)$, $x \neq a \cdot h > 0$ (very small quantity)

In such a case $f(a)$ is said to be the maximum value of $f(x)$ at $x = a$.

A function $f(x)$ is said to attain a **minimum** at $x = a$, if there exists a neighbourhood $(a - \delta, a + \delta)$ such that $f(x) > f(a), \forall x \in (a - \delta, a + \delta)$, $x \neq a$.



Graph of a continuous function explained local maxima (minima) and absolute maxima (minima). In such a case $f(a)$ is said to be the minimum value of $f(x)$ at $x = a$.

The points at which a function attains either the maximum or the minimum values are known as the **extreme points** or **turning points** and both minimum and maximum values of $f(x)$ are called extreme values. The turning points A and C are called **local maximum** and points B and D are called **local minimum**.

Critical Point

- A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a **critical point** of f. Note that, if f is continuous at point c and $f'(c) = 0$, then there exists $h > 0$ such that f is differentiable in the interval $(c - h, c + h)$.
- The converse of above theorem need not be true, that is a point at which the derivative vanishes need not be a point of local maxima or local minima.

Method to Find Local Maxima or Local Minima

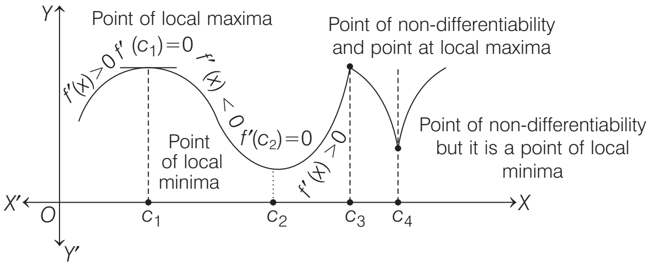
First Derivative Test

Let f be a function defined on an open interval I and f be continuous at a critical point c in I. Then,

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e. if $f'(x) > 0$ at every point sufficiently close to and to the left of c and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of **local maxima**.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through point c, i.e. if $f'(x) < 0$ at every point sufficiently close to and to the left of c and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of **local minima**.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called **point of inflection**.



Graph of f around c explained following points.

(iv) If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then $f(c)$ is a local minimum value of f.

Second or Higher Order Derivative Test

(i) Find $f'(x)$ and equate it to zero. Solve $f'(x) = 0$ let its roots be $x = a_1, a_2, \ldots$

(ii) Find $f''(x)$ and at $x = a_1$,

(a) if $f''(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$.

(b) if $f''(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$.

(iii) (a) If at $x = a_1, f''(a_1) = 0$, then find $f'''(x)$. If $f'''(a_1) \neq 0$, then $f(x)$ is neither maximum nor minimum at $x = a$.

(b) If $f'''(a_1) = 0$, then find $f^{\text{iv}}(x)$.

(c) If $f^{\text{iv}}(x)$ is positive (minimum value) and $f^{\text{iv}}(x)$ is negative (maximum value).

(iv) If at $x = a_1, f^{\text{iv}}(a_1) = 0$, then find $f^{\text{v}}(x)$ and proceed similarly.

Point of Inflection

At point of inflection

(i) It is not necessary that 1st derivative is zero.

(ii) 2nd derivative must be zero or 2nd derivative changes sign in the neighbourhood of point of inflection.

*n*th Derivative Test

Let f be a differentiable function on an interval I and a be an interior point of I such that

(i) $f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ and

(ii) $f^n(a)$ exists and is non-zero.

Important Results

- If n is even and $f^n(a) < 0 \Rightarrow x = a$ is a point of local maximum.
- If n is even and $f^n(a) > 0 \Rightarrow x = a$ is a point of local minimum.
- If n is odd $\Rightarrow x = a$ is a point of neither local maximum nor a point of local minimum.
- The function $f(x) = \dfrac{ax+b}{cx+d}$ has no local maximum or minimum regardless of values of a, b, c and d.
- The function $f(\theta) = \sin^m \theta \cdot \cos^n \theta$ attains maximum values at $\theta = \tan^{-1}\left(\sqrt{\dfrac{m}{n}}\right)$.
- If AB is diameter of circle and C is any point on the circumference, then area of the ΔABC will be maximum, if triangle is isosceles.

Concept of Global Maximum/Minimum

- Let $y = f(x)$ be a given function with domain D and $[a, b] \subseteq D$, then global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest / least value of $f(x)$ in $[a, b]$.
- Global maxima/minima in $[a, b]$ would always occur at critical points of $f(x)$ within $[a, b]$ or at end points of the interval.

Global Maximum/Minimum in $[a, b]$

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$.

Step I Find out all critical points of $f(x)$ in $[a, b]$
[i.e. all points at which $f'(x) = 0$] and let these points are $c_1, c_2, \ldots, c_n$.

Step II Find the value of $f(c_1), f(c_2), \ldots, f(c_n)$ and also at the end points of domain i.e. $f(a)$ and $f(b)$.

Step III Find $M_1 \rightarrow$ Global maxima or greatest value
and $M_2 \rightarrow$ Global minima or least value.
where, $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$
and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

Some Important Results on Maxima and Minima

(i) Maxima and minima occur alternatively i.e. between two maxima there is one minimum and *vice-versa*.

(ii) If $f(x) \rightarrow \infty$ as $x \rightarrow a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the minimum and the least value.

(iii) If $f(x) \rightarrow -\infty$ as $x \rightarrow a$ or b, then $f(c)$ is necessarily the maximum and greatest value.

(iv) The **stationary points** are the points of the domain, where $f'(x) = 0$.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If f is defined as $f(x) = x + \frac{1}{x}$, then which of following is true? → **NCERT Exemplar**

(a) Local maximum value of $f(x)$ is -2
(b) Local minimum value of $f(x)$ is 2
(c) Local maximum value of $f(x)$ is less than local minimum value of $f(x)$
(d) All the above are true

2 If the sum of two numbers is 3, then the maximum value of the product of the first and the square of second is → **NCERT Exemplar**

(a) 4 (b) 1 (c) 3 (d) 0

3 If $y = a\log x + bx^2 + x$ has its extremum value at $x = 1$ and $x = 2$, then (a, b) is equal to

(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 2\right)$ (c) $\left(2, \frac{-1}{2}\right)$ (d) $\left(\frac{-2}{3}, \frac{-1}{6}\right)$

4 The function $f(x) = a\cos x + b\tan x + x$ has extreme values at $x = 0$ and $x = \frac{\pi}{6}$, then

(a) $a = -\frac{2}{3}, b = -1$ (b) $a = \frac{2}{3}, b = -1$
(c) $a = -\frac{2}{3}, b = 1$ (d) $a = \frac{2}{3}, b = 1$

5 The minimum radius vector of the curve
$$\frac{4}{x^2} + \frac{9}{y^2} = 1 \text{ is of length}$$

(a) 1 (b) 5 (c) 7 (d) None of these

6 The function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has → **NCERT Exemplar**

(a) one local maxima
(b) one local minima
(c) one local maxima and two local minima
(d) neither maxima nor minima

7 The function $f(x) = \frac{x^2 - 2}{x^2 - 4}$ has

(a) no point of local minima
(b) no point of local maxima
(c) exactly one point of local minima
(d) exactly one point of local maxima

8 Let $f : R \to R$ be defined by $f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$.

If f has a local minimum at $x = -1$, then a possible value of k is → **AIEEE 2010**

(a) 1 (b) 0 (c) $-\frac{1}{2}$ (d) -1

9 The minimum value of $9x + 4y$, where $xy = 16$ is

(a) 48 (b) 28 (c) 38 (d) 18

10 If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a is equal to

(a) 3 (b) 1 (c) 2 (d) $\frac{1}{2}$

11 If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that minimum $f(x) >$ maximum $g(x)$, then the relation between b and c is

(a) $0 < c < b\sqrt{2}$ (b) $|c| < |b|\sqrt{2}$
(c) $|c| > |b|\sqrt{2}$ (d) No real values of b and c

12 Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to → **JEE Mains 2015**

(a) -8 (b) -4
(c) 0 (d) 4

13 If a differential function $f(x)$ has a relative minimum at $x = 0$, then the function $\phi(x) = f(x) + ax + b$ has a relative minimum at $x = 0$ for

(a) all a and all b (b) all b, if $a = 0$
(c) all $b > 0$ (d) all $a > 0$

14 The denominator of a fraction is greater than 16 of the square of numerator, then least value of fraction is

(a) $-1/4$ (b) $-1/8$
(c) $1/12$ (d) $1/16$

15 The function $f(x) = ax + \frac{b}{x}$, b, $x > 0$ takes the least value at x equal to

(a) b (b) $\sqrt{a}$ (c) $\sqrt{b}$ (d) $\sqrt{\frac{b}{a}}$

16 Let f be a function defined by $f(x) = \begin{cases} \frac{\tan x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

Statement I $x = 0$ is point of minima of f.

Statement II $f'(0) = 0$. → **AIEEE 2011**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

17 The absolute maximum and minimum values of the function f given by $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$ → **NCERT Exemplar**

(a) 2.25 and 2 (b) 1.25 and 1
(c) 1.75 and 1.5 (d) None of these

18 The maximum value of $f(x)=\frac{x}{4+x+x^2}$ on $[-1,1]$ is

(a) $-\frac{1}{4}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{5}$

19 In interval $[1,e]$, the greatest value of $x^2 \log x$ is

(a) e^2 (b) $\frac{1}{e}\log\frac{1}{\sqrt{e}}$ (c) $e^2 \log\sqrt{e}$ (d) None of these

20 If x is real, the maximum value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is

➔ **AIEEE 2007**

(a) 41 (b) 1 (c) $\frac{17}{7}$ (d) $\frac{1}{4}$

21 The maximum and minimum values of $f(x)=\sec x+\log\cos^2 x$, $0<x<2\pi$ are respectively

➔ **NCERT Exemplar**

(a) $(1,-1)$ and $\{2(1-\log 2), 2(1+\log 2)\}$
(b) $(1,-1)$ and $\{2(1-\log 2), 2(1-\log 2)\}$
(c) $(1,-1)$ and $(2,-3)$
(d) None of the above

22 The difference between greatest and least values of the function $f(x)=\sin 2x-x$, on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is

➔ **NCERT Exemplar**

(a) π (b) 2π (c) 3π (d) $\frac{\pi}{2}$

23 The point of inflection for the curve $y=x^{5/2}$ is

(a) (1, 1) (b) (0, 0) (c) (1, 0) (d) (0, 1)

24 The maximum area of a right angled triangle with hypotenuse h is

➔ **JEE Main 2013**

(a) $\frac{h^3}{2\sqrt{2}}$ (b) $\frac{h^2}{2}$ (c) $\frac{h^2}{\sqrt{2}}$ (d) $\frac{h^2}{4}$

25 A straight line is drawn through the point $P(3,4)$ meeting the positive direction of coordinate axes at the points A and B. If O is the origin, then minimum area of ΔOAB is equal to

(a) 12 sq units (b) 6 sq units
(c) 24 sq units (d) 48 sq units

26 Suppose the cubic x^3-px+q has three distinct real roots, where $p>0$ and $q>0$. Then, which one of the following holds?

(a) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$
(b) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$
(c) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$
(d) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

27 If $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of a ΔABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segment BC, CA and AB, respectively. Then, maximum area of such parallelogram is

(a) $\frac{1}{2}$ (area of ΔABC) (b) $\frac{1}{4}$ (area of ΔABC)
(c) $\frac{1}{6}$ (area of ΔABC) (d) $\frac{1}{8}$ (area of ΔABC)

28 If $y=f(x)$ is a parametrically defined expression such that $x=3t^2-18t+7$ and $y=2t^3-15t^2+24t+10$, $\forall\, x\in[0,6]$.
Then, the maximum and minimum values of $y=f(x)$ are

(a) 36, 3 (b) 46, 6 (c) 40, –6 (d) 46, –6

29 The value of a, so that the sum of the squares of the roots of the equation $x^2-(a-2)x-a+1=0$ assume the least value is

(a) 2 (b) 1 (c) 3 (d) 0

30 The minimum intercepts made by the axes on the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

(a) 25 (b) 7 (c) 1 (d) None of these

31 The curved surface of the cone inscribed in a given sphere is maximum, if

(a) $h=\frac{4R}{3}$ (b) $h=\frac{R}{3}$ (c) $h=\frac{2R}{3}$ (d) None of these

32 The volume of the largest cone that can be inscribed in a sphere of radius R is

➔ **NCERT**

(a) $\frac{3}{8}$ of the volume of the sphere
(b) $\frac{8}{27}$ of the volume of the sphere
(c) $\frac{2}{7}$ of the volume of the sphere
(d) None of the above

33 Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(a) $\sqrt{ab}$ (b) $\frac{a}{b}$ (c) $2ab$ (d) ab

34 The real number x when added to its inverse gives the minimum value of the sum at x equal to

➔ **AIEEE 2003**

(a) 2 (b) 1 (c) –1 (d) –2

35 The greatest value of $f(x)=(x+1)^{1/3}-(x-1)^{1/3}$ on [0, 1] is

➔ **AIEEE 2002**

(a) 1 (b) 2 (c) 3 (d) 1/3

36 The coordinate of a point on the parabola $y^2=8x$ whose distance from the circle $x^2+(y+6)^2=1$ is minimum, is

(a) (2,–4) (b) (2,4) (c) (18,–12) (d) (8, 8)

37 The volume of the largest cylinder that can be inscribed in a sphere of radius r cm is

(a) $\frac{4\pi r^3}{\sqrt{3}}$ (b) $\frac{4\pi r^3}{3\sqrt{3}}$ (c) $\frac{4\pi r^3}{2\sqrt{3}}$ (d) $\frac{4\pi r^3}{5\sqrt{2}}$

38 Maximum slope of the curve $y=-x^3+3x^2+9x-27$ is

(a) 0 (b) 12 (c) 16 (d) 32

39 If $ab = 2a + 3b$, $a > 0$, $b > 0$, then the minimum value of ab is

(a) 12
(b) 24
(c) $\frac{1}{4}$
(d) None of these

40 The perimeter of a sector is p. The area of the sector is maximum, when its radius is

(a) $\sqrt{p}$
(b) $\frac{1}{\sqrt{p}}$
(c) $\frac{p}{2}$
(d) $\frac{p}{4}$

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 The minimum radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ is of length

(a) $a - b$
(b) $a + b$
(c) $2a + b$
(d) None of these

2 $f(x) = x^2 - 4|x|$ and

$$g(x) = \begin{cases} \min\{f(t) : -6 \le t \le x\}, & x \in [-6, 0] \\ \max\{f(t) : 0 < t \le x\}, & x \in (0, 6] \end{cases}$$, then $g(x)$ has

(a) exactly one point of local minima
(b) exactly one point of local maxima
(c) no point to local maxima but exactly one point of local minima
(d) neither a point of local maxima nor minima

3 $$f(x) = \begin{cases} 4x - x^3 + \log(a^2 - 3a + 3), & 0 \le x < 3 \\ x - 18, & x \ge 3 \end{cases}$$

Complete the set of values of a such that $f(x)$ has a local maxima at $x = 3$, is

(a) $[-1, 2]$
(b) $(-\infty, 1) \cup (2, \infty)$
(c) $[1, 2]$
(d) $(-\infty, -1) \cup (2, \infty)$

4 The point in the interval $[0, 2\pi]$, where $f(x) = e^x \sin x$ has maximum slope is

(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{2}$
(c) π
(d) None of these

5 The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2 + x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

(a) 0
(b) 1
(c) 2
(d) 3

6 If 20 m of wire is available for fencing off a flower-bed in the form of a circular sector, then the maximum area (in sq m) of the flower-bed is ➜ **JEE Mains 2017**

(a) 12.5
(b) 10
(c) 25
(d) 30

7 The cost of running a bus from A to B, is ₹ $\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be ₹ 75 while at 40 km/h, it is ₹ 65. Then, the most economical speed (in km/h) of the bus is ➜ **JEE Mains 2013**

(a) 45
(b) 50
(c) 60
(d) 40

8 If $f(x) = \begin{cases} |x^2 - 2|, & -1 \le x < \sqrt{3} \\ \frac{x}{\sqrt{3}}, & \sqrt{3} \le x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \le x \le 4 \end{cases}$, then the points, where $f(x)$ takes maximum and minimum values, are

(a) 1, 4
(b) 0, 4
(c) 2, 4
(d) None of these

9 Let $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x| \left(3 + \sin\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$, then number of points [where, $f(x)$ attains its minimum value] is

(a) 1
(b) 2
(c) 3
(d) infinite many

10 A wire of length 2 units is cut into two parts which are bent respectively to form a square of side $= x$ units and a circle of radius $= r$ units. If the sum of the areas of the square and the circle so formed is minimum, then

(a) $2x = (\pi + 4)r$
(b) $(4 - \pi)x = \pi r$
(c) $x = 2r$
(d) $2x = r$

11 Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is ➜ **JEE Mains 2018**

(a) 3
(b) -3
(c) $-2\sqrt{2}$
(d) $2\sqrt{2}$

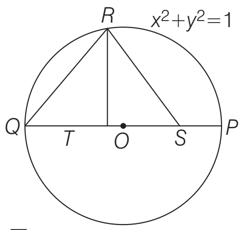
12 The largest term in the sequence $a_n = \frac{n^2}{n^3 + 200}$ is given by

(a) $\frac{529}{49}$
(b) $\frac{8}{89}$
(c) $\frac{49}{543}$
(d) None of these

13 All possible values of the parameter a so that the function $f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$ has a negative point of local minimum are

(a) all real values
(b) no real values
(c) $(0, \infty)$
(d) $(-\infty, 0)$

14 The circle $x^2 + y^2 = 1$ cuts the X-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the X-axis and the line segment PQ at S. Then, the maximum area of the ΔQSR is



(a) $4\sqrt{3}$ sq units (b) $14\sqrt{3}$ sq units
(c) $\frac{4\sqrt{3}}{9}$ sq units (d) $15\sqrt{3}$ sq units

15 Given, $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$. ➡ **AIEEE 2009**

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P
(d) Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

ANSWERS

SESSION 1	**1** (d)	**2** (a)	**3** (d)	**4** (a)	**5** (b)	**6** (d)	**7** (d)	**8** (d)	**9** (a)	**10** (c)
	11 (c)	**12** (c)	**13** (b)	**14** (b)	**15** (d)	**16** (b)	**17** (b)	**18** (c)	**19** (a)	**20** (a)
	21 (b)	**22** (a)	**23** (b)	**24** (d)	**25** (c)	**26** (b)	**27** (a)	**28** (d)	**29** (b)	**30** (b)
	31 (a)	**32** (b)	**33** (c)	**34** (b)	**35** (b)	**36** (a)	**37** (b)	**38** (b)	**39** (b)	**40** (d)
SESSION 2	**1** (b)	**2** (d)	**3** (c)	**4** (b)	**5** (c)	**6** (c)	**7** (c)	**8** (b)	**9** (a)	**10** (c)
	11 (d)	**12** (c)	**13** (b)	**14** (c)	**15** (b)					

Hints and Explanations

SESSION 1

1 Let $y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$

Now, $\frac{dy}{dx} = 0 \Rightarrow x^2 = 1$

$\Rightarrow \quad x = \pm 1$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{x^3}$, therefore

$\frac{d^2y}{dx^2}(\text{at } x = 1) > 0$

and $\frac{d^2y}{dx^2}(\text{at } x = -1) < 0$

Hence, local maximum value of y is at $x = -1$ and the local maximum value $= -2$.

Local minimum value of y is at $x = 1$ and local minimum value $= 2$.

Therefore, local maximum value -2 is less than local minimum value 2.

2 Let two numbers be x and $(3 - x)$.

Then, product $P = x(3 - x)^2$

$$\frac{dP}{dx} = -2x(3 - x) + (3 - x)^2$$

$\frac{dP}{dx} = (3 - x)(3 - 3x)$ and $\frac{d^2P}{dx^2} = 6x - 12$

For maxima or minima, put $\frac{dP}{dx} = 0$

$\Rightarrow (3 - x)(3 - 3x) = 0 \Rightarrow x = 3, 1$

At $x = 3$,

$$\frac{d^2P}{dx^2} = 18 - 12 = 6 > 0 \quad \text{[minima]}$$

At $x = 1$,

$$\frac{d^2P}{dx^2} = -6 < 0$$

So, P is maximum at $x = 1$.

$\therefore$ Maximum value of $P = 1(3 - 1)^2 = 4$

3 $\because \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0$

$\Rightarrow \quad a = -2b - 1$

and $\left(\frac{dy}{dx}\right)_{x=2} = \frac{a}{2} + 4b + 1 = 0$

$\Rightarrow \quad \frac{-2b - 1}{2} + 4b + 1 = 0$

$\Rightarrow -b + 4b + \frac{1}{2} = 0 \Rightarrow 3b = \frac{-1}{2}$

$\Rightarrow \quad b = \frac{-1}{6}$ and $a = \frac{1}{3} - 1 = \frac{-2}{3}$

4 $f'(x) = -a \sin x + b \sec^2 x + 1$

Now, $f'(0) = 0$ and $f'\left(\frac{\pi}{6}\right) = 0$

$\Rightarrow b + 1 = 0$ and $-\frac{a}{2} + \frac{4b}{3} + 1 = 0$

$\Rightarrow \quad b = -1, a = -\frac{2}{3}$

5 The given curve is $\frac{4}{x^2} + \frac{9}{y^2} = 1$

Put $x = r\cos\theta$, $y = r\sin\theta$, we get

$$r^2 = (2\sec\theta)^2 + (3\,\text{cosec}\,\theta)^2$$

So, r^2 will have minimum value $(2 + 3)^2$.

or r have minimum value equal to 5.

6 $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27$

$= 3(4x^2 - 12x + 9) = 3(2x - 3)^2$

$f'(x) = 0 \Rightarrow x = \frac{3}{2}$ (critical point)

Since, $f'(x) > 0$ for all $x < \frac{3}{2}$ and for all $x > \frac{3}{2}$

Hence, $x = \frac{3}{2}$ is a point of inflection *i.e.*, neither a point of maxima nor a point of minima.

$x = \frac{3}{2}$ is the only critical point and f has neither maxima nor minima.

7 For $y = \frac{x^2 - 2}{x^2 - 4} \Rightarrow \frac{dy}{dx} = \frac{-4x}{(x^2 - 4)^2}$

$\Rightarrow \quad \dfrac{dy}{dx} > 0$, for $x < 0$

and $\quad \dfrac{dy}{dx} < 0$, for $x > 0$

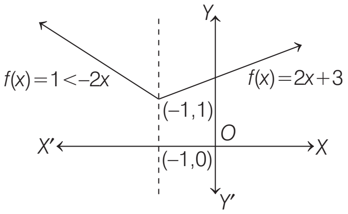
Thus, $x = 0$ is the point of local maxima for y. Now, $(y)_{x=0} = \dfrac{1}{2}$ (positive). Thus, $x = 0$ is also the point of local maximum for $y = \left|\dfrac{x^2 - 2}{x^2 - 4}\right|$.

8 If $f(x)$ has a local minimum at $x = -1$, then

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$$

$\Rightarrow \quad \lim_{x \to -1^+} 2x + 3 = \lim_{x \to -1^-} 1 < -2x$

$\Rightarrow \quad -2 + 3 = k + 2 \quad \Rightarrow \quad k = -1$



9 Let $S = 9x + 4y$

Since, $xy = 16$ is given.

$\therefore \quad y = \dfrac{16}{x} \quad$ or $\quad S = 9x + \dfrac{64}{x}$

On differentiating both sides, we get

$$\frac{dS}{dx} = 9 - \frac{64}{x^2} \quad \text{...(i)}$$

$\because \quad \dfrac{dS}{dx} = 0 \Rightarrow \dfrac{64}{x^2} = 9 \Rightarrow x = \pm \dfrac{8}{3}$

Again, on differentiating Eq. (i) w.r.t. x, we get $\dfrac{d^2S}{dx^2} = \dfrac{128}{x^3}$

Hence, it is minimum at $x = \dfrac{8}{3}$ and minimum value of S is

$$S_{\min} = 9\left(\frac{8}{3}\right) + 4(6) = 48$$

10 We have,

$$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$$
$$f'(x) = 6x^2 - 18ax + 12a^2$$
$$f''(x) = 12x - 18a$$

For maximum and minimum,

$$6x^2 - 18ax + 12a^2 = 0$$

$\Rightarrow \quad x^2 - 3ax + 2a^2 = 0$

$\Rightarrow \quad x = a$ or $x = 2a$

At $x = a$ maximum and at $x = 2a$ minimum.

$\because \quad p^2 = q$

$\therefore \quad a^2 = 2a \Rightarrow a = 2$ or $a = 0$

But $a > 0$, therefore $a = 2$

11 Minimum of $f(x) = -\dfrac{D}{4a}$

$$= \frac{-(4b^2 - 8c^2)}{4}$$
$$= 2c^2 - b^2$$

and maximum of $g(x) = -\dfrac{(4c^2 + 4b^2)}{4(-1)}$

$= b^2 + c^2$

Since, min $f(x) >$ max $g(x)$

$\Rightarrow \quad 2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \quad c^2 > 2b^2$

$\Rightarrow \quad |c| > \sqrt{2}\,|b|$

12 **Central Idea** Any function have extreme values (maximum or minimum) at its critical points, where $f'(x) = 0$.

Since, the function have extreme values at $x = 1$ and $x = 2$.

$\therefore \quad f'(x) = 0$ at $x = 1$ and $x = 2$

$\Rightarrow \quad f'(1) = 0$ and $f'(2) = 0$

Also it is given that

$$\lim_{x \to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3 \Rightarrow 1 + \lim_{x \to 0} \frac{f(x)}{x^2} = 3$$

$\Rightarrow \quad \lim_{x \to 0} \dfrac{f(x)}{x^2} = 2$

$\Rightarrow \quad f(x)$ will be of the form

$$ax^4 + bx^3 + 2x^2$$

[$\because f(x)$ is of four degree polynomial]

Let $f(x) = ax^4 + bx^3 + 2x^2 \Rightarrow f'(x)$

$= 4ax^3 + 3bx^2 + 4x$

$\Rightarrow \quad f'(1) = 4a + 3b + 4 = 0 \quad$ ...(i)

and $f'(2) = 32a + 12b + 8 = 0$

$\Rightarrow \quad 8a + 3b + 2 = 0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$a = \dfrac{1}{2}, b = -2$

$\therefore \quad f(x) = \dfrac{x^4}{2} - 2x^3 + 2x^2$

$\Rightarrow \quad f(2) = 8 - 16 + 8 = 0$

13 $\phi'(x) = f'(x) + a$

$\because \quad \phi'(0) = 0 \Rightarrow f'(0) + a = 0$

$\Rightarrow \quad a = 0 \quad [\because f'(0) = 0]$

Also, $\phi'(0) > 0 \quad [\because f''(0) > 0]$

$\Rightarrow \phi(x)$ has relative minimum at $x = 0$ for all b, if $a = 0$

14 Let the number be x, then

$$f(x) = \frac{x}{x^2 + 16}$$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{(x^2 + 16) \cdot 1 - x(2x)}{(x^2 + 16)^2}$$

$$= \frac{x^2 + 16 - 2x^2}{(x^2 + 16)^2} = \frac{16 - x^2}{(x^2 + 16)^2} \quad \text{...(i)}$$

Put $f'(x) = 0$ for maxima or minima

$f'(x) = 0 \quad \Rightarrow \quad 16 - x^2 = 0$

$\Rightarrow \quad x = 4, -4$

Again, on differentiating w.r.t. x, we get

$$f''(x) = \frac{(x^2 + 16)^2(-2x) - (16 - x^2)\,2(x^2 + 16)\,2x}{(x^2 + 16)^4}$$

At $x = 4$, $f''(x) < 0$

$\therefore f(x)$ is maximum at $x = 4$.

and at $x = -4$, $f''(x) > 0$, $f(x)$ is minimum.

$\therefore$ Least value of

$$f(x) = \frac{-4}{16 + 16} = -\frac{1}{8}$$

15 Given, $f(x) = ax + \dfrac{b}{x}$

On differentiating w.r.t. x, we get

$$f'(x) = a - \frac{b}{x^2}$$

For maxima or minima, put $f'(x) = 0$

$\Rightarrow \quad x = \sqrt{\dfrac{b}{a}}$

Again, differentiating w.r.t. x, we get

$$f''(x) = \frac{2b}{x^3}$$

At $x = \sqrt{\dfrac{b}{a}}$, $f''(x) =$ positive

$\Rightarrow f(x)$ is minimum at $x = \sqrt{\dfrac{b}{a}}$.

$\therefore f(x)$ has the least value at $x = \sqrt{\dfrac{b}{a}}$.

16 $f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

As $\quad \dfrac{\tan x}{x} > 1, \forall\ x \neq 0$

$\therefore f(0 + h) > f(0)$ and $f(0 - h) > f(0)$

At $x = 0$, $f(x)$ attains minima.

Now, $\quad f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{\dfrac{\tan h}{h} - 1}{h} = \lim_{h \to 0} \frac{\tan h - h}{h^2}$$

[using L' Hospital's rule]

$= \lim_{h \to 0} \dfrac{\sec^2 h - 1}{2h} \quad [\because \tan^2\theta = \sec^2\theta - 1]$

$= \lim_{h \to 0} \dfrac{\tan^2 h}{2h^2} \cdot h = \dfrac{1}{2} \cdot 0 = 0$

Therefore, Statement II is true.

Hence, both statements are true but Statement II is not the correct explanation of Statement I.

17 Given, $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$

Now,

$f'(x) = 2\cos x(-\sin x) + \cos x$

$= -2\sin x \cos x + \cos x$

For maximum or minimum put $f'(x) = 0$

$\Rightarrow \quad -2\sin x\cos x + \cos x = 0$

$\Rightarrow \quad \cos x(-2\sin x + 1) = 0$

$\Rightarrow \quad \cos x = 0$ or $\sin x = \dfrac{1}{2}$

$\Rightarrow \quad x = \dfrac{\pi}{6}, \dfrac{\pi}{2}$

For absolute maximum and absolute minimum, we have to evaluate

$$f(0), f\left(\frac{\pi}{6}\right), f\left(\frac{\pi}{2}\right), f(\pi)$$

At $x = 0$,

$f(0) = \cos^2 0 + \sin 0 = 1^2 + 0 = 1$

At $x = \frac{\pi}{6}$, $f\left(\frac{\pi}{6}\right) = \cos^2\left(\frac{\pi}{6}\right) + \sin\frac{\pi}{6}$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4} = 1.25$$

At $x = \frac{\pi}{2}$,

$f\left(\frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{2}\right) + \sin\frac{\pi}{2} = 0^2 + 1 = 1$

At $x = \pi$,

$f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1$

Hence, the absolute maximum value of f is 1.25 occurring at $x = \frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x = 0, \frac{\pi}{2}$ and π.

Note If close interval is given, to determine global maximum (minimum), check the value at all critical points as well as end points of a given interval.

18 $\because \quad f(x) = \frac{x}{4 + x + x^2}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{4 + x + x^2 - x(1 + 2x)}{(4 + x + x^2)^2}$$

For maximum, put $f'(x) = 0$

$\Rightarrow \quad \frac{4 - x^2}{(4 + x + x^2)^2} = 0 \Rightarrow x = 2, -2$

Both the values of x are not in the interval $[-1, 1]$.

$\therefore \quad f(-1) = \frac{-1}{4 - 1 + 1} = \frac{-1}{4}$

$f(1) = \frac{1}{4 + 1 + 1} = \frac{1}{6}$ (maximum)

19 Given, $f(x) = x^2 \log x$

On differentiating w.r.t. x, we get

$f'(x) = (2\log x + 1)x$

For a maximum, put $f'(x) = 0$

$\Rightarrow \quad (2\log x + 1)x = 0$

$\Rightarrow \quad x = e^{-1/2}, 0$

$\because \quad 0 < e^{-1/2} < 1$

None of these critical points lies in the interval $[1, e]$.

So, we only compute the value of $f(x)$ at the end points 1 and e.

We have, $f(1) = 0, \quad f(e) = e^2$

Hence, greatest value of $f(x) = e^2$

20 Let $\quad f(x) = 1 + \frac{10}{3\left(x^2 + 3x + \frac{7}{3}\right)}$

$$= 1 + \frac{10}{3\left[\left(x + \frac{3}{2}\right)^2 + \frac{1}{12}\right]}$$

So, the maximum value of $f(x)$ at $x = -\frac{3}{2}$ is

$$f\left(-\frac{3}{2}\right) = 1 + \frac{10}{3\left(\frac{1}{12}\right)} = 1 + 40 = 41$$

21 Given, $f(x) = \sec x + \log\cos^2 x$

$\Rightarrow \quad f(x) = \sec x + 2\log(\cos x)$

Therefore,

$f'(x) = \sec x \tan x - 2\tan x$

$= \tan x(\sec x - 2)$

$f'(x) = 0$

$\Rightarrow \quad \tan x = 0$ or $\sec x = 2 \Rightarrow \cos x = \frac{1}{2}$

Therefore, possible values of x are $x = 0$, $x = \pi$ and $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$.

Again, $f''(x) = \sec^2 x(\sec x - 2) + \tan x(\sec x \tan x)$

$= \sec^3 x + \sec x \tan^2 x - 2\sec^2 x$

$= \sec x(\sec^2 x + \tan^2 x - 2\sec x)$

$\Rightarrow f''(0) = 1(1 + 0 - 2) = -1 < 0$

Therefore, $x = 0$ is a point of maxima.

$f''(\pi) = -1(1 + 0 + 2) = -3 < 0$

Therefore, $x = \pi$ is a point of maxima.

$f''\left(\frac{\pi}{3}\right) = 2(4 + 3 - 4) = 6 > 0$

Therefore, $x = \frac{\pi}{3}$ is a point of minima.

$f''\left(\frac{5\pi}{3}\right) = 2(4 + 3 - 4) = 6 > 0$

Therefore, $x = \frac{5\pi}{3}$ is a point of minima.

Maximum value of y at $x = 0$ is $1 + 0 = 1$.

Maximum value of y at $x = \pi$ is $-1 + 0 = -1$.

Minimum value of y at $x = \frac{\pi}{3}$ is $2 + 2\log\frac{1}{2} = 2(1 - \log 2)$.

Minimum value of y at $x = \frac{5\pi}{3}$ is $2 + 2\log\frac{1}{2} = 2(1 - \log 2)$.

22 Given, $f(x) = \sin 2x - x$

$\Rightarrow f'(x) = 2\cos 2x - 1$

Put $f'(x) = 0 \quad \Rightarrow \quad \cos 2x = \frac{1}{2}$

$\Rightarrow \quad 2x = -\frac{\pi}{3}$ or $\frac{\pi}{3} \Rightarrow x = -\frac{\pi}{6}$ or $\frac{\pi}{6}$

Now, $f\left(-\frac{\pi}{2}\right) = \sin(-\pi) + \frac{\pi}{2} = \frac{\pi}{2}$

$f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{2\pi}{6}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

and $f\left(\frac{\pi}{2}\right) = \sin(\pi) - \frac{\pi}{2} = -\frac{\pi}{2}$

Clearly, $\frac{\pi}{2}$ is the greatest value and $-\frac{\pi}{2}$ is the least.

Therefore, difference $= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

23 Given, $y = x^{5/2}$

$\therefore \quad \frac{dy}{dx} = \frac{5}{2}x^{3/2}, \frac{d^2y}{dx^2} = \frac{15}{4}x^{1/2}$

At $x = 0$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$

and $\frac{d^3y}{dx^3}$ is not defined,

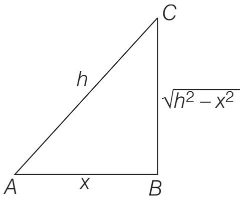
when $x = 0$, $y = 0$

$\therefore (0, 0)$ is a point of inflection.

24 Area of triangle, $\Delta = \frac{1}{2}x\sqrt{h^2 - x^2}$

$$\frac{d\Delta}{dx} = \frac{1}{2}\left[\sqrt{h^2 - x^2} + \frac{x(-2x)}{2\sqrt{h^2 - x^2}}\right] = 0$$

$\Rightarrow \quad x = \frac{h}{\sqrt{2}}$



$\Rightarrow \quad \frac{d^2\Delta}{dx^2} < 0$ at $x = \frac{h}{\sqrt{2}}$

$\therefore \quad \Delta = \frac{1}{2} \times \frac{h}{\sqrt{2}}\sqrt{h^2 - \frac{h^2}{2}} = \frac{h^2}{4}$

25 Let the equation of drawn line be $\frac{x}{a} + \frac{y}{b} = 1$, where $a > 3$, $b > 4$, as the line passes through $(3, 4)$ and meets the positive direction of coordinate axes.

We have, $\frac{3}{a} + \frac{4}{b} = 1 \Rightarrow \quad b = \frac{4a}{(a - 3)}$

Now, area of ΔAOB,

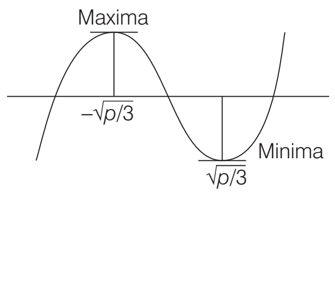
$$\Delta = \frac{1}{2}ab = \frac{2a^2}{(a - 3)}$$

$$\frac{d\Delta}{da} = \frac{2a(a - 6)}{(a - 3)^2}$$

Clearly, $a = 6$ is the point of minima for Δ.

Thus, $\quad \Delta_{\min} = \frac{2 \times 36}{3} = 24$ sq units

26 Let $f(x) = x^3 - px + q$



Then, $\quad f'(x) = 3x^2 - p$

Put $\quad f'(x) = 0$

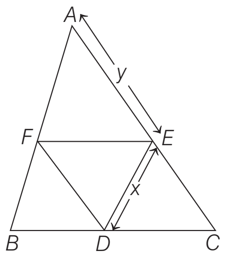
$\Rightarrow \quad x = \sqrt{\frac{p}{3}}, -\sqrt{\frac{p}{3}}$

Now, $\quad f''(x) = 6x$

At $\quad x = \sqrt{\frac{p}{3}}$, $f''(x) = 6\sqrt{\frac{p}{3}} > 0$ [minima]

and at $x = -\sqrt{\frac{p}{3}}$ $f''(x) < 0$ [maxima]

27 We have, $AF \parallel DE$ and $AE \parallel FD$



Now, in ΔABC and ΔEDC,

$\angle DEC = \angle BAC$, $\angle ACB$ is common.

$\Rightarrow \quad \Delta ABC \cong \Delta EDC$

Now, $\frac{b-y}{b} = \frac{x}{c} \Rightarrow x = \frac{c}{b}(b-y)$

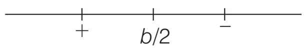
Now, S = Area of parallelogram $AFDE$ = 2 (area of ΔAEF)

$$\Rightarrow \quad S = 2\left(\frac{1}{2}xy\sin A\right)$$

$$= \frac{c}{b}(b-y)y\sin A$$

$$\frac{dS}{dy} = \left(\frac{c}{b}\sin A\right)(b-2y)$$

Sign scheme of $\frac{dS}{dy}$,



Hence, S is maximum when $y = \frac{b}{2}$.

$$\therefore \ S_{\max} = \frac{c}{b}\left(\frac{b}{2}\right) \times \frac{b}{2}\sin A$$

$$= \frac{1}{2}\left(\frac{1}{2}bc\sin A\right) = \frac{1}{2}(\text{area of } \Delta ABC)$$

28 We have,

$$\frac{dy}{dt} = 6t^2 - 30t + 24 = 6(t-1)(t-4)$$

and $\frac{dx}{dt} = 6t - 18 = 6(t-3)$

Thus, $\frac{dy}{dx} = \frac{(t-1)(t-4)}{(t-3)}$

which indicates that $t = 1, 3$ and 4 are the critical points of $y = f(x)$.

Now, $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$

$$= \frac{t^2 - 6t + 11}{(t-3)^2} \times \frac{1}{6(t-3)}$$

At $(t = 1)$, $\frac{d^2y}{dx^2} < 0$

$\Rightarrow t = 1$ is a point of local maxima.

At $(t = 4)$, $\frac{d^2y}{dx^2} > 0$

$\Rightarrow t = 4$ is a point of local minima.

At $(t = 3)$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are not defined and change its sign.

$\frac{d^2y}{dx^2}$ is unknown in the vicinity of $t = 3$, thus $t = 3$ is a point of neither maxima nor minima.

Finally, maximum and minimum values of expression $y = f(x)$ are 46 and -6, respectively.

29 Let α and β be the roots of the equation

$$x^2 - (a-2)x - a + 1 = 0$$

Then, $\quad \alpha + \beta = a - 2, \alpha\beta = -a + 1$

Let $\quad z = \alpha^2 + \beta^2$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (a-2)^2 + 2(a-1)$$

$$= a^2 - 2a + 2$$

$\Rightarrow \quad \frac{dz}{da} = 2a - 2$

Put $\frac{dz}{da} = 0$, then

$\Rightarrow \quad a = 1$

$\therefore \quad \frac{d^2z}{da^2} = 2 > 0$

So, z has minima at $a = 1$.

So, $\alpha^2 + \beta^2$ has least value for $a = 1$.

This is because we have only one stationary value at which we have minima.

Hence, $a = 1$.

30 Any tangent to the ellipse is $\frac{x}{4}\cos t + \frac{y}{3}\sin t = 1$, where the point of contact is $(4\cos t, 3\sin t)$

or $\quad \frac{x}{4\sec t} + \frac{y}{3\,\text{cosec}\,t} = 1$,

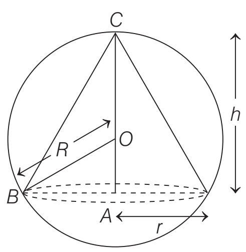
It means the axes $Q\ (4\sec t, 0)$ and $R(0, 3\,\text{cosec}\,t)$.

$\therefore$ The distance of the line segment QR is

$$QR^2 = D = 16\sec^2 t + 9\,\text{cosec}^2 t$$

So, the minimum value of D is $(4+3)^2$ or $QR = 7$.

31 Let S be the curved surface area of a cone.



$OA = AC - OC = h - R$

In ΔOAB, $R^2 = r^2 + (h-R)^2$

$\Rightarrow \quad r = \sqrt{2Rh - h^2}$

$\therefore \quad S = \pi r l = \pi(\sqrt{2Rh - h^2})(\sqrt{h^2 + r^2})$

$$= (\pi\sqrt{2Rh - h^2})(\sqrt{2Rh})$$

Let $S^2 = P$

$\therefore \quad P = \pi^2\, 2R(2Rh^2 - h^3)$

Since, S is maximum, if P is maximum, then

$$\frac{dP}{dh} = 2\pi^2 R(4Rh - 3h^2) = 0$$

$\therefore \quad h = 0, \frac{4R}{3}$

Again, on differentiating $\frac{dP}{dh}$, we get

$$\frac{d^2P}{dh^2} = 2\pi^2 R(4R - 6h)$$

$$\frac{d^2P}{dh^2} < 0 \text{ at } h = \frac{4R}{3}$$

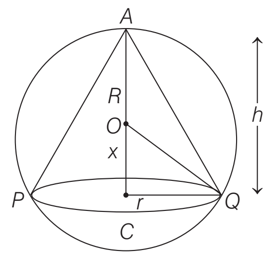
32 Let $OC = x$, $CQ = r$

Now, $OA = R$ [given]

Height of the cone $= h = x + R$

$\therefore$ Volume of the cone

$$= V = \frac{1}{3}\pi r^2 h \qquad \ldots\text{(i)}$$



Also, in right angled ΔOCQ,

$$OC^2 + CQ^2 = OQ^2$$

$\Rightarrow \quad x^2 + r^2 = R^2$

$\Rightarrow \quad r^2 = R^2 - x^2 \qquad \ldots\text{(ii)}$

From Eqs. (i) and (ii),

$$V = \frac{1}{3}\pi(R^2 - x^2)(x + R) \qquad \ldots\text{(iii)}$$

$[\because h = x + R]$

On differentiating Eq. (iii) w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{3}\pi[(R^2 - x^2) - 2x(x + R)]$$

$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R^2 - x^2 - 2x^2 - 2xR)$

$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R^2 - 2xR - 3x^2)$

$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R - 3x)(R + x) \qquad \ldots\text{(iv)}$

For maxima, put $\quad \frac{dV}{dx} = 0$

$\Rightarrow \quad \frac{\pi}{3}(R - 3x)(R + x) = 0$

$\Rightarrow \quad x = \frac{R}{3}$ or $x = -R \Rightarrow x = \frac{R}{3}$

[since, x cannot be negative]

On differentiating Eq. (iv) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}[(-3)(R+x)+(R-3x)]$$

$$= \frac{\pi}{3}(-2R-6x) = -\frac{\pi}{3}(2R+6x)$$

At $x = \frac{R}{3}$, $\frac{d^2V}{dx^2} = \frac{-\pi}{3}\left(2R+\frac{6R}{3}\right)$

$$= -\frac{4\pi}{3}R < 0$$

So, V has a local maxima at $x = R/3$.

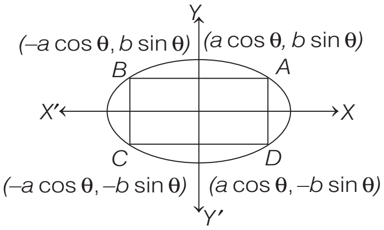
Now, on substituting the value of x in Eq. (iii), we get

$$V = \frac{\pi}{3}\left(R^2-\frac{R^2}{9}\right)\left(R+\frac{R}{3}\right)$$

$$= \frac{\pi}{3}\cdot\frac{8R^2}{9}\cdot\frac{4R}{3} = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$\Rightarrow$ $V = \frac{8}{27}\times$ Volume of sphere

33



Area of rectangle $ABCD$

$= (2a\cos\theta)(2b\sin\theta) = 2ab\sin 2\theta$

Hence, area of greatest rectangle is equal to $2ab$ when $\sin 2\theta = 1$.

34 Let $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2}$$

For maxima and minima, put $f'(x) = 0$

$\Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$

Now, $f''(x) = \frac{2}{x^3}$

At $x = 1$, $f''(x) = +\text{ve}$ [minima]

and at $x = -1$, $f''(x) = -\text{ve}$ [maxima]

Thus, $f(x)$ attains minimum value at $x = 1$.

35 Given that, $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{3}\left[\frac{1}{(x+1)^{2/3}} - \frac{1}{(x-1)^{2/3}}\right]$$

$$= \frac{(x-1)^{2/3}-(x+1)^{2/3}}{3(x^2-1)^{2/3}}$$

Clearly, $f'(x)$ does not exist at $x = \pm 1$.

Now, put $f'(x) = 0$, then

$$(x-1)^{2/3} = (x+1)^{2/3} \Rightarrow x = 0$$

At $x = 0$

$f(x) = (0+1)^{1/3} - (0-1)^{1/3} = 2$

Hence, the greatest value of $f(x)$ is 2.

36 $\because$ $y^2 = 8x$. But $y^2 = 4ax$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

Any point on parabola is $(at^2, 2at)$, i.e., $(2t^2, 4t)$.

For its minimum distance from the circle means its distance from the centre $(0, -6)$ of the circle.

Let $z = (2t^2)^2 + (4t+6)^2$

$= 4(t^4 + 4t^2 + 12t + 9)$

$\therefore$ $\frac{dz}{dt} = 4(4t^3 + 8t + 12)$

$\Rightarrow$ $16(t^3 + 2t + 3) = 0$

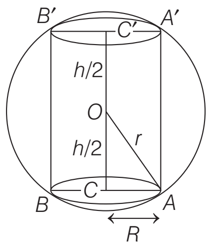
$\Rightarrow$ $(t+1)(t^2 - t + 3) = 0$

$\Rightarrow$ $t = -1$

$\Rightarrow$ $\frac{d^2z}{dt^2} = 16(3t^2+2) > 0$, hence minimum.

So, point is $(2, -4)$.

37 We know that, volume of cylinder, $V = \pi R^2 h$



In ΔOCA, $r^2 = \left(\frac{h}{2}\right)^2 + R^2$

$\Rightarrow$ $R^2 = r^2 - \frac{h^2}{4}$

$\therefore$ $V = \pi\left(r^2 - \frac{h^2}{4}\right)h$

$\Rightarrow$ $V = \pi r^2 h - \frac{\pi}{4}h^3$ ...(i)

On differentiating Eq. (i) both sides w.r.t. h, we get

$$\frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}$$

$\Rightarrow$ $\frac{d^2V}{dh^2} = \frac{-3\pi h}{2}$

For maximum or minimum value of V,

$\frac{dV}{dh} = 0 \Rightarrow \pi r^2 - \frac{3\pi h^2}{4} = 0$

$\Rightarrow$ $h^2 = \frac{4r^2}{3} \Rightarrow h = \frac{2}{\sqrt{3}}r$

Now, $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{2r}{\sqrt{3}}} = -\sqrt{3}\pi r < 0$

Thus, V is maximum when $h = \frac{2r}{\sqrt{3}}$, then

$$R^2 = r^2 - \frac{h^2}{4} = r^2 - \frac{1}{4}\left(\frac{2r}{\sqrt{3}}\right)^2 = \frac{2}{3}r^2$$

$$\text{Max } V = \pi R^2 h = \frac{4\pi r^3}{3\sqrt{3}}$$

38 Let $f(x) = -x^3 + 3x^2 + 9x - 27$

The slope of this curve

$$f'(x) = -3x^2 + 6x + 9$$

Let $g(x) = f'(x) = -3x^2 + 6x + 9$

On differentiating w.r.t. x, we get

$$g'(x) = -6x + 6$$

For maxima or minima put $g'(x) = 0$

$\Rightarrow$ $x = 1$

Now, $g''(x) = -6 < 0$ and hence, at $x = 1$, $g(x)$ (slope) will have maximum value.

$\therefore [g(1)]_{\max} = -3\times 1 + 6(1) + 9 = 12$

39 Given,

$ab = 2a + 3b \Rightarrow (a-3)b = 2a$

$\Rightarrow$ $b = \frac{2a}{a-3}$

Now, let $z = ab = \frac{2a^2}{a-3}$

On differentiating w.r.t. x, we get

$$\frac{dz}{da} = \frac{2[(a-3)2a - a^2]}{(a-3)^2} = \frac{2[a^2-6a]}{(a-3)^2}$$

For a minimum, put $\frac{dz}{da} = 0$

$\Rightarrow$ $a^2 - 6a = 0$

$\Rightarrow$ $a = 0, 6$

At $a = 6$, $\frac{d^2z}{da^2}$ = positive

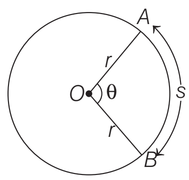
When $a = 6$, $b = 4$

$\therefore$ $(ab)_{\min} = 6\times 4 = 24$

40 $\because$ Perimeter of a sector $= p$

Let AOB be the sector with radius r.

If angle of the sector be θ radians, then area of sector,



$$A = \frac{1}{2}r^2\theta \quad \text{...(i)}$$

and length of arc, $s = r\theta \Rightarrow \theta = \frac{s}{r}$

$\therefore$ Perimeter of the sector

$$p = r + s + r = 2r + s \quad \text{...(ii)}$$

On substituting $\theta = \frac{s}{r}$ in Eq. (i), we get

$$A = \left(\frac{1}{2}r^2\right)\left(\frac{s}{r}\right) = \frac{1}{2}rs \Rightarrow s = \frac{2A}{r}$$

Now, on substituting the value of s in Eq. (ii), we get

$$p = 2r + \left(\frac{2A}{r}\right) \Rightarrow 2A = pr - 2r^2$$

On differentiating w.r.t. r, we get

$$2\frac{dA}{dr} = p - 4r$$

For the maximum area, put

$$\frac{dA}{dr} = 0$$

$\Rightarrow$ $p - 4r = 0$

$\Rightarrow$ $r = \frac{p}{4}$

SESSION 2

1 Let radius vector is r.

$\therefore\ r^2 = x^2 + y^2$

$\Rightarrow r^2 = \dfrac{a^2y^2}{y^2 - b^2} + y^2 \quad \left(\because \dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1\right)$

For minimum value of r,

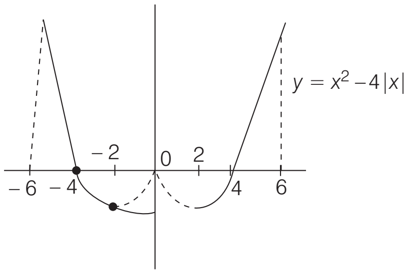
$\dfrac{d(r^2)}{dy} = 0 \Rightarrow \dfrac{-2yb^2a^2}{(y^2 - b^2)^2} + 2y = 0$

$\Rightarrow \quad y^2 = b(a + b)$

$\therefore \quad x^2 = a(a + b)$

$\Rightarrow \quad r^2 = (a + b)^2 \Rightarrow r = a + b$

2 Bold line represents the graph of $y = g(x)$, clearly $g(x)$ has neither a point of local maxima nor a point of local minima.



3 Clearly, $f(x)$ in increasing just before $x = 3$ and decreasing after $x = 3$. For $x = 3$ to be the point of local maxima.

$f(3) \ge f(3 - 0)$

$\Rightarrow -15 \ge 12 - 27 + \log(a^2 - 3a + 3)$

$\Rightarrow \quad 0 < a^2 - 3a + 3 \le 1 \Rightarrow 1 \le a \le 2$

4 (Slope) $f'(x) = e^x \cos x + \sin x e^x$

$= e^x\sqrt{2}\sin(x + \pi/4)$

$f''(x) = \sqrt{2}e^x\{\sin(x + \pi/4) + \cos(x + \pi/4)\}$

$= 2e^x \cdot \sin(x + \pi/2)$

For maximum slope, put $f''(x) = 0$

$\Rightarrow \quad \sin(x + \pi/2) = 0$

$\Rightarrow \quad \cos x = 0$

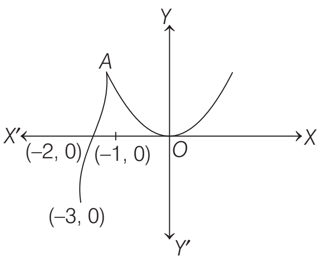
$\therefore \quad x = \pi/2, 3\pi/2$

$f'''(x) = 2e^x \cos(x + \pi/2)$

$f'''(\pi/2) = 2e^x \cdot \cos\pi = -\text{ve}$

Maximum slope is at $x = \pi/2$.

5 $f'(x) = \begin{cases} 3(2 + x)^2, & -3 < x \le -1 \\ \dfrac{2}{3}x^{-1/3}, & -1 < x < 2 \end{cases}$

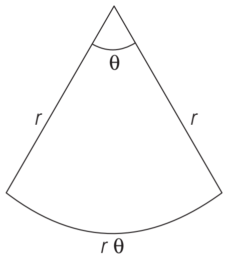


Clearly, $f'(x)$ changes its sign at $x = -1$ from positive to negative and so $f(x)$ has local maxima at $x = -1$.

Also, $f'(0)$ does not exist but $f'(0^-) < 0$ and $f'(0^+) > 0$. It can only be inferred that $f(x)$ has a possibility of a minimum at $x = 0$. Hence, it has one local maxima at $x = -1$ and one local minima at $x = 0$

So, total number of local maxima and local minima is 2.

6 Total length $= 2r + r\theta = 20$



$\Rightarrow \quad \theta = \dfrac{20 - 2r}{r}$

Now, area of flower-bed,

$A = \dfrac{1}{2}r^2\theta$

$\Rightarrow \quad A = \dfrac{1}{2}r^2\left(\dfrac{20 - 2r}{r}\right)$

$\Rightarrow \quad A = 10r - r^2$

$\therefore \quad \dfrac{dA}{dr} = 10 - 2r$

For maxima or minima, put $\dfrac{dA}{dr} = 0$.

$\Rightarrow \quad 10 - 2r = 0 \Rightarrow r = 5$

$\therefore \quad A_{\max} = \dfrac{1}{2}(5)^2\left[\dfrac{20 - 2(5)}{5}\right]$

$= \dfrac{1}{2} \times 25 \times 2 = 25$ sq m

7 Let $\quad c = av + \dfrac{b}{v}$ ...(i)

When $v = 30$ km/ h, then $c =$ ₹ 75

$\therefore \quad 75 = 30a + \dfrac{b}{30}$ ...(ii)

When $v = 40$ km/h, then $c =$ ₹ 65

$\therefore \quad 65 = 40a + \dfrac{b}{40}$ ...(iii)

On solving Eqs. (ii) and (iii), we get

$a = \dfrac{1}{2}$ and $b = 1800$

On differentiating w.r.t. v in Eq. (i),

$\dfrac{dc}{dv} = a - \dfrac{b}{v^2}$

For maximum or minimum c,3

$\dfrac{dc}{dv} = 0 \Rightarrow v = \pm\sqrt{\dfrac{b}{a}}$

$\Rightarrow \quad \dfrac{d^2c}{dv^2} = \dfrac{2b}{v^3}$ at $v = \sqrt{\dfrac{b}{a}}, \dfrac{dx}{dv^2} > 0$

So, at $v = \sqrt{\dfrac{b}{a}}$ the speed is most economical.

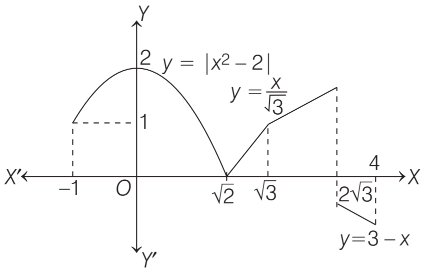
$\therefore$ Most economical speed is

$c = a\sqrt{\dfrac{b}{a}} + b\sqrt{\dfrac{a}{b}} = 2\sqrt{ab}$

$c = 2\sqrt{\dfrac{1}{2} \times 1800} = 2 \times 30$

$\Rightarrow \quad c = 60$

8 $f(x) = \begin{cases} |x^2 - 2|, & -1 \le x < \sqrt{3} \\ \dfrac{x}{\sqrt{3}}, & \sqrt{3} \le x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \le x \le 4 \end{cases}$



From the above graph,

Maximum occurs at $x = 0$ and minimum at $x = 4$.

9 $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\left(\dfrac{1}{x}\right)\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$

Let $\quad g(x) = x^3 + x^2 + 3x + \sin x$

$g'(x) = 3x^2 + 2x + 3 + \cos x$

$= 3\left(x^2 + \dfrac{2x}{3} + 1\right) + \cos x$

$= 3\left\{\left(x + \dfrac{1}{3}\right)^2 + \dfrac{8}{9}\right\} + \cos x > 0$

and $2 < 3 + \sin\left(\dfrac{1}{x}\right) < 4$

Hence, minimum value of $f(x)$ is 0 at $x = 0$.

Hence, number of points = 1

10 According to given information, we have

Perimeter of square + Perimeter of circle = 2 units

$\Rightarrow \quad 4x + 2\pi r = 2$

$\Rightarrow \quad r = \dfrac{1 - 2x}{\pi}$ ...(i)

Now, let A be the sum of the areas of the square and the circle. Then,

$A = x^2 + \pi r^2$

$= x^2 + \pi\dfrac{(1 - 2x)^2}{\pi^2}$

$\Rightarrow \quad A(x) = x^2 + \dfrac{(1 - 2x)^2}{\pi}$

Now, for minimum value of $A(x)$,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad 2x + \frac{2(1-2x)}{\pi} \cdot (-2) = 0$$

$$\Rightarrow \quad x = \frac{2-4x}{\pi}$$

$$\Rightarrow \quad \pi x + 4x = 2$$

$$\Rightarrow \quad x = \frac{2}{\pi + 4} \quad \text{...(ii)}$$

Now, from Eq. (i), we get

$$r = \frac{1 - 2 \cdot \frac{2}{\pi + 4}}{\pi}$$

$$= \frac{\pi + 4 - 4}{\pi(\pi + 4)} = \frac{1}{\pi + 4} \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$x = 2r$$

11 We have,

$$f(x) = x^2 + \frac{1}{x^2} \text{ and } g(x) = x - \frac{1}{x}$$

$$\Rightarrow \quad h(x) = \frac{f(x)}{g(x)}$$

$$\therefore \quad h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{x - \frac{1}{x}}$$

$$\Rightarrow h(x) = \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}}$$

$$x - \frac{1}{x} > 0, \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in [2\sqrt{2}, \infty)$$

$$x - \frac{1}{x} < 0,$$

$$\left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in (-\infty, 2\sqrt{2}]$$

$\therefore$ Local minimum value is $2\sqrt{2}$.

12 Consider the function

$$f(x) = \frac{x^2}{(x^3 + 200)}$$

$$f'(x) = x\frac{(400 - x^3)}{(x^3 + 200)^2} = 0$$

when $x = (400)^{1/3}$, ($\because x \neq 0$)

$$x = (400)^{1/3} - h \Rightarrow f'(x) > 0$$

$$x = (400)^{1/3} + h \Rightarrow f'(x) < 0$$

$\therefore f(x)$ has maxima at $x = (400)^{1/3}$

Since, $7 < (400)^{1/3} < 8$, either a_7 or a_8 is the greatest term of the sequence.

$$\because \quad a_7 = \frac{49}{543}$$

$$\text{and} \quad a_8 = \frac{8}{89}$$

$$\text{and} \quad \frac{49}{543} > \frac{8}{89}$$

$\Rightarrow \quad a_7 = \frac{49}{543}$ is the greatest term.

13 $f(x) = x^3 - 3(7-a)x^2 - 3(9-a^2)x + 2$

$f'(x) = 3x^2 - 6(7-a)x - 3(9-a^2)$

For real root $D \geq 0$,

$\Rightarrow \quad 49 + a^2 - 14a + 9 - a^2 \geq 0$

$\Rightarrow a \leq \frac{58}{14}$

For local minimum

$f''(x) = 6x - 6(7-a) > 0$

$\Rightarrow \quad 7 - x$

has x must be negative

$\Rightarrow \quad 7 - a < 0$

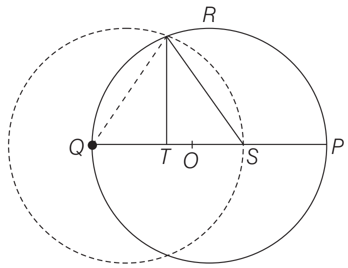
$\Rightarrow \quad a > 7$

Thus constradictory, *i.e.*, for real roots $a \leq \frac{58}{14}$ and for negative point of local minimum $a > 7$.

No possible values of a.

14 From the given figure coordinate of Q is $(-1, 0)$.

The equation of circle centre at Q with variable radius r is



$$(x+1)^2 + y^2 = r^2 \quad \text{...(i)}$$

This circle meets the line segment QP at S, where $QS = r$

It meets the circle $x^2 + y^2 = 1$ at ...(ii)

$$R\left(\frac{r^2 - 2}{2}, \frac{r}{2}\sqrt{4 - r^2}\right)$$

[on solving Eqs. (i) and (ii)]

A = Area of ΔQSR

$$= \frac{1}{2} \times QS \times RT$$

$$= \frac{1}{2} r \left(\frac{r}{2} \cdot \sqrt{4 - r^2}\right)$$

[since, RT is the y-coordinate of R]

$$= \frac{1}{4}[r^2\sqrt{4 - r^2}]$$

$$\therefore \quad \frac{dA}{dr} = \frac{1}{4}\left\{2r\sqrt{4 - r^2} + \frac{r^2(-r)}{\sqrt{4 - r^2}}\right\}$$

$$= \frac{\{2r(4 - r^2) - r^3\}}{4\sqrt{4 - r^2}}$$

$$= \frac{8r - 3r^3}{4\sqrt{4 - r^2}}$$

$\frac{dA}{dr} = 0$, when $r(8 - 3r^2) = 0$ giving

$$r = \sqrt{\frac{8}{3}}$$

$$\Rightarrow \frac{d^2A}{dr^2} = \frac{4\sqrt{4 - r^2}(8 - 9r^2) - (8r - 3r^3)\frac{(-4r)}{\sqrt{4 - r^2}}}{16(4 - r^2)}$$

When $\quad r = \sqrt{\frac{8}{3}}$, then $\frac{d^2A}{dr^2} < 0$

Hence, A is maximum when $r = \sqrt{\frac{8}{3}}$.

Then, maximum area

$$= \frac{8}{4 \times 3}\sqrt{4 - \frac{8}{3}} = \frac{4\sqrt{3}}{9} \text{ sq unit}$$

15 Given,

$P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow \quad P'(x) = 4x^3 + 3ax^2 + 2bx + c$

Since, $x = 0$ is a solution for $P'(x) = 0$, then

$$c = 0$$

$$\therefore \quad P(x) = x^4 + ax^3 + bx^2 + d \quad \text{...(i)}$$

Also, we have $P(-1) < P(1)$

$\Rightarrow 1 - a + b + d < 1 + a + b + d$

$\Rightarrow \quad a > 0$

Since, $P'(x) = 0$, only when $x = 0$ and $P(x)$ is differentiable in $(-1, 1)$, we should have the maximum and minimum at the points $x = -1, 0$ and 1.

Also, we have $P(-1) < P(1)$

$\therefore$ Maximum of $P(x) =$ Max $\{P(0), P(1)\}$ and minimum of $P(x) =$ Min $\{P(-1), P(0)\}$

In the interval $[0, 1]$,

$$P'(x) = 4x^3 + 3ax^2 + 2bx$$

$$= x(4x^2 + 3ax + 2b)$$

Since, $P'(x)$ has only one root $x = 0$, then $4x^2 + 3ax + 2b = 0$ has no real roots.

$$\therefore \quad (3a)^2 - 32b < 0$$

$$\Rightarrow \quad \frac{9a^2}{32} < b$$

$$\therefore \quad b > 0$$

Thus, we have $a > 0$ and $b > 0$.

$\therefore P'(x) = 4x^3 + 3ax^2 + 2bx > 0$, $\forall \ x \in (0, 1)$

Hence, $P(x)$ is increasing in $[0, 1]$.

$\therefore$ Maximum of $P(x) = P(1)$

Similarly, $P(x)$ is decreasing in $[-1, 0]$.

Therefore, minimum $P(x)$ does not occur at $x = -1$.