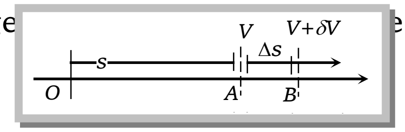
4.1 Application in Mechanics and dy/dx as a Rate Measure

4.1.1 Velocity and Acceleration in Rectilinear Motion

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change ect to time.



Let a particle A moves rectilinearly as shown in figure.

Let s be the displacement from a fixed point O along the path at time t; s is considered to be positive on right of the point O and negative on the left of it.

Also, Δs is positive when s increases *i.e.*, when the particle moves towards right.

Thus, if Δs be the increment in s in time Δt. The **average velocity** in this interval is $\frac{\Delta s}{\Delta t}$

And the instantaneous velocity *i.e.,* velocity at time t is $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$

If the velocity varies, then there is change of velocity Δv in time Δt.

Hence, the acceleration at time $t = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$

Example: 1 The distance travelled s (in metre) by a particle in t second is given by $s = t^3 + 2t^2 + t$. The speed of the particle after 1 *sec.* will be

(a) 8 cm/sec. (b) 6 cm/sec. (c) 2 cm/sec (d) None of these

Solution: (a) $s = t^3 + 2t^2 + t$, $v = \frac{ds}{dt} = 3t^2 + 4t + 1$

Speed of the particle after 1 *second*

$$v_{(t=1)} = \left(\frac{ds}{dt}\right)_{(t=1)} = 3 \times 1^2 + 4 \times 1 + 1 = 8 cm/\sec.$$

Example: 2 A particle moves in a straight line in such a way that its velocity at any point is given by $v^2 = 2 - 3x$, where x is measured from a fixed point. The acceleration is

(a) Zero (b) Uniform (c) Non-uniform (d) Indeterminate

Solution: (b) Velocity, $v^2 = 2 - 3x$

Differentiating with respect to t, we get

$$2v\frac{dv}{dt} = -3.\frac{dx}{dt} \Rightarrow 2v\frac{dv}{dt} = -3v \Rightarrow \frac{dv}{dt} = -\frac{3}{2}$$

ition of a point in time 't' is given by $x = a + bt - ct^2$, $y = at + bt^2$. Its acceleration at time 't' is **[MP PET 2**
Hence, acceleration is uniform.

c (b) $(b+c)$ (c) $2b-2c$ (d) $2\sqrt{b^2+c^2}$

Solution: (d) Acceleration in x-direction = $\frac{d^2x}{dt^2} = -2c$ and acceleration in y-direction = $\frac{d^2y}{dt^2} = 2b$

Resultant acceleration is = $\sqrt{(-2c)^2 + (2b)^2}$ = $2\sqrt{b^2+c^2}$

Example: 4 If the path of a moving point is the curve $x = at \quad y = b \sin at$, then its acceleration at any instant **[SCRA 1996]**

(a) Is constant
(b) Varies as the distance from the axis of x
(c) Varies as the distance from the axis of y
(d) Varies as the of the point from the origin

Solution: (c) $\frac{dx}{dt} = v_x = a \Rightarrow \frac{d^2x}{dt^2} = 0 = a_x$

a_x is acceleration in x-axis

$\frac{d^2y}{dt^2} = -ba^2 \sin at \Rightarrow a_y = -a^2 y$

Hence, a_y changes as y changes.

Example: 5 A stone thrown vertically upwards from the surface of the moon at velocity of 24 m/sec. reaches a height of $s = 24t - 0.8t^2 m$ after t sec. The acceleration due to gravity in m/sec^2 at the surface of the moon is **[MP PET 1992]**

(a) 0.8 (b) 1.6 (c) 2.4 (d) 4.9

Solution: (b) $\frac{ds}{dt}$ = velocity = 24 = 24 – 1.6 t

So acceleration at t, is $\left[\frac{d^2s}{dt^2}\right] = -1.6$

As stone is thrown upwards, so acceleration due to gravity (which acts downwards) = 1.6.

4.1.2 Derivative as the Rate of Change

If a variable quantity y is some function of time t *i.e.*, $y = f(t)$, then small change in time Δt have a corresponding change Δy in y.

Thus, the average rate of change = $\frac{\Delta y}{\Delta t}$

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to t.

i.e., $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$

Hence, it is clear that the rate of change of any variable with respect to some other variable is derivative of first variable with respect to other variable.

Note : ❑ The differential coefficient of y with respect to x *i.e*, $\frac{dy}{dx}$ is nothing but the rate of increase of y relative to x.

Example: 6 The rate of change of the surface area of a sphere of radius r when the radius is increasing at the rate of 2cm/sec is proportional to

(a) $\frac{l}{r}$ (b) $\frac{l}{r^2}$ (c) r (d) r^2

$\therefore$ — $\frac{}{dt}$ r r

$$\frac{ds}{dt} = 4\pi \times 2r \, dr = 8\pi \times 2 = 16\pi \Rightarrow \frac{ds}{dt} \propto r .$$

Example: 7 If the volume of a spherical balloon is increasing at the rate of 900 cm^2/sec. then the rate of change of radius of balloon at instant when radius is 15 *cm* [in *cm*/*sec*]

(a) $\frac{22}{7}$ (b) 22 (c) $\frac{7}{22}$ (d) None of these

Solution: (c) $V = \frac{4}{3}\pi r^3$

Differentiate with respect to t

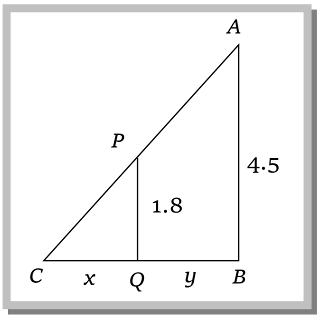
$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 . \frac{dr}{dt} \Rightarrow \frac{dr}{dt} \Rightarrow \frac{1}{4\pi r^2} . \frac{dV}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4 \times \pi \times 15 \times 15} \times 900 = \frac{1}{\pi} = \frac{7}{22} .$$

Example: 8 A man of height 1.8 *m* is moving away from a lamp post at the rate of 1.2 *m*/*sec*. If the height of the lamp post be 4.5 *meter*, then the rate at which the shadow of the man is lengthening

(a) 0.4 *m*/sec (b) 0.8 *m*/sec. (c) 1.2 *m*/*sec*. (d) None of these

Solution: (b) $\frac{dy}{dt} = 1.2$ According to the figure,



$x = \frac{2}{3} y$

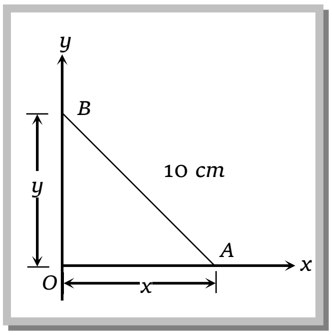
$\Rightarrow \frac{dx}{dt} = \frac{2}{3} . \frac{dy}{dt}$

$\Rightarrow$ Rate of length of shadow $\frac{dx}{dt} = 0.8\, m/s$.

Example: 9 A 10 *cm* long rod *AB* moves with its ends on two mutually perpendicular straight lines *OX* and *OY*. If the end *A* be moving at the rate of 2 *cm*/*sec*. then when the distance of *A* from *O* is 8 *cm*, the rate at which the end *B* is moving, is **[SCRA 1996]**

(a) $\frac{8}{3}$ *cm*/*sec* (b) $\frac{4}{3} cm/\sec$ (c) $\frac{2}{9} cm/\sec$. (d) None of these

Solution: (a) By figure, $x^2 + y^2 = 100$ ......(i)



$\Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$ .....(ii)

$x = 8$

Therefore by (i) and (ii), $\frac{dy}{dt} = -\frac{16}{6} = -\frac{8}{3} cm/\sec$.

$\therefore$ *B* is moving at the rate $\frac{8}{3} cm/\sec$.

Assignment

Application in Mechanics

Basic Level

1. The displacement of a particle in time t is given by $s = 2t^2 - 3t + 1$. The acceleration is

(a) 1 (b) 3 (c) 4 (d) 5

2. A stone is falling freely and describes a distance s in t seconds given by equation $s = \frac{1}{2}gt^2$. The acceleration of the stone is

(a) Uniform (b) Zero (c) Non-uniform (d) Indeterminate

3. The velocity of a particle at time t is given by the relation $v = 6t - \frac{t^2}{6}$. The distance travelled in 3 *seconds* is, if $s = 0$ at $t = 0$

(a) $\frac{39}{2}$ (b) $\frac{57}{2}$ (c) $\frac{51}{2}$ (d) $\frac{33}{2}$

4. The equation of motion of a car is $s = t^2 - 2t$, where t is measured in *hours* and s in *kilometers*. when the distance travelled by the car is 15 *km*, the velocity of the car is

(a) $2 km/h$ (b) $4 km/h$ (c) $6 km/h$ (d) $8 km/h$

5. A particle is moving in a straight line according as $s = 45t + 11t^2 - t^3$, then the time when it will come to rest, is

(a) – 9 *seconds* (b) $\frac{5}{3}$ *seconds* (c) 9 *seconds* (d) $-\frac{5}{3}$ *seconds*

6. If $t = \frac{v^2}{2}$, then $\left(-\frac{df}{dt}\right)$ is equal to (where f is acceleration) **[MP PET 1991]**

(a) f^2 (b) f^3 (c) $-f^3$ (d) $-f^2$

7. A particle is moving in a straight line according to the formula $s = t^2 + 8t + 12$. If s be measured in *meters* and t be measured in *seconds* then the average velocity of the particle in third second is

(a) 14 *m/sec* (b) 13 *m/sec* (c) 15 *m/sec* (d) None of these

8. If $2t = v^2$, then dv/dt is equal to

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{v}$

9. The equation of motion of a particle moving along a straight line is $s = 2t^3 - 9t^2 + 12t$, where the units of s and t are *cm* and *sec*. The acceleration of the particle will be zero after

(a) $\frac{3}{2}$ *sec* (b) $\frac{2}{3}$ *sec* (c) $\frac{1}{2}$ *sec* (d) Never

10. A body moves according to the formula $v = 1 + t^2$, where *v* is the velocity at time *t*. The acceleration after 3 *sec* will be (*v* in *cm/sec*)

[MP PET 1988]

(a) 24 cm/sec^2 (b) 12 cm/sec^2 (c) 6 cm/sec^2 (d) None of these

11. A particle moves in a straight line so that its velocity at any point is given by $v^2 = a + bx$, where $a, b \neq 0$ are constant. The acceleration is

(a) Zero (b) Uniform (c) Non-uniform (d) Indeterminate

12. The distance in seconds, described by a particle in *t* seconds is given by $s = ae^t + \frac{b}{e^t}$. The acceleration of the particle at time *t* is

(a) Proportional to *t* (b) Proportional to *s* (c) *s* (d) Constant

13. A stone thrown vertically upwards rises '*s*' *metre* in *t* seconds, where $s = 80t - 16t^2$, then velocity after 2 seconds is **[SCRA 1996]**

(a) 8 *m per sec.* (b) 16 *m per sec.* (c) 32 *m per sec.* (d) 64 *m per sec.*

14. If the distance '*s*' travelled by a particle in time *t* is $s = a\sin t + b\cos 2t,$ then the acceleration at $t = 0$ is

(a) *a* (b) – *a* (c) 4*b* (d) – 4*b*

15. If the distance travelled by a point in time *t* is $s = 180t - 16t^2$, then the rate of change in velocity is

(a) – 16 *t unit* (b) 48 *unit* (c) – 32 *unit* (d) None of these

16. The motion of stone thrown up vertically is given by $s = 13.8t - 4.9t^2$, where *s* is in *metres* and *t* is in seconds. Then its velocity at $t = 1$ second is

(a) 3 *m/s* (b) 5 *m/s* (c) 4 *m/s* (d) None of these

17. A particle is moving in a straight line. Its displacement at time *t* is given by $s = -4t^2 + 2t$, then its velocity and acceleration at time $t = \frac{1}{2}$ second are

(a) –2, –8 (b) 2, 6 (c) –2, 8 (d) 2, 8

18. A ball thrown vertically upwards falls back on the ground after 6 *seconds*. Assuming that the equation of motion is of the form $s = ut - 4.9t^2$, where *s* is in *metres* and *t* is in seconds, find the velocity at $t = 0$

(a) 0 *m/s* (b) 1 *m/s* (c) 29.4 *m/s* (d) None of these

19. A particle is moving in a straight line according as $s = \sqrt{1+t}$, then the relation between its acceleration (*a*) and velocity (*v*) is

(a) $a \propto v^2$ (b) $a \propto v^3$ (c) $a \propto \frac{1}{v^3}$ (d) $a \propto v$

20. The distance travelled by a particle moving in a straight line in time *t* is $s = \sqrt{at^2 + bt + c}$. Acceleration of the particle is

[Kerala (Engg.) 2002]

(a) Proportional to *t* (b) Proportional to *s* (c) Proportional to s^{-3} (d) None of these

Advance Level

21. A particle is moving along the curve $x = at^2 + bt + c$. If $ac = b^2$, then the particle would be moving with uniform**[Orissa JE**

(a) Rotation (b) Velocity (c) Acceleration (d) Retardation

22. The equations of motion of two stones thrown vertically upwards simultaneously are $s = 19.6t - 4.9t^2$ and $s = 9.8t - 4.9t^2$ respectively and the maximum height attained by the first one is h. When the height of the first stone is maximum, the height of the second stone will be

(a) $h/3$ (b) $2h$ (c) h (d) 0

23. A particle is moving on a straight line, where its position s (in *metres*) is a function of time t (in *seconds*) given by $s = at^2 + bt + 6,\ t \geq 0$. If it is known that the particle comes to rest after 4 *seconds* at a distance of 16 *metres* from the starting position $(t = 0)$, then the retardation in its motion is

(a) $-1 m/\sec^2$ (b) $\frac{5}{4} m/\sec^2$ (c) $-\frac{1}{2} m/\sec^2$ (d) $-\frac{5}{4} m/\sec^2$

24. A point moves in a straight line during the time $t = 0$ to $t = 3$ according to the law $s = 15t - 2t^2$. The average velocity is

[MP PET 1992]

(a) 3 (b) 9 (c) 15 (d) 27

25. The equation of motion of a stone, thrown vertically upwards is $s = ut - 6.3t^2$, where the units of s and t are *cm* and *sec*. If the stone reaches at maximum height in 3*sec*. then u =

(a) 18.9 *cm/sec* (b) 12.6 *cm/sec* (c) 37.8 *cm/sec* (d) None of these

Rate Measures

Basic Level

26. Radius of a circle is increasing uniformly at the rate of 3 *cm/sec*. The rate of increase of area when radius is 10 *cm*, will be

(a) $\pi cm^2/s$ (b) $2\pi\, cm^2/s$ (c) $10\pi\, cm^2/s$ (d) None of these

27. A 10 *cm* long rod *AB* moves with its ends on two mutually perpendicular straight lines *OX* and *OY*. If the end *A* be moving at the rate of 2 *cm/sec*, then when the distance of *A* from *O* is 8 *cm*, the rate at which the end *B* is moving , is **[SCRA 1996]**

(a) $\frac{8}{3}$ *cm/sec* (b) $\frac{4}{3}$ *cm/sec* (c) $\frac{2}{9}$ *cm/sec* (d) None of these

28. If $y = x^3 + 5$ and x changes from 3 to 2.99, then the approximate change in y is

(a) 2.7 (b) – .27 (c) 27 (d) None of these

29. The volume of a spherical balloon is increasing at the rate of 40 cubic centimetre per minute. The rate of change of the surface of the balloon at the instant when its radius is 8 centimetres, is

(a) $\frac{5}{2}$ *sq cm/min.* (b) 5 *sq cm/min.* (c) 10 *sq cm/min.* (d) 20 *sq cm/min.*

30. A ladder 5 *m* in length is resting against vertical wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 1.5 *m/sec*. The length of the highest point of the ladder when the foot of the ladder is 4.0 *m* away from the wall decreases at the rate of

(a) 2 *m/sec* (b) 3 *m/sec* (c) 2.5 *m/sec* (d) 1.5 *m/sec*

31. If the rate of increase of area of a circle is not constant but the rate of increase of perimeter is constant, then the rate of increase of area varies

(a) As the square of the perimeter(b) Inversely as the perimeter (c) As the radius (d)

Advance Level

32. Gas is being pumped into a spherical balloon at the rate of 30 ft^3/min. Then the rate at which the radius increases when its reaches the value 15 *ft* is

(a) $\frac{1}{30\pi} ft/\min$.   (b) $\frac{1}{15\pi} ft/\min$.   (c) $\frac{1}{20} ft/\min$.   (d) $\frac{1}{25} ft/\min$.

33. On dropping a stone in stationary water circular ripples are observed. Rate of flow of ripples is 6 *cm*/sec. When radius of the circle is 10 *cm*, then fluid rate of increase in its area is

(a) $120\pi\, cm/\sec$ (b) $120\, sq\, cm/\sec$ (c) $\pi\, sq\, cm/\sec$ (d) $120\,\pi\, sq\, cm/\sec$

34. If the edge of a cube increases at the rate of 60 *cm per second,* at what rate the volume is increasing when the edge is 90 *cm*

(a) 486000 *cu cm per sec*   (b) 1458000 *cu cm per sec*   (c) 43740000 *cu cm per sec*   (d) None of these

35. If a spherical balloon has a variable diameter $3x+\frac{9}{2}$, then the rate of change of its volume with respect to *x* is

(a) $27\,\pi(2x+3)^2$ (b) $\frac{27\pi}{16}(2x+3)^2$ (c) $\frac{27\pi}{8}(2x+3)^2$ (d) None of these

36. Two cyclists start from the junction of two perpendicular roads, their velocities being 3*v* metres/minute and 4*v* metres/minute. The rate at which the two cyclists are separating is

(a) $\frac{7}{2}\, vm/\min$ (b) $5\, vm/\min$ (c) $vm/\min$ (d) None of these

37. A stick of length *a cm* rests against a vertical wall and the horizontal floor. If the foot of the stick slides with a constant velocity of *b cm/s* then the magnitude of the velocity of the middle point of the stick when it is equally inclined with the floor and the wall, is

(a) $\frac{b}{\sqrt{2}} cm/s$ (b) $\frac{b}{2} cm/s$ (c) $\frac{ab}{2} cm/s$ (d) None of these

38. If $y=\int_0^x \frac{t^2}{\sqrt{t^2+1}}\,dt$ then the rate of change of *y* with respect to *x* when $x=1$, is

(a) $\sqrt{2}$ (b) 1/2   (c) $1/\sqrt{2}$ (d) None of these

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Answer Sheet

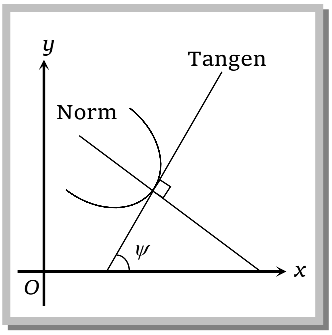
Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	a	c	d	c	b	b	d	a	c	b	c	b	d	c	c	a	c	b	c
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38		
c	d	b	b	c	d	a	b	c	a	c	a	d	b	c	b	a	c		

4.2 Tangent and Normal

4.2.1 Slope of the Tangent and Normal

(1) **Slope of the tangent :** If tangent is drawn on the curve $y = f(x)$ at point $P(x_1, y_1)$ and this tangent makes an angle ψ with positive x-direction then,



$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \tan\psi = \text{slope of the tangent}$$

Note : ❑ If tangent is parallel to x-axis $\psi = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$

❑ If tangent is perpendicular to x-axis $\psi = \frac{\pi}{2} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$

(2) **Slope of the normal :** The normal to a curve at $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P and slope of the normal $= \frac{-1}{\text{Slope of tangent}} =$

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{P(x_1,y_1)}} = -\left(\frac{dx}{dy}\right)_{P(x_1,y_1)}$$

Note : ❑ If normal is parallel to x-axis

$$\Rightarrow -\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = 0 \text{ or } \left(\frac{dx}{dy}\right)_{(x_1,y_1)} = 0$$

❑ If normal is perpendicular to x-axis (for parallel to y-axis)

$$\Rightarrow -\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

Example: 1 The slope of the tangent to the curve $x^2 + y^2 = 2c^2$ at point (c, c) is **[AMU 1998]**

(a) 1 (b) – 1 (c) 0 (d) 2

Solution: (b) Given $x^2 + y^2 = 2c^2$

Differentiating *w.r.t. x*, $2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(c,c)} = -1$

Example: 2 The line $x + y = 2$ is tangent to the curve $x^2 = 3 - 2y$ at its point **[MP PET 1998]**

(a) (1, 1) (b) (−1, 1) (c) $(\sqrt{3}, 0)$ (d) (3, − 3)

Solution: (a) Given curve $x^2 = 3 - 2y$

diff. *w.r.t. x*, $2x = -\frac{2dy}{dx}$; $\frac{dy}{dx} = -x$

Slope of the line = − 1

$\frac{dy}{dx} = -x = -1$; $x = 1$

$\therefore y = 1$ point (1, 1)

Example: 3 The tangent to the curve $y = 2x^2 - x + 1$ at a point *P* is parallel to $y = 3x + 4$, the co-ordinate of *P* are **[Rajasthan P**

(a) (2, 1) (b) (1, 2) (c) (− 1, 2) (d) (2, − 1)

Solution: (b) Given $y = 2x^2 - x + 1$

Let the co-ordinate of *P* is (*h*, *k*) then $\left(\frac{dy}{dx}\right)_{(h,k)} = 4h - 1$

Clearly $4h - 1 = 3$

$h = 1 \Rightarrow k = 2$. *P* is (1, 2).

4.2.2 Equation of the Tangent and Normal

(1) **Equation of the tangent :** We know that the equation of *a* line passing through a point $P(x_1, y_1)$ and having slope *m* is $y - y_1 = m(x - x_1)$

Slope of the tangent at (x_1, y_1) is = $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

The equation of the tangent to the curve $y = f(x)$ at point $P(x_1, y_1)$ is

$$\boxed{y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1)}$$

(2) **Equation of the normal :** Slope of the Normal = $\dfrac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}$

Thus equation of the normal to the curve $y = f(x)$ at point $P(x_1, y_1)$

$$\boxed{y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x - x_1)}$$

Note : ❑ If at any point $P(x_1, y_1)$ on the curve $y = f(x)$, the tangent makes equal angle with the axes, then at the point P, $\psi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Hence, at P $\tan\psi = \frac{dy}{dx} = \pm 1$.

Example: 4 The equation of the tangent at $(-4, -4)$ on the curve $x^2 = -4y$ is **[Karnataka CET 2001]**

(a) $2x + y + 4 = 0$ (b) $2x - y - 12 = 0$ (c) $2x + y - 4 = 0$ (d) $2x - y + 4 = 0$

Solution: (d) $x^2 = -4y \Rightarrow 2x = -4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-x}{2} \Rightarrow \left(\frac{dy}{dx}\right)_{(-4,-4)} = 2$.

We know that equation of tangent is $(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1) \Rightarrow y + 4 = 2(x + 4) \Rightarrow 2x - y + 4 = 0$.

Example: 5 The equation of the normal to the curve $y = \sin\frac{\pi x}{2}$ at (1, 1) is

(a) $y = 1$ (b) $x = 1$ (c) $y = x$ (d) $y - 1 = \frac{-2}{\pi}(x - 1)$

Solution: (b) $y = \sin\frac{\pi x}{2} \Rightarrow \frac{dy}{dx} = \frac{\pi}{2}\cos\frac{\pi}{2}x \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 0$

$\therefore$ Equation of normal is $y - 1 = \frac{1}{0}(x - 1) \Rightarrow x = 1$.

Example: 6 The equation of the tangent to the curve $y = be^{-x/a}$ at the point where it crosses y-axis is

(a) $ax + by = 1$ (b) $ax - by = 1$ (c) $\frac{x}{a} - \frac{y}{b} = 1$ (d) $\frac{x}{a} + \frac{y}{b} = 1$

Solution: (d) Curve is $y = be^{-x/a}$

Since the curve crosses y-axis (*i.e.*, $x = 0$) $\therefore$ $y = b$

Now $\frac{dy}{dx} = \frac{-b}{a}e^{-x/a}$. At point $(0, b)$, $\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}$

$\therefore$ Equation of tangent is $y - b = \frac{-b}{a}(x - 0) \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

Example: 7 If the normal to the curve $y = f(x)$ at the point $(3, 4)$ makes an angle $\frac{3\pi}{4}$ with the positive x-axis then $f'(3)$ is equal to

[IIT Screening 2000; DCE 2001]

(a) – 1 (b) $\frac{-3}{4}$ (c) $\frac{4}{3}$ (d) 1

Solution: (d) Slope of the normal $= \frac{-1}{dy/dx} \Rightarrow \tan\frac{3\pi}{4} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(3,4)}}$

$\therefore \left(\frac{dy}{dx}\right)_{(3,4)} = 1$; $f'(3) = 1$.

Example: 8 The point (s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical (parallel to y-axis), is are**[IIT Screenin**

(a) $\left[\pm\frac{4}{\sqrt{3}}, -2\right]$ (b) $\left(\pm\frac{\sqrt{11}}{3}, 1\right)$ (c) $(0, 0)$ (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

Solution: (d) $y^3 + 3x^2 = 12y$

$\Rightarrow 3y^2.\frac{dy}{dx}+6x=12.\frac{dy}{dx} \Rightarrow \frac{dy}{dx}(3y^2-12)+6x=0 \Rightarrow \frac{dy}{dx}=\frac{6x}{12-3y^2} \Rightarrow \frac{dx}{dy}=\frac{12-3y^2}{6x}$

Tangent is parallel to y-axis, $\frac{dx}{dy}=0 \Rightarrow 12-3y^2$ = 0 or $y=\pm 2$. Then $x=\pm\frac{4}{\sqrt{3}}$, for $y=2$

$y=-2$ does not satisfy the equation of the curve, $\therefore$ The point is $\left(\pm\frac{4}{\sqrt{3}},2\right)$

Example: 9 At which point the line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=be^{-x/a}$ **[Rajasthan PET 1999]**

(a) (0, 0) (b) (0, a) (c) (0, b) (d) (b, 0)

Solution: (c) Let the point be (x_1,y_1) $\therefore y_1=be^{-x_1/a}$ ......(i)

Also, curve $y=be^{-x/a} \Rightarrow \frac{dy}{dx}=\frac{-b}{a}e^{-x/a}$

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b}{a}e^{-x_1/a}=\frac{-y_1}{a}$ (by (i))

Now, the equation of tangent of given curve at point (x_1,y_1) is $y-y_1=\frac{-y_1}{a}(x-x_1) \Rightarrow \frac{x}{a}+\frac{y}{y_1}=\frac{x_1}{a}+1$

Comparing with $\frac{x}{a}+\frac{y}{b}=1$, we get, $y_1=b$ and $1+\frac{x_1}{a}=1 \Rightarrow x_1=0$

Hence, the point is (0, b).

Example: 10 The abscissa of the point, where the tangent to curve $y=x^3-3x^2-9x+5$ is parallel to x-axis are **[Karnataka C**

(a) 0 and 0 (b) $x=1$ and -1 (c) $x=1$ and -3 (d) $x=-1$ and 3

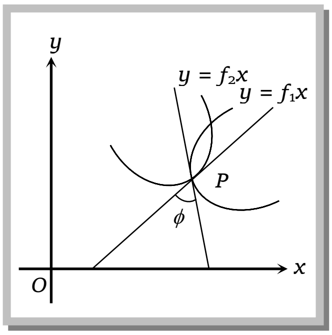
Solution: (d) $y=x^3-3x^2-9x+5 \Rightarrow \frac{dy}{dx}=3x^2-6x-9$.

We know that this equation gives the slope of the tangent to the curve. The tangent is parallel to x-axis $\frac{dy}{dx}=0$

Therefore, $3x^2-6x-9=0 \Rightarrow x=-1,3$.

4.2.3 Angle of Intersection of Two Curves

The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection.



We know that the angle between two straight lines having slopes

$$\phi=\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}$$

Also slope of the tangent at $P(x_1,y_1)$

$$m_1=\left(\frac{dy}{dx}\right)_{1(x_1,y_1)}, \quad m_2=\left(\frac{dy}{dx}\right)_{2(x_1,y_1)}$$

Thus the angle between the tangents of the two curves $y=f_1(x)$ and $y=f_2(x)$

$$\boxed{\tan\phi = \frac{\left(\frac{dy}{dx}\right)_{1(x_1,y_1)} - \left(\frac{dy}{dx}\right)_{2(x_1,y_1)}}{1+\left(\frac{dy}{dx}\right)_{1(x_1,y_1)}\left(\frac{dy}{dx}\right)_{2(x_1,y_1)}}}$$

Orthogonal curves : If the angle of intersection of two curves is right angle, the two curves are said to intersect orthogonally. The curves are called orthogonal curves. If the curves are orthogonal, then $\phi = \frac{\pi}{2}$

$$m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -1$$

Example: 11 The angle between the curves $y^2 = x$ and $x^2 = y$ at (1, 1) is

(a) $\tan^{-1}\frac{4}{3}$ (b) $\tan^{-1}\frac{3}{4}$ (c) 90^o (d) 45^o

Solution: (b) Given curve $y^2 = x$ and $x^2 = y$

Differentiating *w.r.t. x*, $2y\frac{dy}{dx} = 1$ and $2x = \frac{dy}{dx}$

$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2}$ and $\left(\frac{dy}{dx}\right)_{(1,1)} = 2$

Angle between the curve

$\Rightarrow \tan\phi = \frac{2-\frac{1}{2}}{1+\frac{1}{2}.2} \Rightarrow \tan\phi = \frac{3}{4} \Rightarrow \phi = \tan^{-1}\frac{3}{4}$.

Example: 12 If the two curves $y = a^x$ and $y = b^x$ intersect at α, then $\tan\alpha$ equal **[MP PET 2001]**

(a) $\frac{\log a - \log b}{1+\log a \log b}$ (b) $\frac{\log a + \log b}{1-\log a \log b}$ (c) $\frac{\log a - \log b}{1-\log a \log b}$ (d) None of these

Solution: (a) Clearly the point of intersection of curves is (0, 1)

Now, slope of tangent of first curve, $m_1 = \frac{dy}{dx} = a^x \log a \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = m_1 = \log a$

Slope of tangent of second curve, $m_2 = \frac{dy}{dx} = b^x \log b \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(0,1)} = \log b$

$\therefore \tan\alpha = \frac{m_1 - m_2}{1+m_1 m_2} = \frac{\log a - \log b}{1+\log a \log b}$.

Example: 13 The angle of intersection between curve $xy = 6$ and $x^2 y = 12$

(a) $\tan^{-1}\left(\frac{3}{4}\right)$ (b) $\tan^{-1}\left(\frac{3}{11}\right)$ (c) $\tan^{-1}\left(\frac{11}{3}\right)$ (d) 0^o

Solution: (b) The equation of two curves are $xy = 6$ and $x^2 y = 12$ from (i) we obtain $y = \frac{6}{x}$ putting this value of y in equation (ii) to obtain $x^2\left(\frac{6}{x}\right) = 12 \Rightarrow 6x = 12 \Rightarrow x = 2$

Putting $x = 2$ in (i) or (ii) we get, $y = 3$. Thus, the two curves intersect at P(2, 3)

Differentiating (i) *w.r.t. x*, we get $x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = -\frac{3}{2} = m_1$

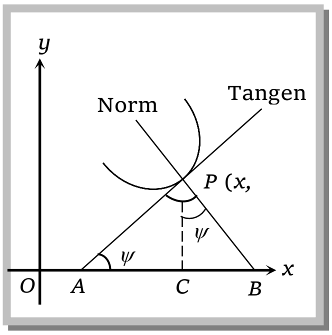
Differentiating (ii) *w.r.t. x*, we get $x^2\frac{dy}{dx}+2xy=0 \Rightarrow \frac{dy}{dx}=\frac{-2y}{x}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)}=-3=m_2 \Rightarrow \tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\left(\frac{-3}{2}+3\right)\Big/\left(1+\left(\frac{-3}{2}\right)(-3)\right)=\frac{3}{11} \Rightarrow \theta=\tan^{-1}\frac{3}{11}.$$

4.2.4 Length of Tangent, Normal, Subtangent and Subnormal

Let the tangent and normal at point $P(x,y)$ on the curve $y=f(x)$ meet the x-axis at points A and B respectively. Then PA and PB are called length of tangent and normal respectively at point P. If PC be the perpendicular from P on x-axis, the AC and BC are called length of subtangent and subnormal respectively at P. If PA makes angle ψ with x-axis, then $\tan\psi=\frac{dy}{dx}$ from fig., we find that

(1) Length of tangent $PA=y\operatorname{cosec}\psi=y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)}$



(2) Length of normal $PB=y\sec\psi=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(3) Length of subtangent $AC=y\cot\psi=\frac{y}{\left(\frac{dy}{dx}\right)}$

(4) Length of subnormal $BC=y\tan\psi=y\left(\frac{dy}{dx}\right)$

Example: 14 The length of subtangent to the curve $x^2y^2=a^4$ at the point $(-a,a)$ is **[Karnataka CET 2001]**

(a) $3a$ (b) $2a$ (c) a (d) $4a$

Solution: (c) Equation of the curve $x^2y^2=a^4$.

Differentiating the given equation,

$$x^2 2y\frac{dy}{dx}+y^2 2x=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(-a,a)}=-\left(\frac{a}{-a}\right)=1$$

Therefore, sub-tangent $=\frac{y}{\left(\frac{dy}{dx}\right)}=a$.

Example: 15 For the curve $y^n=a^{n-1}x$, the sub-normal at any point is constant, the value of n must be

(a) 2 (b) 3 (c) 0 (d) 1

Solution: (a) $y^n=a^{n-1}x \Rightarrow ny^{n-1}\frac{dy}{dx}=a^{n-1} \Rightarrow \left(\frac{dy}{dx}\right)=\frac{a^{n-1}}{ny^{n-1}}$

$\therefore$ Length of the subnormal $= y\dfrac{dy}{dx} = \dfrac{ya^{n-1}}{ny^{n-1}} = \dfrac{a^{n-1}y^{2-n}}{n}$

We also know that if the subnormal is constant, then $\dfrac{a^{n-1}}{n}.y^{2-n}$ should not contain y.

Therefore, $2-n=0$ or $n=2$.

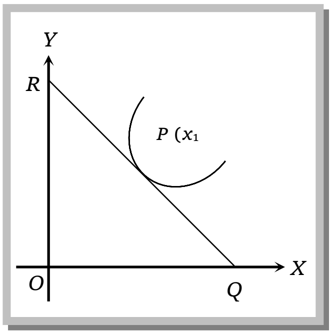
4.2.5 Length of Intercept made on Axis by the Tangent

Equation of tangent at any point (x_1, y_1) to the curve $y = f(x)$ is $y - y_1 = \left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$ ......(i)

Equation of x-axis $y = 0$ ......(ii)

and Equation of y-axis $x = 0$ ......(iii)

Solving (i) and (ii) we get $x = x_1 - \left[\dfrac{y_1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}}\right]$



x-intercept $OQ = x_1 - \left[\dfrac{y_1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}}\right]$

Similarly solving (i) and (iii) we get, y-intercept OR $= y_1 - \left[x_1\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}\right]$

Example: 16 The sum of intercepts on co-ordinate axes made by tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is

(a) a (b) $2a$ (c) $2\sqrt{a}$ (d) None of these

Solution: (a) $\sqrt{x} + \sqrt{y} = \sqrt{a} \Rightarrow \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0 \Rightarrow \therefore \dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$

Hence tangent at (x, y) is $Y - y = -\dfrac{\sqrt{y}}{\sqrt{x}}(X - x)$ or $X\sqrt{y} + Y\sqrt{x} = \sqrt{xy}(\sqrt{x} + \sqrt{y}) = \sqrt{axy}$ or $\dfrac{X}{\sqrt{a}\sqrt{x}} + \dfrac{Y}{\sqrt{a}\sqrt{y}} = 1$.

Clearly its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of the intercepts $= \sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{a}.\sqrt{a} = a$.

4.2.6 Length of Perpendicular from Origin to the Tangent

Length of perpendicular from origin (0, 0) to the tangent drawn at point $P(x_1, y_1)$ of the curve $y = f(x)$

$$p = \left|\frac{y_1 - x_1\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}\right|$$

Example: 17 The length of perpendicular from (0, 0) to the tangent drawn to the curve $y^2 = 4(x+2)$ at point (2, 4) is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{3}{\sqrt{5}}$ (c) $\frac{6}{\sqrt{5}}$ (d) 1

Solution: (c) Differentiating the given equation *w.r.t.* x, $2y\frac{dy}{dx} = 4$ at point (2, 4) $\frac{dy}{dx} = \frac{1}{2}$

$$P = \frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{4-2\left(\frac{1}{2}\right)}{\sqrt{1+\frac{1}{4}}} = \frac{6}{\sqrt{5}}.$$

Assignment

Tangent and Normal

Basic Level

1. If the line $y = 2x + k$ is a tangent to the curve $x^2 = 4y$, then k is equal to **[AMU 2002]**

(a) 4 (b) $\frac{1}{2}$ (c) – 4 (d) $-\frac{1}{2}$

2. The point on the curve $y^2 = x$ where tangent makes 45^o angle with x-axis is **[Rajasthan PET 1990, 92]**

(a) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (c) (4, 2) (d) (1, 1)

3. If $x = t^2$ and $y = 2t$, then equation of the normal at $t = 1$ is

(a) $x + y - 3 = 0$ (b) $x + y - 1 = 0$ (c) $x + y + 1 = 0$ (d) $x + y + 3 = 0$

4. If normal to the curve $y = f(x)$ is parallel to x-axis, then correct statement is **[Rajasthan PET 2000]**

(a) $\frac{dy}{dx} = 0$ (b) $\frac{dy}{dx} = 1$ (c) $\frac{dx}{dy} = 0$ (d) None of these

5. The equation of the tangent to the curve $(1 + x^2)y = 2 - x$, where it crosses the x-axis, is

(a) $x + 5y = 2$ (b) $x - 5y = 2$ (c) $5x - y = 2$ (d) $5x + y - 2 = 0$

6. The equation of tangent to the curve $y = 2\cos x$ at $x = \frac{\pi}{4}$ is

(a) $y - \sqrt{2} = 2\sqrt{2}\left(x - \frac{\pi}{4}\right)$ (b) $y + \sqrt{2} = \sqrt{2}\left(x + \frac{\pi}{4}\right)$ (c) $y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$ (d) $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$

7. For the curve $x = t^2 - 1$, $y = t^2 - t$, the tangent line is perpendicular to x-axis where **[MNR 1980]**

(a) $t = 0$ (b) $t = \infty$ (c) $t = \frac{1}{\sqrt{3}}$ (d) $t = -\frac{1}{\sqrt{3}}$

8. If at any point on a curve the sub-tangent and subnormal are equal, then the tangent is equal to

(a) Ordinate (b) $\sqrt{2}$ ordinate (c) $\sqrt{2\,(\text{ordinate})}$ (d) None of these

9. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts, α and β on the coordinate axes such that $\alpha^2 + \beta^2 = 61$, then a =

(a) ± 30 (b) ± 5 (c) ± 6 (d) ± 61

10. If the tangent to the curve $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ at $\theta = \frac{\pi}{3}$ makes an angle α with x-axis, then α =

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{6}$

11. If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle $\tan^{-1} 2$ with x-axis, then

(a) $a = 1, b = 2$ (b) $a = 1, b = -2$ (c) $a = -1, b = 2$ (d) $a = -1, b = -2$

12. The fixed point P on the curve $y = x^2 - 4x + 5$ such that the tangent at P is perpendicular to the line $x + 2y - 7 = 0$ is given by

(a) (3, 2) (b) (1, 2) (c) (2, 1) (d) None of these

13. The points of contact of the tangents drawn from the origin to the curve $y = \sin x$ lie on the curve

(a) $x^2 - y^2 = xy$ (b) $x^2 + y^2 = x^2y^2$ (c) $x^2 - y^2 = x^2y^2$ (d) None of these

14. The slope of the tangent to the curve $y^2 = 4ax$ drawn at point $(at^2, 2at)$ is **[Rajasthan PET 1993]**

(a) t (b) $\frac{1}{t}$ (c) $-t$ (d) $\frac{-1}{t}$

15. The slope of the curve $y = \sin x + \cos^2 x$ is zero at the point, where

(a) $x = \frac{\pi}{4}$ (b) $x = \frac{\pi}{2}$ (c) $x = \pi$ (d) No where

16. The equation of tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point (x_1, y_1) is

(a) $\frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \frac{1}{\sqrt{a}}$ (b) $\frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{a}$ (c) $x\sqrt{x_1} + y\sqrt{y_1} = \sqrt{a}$ (d) None of these

17. A tangent to the curve $y = x^2 + 3x$ passes through a point (0, – 9) if it is drawn at the point

(a) (– 3, 0) (b) (1, 4) (c) (0, 0) (d) (– 4, 4)

18. The sum of the intercepts made by a tangent to the curve $\sqrt{x} + \sqrt{y} = 4$ at point (4, 4) on coordinate axes is

(a) $4\sqrt{2}$ (b) $6\sqrt{3}$ (c) $8\sqrt{2}$ (d) $\sqrt{256}$

19. The angle of intersection between the curve $y^2 = 16x$ and $2x^2 + y^2 = 4$ is **[Rajasthan PET 1993]**

(a) 0^o (b) 30^o (c) 45^o (d) 90^o

20. The equation of normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$ is

(a) $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ (b) $\frac{ax}{\sec\theta} - \frac{by}{\tan\theta} = a^2 - b^2$ (c) $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 - b^2$ (d) $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a - b$

21. If tangent to a curve at a point is perpendicular to x-axis, then at the point

(a) $\frac{dy}{dx} = 0$ (b) $\frac{dx}{dy} = 0$ (c) $\frac{dy}{dx} = 1$ (d) $\frac{dy}{dx} = -1$

22. If m be the slope of a tangent to the curve $e^y = 1 + x^2$ then

(a) $|m| > 1$ (b) $m < 1$ (c) $|m| < 1$ (d) $|m| \le 1$

23. The equation of the tangent to the curve $y = e^{-|x|}$ at the point where the curve cuts the line $x = 1$ is

(a) $x + y = e$ (b) $e(x + y) = 1$ (c) $y + ex = 1$ (d) None of these

24. The slope of the tangent to the curve $y=\int_0^x \frac{dx}{1+x^3}$ at the point where $x=1$ is

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{1}{4}$ (d) None of these

25. The angle of intersection between the curves $x^2=4ay$ and $y^2=4ax$ at origin is **[Rajasthan PET 1997]**

(a) 30^o (b) 45^o (c) 60^o (d) 90^o

26. The equation of the normal to the curve $y=x(2-x)$ at the point (2, 0) is **[Rajasthan PET 1989, 1992]**

(a) $x-2y=2$ (b) $x-2y+2=0$ (c) $2x+y=4$ (d) $2x+y-4=0$

27. The angle of intersection of the curve $y=4-x^2$ and $y=x^2$ is
[Rajasthan PET 1989, 1993; MNR 1978]

(a) $\frac{\pi}{2}$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$ (c) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ (d) None of these

28. Tangent to the curve $y=e^{2x}$ at point (0, 1) meets x-axis at the point

(a) (0, a) (b) (2, 0) (c) $\left(-\frac{1}{2},0\right)$ (d) Non where

29. The equation of the tangent to the curve $x=a\cos^3 t, y=a\sin^3 t$ at 't' point is **[Rajasthan PET 1988]**

(a) $x\sec t-y\operatorname{cosec} t=a$ (b) $x\sec t+y\operatorname{cosec} t=a$ (c) $x\operatorname{cosec} t-y\sec t=a$ (d) $x\operatorname{cosec} t+y\sec t=a$

30. The length of the tangent to the curve $x=a\left(\cos t+\log\tan\frac{t}{2}\right), y=a\sin t$ is

(a) ax (b) ay (c) a (d) xy

31. The point at the curve $y=12x-x^3$ where the slope of the tangent is zero will be **[Rajasthan PET 1992]**

(a) (0, 0) (b) (2, 16) (c) (3, 9) (d) None of these

32. The angle of intersection between the curves $y=x^2$ and $4y=7-3x^3$ at point (1, 1) is **[Andhra CEE 1992]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) None of these

Advance Level

33. Consider the following statements:

Assertion (A) : The circle $x^2+y^2=1$ has exactly two tangents parallel to the x-axis

Reason (R) : $\frac{dy}{dx}=0$ on the circle exactly at the points $(0,\pm 1)$. Of these statements **[SCRA 1996]**

(a) Both A and R and true and R is the correct explanation of A

(b) Both A and R are true but R is not the correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

34. The slope of the tangent to the curve $x=3t^2+1, y=t^3-1$ at $x=1$ is

(a) 0 (b) $\frac{1}{2}$ (c) ∞ (d) –2

35. The slope of tangent to the curve $x=t^2+3t-8,\ y=2t^2-2t-5$ at the point $(2,-1)$ is **[MNR 1994]**

(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) –6 (d) None of these

36. At what points of the curve $y=\frac{2}{3}x^3+\frac{1}{2}x^2$, tangent makes the equal angle with axis **[UPSEAT 1999]**

(a) $\left(\frac{1}{2},\frac{5}{24}\right)$ and $\left(-1,-\frac{1}{6}\right)$ (b) $\left(\frac{1}{2},\frac{4}{9}\right)$ and (-1, 0) (c) $\left(\frac{1}{3},\frac{1}{47}\right)$ and $\left(-1,\frac{1}{3}\right)$ (d) $\left(\frac{1}{3},\frac{1}{7}\right)$ and $\left(-3,\frac{1}{2}\right)$

37. For the curve $xy=c^2$ the subnormal at any point varies as

(a) x^2 (b) x^3 (c) y^2 (d) y^3

38. The point of the curve $y^2=2(x-3)$ at which the normal is parallel to the line $y-2x+1=0$ is

(a) (5, 2) (b) $\left(-\frac{1}{2},-2\right)$ (c) (5, – 2) (d) $\left(\frac{3}{2},2\right)$

39. Coordinates of a point on the curve $y=x\log x$ at which the normal is parallel to the line $2x-2y=3$ are **[Rajasthan PET**

(a) (0, 0) (b) (e, e) (c) $(e^2, 2e^2)$ (d) $(e^{-2},-2e^{-2})$

40. The abscissa of the points of curve $y=x(x-2)(x-4)$ where tangents are parallel to x-axis is obtained as

(a) $x=2\pm\frac{2}{\sqrt{3}}$ (b) $x=1\pm\frac{1}{\sqrt{3}}$ (c) $x=2\pm\frac{1}{\sqrt{3}}$ (d) $x=\pm 1$

41. The length of the normal at point 't' of the curve $x=a(t+\sin t), y=a(1-\cos t)$ is **[Rajasthan PET 2001]**

(a) $a\sin t$ (b) $2a\sin^3\left(\frac{t}{2}\right)\sec\left(\frac{t}{2}\right)$ (c) $2a\sin\left(\frac{t}{2}\right)\tan\left(\frac{t}{2}\right)$ (d) $2a\sin\left(\frac{t}{2}\right)$

42. The length of normal to the curve $x=a(\theta+\sin\theta), y=a(1-\cos\theta)$ at the point $\theta=\frac{\pi}{2}$ is **[Rajasthan PET 1999; AIEEE 2004]**

(a) $2a$ (b) $\frac{a}{2}$ (c) $\sqrt{2}a$ (d) $\frac{a}{\sqrt{2}}$

43. The area of the triangle formed by the coordinate axes and a tangent to the curve $xy=a^2$ at the point (x_1, y_1) on it is **[DCE 2001]**

(a) $\frac{a^2x_1}{y_1}$ (b) $\frac{a^2y_1}{x_1}$ (c) $2a^2$ (d) $4a^2$

44. The normal of the curve $x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$ at any θ is such that **[DCE 2000]**

(a) It makes a constant angle with x-axis (b) It passes through the origin

(c) It is at a constant distance from the origin (d) None of these

45. An equation of the tangent to the curve $y=x^4$ from the point (2, 0)not on the curve is

(a) $y=0$ (b) $x=0$ (c) $x+y=0$ (d) None of these

46. For the curve $by^3(x+a)^3$ the square of subtangent is proportional to

(a) $(\text{Subnormal})^{1/2}$ (b) Subnormal (c) $(\text{Subnormal})^{3/2}$ (d) None of these

47. The tangent to the curve $y = ax^2 + bx$ at $(2, -8)$ is parallel to x-axis. Then **[AMU 1999]**

(a) $a = 2, b = -2$ (b) $a = 2, b = -4$ (c) $a = 2, b = -8$ (d) $a = 4, b = -4$

48. If the area of the triangle include between the axes and any tangent to the curve $x^n y = a^n$ is constant, then n is equal to

(a) 1 (b) 2 (c) $\frac{3}{2}$ (d) $\frac{1}{2}$

49. All points on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ at which the tangents are parallel to the axis of x, lie on a

(a) Circle (b) Parabola (c) Line (d) None of these

50. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$ cut each other orthogonally, then

(a) $a^2 + b^2 = l^2 + m^2$ (b) $a^2 - b^2 = l^2 - m^2$ (c) $a^2 - b^2 = l^2 + m^2$ (d) $a^2 + b^2 = l^2 - m^2$

51. The length of the normal at any point on the catenary $y = c\cos h\left(\frac{x}{c}\right)$ varies as

(a) $(\text{abscissa})^2$ (b) $(\text{Ordinate})^2$ (c) abscissa (d) ordinate

52. The point P of the curve $y^2 = 2x^3$ such that the tangent at P is perpendicular to the line $4x - 3y + 2 = 0$ is given by

(a) (2, 4) (b) $(1, \sqrt{2})$ (c) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ (d) $\left(\frac{1}{8}, -\frac{1}{16}\right)$

53. The length of the normal to the curve $y = a\left(\frac{e^{-x/a} + e^{x/a}}{2}\right)$ at any point varies as the

(a) Abscissa of the point (b) Ordinate of the point

(c) Square of the abscissa of the point (d) Square of the ordinate of the point

54. If the parametric equation of a curve given by $x = e^t \cos t, y = e^t \sin t,$ then the tangent to the curve at the point $t = \frac{\pi}{4}$ makes with axes of x the angle

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

55. For the parabola $y^2 = 4ax$, the ratio of the subtangent to the abscissa is

(a) 1 : 1 (b) 2 : 1 (c) $x : y$ (d) $x^2 : y$

56. Tangents are drawn from the origin to the curve $y = \cos x$. Their points of contact lie on

(a) $x^2y^2 = y^2 - x^2$ (b) $x^2y^2 = x^2 + y^2$ (c) $x^2y^2 = x^2 - y^2$ (d) None of these

57. If $y = 4x - 5$ is a tangent to the curve $y^2 = px^3 + q$ at (2, 3) then

(a) $p = 2, q = -7$ (b) $p = -2, q = 7$ (c) $p = -2, q = -7$ (d) $p = 2, q = 7$

58. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point

(a) (1, 1) (b) At no point (c) (0, 1) (d) (1, 0)

59. If the tangent and normal at any point P of parabola meet the axes at T and G respectively then

(a) ST = SG.SP (b) ST = SG = SP (c) ST $\neq$ SG = SP (d) ST = SG $\neq$ SP

60. Slope of the tangent to the curve $y=|x^3|$ at origin is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) 0

61. The line $\left(\frac{x}{a}\right)+\left(\frac{y}{b}\right)=2$, touches the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at point (a, b) then $n=$ **[Rajasthan PET 1998]**

(a) 1 (b) 2 (c) 3 (d) For non-zero values of n

62. The sum of the squares of intercepts made by a tangent to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ with coordinate axes is **[Rajastha**

(a) a (b) $2a$ (c) a^2 (d) $2a^2$

63. The point of the curve $y=x^2-3x+2$ at which the tangent is perpendicular to the $y=x$ will be

(a) (0, 2) (b) (1, 0) (c) (–1, 6) (d) (2, –2)

64. The equation of normal to the curve $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point $(8, 3\sqrt{3})$ is **[MP PET 1996]**

(a) $\sqrt{3}x+2y=25$ (b) $x+y=25$ (c) $y+2x=25$ (d) $2x+\sqrt{3}y=25$

65. The angle of intersection between the curves $xy=a^2$ and $x^2+y^2=2a^2$ is **[Rajasthan PET 1998]**

(a) 0^o (b) 30^o (c) 45^o (d) 90^o

66. The subtangent to the curve $x^m y^n = a^{m+n}$ at any point is proportional to **[Rajasthan PET 1998]**

(a) Ordinate (b) Abscissa (c) $(\text{Ordinate})^n$ (d) $(\text{Abscissa})^n$

67. If tangents drawn on the curve $x=at^2, y=2at$ is perpendicular to x-axis then its point of contact is

(a) (a, a) (b) $(a, 0)$ (c) $(0, a)$ (d) (0, 0)

68. Tangents are drawn to the curve $y=x^2-3x+2$ at the points where it meets x-axis. Equations of these tangents are

[Rajasthan PET 1993]

(a) $x-y+2=0, x-y-1=0$ (b) $x+y-1=0, x-y=2$ (c) $x-y-1=0, x-y=0$ (d) $x-y=0, x+y=0$

69. If the tangents at any point on the curve $x^4+y^4=a^4$ cuts off intercept p and q on the axes, the value of $p^{-4/3}+q^{-4/3}$ is

(a) $a^{-4/3}$ (b) $a^{-1/2}$ (c) $a^{1/2}$ (d) None of these

70. At any point (x_1, y_1) of the curve $y=ce^{x/a}$

(a) Subtangent is constant

(b) Subnormal is proportional to the square of the ordinate of the point

(c) Tangent cuts x-axis at (x_1-a) distance from the origin

(d) All the above

71. The equation of the tangent to the curve $y=1-e^{x/2}$ at the point where it meets y-axis is

(a) $x+2y=2$ (b) $2x+y=0$ (c) $x-y=2$ (d) None of these

72. The coordinates of the points on the curve $x=a(\theta+\sin\theta), y=a(1-\cos\theta)$, where tangent is inclined an angle $\frac{\pi}{4}$ to the *x*-axis are

(a) (a, a) (b) $\left(a\left(\frac{\pi}{2}-1\right), a\right)$ (c) $\left(a\left(\frac{\pi}{2}+1\right), a\right)$ (d) $\left(a, a\left(\frac{\pi}{2}+1\right)\right)$

73. If equation of normal at a point (m^2-m^3) on the curve $x^3-y^2=0$ is $y=3mx-4m^3$, then m^2 equals

(a) 0 (b) 1 (c) $-\frac{2}{9}$ (d) $\frac{2}{9}$

74. For a curve $\frac{(\text{Length of normla})^2}{(\text{Length of tangent})^2}$ is equal to

(a) (Subnormal)/(Subtangent) (b) (Subtangent)/(Subnormal) (c) (Subtangent/Subnormal)^2 (d) Constant

75. If the curve $y=x^2+bx+c$, touches the line $y=x$ at the point (1, 1), the values of *b* and *c* are

(a) – 1, 2 (b) –1, 1 (c) 2, 1 (d) –2, 1

76. Let *C* be the curve $y^3-3xy+2=0$. If H and *V* be the set of points on the curve *C* where tangent to the curve is horizontal and vertical respectively, then

(a) $H=\{(1,1)\}, V=\phi$ (b) $H=\phi, V=\{(1,1)\}$ (c) $H=\{(0,0)\}, V=\{(1,1)\}$ (d) None of these

77. If the line $ax+by+c=0$ is a normal to the curve $xy=1$ then

(a) $a, b \in R$ (b) $a>0, b>0$ (c) $a<0, b>0$ or $a>0, b<0$ (d) $a<0, b<0$

78. If the tangent to the curve $f(x)=x^2$ at any pint $(c, f(c))$ is parallel to line joining the points $(a, f(a))$ and $(b, f(b))$ on the curve, then *a, c, b* are in

(a) H.P. (b) G.P. (c) A.P. (d) A.P. and G.P. both

79. The area of triangle formed by tangent to the hyperbola $2xy=a^2$ and coordinates axes is

(a) a^2 (b) $2a^2$ (c) $\frac{a^2}{2}$ (d) $\frac{3a^2}{2}$

80. The angle of intersection between the curves $r=a\sin(\theta-\alpha)$ and $r=b\cos(\theta-\beta)$ is

(a) $\alpha-\beta$ (b) $\alpha+\beta$ (c) $\frac{\pi}{2}+\alpha+\beta$ (d) $\frac{\pi}{2}+\alpha-\beta$

81. The distance between the origin and the normal to the cure $y=e^{2x}+x^2$ at the point $x=0$ is

(a) $2\sqrt{5}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\sqrt{5}$ (d) None of these

82. If the curve $y=ax^2-6x+b$ passes through (0, 2) and has its tangent parallel to *x*-axis at $x=\frac{3}{2}$, then the value of *a* and *b* are

[SCRA 1999]

(a) 2, 2 (b) –2, –2 (c) –2, 2 (d) 2, –2

83. If at any point S of the curve $by^2 = (x+a)^3$ the relation between subnormal SN and subtangent ST be $p(SN) = q(ST)^2$ then p/q is equal to

[Rajasthan PET 1999; EAMCET 1991]

(a) $\frac{8b}{27}$ (b) $\frac{8a}{27}$ (c) $\frac{b}{a}$ (d) None of these

84. The points on the curve $9y^2 = x^3$ where the normal to the curve cuts equal intercepts from the axes are

(a) (4, 8/3), (4, –8/3) (b) (1, 1/3) (1, –1/3) (c) (0, 0) (d) None of these

85. The equation of the normal to the curve $y^2 = x^3$ at the point whose abscissa is 8, will be

(a) $x \pm \sqrt{2}y = 104$ (b) $x \pm 3\sqrt{2}y = 104$ (c) $3\sqrt{2}x \pm y = 104$ (d) None of these

86. At any point (except vertex) of the parabola $y^2 - 4ax$ subtangent, ordinate and subnormal are in

(a) AP (b) GP (c) HP (d) None of these

87. At what point the slope of the tangent to the curve $x^2 + y^2 - 2x - 3 = 0$ is zero **[Rajasthan PET 1989, 1995]**

(a) (3 0); (–1, 0) (b) $(3,0); (1,2)$ (c) (–1, 0); (1, 2) (d) (1, 2); (1, –2)

88. Let the equation of a curve be $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$. If θ changes at a constant rate k then the rate of change of the slope of the tangent to the curve at $\theta = \frac{\pi}{3}$ is

(a) $\frac{2k}{\sqrt{3}}$ (b) $\frac{k}{\sqrt{3}}$ (c) k (d) None of these

89. The equation of a curve is $y = f(x)$. The tangents at $(1, f(1)), (2, f(2))$ and $(3, f(3))$ makes angles $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively with the positive direction of the x-axis. Then the value of $\int_2^3 f'(x) f''(x) dx + \int_1^3 f''(x) dx$ is equal to

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) 0 (d) None of these

90. $P(2,2)$ and $Q\left(\frac{1}{2}, -1\right)$ are two points on the parabolas $y^2 = 2x$. the coordinates of the point R on the parabola, where the tangent to the curve is parallel to the chord PQ, is

(a) $\left(\frac{5}{4}, \sqrt{\frac{5}{2}}\right)$ (b) (2, – 1) (c) $\left(\frac{1}{8}, \frac{1}{2}\right)$ (d) None of these

91. The number of tangents to the curve $x^{3/2} + y^{3/2} = a^{3/2}$,. where the tangents are equally inclined to the axes, is

(a) 2 (b) 1 (c) 0 (d) 4

92. If at each point of the curve $y = x^3 - ax^2 + x + 1$ the tangent is inclined at an acute angle with the positive direction of the x-axis then

(a) $a > 0$ (b) $a \le \sqrt{3}$ (c) $-\sqrt{3} \le a \le \sqrt{3}$ (d) None of these

***

Answer Sheet

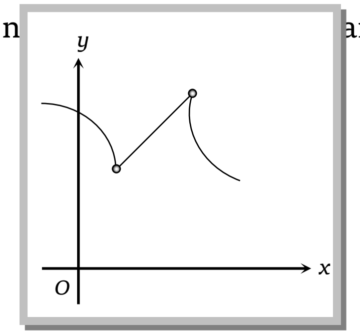
Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	b	a	c	a	c	a	b	a	d	b	a	c	b	b	b	a	d	d	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
b	d	d	a	d	a	c	c	b	c	b	d	a	a	b	a	d	c	d	a
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
c	c	c	c	a	b	c	a	b	c	b	d	d	d	b	c	a	d	b	d
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
d	c	b	d	a	b	d	b	a	d	d	c	d	a	b	b	c	c	a	d
81	82	83	84	85	86	87	88	89	90	91	92								
b	a	a	a	b	b	d	d	a	c	b	c								

4.3 Maxima and Minima

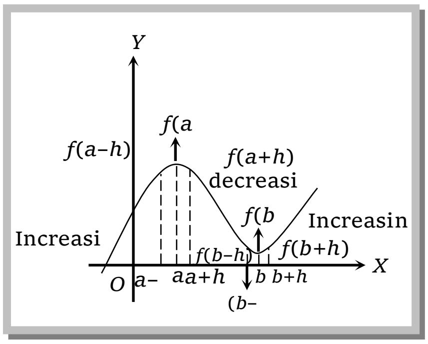
4.3.1 Introduction

In this chapter we shall study those points of the domain of a function where its graph changes its direction from upwards to downwards or from downwards to upwards. At such points the derivative of the function arily zero.



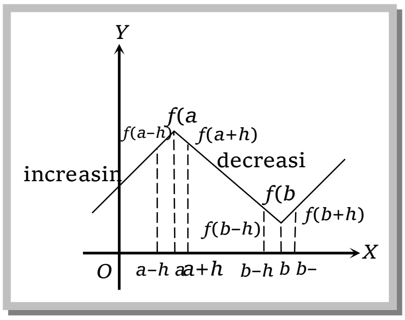
4.3.2 Maximum and Minimum Values of a Function

By the maximum / minimum value of function $f(x)$ we should mean local or regional maximum/minimum and not the greatest / least value attainable by the function. It is also possible in a function that local maximum at one point is smaller than local minimum at another point. Sometimes we use the word extreme for maxima and minima.

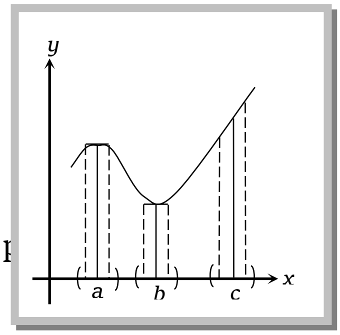


Definition: A function $f(x)$ is said to have a maximum at $x = a$ if $f(a)$ is greatest of all values in the suitably small neighbourhood of a where $x = a$ is an interior point in the domain of $f(x)$. Analytically this means $f(a) \geq f(a+h)$ and $f(a) \geq f(a-h)$ where $h \geq 0$. (very small quantity).

Similarly, a function $y = f(x)$ is said to have a minimum at $x = b$. If $f(b)$ is smallest of all values in the suitably small neighbourhood of b where $x = b$ is an interior point in the domain of $f(x)$. Analytically, $f(b) \leq f(b+h)$ and $f(b) \leq f(b-h)$ where $h \geq 0$. (very small quantity).



Hence we find that,



(i) $x = a$ is a maximum point of $f(x)$ $\begin{cases} f(a) - f(a+h) > 0 \\ f(a) - f(a-h) > 0 \end{cases}$

(ii) $x = b$ is a minimum point of $f(x)$ $\begin{cases} f(b) - f(b+h) < 0 \\ f(b) - f(b-h) < 0 \end{cases}$

(iii) $x = c$ is neither a maximum point nor a minimum p

$\left.\begin{cases} f(c) - f(c+h) \text{ and} \\ f(c) - f(c-h) \end{cases}\right\}$ have opposite signs .

4.3.3 Local Maxima and Local Minima

(1) **Local maximum :** A function $f(x)$ is said to attain a local maximum at $x = a$ if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that $f(x) < f(a)$ for all $x \in (a-\delta, a+\delta), x \neq a$

or $f(x) - f(a) < 0$ for all $x \in (a-\delta, a+\delta), x \neq a$.

In such a case $f(a)$ is called the local maximum value of $f(x)$ at $x = a$.

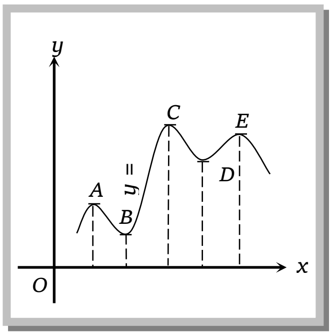
(2) **Local minimum:** A function $f(x)$ is said to attain a local minimum at $x = a$ if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that

$f(x) > f(a)$ for all $x \in (a-\delta, a+\delta), x \neq a$

or $f(x) - f(a) > 0$ for all $x \in (a-\delta, a+\delta), x \neq a$

The value of function at $x = a$ *i.e.,* $f(a)$ is called the local minimum value of $f(x)$ at $x = a$.

The points at which a function attains either the local maximum values or local minimum values are known as the extreme points or turning points and both local maximum and local minimum values are called the extreme values of $f(x)$. Thus, a function attains an extreme value at $x = a$ if $f(a)$ is either a local maximum value or a local minimum value. Consequently at an extreme point 'a' $f(x) - f(a)$ keeps the same sign for all values of *x* in a deleted *nbd* of *a*.



In fig. we observe that the *x*-coordinates of the points *A, C, E* are points of local maximum and the values at these points *i.e.,* their *y*-coordinates are the local maximum values of $f(x)$. The *x*-coordinates of points *B* and *D* are points of local minimum and their *y*-coordinates are the local minimum values of $f(x)$.

Note : ❑ By a local maximum (or local minimum) value of a function at a point $x = a$ we mean the greatest (or the least) value in the neighbourhood of point $x = a$ and not the absolute maximum (or the absolute minimum). In fact a function may have any number of points of local maximum (or local minimum) and even a local minimum value may be greater than a local maximum value. In fig. the minimum value at D is greater than the maximum value at A. Thus, a local maximum value may not be the greatest value and a local minimum value may not be the least value of the function in its domain.

❑ The maximum and minimum points are also known as extreme points.

❑ A function may have more than one maximum and minimum points.

❑ A maximum value of a function $f(x)$ in an interval $[a, b]$ is not necessarily its greatest value in that interval. Similarly, a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.

❑ If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.

❑ Monotonic functions do not have extreme points.

4.3.4 Conditions for Maxima and Minima of a Function

(1) **Necessary condition:** A point $x = a$ is an extreme point of a function $f(x)$ if $f'(a) = 0$, provided $f'(a)$ exists. Thus, if $f'(a)$ exists, then

$$\boxed{\begin{array}{c} x = a \text{ is an extreme point } \Rightarrow f'(a) = 0 \\ \text{or} \\ f'(a) \neq 0 \Rightarrow x = a \text{ is not an extreme point} \end{array}}$$

But its converse is not true *i.e.,* $f'(a) = 0, x = a$ is not an extreme point.

For example if $f(x) = x^3$, then $f'(0) = 0$ but $x = 0$ is not an extreme point.

(2) **Sufficient condition:**

(i) The value of the function $f(x)$ at $x = a$ is maximum, if $f'(a) = 0$ and $f''(a) < 0$.

(ii) The value of the function $f(x)$ at $x = a$ is minimum if $f'(a) = 0$ and $f''(a) > 0$.

Note : ❑ If $f'(a) = 0$, $f''(a) = 0$, $f'''(a) \neq 0$ then $x = a$ is not an extreme point for the function $f(x)$.

❑ If $f'(a) = 0, f''(a) = 0, f'''(a) = 0$ then the sign of $f^{(iv)}$ (a) will determine the maximum and minimum value of function *i.e.,* $f(x)$ is maximum, if $f^{(iv)}(a) < 0$ and minimum if $f^{(iv)}(a) > 0$.

4.3.5 Working rule for Finding Maxima and Minima

(1) Find the differential coefficient of $f(x)$ with respect to x, *i.e.,* $f'(x)$ and equate it to zero.

(2) Find differential real values of x by solving the equation $f'(x)=0$. Let its roots be *a, b, c*......

(3) Find the value of $f''(x)$ and substitute the value of $a_1, a_2, a_3......$ in it and get the sign of $f''(x)$ for each value of x.

(4) If $f''(a)<0$ then the value of $f(x)$ is maximum at $x=a$ and if $f''(a)>0$ then value of $f(x)$ will be minimum at $x=a$. Similarly by getting the signs of $f''(x)$ at other points *b, c*.....we can find the points of maxima and minima.

Example: 1 What are the minimum and maximum values of the function $x^5-5x^4+5x^3-10$ **[DCE 1999; Rajasthan PET 1995]**

(a) $-37, -9$ (b) 10, 0

(c) It has 2 minimum and 1 maximum values (d) It has 2 maximum and 1 minimum values

Solution: (a) $y = x^5 - 5x^4 + 5x^3 - 10$

$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2-4x+3) = 5x^2(x-3)(x-1)$

$\frac{dy}{dx}=0$, gives $x=0,1,3$ ......(i)

Now, $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x = 10x(2x^2-6x+3)$ and $\frac{d^3y}{dx^3} = 10(6x^2-12x+3)$

For $x=0$: $\frac{dy}{dx}=0, \frac{d^2y}{dx^2}=0, \frac{d^3y}{dx^3}\neq 0$, $\therefore$ Neither minimum nor maximum

For $x=1$, $\frac{d^2y}{dx^2}=-10$ =negative, $\therefore$ Maximum value $y_{\max.}=-9$

For $x=3$, $\frac{d^2y}{dx^2}=90$ =positive, $\therefore$ Minimum value $y_{\min.}=-37$.

Example: 2 The maximum value of $\sin x(1+\cos x)$ will be at **[UPSEAT 1999]**

(a) $x=\frac{\pi}{2}$ (b) $x=\frac{\pi}{6}$ (c) $x=\frac{\pi}{3}$ (d) $x=\pi$

Solution: (c) $y = \sin x(1+\cos x) = \sin x + \frac{1}{2}\sin 2x$

$\therefore \frac{dy}{dx} = \cos x + \cos 2x$ and $\frac{d^2y}{dx^2} = -\sin x - 2\sin 2x$

On putting $\frac{dy}{dx}=0, \cos x + \cos 2x = 0 \Rightarrow \cos x = -\cos 2x = \cos(\pi - 2x) \Rightarrow x = \pi - 2x$

$\therefore x = \frac{\pi}{3}$, $\therefore \left(\frac{d^2y}{dx^2}\right)_{x=\pi/3} = -\sin\left(\frac{1}{3}\pi\right) - 2\sin\left(\frac{2}{3}\pi\right) = \frac{-\sqrt{3}}{2} - 2.\frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2}$ which is negative.

$\therefore$ at $x=\frac{\pi}{3}$ the function is maximum.

Example: 3 If $y = a\log x + bx^2 + x$ has its extremum value at $x=1$ and $x=2$, then $(a,b)=$

(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 2\right)$ (c) $\left(2, -\frac{1}{2}\right)$ (d) $\left(-\frac{2}{3}, -\frac{1}{6}\right)$

Solution: (d) $\frac{dy}{dx} = \frac{a}{x} + 2bx + 1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0 \Rightarrow a = -2b - 1$

and $\left(\frac{dy}{dx}\right)_{x=2} = \frac{a}{2} + 4b + 1 = 0 \Rightarrow \frac{-2b-1}{2} + 4b + 1 = 0 \Rightarrow -b + 4b + \frac{1}{2} = 0 \Rightarrow 3b = \frac{-1}{2} \Rightarrow b = \frac{-1}{6}$ and $a = \frac{1}{3} - 1 = \frac{-2}{3}$.

Example: 4 Maximum value of $\left(\frac{1}{x}\right)^x$ is **[DCE 1999; Karnataka CET 1999; UPSEAT 2003]**

(a) $(e)^e$ (b) $(e)^{1/e}$ (c) $(e)^{-e}$ (d) $\left(\frac{1}{e}\right)^e$

Solution: (b) $f(x) = \left(\frac{1}{x}\right)^x \Rightarrow f'(x) = \left(\frac{1}{x}\right)^x \left(\log \frac{1}{x} - 1\right)$

$f'(x) = 0 \Rightarrow \log \frac{1}{x} = 1 = \log e \Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$. Therefore, maximum value of function is $e^{1/e}$.

Example: 5 Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12 (c) 16 (d) 32

Solution: (b) $y = f(x) = -x^3 + 3x^2 + 9x - 27$

The slope of this curve $f'(x) = -3x^2 + 6x + 9$

Let $g(x) = f'(x) = -3x^2 + 6x + 9$

Differentiate with respect to x, $g'(x) = -6x + 6$

Put $g'(x) = 0 \Rightarrow x = 1$

Now, $g''(x) = -6 < 0$ and hence at $x = 1, g(x)$

(Slope) will have maximum value.

$\therefore [g(1)]_{\text{max.}} = -3 \times 1 + 6 + 9 = 12$.

Example: 6 The function $f(x) = \int_1^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$ has a local minimum at $x =$ **[IIT1999]**

(a) 0 (b) 1 (c) 2 (d) 3

Solution: (b, d) $f(x) = \int_{-1}^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$, $\therefore f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$

For local minima, slope *i.e.,* $f'(x)$ should change sign from – *ve* to +*ve*

$f'(x) = 0 \Rightarrow x = 0, 1, 2, 3$

If $x = 0 - h,$ where *h* is a very small number, then $f'(x) = (-)(-)(-1)(-1)(-1) = -ve$

If $x = 0 + h$, $f'(x) = (+)(+)(-)(-1)(-1) = -ve$

Hence at $x = 0$ neither maxima nor minima.

If $x = 1 - h$, $f'(x) = (+)(+)(-)(-1)(-1) = -ve$

If $x = 1 + h$, $f'(x) = (+)(+)(+)(-1)(-1) = +ve$

Hence, at $x = 1$ there is a local minima.

If $x = 2 - h$, $f'(x) = (+)(+1)(+)(-)(-) = +ve$

If $x = 2+h,\ f'(x) = (+)(+)(+)(+)(-1) = -ve$

Hence at $x=2$ there is a local maxima.

If $x = 3-h\,,\ f'(x) = (+)(+)(+)(+)(-) = -ve$

If $x = 3+h\,,\ f'(x) = (+)(+)(+)(+)(+) = +ve$

Hence at $x=3$ there is a local minima.

Example: 7 If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$ attains its maximum and minimum at *p* and *q* respectively such that $p^2 = q$, then a equals

(a) 3 (b) 1 (c) 2 (d) $\frac{1}{2}$

Solution: (c) $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2$

$f''(x) = 12x - 18a$

For maximum and minimum, $6x^2 - 18ax + 12a^2 = 0 \Rightarrow x^2 - 3ax + 2a^2 = 0$

$x = a$ or $x = 2a$ at $x = a$ maximum and at $x = 2a$ minimum

$\because\ p^2 = q$

$a^2 = 2a \Rightarrow a = 2$ or $a = 0$ but $a > 0,$ therefore $a = 2$.

Example: 8 The points of extremum of the function $\phi(x) = \int_1^x e^{-t^2/2}(1-t^2)dt$ are

(a) $x = 0$ (b) $x = 1$ (c) $x = \frac{1}{2}$ (d) $x = -1$

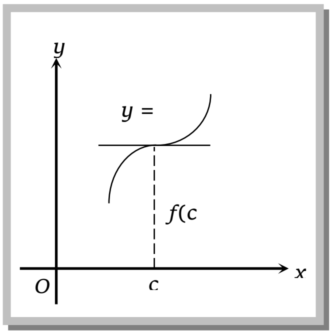
Solution: (b,d) $\phi(x) = \int_1^x e^{-t^2/2}(1-t^2)dt \Rightarrow \phi'(x) = e^{-x^2/2}(1-x^2)$

Now $\phi'(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

Hence, $x = \pm 1$ are points of extremum of $\phi(x)$.

4.3.6 Point of Inflection

A point of inflection is a point at which a curve is changing concave upward to concave downward or vice-versa. A curve $y = f(x)$ has one of its points $x = c$ as an inflection point, if $f''(c) = 0$ or is not defined and if $f''(x)$ changes sign as *x* increases through $x = c$.



The later condition may be replaced by $f'''(c) \neq 0$, when $f'''(c)$ exists.

Thus, $x = c$ is a point of inflection if $f''(c) = 0$ and $f'''(c) \neq 0$.

Properties of maxima and minima

(i) If $f(x)$ is continuous function in its domain, then at least one maxima and one minima must lie between two equal values of *x*.

(ii) Maxima and minima occur alternately, that is, between two maxima there is one minimum and vice-versa.

(iii) If $f(x) \to \infty$ as $x \to a$ or *b* and $f'(x) = 0$ only for one value of *x* (say *c*) between *a* and *b*, then $f(c)$ is necessarily the minimum and the least value.

If $f(x) \to -\infty$ as $x \to a$ or b, then $f(c)$ is necessarily the maximum and the greatest value.

4.3.7 Greatest and Least Values of a Function in a given Interval

If a function $f(x)$ is defined in an interval [a, b], then greatest or least values of this function occurs either at $x = a$ or $x = b$ or at those values of x where $f'(x) = 0$.

Remember that a maximum value of the function $f(x)$ in any interval [a, b] is not necessarily its greatest value in that interval. Thus greatest value of $f(x)$ in interval [a, b] = max. $[f(a), f(b), f(c)]$

Least value of $f(x)$ interval [a, b]= min. $[f(a), f(b), f(c)]$

Where $x = c$ is a point such that $f'(c) = 0$

Example: 9 The maximum and minimum values of $x^3 - 18x^2 + 96$ in interval (0, 9) are **[RPET 1999]**

(a) 160, 0 (b) 60, 0 (c) 160, 128 (d) 120, 28

Solution: (c) Let $y = x^3 - 18x^2 + 96x \Rightarrow \frac{dy}{dx} = 3x^2 - 36x + 96 = 0$

$\therefore$ $x^2 - 12x + 32 = 0 \Rightarrow (x-4)(x-8) = 0, x = 4, 8$

Now, $\frac{d^2y}{dx^2} = 6x - 36$ at $x = 4, \frac{d^2y}{dx^2} = 24 - 36 = -12 < 0$

$\therefore$ at $x = 4$ function will be maximum and $[f(x)]_{\text{max.}} = 64 - 288 + 384 = 160$ at $x = 8\,\frac{d^2y}{dx^2} = 48 - 36 = 12 > 0$

$\therefore$ at $x = 8$ function will be minimum and $[f(x)]_{\text{min.}} = 128$.

Example: 10 The minimum value of the function $2\cos 2x - \cos 4x$ in $0 \le x \le \pi$ is

(a) 0 (b) 1 (c) $\frac{3}{2}$ (d) – 3

Solution: (d) $y = 2\cos 2x - \cos 4x = 2\cos 2x(1 - \cos 2x) + 1 = 4\cos 2x \sin^2 x + 1$

Obviously, $\sin^2 x \ge 0$

Therefore, to be least value of y, cos $2x$ should be least *i.e.,* – 1. Hence least value of y is – 4 + 1 = –3.

Example: 11 On [1, e] the greatest value of $x^2 \log x$ **[AMU 2002]**

(a) e^2 (b) $\frac{1}{e}\log\frac{1}{\sqrt{e}}$ (c) $e^2 \log\sqrt{e}$ (d) None of these

Solution: (a) $f(x) = x^2 \log x \Rightarrow f'(x) = (2\log x + 1)x$

Now $f'(x) = 0 \Rightarrow x = e^{-1/2}, 0$

$\because$ $0 < e^{-1/2} < 1$, $\because$ None of these critical points lies in the interval $[1, e]$

$\therefore$ So we only compare the value of $f(x)$ at the end points 1 and e. We have $f(1) = 0, f(e) = e^2$

$\therefore$ greatest value = e^2

4.3.8 Maxima and Minima of Functions of Two Variables

If a function is defined in terms of two variables and if these variables are associated with a given relation then by eliminating one variable, we convert function in terms of one variable and then find maxima and minima by known methods.

Example: 12 x and y be two variables such that $x > 0$ and $xy = 1$. Then the minimum value of $x + y$ is

[Kurukshetra CEE 1988; MP PET 2002]

(a) 2 (b) 3 (c) 4 (d) 0

Solution: (a) $xy = 1 \Rightarrow y = \frac{1}{x}$ and let $z = x + y$

$z = x + \frac{1}{x} \Rightarrow \frac{dz}{dx} = 1 - \frac{1}{x^2} \Rightarrow \frac{dz}{dx} = 0 \Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = -1, +1$ and $\frac{d^2z}{dx^2} = \frac{2}{x^3}$

$\left(\frac{d^2z}{dx^2}\right)_{x=1} = \frac{2}{1} = 2 = +ve$, $\therefore$ $x = 1$ is point of minima.

$x = 1, y = 1$, $\therefore$ minimum value = $x + y = 2$.

Example: 13 The sum of two non-zero numbers is 4. The minimum value of the sum of their reciprocals is

(a) $\frac{3}{4}$ (b) $\frac{6}{5}$ (c) 1 (d) None of these

Solution: (c) Let $x + y = 4$ or $y = 4 - x$

$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$ or $f(x) = \frac{4}{xy} = \frac{4}{x(4-x)}$

$f(x) = \frac{4}{4x - x^2}$, $f'(x) = \frac{-4}{(4x - x^2)^2}.(4 - 2x)$

Put $f'(x) = 0 \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$ and $y = 2$

$\therefore$ min. $\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{1}{2} + \frac{1}{2} = 1$.

Example: 14 The real number which most exceeds its cube is **[MP PET 2000]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) None of these

Solution: (b) Let number = x, then cube = x^3

Now $f(x) = x - x^3$ (Maximum) $\Rightarrow f'(x) = 1 - 3x^2$

Put $f'(x) = 0 \Rightarrow 1 - 3x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

Because $f''(x) = -6x = -ve$. when $x = +\frac{1}{\sqrt{3}}$.

4.3.9 Geometrical Results related to Maxima and Minima

The following results can easily be established.

(1) The area of rectangle with given perimeter is greatest when it is a square.

(2) The perimeter of a rectangle with given area is least when it is a square.

(3) The greatest rectangle inscribed in a given circle is a square.

(4) The greatest triangle inscribed in given circle is equilateral.

(5) The semi vertical angle of a cone with given slant height and maximum volume is $\tan^{-1}\sqrt{2}$

(6) The height of a cylinder of maximum volume inscribed in a sphere of radius a is a $2a/\sqrt{3}$.

Important Tips

☞ ***Equilateral triangle:*** *Area = $(\sqrt{3}/4)x^2$, where x is its side.*

- ☞ ***Square:*** *Area =* a^2, *perimeter =* $4a$, *where a is its side.*
- ☞ ***Rectangle:*** *Area = ab, perimeter =* $2(a+b)$, *where a, b are its sides.*
- ☞ ***Trapezium:*** *Area =* $\frac{1}{2}(a+b)h$, *where a, b are lengths of parallel sides and h be the distance between them.*
- ☞ ***Circle:*** *Area =* πa^2, *perimeter =* $2\pi a$, *where a is its radius.*
- ☞ ***Sphere:*** *Volume =* $\frac{4}{3}\pi a^3$, *surface area =* $4\pi a^2$, *where a is its radius.*
- ☞ ***Right circular cone:*** *Volume =* $\frac{1}{3}\pi r^2 h$, *curved surface =* $\pi r l$, *where r is the radius of its base, h is its height and l is its slant height.*
- ☞ ***Cylinder:*** *Volume =* $\pi r^2 h$, *whole surface =* $2\pi r(r+h)$, *where r is the radius of the base and h is its height.*

Example: 15 The adjacent sides of a rectangle with given perimeter as 100 *cm* and enclosing maximum area are **[MP PET 1**

(a) 10 *cm* and 40 *cm* (b) 20 *cm* and 30 *cm* (c) 25 *cm* and 25 *cm* (d) 15 *cm* and 35 *cm*

Solution: (c) $2x+2y=100 \Rightarrow x+y=50$ .....(i)

Let area of rectangle is *A*, $\therefore\ A=xy \Rightarrow y=\frac{A}{x}$

From (i), $x+\frac{A}{x}=50 \Rightarrow A=50x-x^2 \Rightarrow \frac{dA}{dx}=50-2x$

for maximum area $\frac{dA}{dx}=0$

$\therefore\ 50-2x=0 \Rightarrow x=25$ and $y=25$

$\therefore$ adjacent sides are 25 *cm* and 25 *cm*.

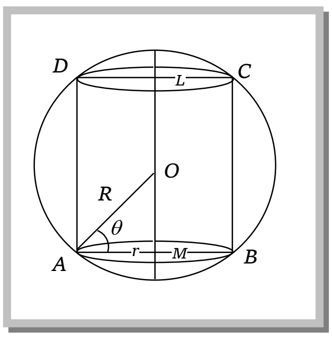
Example: 16 The radius of the cylinder of maximum volume, which can be inscribed a sphere of radius *R* is **[AMU 1999]**

(a) $\frac{2}{3}R$ (b) $\sqrt{\frac{2}{3}}R$ (c) $\frac{3}{4}R$ (d) $\sqrt{\frac{3}{4}}R$

Solution: (b) If *r* be the radius and *h* the height, the from the figure, $r^2+\left(\frac{h}{2}\right)^2=R^2 \Rightarrow h^2=4(R^2-r^2)$

Now, $V=\pi r^2 h=2\pi r^2\sqrt{R^2-r^2}$

$\therefore\ \frac{dV}{dr}=4\pi r\sqrt{R^2-r^2}+2\pi r^2.\frac{1}{2}\frac{(-2r)}{\sqrt{R^2-r^2}}$



For max. or min., $\frac{dV}{dr}=0$

$\Rightarrow 4\pi r\sqrt{R^2-r^2}=\frac{2\pi r^3}{\sqrt{R^2-r^2}} \Rightarrow 2(R^2-r^2)=r^2$

$\Rightarrow 2R^2=3r^2 \Rightarrow r=\sqrt{\frac{2}{3}}R \Rightarrow \frac{d^2V}{dr^2}=-ve$. Hence *V* is max. when $r=\sqrt{\frac{2}{3}}R$.

Example: 17 The ratio of height of a cone having maximum volume which can be inscribed in a sphere with the diameter of sphere is

[MNR 1985]

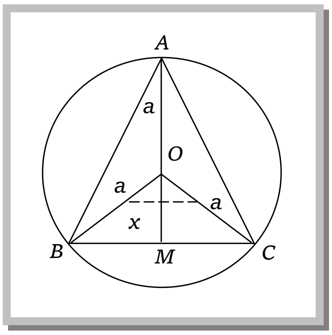
(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{1}{4}$

Solution: (a) Let $OM = x$

Then height of cone *i.e.,* $h = x + a$ (where *a* is radius of sphere)

Radius of base of cone = $\sqrt{a^2 - x^2}$

Therefore, volume $V = \frac{1}{3}\pi(a^2 - x^2)(x + a) \Rightarrow \frac{dV}{dx} = \frac{\pi}{3}(a + x)(a - 3x)$



Now, $\frac{dV}{dx} = 0 \Rightarrow x = -a, \frac{a}{3}$

But $x \neq -a$, So, $x = \frac{a}{3}$

The volume is maximum at $x = \frac{a}{3}$

Height of a cone $h = a + \frac{a}{3} = \frac{4}{3}a$

Therefore ratio of height and diameter = $\frac{\frac{4}{3}a}{2a} = \frac{2}{3}$.

Assignment

Maxima and Minima

Basic Level

1. The maximum value of $f(x) = \frac{x}{4 + x + x^2}$ on [– 1, 1] is **[MP PET 2000]**

(a) $\frac{-1}{4}$ (b) $\frac{-1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{5}$

2. Maximum value of $x(1-x)^2$ when $0 \le x \le 2$, is **[MP PET 1997]**

(a) 2 (b) $\frac{4}{27}$ (c) 5 (d) 0

3. The maximum value of $2x^3 - 24x + 107$ in the interval [– 3, 3] is

(a) 75 (b) 89 (c) 125 (d) 139

4. The maximum value of the function $f(x) = 3\sin x + 4\cos x$ is

(a) 3 (b) 4 (c) 5 (d) 7

5. If the function $f(x) = x^4 - 62x^2 + ax + 9$ is maximum at $x = 1$, then the value of *a* is

(a) 120 (b) – 120 (c) 52 (d) 128

6. The maximum value of $f(\theta) = a\sin\theta + b\cos\theta$ is **[MP PET 1999; UPSEAT 2000]**

(a) $\frac{a}{b}$ (b) $\frac{a}{\sqrt{a^2+b^2}}$ (c) $\sqrt{ab}$ (d) $\sqrt{a^2+b^2}$

7. The minimum value of the function $y = 2x^3 - 21x^2 + 36x - 20$ is

(a) – 128 (b) – 126 (c) – 120 (d) None of these

8. $\frac{x}{1 + x\tan x}$ is maximum at **[UPSEAT 1999]**

(a) $x = \sin x$ (b) $x = \cos x$ (c) $x = \frac{\pi}{3}$ (d) $x = \tan x$

9. The minimum value of the expression $7 - 20x + 11x^2$ is

(a) $\frac{177}{11}$ (b) $-\frac{177}{11}$ (c) $-\frac{23}{11}$ (d) $\frac{23}{11}$

10. The minimum value of $2x^2 + x - 1$ is **[EAMCET 2003]**

(a) $\frac{-1}{4}$ (b) $\frac{3}{2}$ (c) $\frac{-9}{8}$ (d) $\frac{9}{4}$

11. The maximum value of *xy* subject to $x + y = 8$, is **[MNR 1995]**

(a) 8 (b) 16 (c) 20 (d) 24

12. If $A + B = \frac{\pi}{2}$, the maximum value of $\cos A \cos B$ is **[AMU 1999]**

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) 1 (d) $\frac{4}{3}$

13. If $xy = c^2$, then minimum value of $ax + by$ is **[Rajasthaan PET 2001]**

(a) $c\sqrt{ab}$ (b) $2c\sqrt{ab}$ (c) $-c\sqrt{ab}$ (d) $-2c\sqrt{ab}$

14. If $a^2x^4 + b^2y^4 = c^6$, then maximum value of xy is **[Rajasthan PET 2001]**

(a) $\frac{c^2}{\sqrt{ab}}$ (b) $\frac{c^3}{ab}$ (c) $\frac{c^3}{\sqrt{2ab}}$ (d) $\frac{c^3}{2ab}$

15. The function $f(x) = 2x^3 - 15x^2 + 36x + 4$ is maximum at **[Karnataka CET 2001]**

(a) $x = 2$ (b) $x = 4$ (c) $x = 0$ (d) $x = 3$

16. The function $f(x) = x^{-x}, (x \in R)$ attains a maximum value at $x =$

(a) 2 (b) 3 (c) $\frac{1}{e}$ (d) 1

17. The function $y = a(1 - \cos x)$ is maximum when $x =$ **[Kerala (Engg.) 2002]**

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{-\pi}{2}$ (d) $\frac{-\pi}{6}$

18. The minimum value of $\left(x^2 + \frac{250}{x}\right)$ is **[Haryana CEE 2002]**

(a) 75 (b) 50 (c) 25 (d) 55

19. In the graph of the function $\sqrt{3}\sin x + \cos x$ the maximum distance of a point from *x*-axis is

(a) 4 (b) 2 (c) 1 (d) $\sqrt{3}$

20. The function $f(x)=x+\sin x$ has **[AMU 2000]**

(a) A minimum but no maximum (b) A maximum but no minimum

(c) Neither maximum nor minimum (d) Both maximum and minimum

21. The point for the curve $y=xe^x$

(a) $x=-1$ is minimum (b) $x=0$ is minimum (c) $x=-1$ is maximum (d) $x=0$ is maximum

22. 36 is factorized into two factors in such a way that sum of factors is minimum, then the factors are

(a) 2, 18 (b) 9, 4 (c) 3, 12 (d) None of these

23. The necessary condition to be maximum or minimum for the function is

(a) $f'(x)=0$ and it is sufficient (b) $f''(x)=0$ and it is sufficient

(c) $f'(x)=0$ but it is not sufficient (d) $f'(x)=0$ and $f''(x)=-ve$

24. The maximum and minimum value of the function $3x^4-8x^3+12x^2-48x+25$ in the interval [1, 3]

(a) 16, – 39 (b) – 16, 39 (c) 6, – 9 (d) None of these

25. If $f(x)=2x^3-3x^2-12x+5$ and $x\in[-2,4]$, then the maximum value of function is at the following value of x **[MP PET 1987**

(a) 2 (b) – 1 (c) – 2 (d) 4

26. The minimum value of $|x|+|x+\frac{1}{2}|+|x-3|+|x-\frac{5}{2}|$ is

(a) 0 (b) 2 (c) 4 (d) 6

27. The maximum value of the function x^3+x^2+x-4 is

(a) 127 (b) 4

(c) Does not have a maximum value (d) None of these

28. The function $x^5-5x^4+5x^3-10$ has a maximum when x =

(a) 3 (b) 2 (c) 1 (d) 0

29. If $x-2y=4$, the minimum value of xy is **[UPSEAT 2003]**

(a) – 2 (b) 2 (c) 0 (d) – 3

30. The minimum value of $x^2+\frac{1}{1+x^2}$ is at **[UPSEAT 2003]**

(a) $x=0$ (b) $x=1$ (c) $x=4$ (d) $x=3$

31. The maximum and minimum value of the function $|\sin 4x+3|$ are

(a) 1, 2 (b) 4, 2 (c) 2, 4 (d) – 1, 1

32. The maximum value of function $x^3-12x^2+36x+17$ in the interval [1, 10] is

(a) 17 (b) 177 (c) 77 (d) None of these

33. Let $f(x)=(x-p)^2+(x-q)^2+(x-r)^2$. Then $f(x)$ has a minimum at $x=\lambda$, where λ is equal to

(a) $\frac{p+q+r}{3}$ (b) $3\sqrt{pqr}$ (c) $\frac{3}{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}}$ (d) None of these

34. The function $x^2\log x$ in the interval $(1, e)$ has

(a) A point of maximum (b) A point of minimum

(c) Points of maximum as well as of minimum (d) Neither a point of maximum nor minimum

35. The two parts of 100 for which the sum of double of first and square of second part is minimum, are

(a) 50, 50 (b) 99, 1 (c) 98, 2 (d) None of these

36. Of the given perimeter, the triangle having maximum area is

(a) Isosceles triangle (b) Right angled triangle (c) Equilateral (d) None of these

37. The function $x^5 - 5x^4 + 5x^3 - 1$ is

(a) Maximum at $x = 3$ and minimum at $x = 1$ (b) Minimum at $x = 1$

(c) Neither maximum nor minimum at $x = 0$ (d) Maximum at $x = 0$

38. If $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ then the difference between the maximum and minimum values of u^2 is given by

[AIEEE 2004]

(a) $(a-b)^2$ (b) $2\sqrt{a^2 + b^2}$ (c) $(a+b)^2$ (d) $2(a^2 + b^2)$

39. The minimum value of $2x + 3y$, when $xy = 6$, is **[MP PET 2003]**

(a) 12 (b) 9 (c) 8 (d) 6

40. The real number x when added to its inverse gives the minimum value of the sum at x equal to **[Rajasthan PET 2000; AIEEE 2003]**

(a) – 2 (b) 2 (c) 1 (d) – 1

41. $x + \frac{1}{x}$ is maximum at **[Rajasthan PET 1991]**

(a) $x = 1$ (b) $x = -1$ (c) $x = 2$ (d) $x = -2$

42. $f(x) = (1-x)^2 e^x$ is minimum at

(a) $x = 1$ (b) $x = -1$ (c) $x = 0$ (d) $x = 2$

43. The maximum value of the function $x^3 - 12x^2 + 45x$ is **[Rajasthan PET 1994]**

(a) 0 (b) 50 (c) 54 (d) 70

Advance Level

44. Let $f(x) = \begin{cases} |x| & , \quad 0 < |x| \le 2 \\ 1 & , \quad x = 0 \end{cases}$, then at $x = 0$ f has **[IIT Screening 2000]**

(a) A local maximum (b) No local maximum (c) A local minimum (d) No extremum

45. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f

(a) Does not exist because f is unbounded (b) Is not attained even though f is bounded

(c) Is equal to 1 (d) Is equal to –1

46. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is

(a) 0 (b) 1 (c) 2 (d) Infinite

47. On the interval [0, 1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point **[IIT 1995]**

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

48. x^x has a stationary point at

(a) $x = e$ (b) $x = \frac{1}{e}$ (c) $x = 1$ (d) $x = \sqrt{e}$

49. A minimum value of $\int_0^x te^{-t^2} dt$ is

(a) 1 (b) 2 (c) 3 (d) 0

50. The sum of two numbers is fixed. Then its multiplication is maximum, when

(a) Each number is half of the sum (b) Each number is $\frac{1}{3}$ and $\frac{2}{3}$ respectively of the sum

(c) Each number is $\frac{1}{4}$ and $\frac{3}{4}$ respectively of the sum (d) None of these

51. The value of *a* so that the sum of the squares of the roots of the equation $x^2-(a-2)x-a+1=0$ assume the least value, is

(a) 2 (b) 1 (c) 3 (d) 0

52. If from a wire of length 36 *metre* a rectangle of greatest area is made, then its two adjacent sides in *metre* are**[MP PET**

(a) 6, 12 (b) 9, 9 (c) 10, 8 (d) 13, 5

53. The maximum value of $x^4e^{-x^2}$ is

(a) e^2 (b) e^{-2} (c) $12e^{-2}$ (d) $4e^{-2}$

54. One maximum point of $\sin^p x\cos^q x$ is **[Rajasthan PET 1997; AMU 2000]**

(a) $x=\tan^{-1}\sqrt{(p/q)}$ (b) $x=\tan^{-1}\sqrt{(q/p)}$ (c) $x=\tan^{-1}(p/q)$ (d) $x=\tan^{-1}(q/p)$

55. 20 is divided into two parts so that product of cube of one quantity and square of the other quantity is maximum. The parts are

[Rajasthan PET 1997]

(a) 10, 10 (b) 16, 4 (c) 8, 12 (d) 12, 8

56. The minimum value of $e^{(2x^2-2x+1)\sin^2 x}$ is **[Rorkee 1998]**

(a) e (b) $\frac{1}{e}$ (c) 1 (d) 0

57. Divide 20 into two parts such that the product of one part and the cube of the other is maximum. The two parts are **[DCE 1999]**

(a) (10, 10) (b) (5, 15) (c) (13, 7) (d) None of these

58. The minimum value of $\exp(2+\sqrt{3}\cos x+\sin x)$ is **[AMU 1999]**

(a) exp(2) (b) $\exp(2-\sqrt{3})$ (c) exp(4) (d) 1

59. The minimum value of $\frac{\log x}{x}$ in the interval $[2, \infty)$ **[Roorkee 1999]**

(a) Is $\frac{\log 2}{2}$ (b) Is zero (c) Is $\frac{1}{e}$ (d) Does not exist

60. The function $f(x)=ax+\frac{b}{x}, a, b, x>0$ takes on the least value at *x* equal to **[AMU 2000]**

(a) b (b) $\sqrt{a}$ (c) $\sqrt{b}$ (d) $\sqrt{b/a}$

61. The area of a rectangle of given perimeter is maximum, when ratio of its length and breadth is

(a) 2 : 1 (b) 3 : 2 (c) 4 : 3 (d) 1 : 1

62. The denominator of a fraction number is greater than 16 of the square of numerator, then least value of the number is

[Rajasthan PET 2000]

(a) $\frac{-1}{4}$ (b) $\frac{-1}{8}$ (c) $\frac{1}{12}$ (d) $\frac{1}{16}$

63. If for a function $f(x)$, $f'(a)=0$, $f''(a)=0, f'''(a)>0$, then at $x=a$, $f(x)$ is **[MP PET 1994]**

(a) Minimum (b) Maximum (c) Not an extreme points (d) Extreme point

64. The least value of the sum of any positive real number and its reciprocal is **[MP PET 1994]**

(a) 1 (b) 2 (c) 3 (d) 4

65. If x is real, then greatest and least values of $\frac{x^2-x+1}{x^2+x+1}$ are **[Rajasthan PET 1999; AMU 1999; UPSEAT 2002]**

(a) $3, -\frac{1}{2}$ (b) $3, \frac{1}{3}$ (c) $-3, -\frac{1}{3}$ (d) None of these

66. A wire of constant length is given. In which shape it should be bent to surround maximum area

(a) Circle (b) Square (c) Both (a) and (b) (d) Neither (a) nor (b)

67. The function $x\sqrt{1-x^2}, (x>0)$ has

(a) A local maxima (b) A local minima
(c) Neither a local maxima nor a local minima (d) None of these

68. If $x+y=16$ and x^2+y^2 is minimum, the value of x and y are

(a) 3, 13 (b) 4, 12 (c) 6, 10 (d) 8, 8

69. The area of a rectangle will be maximum for the given perimeter. When rectangle is a

(a) Parallelogram (b) Trapezium (c) Square (d) None of these

70. Local maximum value of the function $\frac{\log x}{x}$ is

[MNR 1984; Rajasthan PET 1997, 2002; DCE 2002; Karnataka CET 2000; UPSEAT 2001; MP PET 2002]

(a) e (b) 1 (c) $\frac{1}{e}$ (d) $2e$

71. Local maximum and local minimum values of the function $(x-1)(x+2)^2$ are

(a) – 4, 0 (b) 0, – 4 (c) 4,0 (d) None of these

72. If $f(x)=2x^3-21x^2+36-30$, then which one of the following is correct

(a) $f(x)$ has minimum at $x=1$ (b) $f(x)$ has maximum at $x=6$
(c) $f(x)$ has maximum at $x=1$ (d) $f(x)$ has no maxima or minima

73. If sum of two numbers is 3, then maximum value of the product of first and the square of second is

(a) 4 (b) 3 (c) 2 (d) 1

74. If $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$ such that min $f(x)>$ max $g(x)$, then the relation between b and c is **[IIT Screening 2003]**

(a) No real value of b and c (b) $0<c<b\sqrt{2}$ (c) $|c|<|b|\sqrt{2}$ (d) $|c|>|b|\sqrt{2}$

75. The minimum value of $[(5+x)(2+x)]/[1+x]$ for non-negative real x is

(a) 12 (b) 1 (c) 9 (d) 8

76. Let $f(x)=\int_0^x \frac{\cos t}{t}dt, x>0$ then $f(x)$ has **[Haryana CEE 2002]**

(a) Maxima when $n=-2,-4,-6.....$ (b) Maxima $n=-1,-3,-5,......$
(c) Minima when $n=0,2,4,......$ (d) Minima when $n=1,3,5,......$

77. The function $f(x)=2x^3-3x^2-12x+4$ has **[DCE 2002]**

(a) No maxima and minima (b) One maximum and one minimum
(c) Two maxima (d) Two minima

78. If $f(x)=\frac{1}{4x^2+2x+1}$, then its maximum value is **[Rajasthan PET 2002]**

(a) $\frac{4}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{3}{4}$

79. If PQ and PR are the two sides of a triangle, then the angle between them which gives maximum area of the triangle is

[Kerala (Engg.) 2002]

(a) π (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

80. If $ab = 2a + 3b, a > 0, b > 0$ then the minimum value of ab is **[Orissa JEE 2002]**

(a) 12 (b) 24 (c) $\frac{1}{4}$ (d) None of these

81. The perimeter of a sector is p. The area of the sector is maximum when its radius is **[Karnataka CET 2002]**

(a) $\sqrt{p}$ (b) $\frac{1}{\sqrt{p}}$ (c) $\frac{p}{2}$ (d) $\frac{p}{4}$

82. The maximum area of the rectangle that can be inscribed in a circle of radius r is **[EAMCET 1994]**

(a) πr^2 (b) r^2 (c) $\frac{\pi r^2}{4}$ (d) $2r^2$

83. If $f(x) = \begin{cases} 3x^2 + 12x - 1 & , \ -1 \le x \le 2 \\ 37 - x & , \ 2 < x \le 3 \end{cases}$, then **[IIT 1993]**

(a) $f(x)$ is increasing $[-1, 2]$ (b) $f(x)$ is continuous in [– 1, 3] (c) $f(x)$ is maximum at $x = 2$ (d)

84. If $f'(x) = (x-a)^{2n}(x-b)^{2p+1}$ when n and p are positive integers, then

(a) $x = a$ is a point of minimum (b) $x = a$ is a point of maximum

(c) $x = a$ is not a point of maximum or minimum (d) None of these

85. N characters of information are held on magnetic tape, in batches of x characters each, the batch processing time is $\alpha + \beta x^2$ seconds, α and β are constants. The optical value of x for fast processing is

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

86. If $f(x) = \sin^6 x + \cos^6 x$, then

(a) $f(x) \le 1$ (b) $f(x) \le 2$ (c) $f(x) > \frac{1}{4}$ (d) $f(x) > \frac{1}{8}$

87. The maximum and minimum values of $y = \frac{ax^2 + 2bx + c}{Ax^2 + 2Bx + C}$ are those for which

(a) $ax^2 + 2bx + c - y(Ax^2 + 2Bx + C)$ is equal to zero (b) $ax^2 + 2bx + c - y(Ax^2 + 2Bx + C)$ is a perfect square

(c) $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \ne 0$ (d) None of these

88. A differentiable function $f(x)$ has a relative minimum at $x = 0$, then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for

(a) All a and all b (b) All b if $a = 0$ (c) All $b > 0$ (d) All $a > 0$

89. An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Then area of the triangle is maximum when θ

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

90. The greatest value of the function $f(x) = \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$ on the interval $\left[0, \frac{\pi}{2}\right]$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) 1 (d) $-\sqrt{2}$

91. The longest distance of the point (a, 0) from the curve $2x^2 + y^2 - 2x = 0$, is given by

(a) $\sqrt{1 - 2a + a^2}$ (b) $\sqrt{1 + 2a + 2a^2}$ (c) $\sqrt{1 + 2a - a^2}$ (d) $\sqrt{1 - 2a + 2a^2}$

92. The function $f(x)=\int_1^x \{2(t-1)(t-2)^3+3(t-1)^2(t-2)^2 dt$ attains its maximum at $x=$

(a) 1 (b) 2 (c) 3 (d) 4

93. If the function $f(x)=x^3+3(a-7)x^2+3(a^2-9)x-1$ has a positive point of maximum, then

(a) $a\in(3,\infty)\cup(-\infty,-3)$ (b) $a\in(-\infty,-3)\cup\left(3,\frac{29}{7}\right)$ (c) $(-\infty,7)$ (d) $\left(-\infty,\frac{29}{7}\right)$

94. The minimum value of $\left(1+\frac{1}{\sin^n\alpha}\right)\left(1+\frac{1}{\cos^n\alpha}\right)$ is

(a) 1 (b) 2 (c) $(1+2^{n/2})^2$ (d) None of these

95. A cubic $f(x)$ vanishes at $x=-2$ and has relative minimum/maximum at $x=-1$ and $x=\frac{1}{3}$ such that $\int_{-1}^{1} f(x)dx=\frac{14}{3}$. Then $f(x)$ is

(a) x^3+x^2-x (b) x^3+x^2-x+1 (c) x^3+x^2-x+2 (d) x^3+x^2-x-2

96. If $A>0,B>0$ and $A+B=\frac{\pi}{3}$, then the maximum value of $\tan A\tan B$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{3}$ (c) 3 (d) $\sqrt{3}$

97. Total number of parallel tangents of f $f_1(x)=x^2-x+1$ and x^3-x^2-2x+1 is equal to

(a) 2 (b) 3 (c) 4 (d) None of these

98. Function $f(x)=|px-q|+r|x|\ (p>0,q>0,r>0)$ attains its minimum value only at one point, if

(a) $p\neq q$ (b) $q\neq r$ (c) $r\neq p$ (d) $p=q=r$

99. The height of right circular cylinder of maximum volume inscribed in a sphere of radius *a* is

(a) $\frac{a}{\sqrt{3}}$ (b) $\sqrt{3}a$ (c) $\frac{2a}{\sqrt{3}}$ (d) $2\sqrt{3}a$

100. A line is drawn through a fixed point (*a*, *b*), $(a>0,b>0)$ to meet the positive direction of the coordinate axes in *P*,*Q* respectively. The minimum value of $OP+OQ$ is

(a) $\sqrt{a}+\sqrt{b}$ (b) $(\sqrt{a}+\sqrt{b})^2$ (c) $(\sqrt{a}+\sqrt{b})^3$ (d) None of these

101. For the curve $\frac{C^4}{r^2}=\frac{a^2}{\sin^2\theta}+\frac{b^2}{\cos^2\theta}$, the maximum value of *r* is

(a) $\frac{c^2}{a+b}$ (b) $\frac{a+b}{c^2}$ (c) $\frac{c^2}{a-b}$ (d) $c^2(a+b)$

102. The coordinates of a point situated on the curve $4x^2+a^2y^2=4a^2(4<a^2<8)$, which are at maximum distance from the point (0, – 2) is

(a) (*a*, 0) (b) $(2a,-4)$ (c) (0, 2) (d) None of these

103. For what value of *k*, the function: $f(x)=kx^2+\frac{2k^2-81}{2}x-12$, is maximum at $x=\frac{9}{4}$

(a) $\frac{9}{2}$ (b) -9 (c) $\frac{-9}{2}$ (d) 9

104. If $\alpha<\beta$, then correct statement is

(a) $\alpha-\sin\alpha>\beta-\sin\beta$ (b) $\alpha-\sin\alpha<\beta-\sin\beta$ (c) $\sin\alpha-\alpha<-\sin\beta+\beta$ (d) None of these

105. The difference between two numbers is a if their product is minimum, then number are

(a) $\frac{-a}{2},\frac{a}{2}$ (b) $-a,2a$ (c) $\frac{-a}{3},\frac{2a}{3}$ (d) $\frac{-a}{3},\frac{4a}{3}$

106. If λ, μ be real numbers such that $x^3 - \lambda x^2 + \mu x - 6 = 0$ has its roots real and positive then the minimum value of μ is

(a) $3 \times \sqrt[3]{36}$ (b) 11 (c) 0 (d) None of these

107. Let the tangent to the graph of $y = f(x)$ at the point $x = a$ be parallel to the x-axis, let $f'(a-h) > 0$ and $f'(a+h) < 0$, where h is a very small positive number. Then the ordinate of the point is

(a) A maximum (b) A minimum

(c) Both a maximum and a minimum (d) Neither a maximum nor a minimum

108. If $a > b > 0$, the minimum value of $a \sec\theta - b\tan\theta$ is

(a) $b - a$ (b) $\sqrt{a^2 + b^2}$ (c) $\sqrt{a^2 - b^2}$ (d) $2\sqrt{a^2 - b^2}$

109. Let the function $f(x)$ be defined as below:

$$f(x) = \sin^{-1}\lambda + x^2, 0 < x < 1; \quad 2x, x \ge 1$$

$f(x)$ can have a minimum at $x = 1$ if the value of λ is

(a) 1 (b) – 1 (c) 0 (d) None of these

110. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extreme at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha).f(\beta) < 0$. Then the equation $f(x) = 0$ has

(a) Three equal real roots (b) Three distinct real roots

(c) One positive root if $f(\alpha) < 0$ and $f(\beta) > 0$ (d) One negative root if $f(\alpha) > 0$ and $f(\beta) < 0$

111. Let $f(x) = 1 + 2x^2 + 2^2x^4 + ... + 2^{10}x^{20}$; Then $f(x)$ has

(a) More than one minimum (b) Exactly one minimum (c) At least one maximum(d)

112. Let the function $f(x)$ be defined as follows:

$$f(x) = x^3 + x^2 - 10x, \quad -1 \le x < 0$$

$$\cos x, \quad 0 \le x < \frac{\pi}{2}; \quad 1 + \sin x, \quad \frac{\pi}{2} \le x \le \pi$$

Then $f(x)$ has

(a) A local minimum at $x = \frac{\pi}{2}$ (b) A local maximum at $x = \frac{\pi}{2}$

(c) An absolute minimum at $x = -1$ (d) An absolute maximum at $x = \pi$

113. Two part of 64 such that the sum of their cubes is minimum will be

(a) 44, 20 (b) 16, 48 (c) 32, 32 (d) 50, 14

114. If x be real then the minimum value of $f(x) = 3^{x+1} + 3^{-(x+1)}$ is

(a) 2 (b) 6 (c) $\frac{2}{3}$ (d) $\frac{7}{9}$

115. The minimum value of $e^{(2x^2 - 2x - 1)\sin^2 x}$ is **[Roorkee 1998]**

(a) e (b) $\frac{1}{e}$ (c) 1 (d) 0

116. The semi-vertical angle of a right circular cone of given slant height and maximum volume is

(a) $\tan^{-1} 2$ (b) $\tan^{-1}\sqrt{2}$ (c) $\tan^{-1} 1/2$ (d) $\tan^{-1} 1/\sqrt{2}$

117. If $0 < a < x$, then the minimum value of $\log_a x + \log_a a$ is **[IIT 1984]**

(a) 2 (b) – 2 (c) 2a (d) Does not exist

118. Which point of the parabola $y = x^2$ is nearest to the point (3, 0)

(a) (– 1, 1) (b) (1, 1) (c) (2, 4) (d) (– 2, 4)

119. The point of inflexion for the curve $y = x^{5/2}$ is **[Rajasthan PET 1989, 1992]**

(a) (1, 1) (b) (0, 0) (c) (1, 0) (d) (0, 1)

***

Answer Sheet

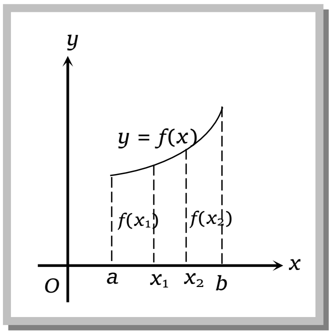
Assignment (Basic & Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	a	d	c	a	d	a	b	c	c	b	a	b	c	a	c	a	a	b	c
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
a	d	c	a	d	d	c	c	a	a	b	b	a	d	b	c	c	a	a	c
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
b	a	c	a	d	b	d	b	d	a	b	b	d	a	d	c	b	d	d	d
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
d	b	c	b	b	a	a	d	c	c	b	c	a	d	c	a,d	b	a	d	b
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
d	d	d	c	c	a,c	b,c	b	a	c	d	a	b	c	c	b	d	d	c	b
101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	
a	c	b	b	a	a	a	c	d	b,c	b	b	c	a	c	b	d	b	b	

4.4 Increasing and Decreasing Function

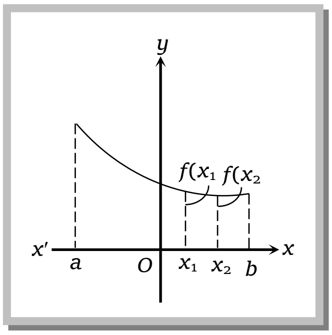
4.4.1 Definition

(1) **Strictly increasing function :** A function $f(x)$ is said to be a strictly increasing function on (a, b), if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all x_1, $x_2 \in (a, b)$.



Thus, $f(x)$ is strictly increasing on (a, b), if the values of $f(x)$ increase with the increase in the values of x.

(2) **Strictly decreasing function :** A function $f(x)$ is said to be a strictly decreasing function on (a, b), if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$. Thus, $f(x)$ is strictly decreasing on (a, b), if the values of $f(x)$ decrease with the increase in the values of x.



Example: 1 On the interval $(1, 3)$ the function $f(x) = 3x + \frac{2}{x}$ is **[AMU 1999]**

(a) Strictly decreasing (b) Strictly increasing

(c) Decreasing in (2, 3) only (d) Neither increasing nor decreasing

Solution: (b) $f(x) = 3x + \frac{2}{x} \Rightarrow f'(x) = 3 - \frac{2}{x^2}$

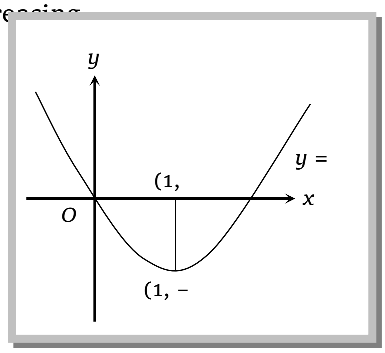
Clearly $f'(x) > 0$ on the interval (1, 3)

$\therefore$ $f(x)$ is strictly increasing.

Example: 2 For which value of x, the function $f(x) = x^2 - 2x$ is decreasing

(a) $x > 1$ (b) $x > 2$

(c) $x < 1$ (d) $x < 2$



Solution: (c) $f(x) = (x-1)^2 - 1$

Hence decreasing in $x < 1$

Alternative method: $f'(x) = 2x - 2 = 2(x-1)$

To be decreasing, $2(x-1) < 0 \Rightarrow (x-1) < 0 \Rightarrow x < 1$.

Example: 3 $2x^3 + 18x^2 - 96x + 45 = 0$ is an increasing function when

(a) $x \le -8, x \ge 2$ (b) $x < -2, x \ge 8$ (c) $x \le -2, x \ge 8$ (d) $0 < x \le -2$

Solution: (a) $f'(x) = 6x^2 + 36x - 96 > 0$, for increasing

$\Rightarrow f'(x) = 6(x+8)(x-2) \ge 0 \Rightarrow x \ge 2, x \le -8$.

Example: 4 The function x^x is increasing, when **[MP PET 2003]**

(a) $x > \frac{1}{e}$ (b) $x < \frac{1}{e}$ (c) $x < 0$ (d) For all real x

Solution: (a) Let $y = x^x \Rightarrow \frac{dy}{dx} = x^x(1 + \log x)$; For $\frac{dy}{dx} > 0$

$x^x(1 + \log x) > 0 \Rightarrow 1 + \log x > 0 \Rightarrow \log_e x > \log_e \frac{1}{e}$

For this to be positive, x should be greater than $\frac{1}{e}$.

4.4.2 Monotonic Function

A function $f(x)$ is said to be monotonic on an interval (a, b) if it is either increasing or decreasing on (a, b).

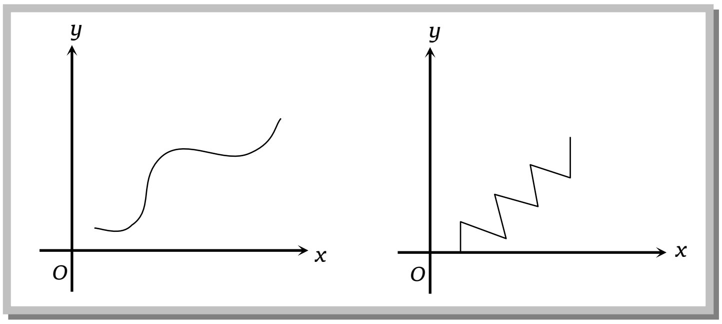
(1) **Monotonic increasing function :** A function is said to be a monotonic increasing function in defined interval if,

$x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$

or $x_1 > x_2 \Rightarrow f(x_1) \nless f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) \ngtr f(x_2)$

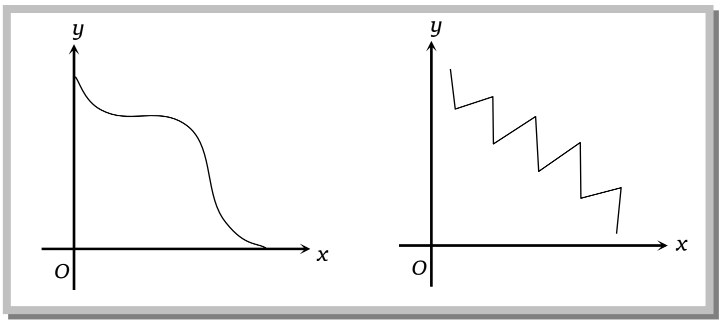


(2) **Monotonic decreasing function:** A function is said to be a monotonic decreasing function in defined interval, if $x_1 > x_2 \Rightarrow f(x_1)$

or $x_1 > x_2 \Rightarrow f(x_1) \ngtr f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

or $x_1 < x_2 \not\Rightarrow f(x_1) < f(x_2)$



Example: 5 The function $f(x) = \cos x - 2px$ is monotonically decreasing for **[MP PET 2002]**

(a) $p < \frac{1}{2}$ (b) $p > \frac{1}{2}$ (c) $p < 2$ (d) $p > 2$

Solution: (b) $f(x)$ will be monotonically decreasing, if $f'(x) < 0$.

$\Rightarrow f'(x) = -\sin x - 2p < 0 \Rightarrow \frac{1}{2}\sin x + p > 0 \Rightarrow p > \frac{1}{2}$ $[\because -1 \le \sin x \le 1]$

Example: 6 If $f(x) = x^5 - 20x^3 + 240x$, then $f(x)$ satisfies which of the following **[Kurukshetra CEE 1996]**

(a) It is monotonically decreasing everywhere (b) It is monotonically decreasing only in $(0, \infty)$

(c) It is monotonically increasing every where (d) It is monotonically increasing only in $(-\infty, 0)$

Solution: (c) $f'(x) = 5x^4 - 60x^2 + 240 = 5(x^4 - 12x^2 + 48) = 5[(x^2 - 6)^2 + 12]$

$\Rightarrow f'(x) > 0 \;\forall\; x \in R$

i.e., $f(x)$ is monotonically increasing everywhere.

Example: 7 The value of a for which the function $(a+2)x^3 - 3ax^2 + 9ax - 1$ decrease monotonically throughout for all real x, are

[Kurukshetra CEE 2002]

(a) $a < -2$ (b) $a > -2$ (c) $-3 < a < 0$ (d) $-\infty < a \le -3$

Solution: (d) If $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \le 0$ for all $x \in R$

$\Rightarrow 3(a+2)x^2 - 6ax + 9a \le 0$ for all $x \in R$ $\Rightarrow (a+2)x^2 - 2ax + 3a \le 0$ for all $x \in R$

$\Rightarrow a + 2 < 0$ and discriminant ≤ 0 $\Rightarrow a < -2$ and $-8a^2 - 24a \le 0$

$\Rightarrow a < -2$ and $a(a+3) \ge 0 \Rightarrow a < -2$ and $a \le -3$ or $a \ge 0 \Rightarrow a \le -3 \Rightarrow -\infty < a \le -3$

Example: 8 Function $f(x) = \frac{\lambda \sin x + 6\cos x}{2\sin x + 3\cos x}$ is monotonic increasing if

(a) $\lambda > 1$ (b) $\lambda < 1$ (c) $\lambda < 4$ (d) $\lambda > 4$

Solution: (d) The function is monotonic increasing if, $f'(x) > 0$

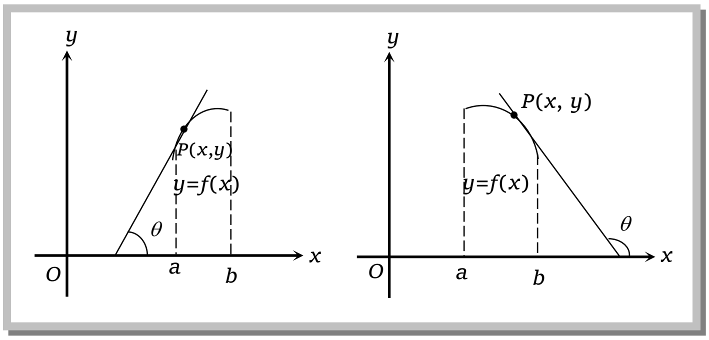
$$\Rightarrow \frac{(2\sin x + 3\cos x)(\lambda\cos x - 6\sin x)}{(2\sin x + 3\cos x)^2} - \frac{(\lambda\sin x + 6\cos x)(2\cos x - 3\sin x)}{(2\sin x + 3\cos x)^2} > 0$$

$\Rightarrow 3\lambda(\sin^2 x + \cos^2 x) - 12(\sin^2 x + \cos^2 x) > 0 \Rightarrow 3\lambda - 12 > 0 \Rightarrow \lambda > 4$.

4.4.3 Necessary and Sufficient Condition for Monotonic Function

In this section we intend to see how we can use derivative of a function to determine where it is increasing and where it is decreasing

(1) **Necessary condition :** From figure we observe that if $f(x)$ is an increasing function on (a, b), then tangent at every point on the curve $y = f(x)$ makes an acute angle θ with the positive direction of x-axis.



$\therefore$ $\tan\theta > 0 \Rightarrow \frac{dy}{dx} > 0$ or $f'(x) > 0$ for all $x \in (a, b)$

It is evident from figure that if $f(x)$ is a decreasing function on (a, b), then tangent at every point on the curve $y = f(x)$ makes an obtuse angle θ with the positive direction of x-axis.

$\therefore$ $\tan\theta < 0 \Rightarrow \dfrac{dy}{dx} < 0$ or $f'(x) < 0$ for all $x \in (a, b)$.

Thus, $f'(x) > 0 (< 0)$ for all $x \in (a,\ b)$ is the necessary condition for a function $f\ (x)$ to be increasing (decreasing) on a given interval (a, b). In other words, if it is given that $f\ (x)$ is increasing (decreasing) on (a, b), then we can say that $f'(x) > 0\ (< 0)$.

(2) **Sufficient condition : Theorem :** Let f be a differentiable real function defined on an open interval (a, b).

(a) If $f'(x) > 0$ for all $x \in (a,\ b)$, then $f(x)$ is increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a,\ b)$, then $f(x)$ is decreasing on (a, b).

Corollary : Let $f\ (x)$ be a function defined on (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a, b)$, except for a finite number of points, where $f'(x) = 0$, then $f\ (x)$ is decreasing on (a, b).

Example: 9 The function $f(x) = \dfrac{\ln(\pi + x)}{\ln(e + x)}$ is

(a) Increasing on [0, ∞) (b) Decreasing on [0, ∞)

(c) Decreasing on $\left[0, \dfrac{\pi}{e}\right)$ and increasing on $\left[\dfrac{\pi}{e}, \infty\right)$ (d) Increasing on $\left[0, \dfrac{\pi}{e}\right)$ and decreasing on $\left[\dfrac{\pi}{e}, \infty\right)$

Solution: (b) Let $f(x) = \dfrac{\ln(\pi + x)}{\ln(e + x)}$

$$\therefore \quad f'(x) = \frac{\ln(e + x) \times \dfrac{1}{\pi + x} - \ln(\pi + x)\dfrac{1}{e + x}}{\{\ln(e + x)\}^2} = \frac{(e + x)\ln(e + x) - (\pi + x)\ln(\pi + x)}{\{\ln(e + x)\}^2 \times (e + x)(\pi + x)}$$

$\Rightarrow f'(x) < 0$ for all $x \geq 0$ $\{\because \pi > e\}$. Hence, $f(x)$ is decreasing in $[0, \infty)$.

Example: 10 Which of the following is not a decreasing function on the interval $\left(0, \dfrac{\pi}{2}\right)$

(a) $\cos x$ (b) $\cos 2x$ (c) $\cos 3x$ (d) $\cot x$

Solution: (c) Obviously, here $\cos 3x$ in not decreasing in $\left(0, \dfrac{\pi}{2}\right)$ because $\dfrac{d}{dx}\cos 3x = -3\sin 3x$.

But at $x = 75^o$, $-3\sin 3x > 0$. Hence the result.

Example: 11 The interval of increase of the function $f(x) = x - e^x + \tan\left(\dfrac{2\pi}{7}\right)$ is

(a) $(0, \infty)$ (b) $(-\infty, 0)$ (c) $(1, \infty)$ (d) $(-\infty, -1)$

Solution: (b, d) We have $f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right) \Rightarrow f'(x) = 1 - e^x$

For $f(x)$ to be increasing, we must have $f'(x) > 0 \Rightarrow 1 - e^x > 0 \Rightarrow e^x < 1 \Rightarrow x < 0 \Rightarrow x \in (-\infty, 0) \Rightarrow (-\infty, -1) \subseteq (-\infty, 0)$

4.4.4 Test for Monotonicity

(1) **At a point :** (i) Function $f(x)$ will be monotonic increasing in domain at a point if and only if, $f'(a) > 0$

(ii) Function $f(x)$ will be monotonic decreasing in domain at a point if and only if, $f'(a) < 0$.

(2) In an interval : Function $f(x)$, defined in $[a, b]$ is

(i) Monotonic increasing in (a, b) if, $f'(x) \geq 0$, $a < x < b$

(ii) Monotonic increasing in $[a, b]$ if, $f'(x) \geq 0$, $a \leq x \leq b$

(iii) Strictly increasing in $[a, b]$, if, $f'(x) > 0$, $a \leq x \leq b$

(iv) Monotonic decreasing in (a, b), if, $f'(x) \leq 0$, $a < x < b$

(v) Monotonic decreasing in $[a, b]$, if, $f'(x) \leq 0$, $a \leq x \leq b$

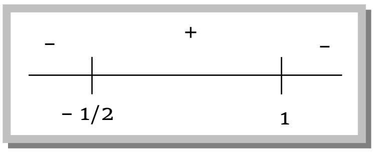
(vi) Strictly decreasing in $[a, b]$, if, $f'(x) < 0$, $a \leq x \leq b$

Example: 12 $f(x) = xe^{x(1-x)}$ then $f(x)$ is **[IIT Screening 2001]**

(a) Increasing on $\left[\frac{-1}{2}, 1\right]$ (b) Decreasing on R (c) Increasing on R (d) Decreasing on $\left[\frac{-1}{2}, 1\right]$

Solution: (a) $f'(x) = e^{x(1-x)} + x \,.\, e^{x(1-x)}.(1-2x) = e^{x(1-x)}\{1 + x(1-2x)\} = e^{x(1-x)}.(-2x^2 + x + 1)$

Now by the sign-scheme for $-2x^2 + x + 1$



$f'(x) \geq 0$, if $x \in \left[-\frac{1}{2}, 1\right]$, because $e^{x(1-x)}$ is always positive. So, $f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$.

Example: 13 x tends 0 to π then the given function $f(x) = x\sin x + \cos x + \cos^2 x$ is

(a) Increasing (b) Decreasing

(c) Neither increasing nor decreasing (d) None of these

Solution: (b) $f(x) = x\sin x + \cos x + \cos^2 x$

$\therefore f'(x) = \sin x + x\cos x - \sin x - 2\cos x \sin x = \cos x(x - 2\sin x)$

Hence $x \to 0$ to π, then $f'(x) \leq 0$, *i.e.*, $f(x)$ is decreasing function.

4.4.5 Properties of Monotonic Function

(1) If $f(x)$ is strictly increasing function on an interval [*a, b*], then f^{-1} exists and it is also a strictly increasing function.

(2) If $f(x)$ is strictly increasing function on an interval [*a, b*] such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$

(3) If $f(x)$ is continuous on [*a, b*] such that $f'(c) \geq 0 (f'(c) > 0)$ for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) increasing function on [*a, b*].

(4) If $f(x)$ is continuous on [*a, b*] such that $f'(c) \leq 0 (f'(c) < 0)$ for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) decreasing function on [*a, b*]

(5) If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) functions on [*a, b*], then *gof*(*x*) is a monotonically (or strictly) increasing function on [*a, b*]

(6) If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then *gof*(*x*) is strictly (monotonically) decreasing on [*a, b*].

Example: 14 The interval in which the function $x^2 e^{-x}$ is non decreasing, is

(a) $(-\infty, 2]$ (b) [0, 2] (c) $[2, \infty)$ (d) None of these

Solution: (b) Let $f(x) = x^2 e^{-x}$

$$\Rightarrow \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2)$$

Hence $f'(x) \geq 0$ for every $x \in [0, 2]$, therefore it is non-decreasing in [0, 2].

Example: 15 The function $\sin^4 x + \cos^4 x$ increase if **[IIT 1999]**

(a) $0 < x < \frac{\pi}{8}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$ (c) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (d) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

Solution: (b) $f(x) = \sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$

$$= 1 - \frac{4\sin^2 x \cos^2 x}{2} = 1 - \frac{\sin^2 2x}{2} = 1 - \frac{1}{4}(2\sin^2 2x)$$

$$= 1 - \left(\frac{1 - \cos 4x}{4}\right) = \frac{3}{4} + \frac{1}{4}\cos 4x$$

Hence function $f(x)$ is increasing when $f'(x) > 0$

$f'(x) = -\sin 4x > 0 \Rightarrow \sin 4x < 0$

Hence $\pi < 4x < \frac{3\pi}{2}$ or $\frac{\pi}{4} < x < \frac{3\pi}{8}$.

Assignment

Increasing and Decreasing Function

Basic Level

1. The function $x+\frac{1}{x}$ ($x \neq 0$) is a non-increasing function in the interval

(a) [– 1, 1] (b) [0, 1] (c) [– 1, 0] (d) [–1, 2]

2. The interval for which the given function $f(x)=2x^3-3x^2-36x+7$ is decreasing, is

(a) (– 2, 3) (b) (2, 3) (c) (2, –3) (d) None of these

3. If $f(x)=\sin x-\frac{x}{2}$ is increasing function, then **[MP PET 1987]**

(a) $0<x<\frac{\pi}{3}$ (b) $-\frac{\pi}{3}<x<0$ (c) $-\frac{\pi}{3}<x<\frac{\pi}{3}$ (d) $x=\frac{\pi}{2}$

4. If the function $f:R\to R$ be defined by $f(x)=\tan x-x$, then $f(x)$

(a) Increases (b) Decreases (c) Remains constant (d) Becomes zero

5. $2x^3 - 6x + 5$ is an increasing function if **[UPSEAT 2003]**

(a) $0 < x < 1$ (b) $-1 < x < 1$ (c) $x < -1$ or $x > 1$ (d) $-1 < x < -1/2$

6. The function $f(x) = 1 - x^3 - x^5$ is decreasing for **[Kerala (Engg.) 2002]**

(a) $1 \le x \le 5$ (b) $x \le 1$ (c) $x \ge 1$ (d) All values of x

7. For which interval, the given function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is decreasing **[MP PET 1993]**

(a) $(-2, \infty)$ (b) $(-2, -1)$ (c) $(-\infty, -1)$ (d) $(-\infty, -2)$ and $(-1, \infty)$

8. The function $f(x) = \tan x - x$ **[MNR 1995]**

(a) Always increases (b) Always decreases

(c) Never decreases (d) Sometimes increases and sometimes decreases

9. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonically increasing in each interval, then **[Rajasthan PET 1992; Kurukshetra CEE 2002]**

(a) $k < 3$ (b) $k \le 3$ (c) $k > 3$ (d) None of these

10. The least value of k for which the function $x^2 + kx + 1$ is an increasing function in the interval $1 < x < 2$ is

(a) – 4 (b) – 3 (c) – 1 (d) – 2

11. The function $f(x) = x + \cos x$ is

(a) Always increasing (b) Always decreasing

(c) Increasing for certain range of x (d) None of these

12. The function $f(x) = x^2$ is increasing in the interval

(a) $(-1, 1)$ (b) $(-\infty, \infty)$ (c) $(0, \infty)$ (d) $(-\infty, 0)$

13. Function $f(x) = x^4 - \frac{x^3}{3}$ is

(a) Increasing for $x > \frac{1}{4}$ and decreasing for $x < \frac{1}{4}$ (b) Increasing for every value of x

(c) Decreasing for every value of x (d) None of these

14. The function $y = 2x^3 - 9x^2 + 12x - 6$ is monotonic decreasing when **[MP PET 1994; Rajasthan PET 1996]**

(a) $1 < x < 2$ (b) $x > 2$ (c) $x < 1$ (d) None of these

15. The interval in which the $x^2 e^{-x}$ is non-decreasing, is

(a) $(-\infty, 2]$ (b) [0, 2] (c) $[2, \infty)$ (d) None of these

16. The function $\frac{1}{1 + x^2}$ is decreasing in the interval

(a) $(-\infty, -1]$ (b) $(-\infty, 0]$ (c) $[1, \infty)$ (d) $(0, \infty)$

17. The function $\sin x - bx + c$ will be increasing in the interval $(-\infty, \infty)$ if

(a) $b \le 1$ (b) $b \le 0$ (c) $b < -1$ (d) $b \ge 0$

18. In the interval [0, 1], the function $x^2 - x + 1$ is

(a) Increasing (b) Decreasing

(c) Neither increasing nor decreasing (d) None of these

19. $f(x)=x^3-27x+5$ is an increasing function, when **[MP PET 1995]**

(a) $x<-3$ (b) $|x|>3$ (c) $x\le -3$ (d) $|x|<3$

20. For the every value of x the function $f(x)=\frac{1}{5^x}$ is

(a) Decreasing (b) Increasing

(c) Neither increasing nor decreasing (d) Increasing for $x>0$ and decreasing for $x<0$

21. In which interval is the given function $f(x)=2x^3-15x^2+36x+1$ is monotonically decreasing

(a) [2, 3] (b) (2, 3) (c) $(-\infty,2)$ (d) $(3,\infty)$

22. The interval of the decreasing function $f(x)=x^3-x^2-x-4$ is

(a) $\left(\frac{1}{3},1\right)$ (b) $\left(-\frac{1}{3},1\right)$ (c) $\left(-\frac{1}{3},\frac{1}{3}\right)$ (d) $\left(-1,-\frac{1}{3}\right)$

23. Let $f(x)=x^3+bx^2+cx+d, 0<b^2<c$. Then f **[IIT JEE Screening 2004]**

(a) Is bounded (b) Has a local maxima (c) Has a local minima (d) Is strictly increasing

24. The function $f(x)=x^3-3x^2-24x+5$ is an increasing function in the interval given below

(a) $(-\infty,-2)\cup(4,\infty)$ (b) $(-2,\infty)$ (c) $(-2,4)$ (d) $(-\infty,4)$

25. Which one is the correct statement about the function $f(x)=\sin 2x$

(a) $f(x)$ is increasing in $\left(0,\frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2},\pi\right)$

(b) $f(x)$ is decreasing in $\left(0,\frac{\pi}{2}\right)$ and increasing in $\left(\frac{\pi}{2},\pi\right)$

(c) $f(x)$ is increasing in $\left(0,\frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

(d) The statement (a), (b) and (c) are all correct

26. If $f(x)=x^3-10x^2+200x-10$, then **[Kurukshetra CEE 1998]**

(a) $f(x)$ is decreasing in $]-\infty,10]$ and increasing in $[10,\infty[$ (b) $f(x)$ is increasing in $]-\infty,10]$ and decreasing in $[10,\infty[$

(c) $f(x)$ is increasing throughout real line (d) $f(x)$ is decreasing throughout real line

27. If f is a strictly increasing function, then $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is equal to

(a) 0 (b) 1 (c) – 1 (d) 2

28. Function x^3-6x^2+9x+1 is monotonic decreasing when **[Rajasthan PET 1991]**

(a) $1<x<3$ (b) $x<3$ (c) $x>1$ (d) $x>3$ or $x<1$

29. The function $f(x)=\frac{1}{3}x^3+\frac{1}{2}x^2-6x+8$ is decreasing in the interval

(a) $x<-3$ (b) $x>2$ (c) $-3<x<2$ (d) None of these

30. The function $f(x)=2\log(x-2)-x^2+4x+1$ increases in the interval

(a) (1, 2) (b) (2, 3) (c) $(-\infty,-1)$ (d) (2, 4)

31. The function $f(x)=\frac{|x|}{x}(x\neq 0), x>0$ is

(a) Monotonically decreasing (b) Monotonically increasing (c) Constant function (d)

32. In the following decreasing function is

(a) ln x (b) $\frac{1}{|x|}$ (c) $e^{1/x}$ (d) None of these

33. If $f(x)=kx-\sin x$ is monotonically increasing, then

(a) $k>1$ (b) $k>-1$ (c) $k<1$ (d) $k<-1$

Advance Level

34. The function f defined by $f(x)=(x+2)e^{-x}$ is **[IIT Screening 1994]**

(a) Decreasing for all x

(b) Decreasing in $(-\infty,-1)$ and increasing in $(-1,\infty)$

(c) Increasing for all x

(d) Decreasing in $(-1,\infty)$ and increasing in $(-\infty,-1)$

35. The value of a in order that $f(x)=\sqrt{3}\ \sin x\ -\cos x-2ax+b$ decreases for all real values of x, is given by

(a) $a<1$ (b) $a\geq 1$ (c) $a\geq\sqrt{2}$ (d) $a<\sqrt{2}$

36. The interval in which the function x^3 increases less rapidly then $6x^2+15x+5$, is

(a) $(-\infty,-1)$ (b) $(-5,1)$ (c) $(-1,5)$ (d) $(5,\infty)$

37. Let $f(x)=\int e^x(x-1)(x-2)dx$. Then f decreases in the interval

(a) $(-\infty,-2)$ (b) $(-2,-1)$ (c) (1, 2) (d) $(2,+\infty)$

38. If $f(x)=2x+\cot^{-1}x+\log(\sqrt{1+x^2}-x)$, then $f(x)$

(a) Increases in $[0,\infty)$

(b) Decreases in $[0,\infty)$

(c) Neither increases nor decreases in $(0,\infty)$

(d) Increases in $(-\infty,\infty)$

39. The function $\frac{(e^{2x}-1)}{(e^{2x}+1)}$ is **[Roorkee 1998]**

(a) Increasing (b) Decreasing (c) Even (d) Odd

40. The function $\frac{a\sin x+b\cos x}{c\sin x+d\cos x}$ is decreasing if **[Rajasthan PET 1999]**

(a) $ad-bc>0$ (b) $ad-bc<0$ (c) $ab-cd>0$ (d) $ab-cd<0$

41. If $f(x)=\sin x-\cos x$, $0\leq x\leq 2\pi$ the function decreasing in **[UPSEAT 2001]**

(a) $\left[\frac{5\pi}{6},\frac{3\pi}{4}\right]$ (b) $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ (c) $\left[\frac{3\pi}{2},\frac{5\pi}{2}\right]$ (d) None of these

42. If $f(x)=\frac{1}{x+1}-\log(1+x), x>0$ then f is **[Rajasthan PET 2002]**

(a) An increasing function (b) A decreasing function

(c) Both increasing and decreasing function (d) None of these

43. The function $f(x)=x^{1/x}$ is **[AMU 2002]**

(a) Increasing in $(1,\infty)$ (b) Decreasing in $(1,\infty)$

(c) Increasing in $(1,e)$, decreasing in (e,∞) (d) Decreasing in $(1,e)$ increasing in (e,∞)

44. The length of the longest interval, in which the function $3\sin x-4\sin^3 x$ is increasing, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

45. The function $f(x)=1-e^{-x^2/2}$ is

(a) Decreasing for all x (b) Increasing for all x

(c) Decreasing for $x<0$ and increasing for $x>0$ (d) Increasing for $x<0$ and decreasing for $x>0$

46. The function $\sin x-\cos x$ is increasing in the interval

(a) $\left[\frac{3\pi}{4},\frac{7\pi}{4}\right]$ (b) $\left[0,\frac{3\pi}{4}\right)$ (c) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$ (d) None of these

47. On the interval $\left(0,\frac{\pi}{2}\right)$, the function log $\sin x$ is

(a) Increasing (b) Decreasing

(c) Neither increasing nor decreasing (d) None of these

48. For all real values of x, increasing function $f(x)$ is **[MP PET 1996]**

(a) x^{-1} (b) x^2 (c) x^3 (d) x^4

49. The function which is neither decreasing nor increasing in $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ is

(a) $\text{cosec}\, x$ (b) $\tan x$ (c) x^2 (d) $|x-1|$

50. For every value of x, function $f(x)=e^x$ is

(a) Decreasing (b) Increasing

(c) Neither increasing nor decreasing (d) None of these

51. Consider the following statements S and R

S : Both $\sin x$ and $\cos x$ are decreasing functions in $\left(\frac{\pi}{2},\pi\right)$

R : If a differentiable function decreases in (a, b) then its derivative also decrease in (a,b)

Which of the following is true

(a) Both S and R are wrong

(b) Both S and R are correct but R is not the correct explanation for S

(c) S is correct and R is the correct explanation for S

(d) S is correct and R is wrong

52. If $f'(x)$ is zero in the interval (a,b) then in this interval it is

(a) Increasing function (b) Decreasing function

(c) Only for $a>0$ and $b>0$ is increasing function (d) None of these

53. The function $\frac{x-2}{x+1}$, $(x \neq -1)$ is increasing on the interval

(a) $(-\infty, 0]$ (b) $[0, \infty)$ (c) R (d) None of these

54. If *f* and *g* are two decreasing functions such that *fog* exists, then *fog*

(a) Is an increasing function (b) Is a decreasing function

(c) Is neither increasing nor decreasing (d) None of these

55. The function $f(x) = \cos(\pi / x)$ is increasing in the interval

(a) $(2n+1, 2n), n \in N$ (b) $\left(\frac{1}{2n+1}, 2n\right)$, $n \in N$ (c) $\left(\frac{1}{2n+2}, \frac{1}{2n+1}\right)$, $n \in N$ (d) None of these

56. The set of all values of *a* for which the function $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right) x^5 - 3x + \log 5$ decreases for all real *x* is

(a) $(-\infty, \infty)$ (b) $\left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$ (c) $\left(-3, 5 - \frac{\sqrt{27}}{2}\right) \cup (2, \infty)$ (d) $[1, \infty)$

57. The function $f(x) = x\sqrt{ax - x^2}, a > 0$

(a) Increases on the interval $\left(0, \frac{3a}{4}\right)$ (b) Decreases on the interval $\left(\frac{3a}{4}, a\right)$

(c) Decreases on the interval $\left(0, \frac{3a}{4}\right)$ (d) Increases on the interval $\left(\frac{3a}{4}, a\right)$

58. The function $f(x) = \frac{|x-1|}{x^2}$ is monotonically decreasing on

(a) $(-2, \infty)$ (b) (0, 1) (c) (0, 1) $\cup (2, \infty)$ (d) $(-\infty, \infty)$

59. The set of values of *a* for which the function $f(x) = x^2 + ax + 1$ is an increasing function on [1, 2] is

(a) $(-2, \infty)$ (b) $[-4, \infty]$ (c) $[-\infty, -2)$ (d) $(-\infty, 2]$

60. On which of the following intervals is the function $x^{100} + \sin x - 1$ decreasing

(a) $\left(0, \frac{\pi}{2}\right)$ (b) (0, 1) (c) $\left(\frac{\pi}{2}, \pi\right)$ (d) None of these

61. If $a < 0$ the function $f(x) = e^{ax} + e^{-ax}$ is a monotonically decreasing function for values of *x* given by

(a) $x > 0$ (b) $x < 0$ (c) $x > 1$ (d) $x < 1$

62. $y = [x(x-3)]^2$ increases for all values of *x* lying in the interval

(a) $0 < x < \frac{3}{2}$ (b) $0 < x < \infty$ (c) $-\infty < x < 0$ (d) $1 < x < 3$

63. The function $f(x) = \frac{\log x}{x}$ is increasing in the interval **[EAMCET 1994]**

(a) $(1, 2e)$ (b) $(0, e)$ (c) $(2, 2e)$ (d) $\left(\frac{1}{e}, 2e\right)$

64. The value of a for which the function $f(x) = \sin x - \cos x - ax + b$ decreases for all real values of x, is given by

(a) $a \geq \sqrt{2}$ (b) $a \geq 1$ (c) $a < \sqrt{2}$ (d) $a < 1$

65. If the function $f(x) = \cos|x| - 2ax + b$ increases along the entire number scale, the range of values of a is given by

(a) $a \leq b$ (b) $a = \frac{b}{2}$ (c) $a \leq -\frac{1}{2}$ (d) $a \geq -\frac{3}{2}$

66. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval

(a) Both $f(x)$ and $g(x)$ are increasing functions (b) Both $f(x)$ and $g(x)$ are decreasing function

(c) $f(x)$ is an increasing function (d) $g(x)$ is an increasing function

67. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x, then

(a) h is increasing whenever f is increasing and decreasing whenever f is decreasing

(b) h is increasing whenever f is decreasing

(c) h is decreasing whenever f is increasing

(d) Nothing can be said in general

68. If $f(x) = \begin{cases} 3x^2 + 12x - 1 & , & -1 \leq x \leq 2 \\ 37 - x & , & 2 < x \leq 3 \end{cases}$ then $f(x)$ is **[IIT 1993]**

(a) Increasing in [–1, 2] (b) Continuous in [–1, 3] (c) Greatest at $x = 2$ (d) All of these

69. If $f'(x) = g(x)(x - \lambda)^2$ where $g(\lambda) \neq 0$ and $g(x)$ is continuous at $x = \lambda$ then function $f(x)$

(a) Increasing near to λ if $g(\lambda) > 0$ (b) Decreasing near to λ if $g(\lambda) > 0$

(c) Increasing near to λ if $g(\lambda) < 0$ (d) Increasing near to λ for every value of $g(\lambda)$

70. Function $\cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x \cos\left(\frac{\pi}{3} + x\right)$ for all real values of x will be

(a) Increasing (b) Constant (c) Decreasing (d) None of these

71. Let $Q(x) = f(x) + f(1 - x)$ and $f''(x) < 0$ whereas $0 \leq x \leq 1$ then function $Q(x)$ is decreasing in

(a) $\left[\frac{1}{2}, 1\right]$ (b) $\left[0, \frac{1}{2}\right]$ (c) $\left(\frac{1}{2}, 1\right)$ (d) (0, 1)

72. If $f(x) = \frac{x}{c} + \frac{c}{x}$ for $-5 \leq x \leq 5$, then $f(x)$ is increasing function in the interval

(a) [c, 5] (b) [0, c] (c) [c, 0] (d) [c, c]

73. If the domain of $f(x) = \sin x$ is $D = \{x : 0 \leq x \leq \pi\}$, then $f(x)$ is

(a) Increasing in D (b) Decreasing in D

(c) Decreasing in $\left[0,\frac{\pi}{2}\right]$ and increasing in $\left[\frac{\pi}{2},\pi\right]$ (d) None of these

74. If $f(x)=(ab-b^2-1)x-\int_0^x(\cos^4\theta+\sin^4\theta)d\theta$ is a decreasing function of x for all $x\in R$ and $b\in R$, b being independent of x, then

(a) $a\in(0,\sqrt{6})$ (b) $a\in(-\sqrt{6},\sqrt{6})$ (c) $a\in(-\sqrt{6},0)$ (d) None of these

75. If $f(x)=\frac{p^2-1}{p^2+1}x^3-3x+\log 2$ is a decreasing function of x in R then the set of possible values of p (independent of x) is

(a) [-1, 1] (b) $[1,\infty)$ (c) $(-\infty,-1]$ (d) None of these

76. Let f $f(x)=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x$, where $a_i^{'}$s are real and $f(x)=0$ has a positive root α_0. Then

(a) $f'(x)=0$ has a root α_1 such that $0<\alpha_1<\alpha_0$ (b) $f'(x)=0$ has at least two real root

(c) $f''(x)=0$ has at least one real roots (d) None of these

77. If *a, b, c* are real, then $f(x)=\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$ is decreasing in

(a) $\left(-\frac{2}{3}(a^2+b^2+c^2),0\right)$ (b) $\left(0,\frac{2}{3}(a^2+b^2+c^2)\right)$ (c) $\left(\frac{a^2+b^2+c^2}{3},0\right)$ (d) None of these

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Answer Sheet

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	a	c	a	c	d	d	a	c	d	a	c	a	a	b	d	c	d	b	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
b	b	d	a	c	c	c	a	c	b	c	c	a	d	b	c	c	a,d	a,d	b
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
d	b	c	a	c	b	a	c	a	b	d	d	b	a	d	b	a,b	c	a	d
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77			
b	a	b	a	c	c	a	d	a	b	a	a	d	b	a	a,b,c	a			

4.5 Rolle's Theorem

4.5.1 Definition

Let f be a real valued function defined on the closed interval $[a, b]$ such that,

(1) $f(x)$ is continuous in the closed interval $[a, b]$

(2) $f(x)$ is differentiable in the open interval $]a,b[$ and

(3) $f(a)=f(b)$

Then there is atleast one value c of x in open interval $]a, b[$ for which $f'(c)=0$.

4.5.2 Analytical Interpretation

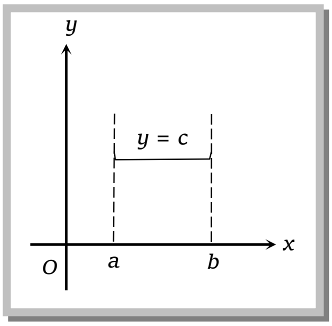
Now, Rolle's theorem is valid for a function such that

(1) $f(x)$ is continuous in the closed interval $[a, b]$

(2) $f(x)$ is differentiable in open interval $]a, b[$ and

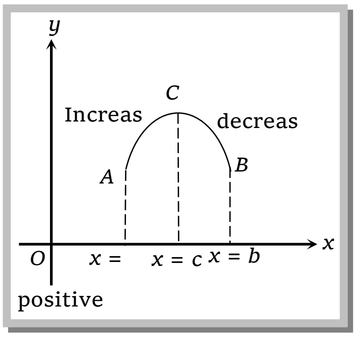
(3) $f(a)=f(b)$

So, generally two cases arises in such circumstances.



Case I: $f(x)$ is constant in the interval $[a, b]$ then $f'(x)=0$ for all $x \in [a,b]$. Hence, Rolle's theorem follows, and we can say, $f'(c)=0$, where $a<c<b$

Case II: $f(x)$ is not constant in the interval $[a, b]$ and since $f(a)=f(b)$.



The function should either increase or decrease when x takes values slightly greater than a.

Now, let $f(x)$ increases for $x>a$

Since, $f(a)=f(b)$, hence the function must seize to increase at some value $x=c$ and decreasing upto $x=b$.

Clearly at $x = c$ function has maximum value.

Now let h be a small positive quantity then, from definition of maximum value of the function,

$$f(c+h) - f(c) < 0 \quad \text{and} \quad f(c-h) - f(c) < 0$$

$$\therefore \frac{f(c+h)-f(c)}{h} < 0 \text{ and } \frac{f(c-h)-f(c)}{-h} > 0$$

So, $$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \le 0 \text{ and } \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} \ge 0 \qquad \text{.....(i)}$$

But, if $$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \ne \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h},$$

The Rolle's theorem cannot be applicable because in such case,

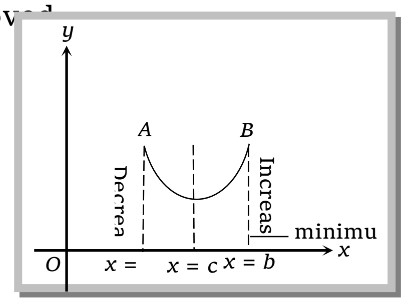
RHD at $x = c \ne$ LHD at $x = c$.

Hence, $f(x)$ is not differentiable at $x = c$, which contradicts the condition of Rolle's theorem.

$\therefore$ Only one possible solution arises, when $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} = 0$

Which implies that, $f'(c) = 0$ where $a < c < b$

Hence, Rolle's theorem is prov



Similarly, the case where $f(x)$ decreases in the interval $a < x < c$ and then increases in the interval $c < x < b, f'(c) = 0$. But when $x = c$, the minimum value of $f(x)$ exists in the interval [*a*, *b*].

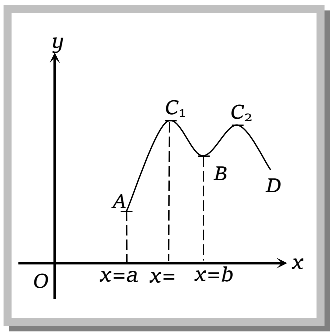
4.5.3 Geometrical Interpretation

Consider the portion *AB* of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(1) It goes continuously from *A* to *B*.

(2) It has tangent at every point between *A* and *B* and

(3) Ordinate of *A* = ordinate of *B*



From figure, it is clear that $f(x)$ increases in the interval AC_1, which implies that $f'(x) > 0$ in this region and decreases in the interval C_1B which implies $f'(x) < 0$ in this region. Now, since there is unique tangent to be drawn on the curve lying in between A and B and since each of them has a unique slope *i.e.*, unique value of $f'(x)$.

$\therefore$ Due to continuity and differentiability of the function $f(x)$ in the region A to B. There is a point $x = c$ where $f'(c) = 0$. Hence, $f'(c) = 0$ where $a < c < b$

Thus Rolle's theorem is proved.

Similarly the other parts of the figure given above can be explained, establishing Rolle's theorem throughout.

Note : On Rolle's theorem generally two types of problems are formulated.

- To check the applicability of Rolle's theorem to a given function on a given interval.
- To verify Rolle's theorem for a given function in a given interval.

In both types of problems we first check whether $f(x)$ satisfies the condition of Rolle's theorem or not.

The following results are very helpful in doing so.

(i) A polynomial function is everywhere continuous and differentiable.

(ii) The exponential function, sine and cosine functions are everywhere continuous and differentiable.

(iii) Logarithmic functions is continuos and differentiable in its domain.

(iv) tan x is not continuous and differentiable at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$

(v) $|x|$ is not differentiable at $x = 0$.

(vi) If $f'(x)$ tends to $\pm\infty$ as $x \to K$, then $f(x)$ is not differentiable at $x = K$.

For example, if $f(x) = (2x-1)^{1/2}$, then $f'(x) = \frac{1}{\sqrt{2x-1}}$ is such that as $x \to \left(\frac{1}{2}\right)^+ \Rightarrow f'(x) \to \infty$

So, $f(x)$ is not differentiable at $x = \frac{1}{2}$.

Example: 1 The function $f(x) = x(x+3)e^{-1/2x}$ satisfies all the condition of Rolle's theorem in [– 3, 0]. The value of c is

(a) 0 (b) 1 (c) – 2 (d) – 3

Solution: (c) To determine 'c' in Rolle's theorem, $f'(c) = 0$

Here $f'(x) = (x^2+3x)e^{-(1/2)x}.\left(-\frac{1}{2}\right) + (2x+3)e^{-(1/2)x} = e^{-(1/2)x}\left\{-\frac{1}{2}(x^2+3x) + 2x + 3\right\} = -\frac{1}{2}e^{-(x/2)}\left\{x^2 - x - 6\right\}$

$\therefore f'(c) = 0 \Rightarrow c^2 - c - 6 = 0 \Rightarrow c = 3, -2$.

But $c = 3 \notin [-3, 0]$, Hence c = –2.

Example: 2 If the function $f(x) = x^3 - 6x^2 + ax + b$ satisfies Rolle's theorem in the interval [1, 3] and $f'\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right) = 0$ then

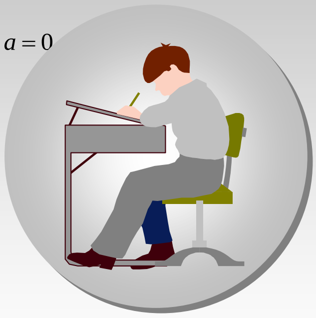
[MP PET 2002]

(a) $a = 11$ (b) $a = -6$ (c) $a = 6$ (d) $a = 1$

Solution: (a) $f(x) = x^3 - 6x^2 + ax + b \Rightarrow f'(x) = 3x^2 - 12x + a$

$$\Rightarrow f'(c) = 0 \Rightarrow f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0 \Rightarrow 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow 3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0 \Rightarrow 12 + 1 + 4\sqrt{3} - 24 - 4\sqrt{3} + a = 0$$



Assignment

Rolle's Theorem

Basic Level

1. Rolle's theorem is true for the function $f(x) = x^2 - 4$ in the interval

(a) [– 2, 0] (b) [– 2, 2] (c) $\left[0, \frac{1}{2}\right]$ (d) [0, 2]

2. For which interval, the function $\frac{x^2 - 3x}{x - 1}$ satisfies all the conditions of Rolle's theorem

(a) [0, 3] (b) [–3, 0] (c) [1.5, 3] (d) For no interval

3. If $f(x)$ satisfies the conditions of Rolle's theorem in [1, 2] and $f(x)$ is continuous in [1, 2] then $\int_1^2 f'(x)dx$ is equal to **[DCE 2002]**

(a) 3 (b) 0 (c) 1 (d) 2

4. Consider the function $f(x) = e^{-2x} \sin 2x$ over the interval $\left(0, \frac{\pi}{2}\right)$. A real number $c \in \left(0, \frac{\pi}{2}\right)$, as guaranteed by Roll's theorem, such that $f'(c) = 0$ is

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

5. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies the conditions of Rolle's theorem for the interval [1, 3] and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then the values of a and b are respectively

(a) 1, – 6 (b) – 2, 1 (c) –1, $\frac{1}{2}$ (d) – 1, 6

6. Rolle's theorem is not applicable to the function $f(x) = |x|$ defined on [– 1, 1] because **[AISSE 1986; MP PET 1994, 95]**

(a) f is not continuous on [– 1, 1]
(b) f is not differentiable on (– 1, 1)
(c) $f(-1) \neq f(1)$
(d) $f(-1) = f(1) \neq 0$

7. Let $f(x) = \begin{cases} x^{\alpha} \ln x & , \quad x > 0 \\ 0 & , \quad x = 0 \end{cases}$ Rolle's theorem is applicable to f for $x \in [0, 1]$, if $\alpha =$ **[IIT-JEE Screening 2004]**

(a) –2
(b) –1
(c) 0
(d) $\frac{1}{2}$

8. The value of a for which the equation $x^3 - 3x + a = 0$ has two distinct roots in [0, 1] is given by

(a) –1
(b) 1
(c) 3
(d) None of these

9. Let a, be two distinct roots of a polynomial $f(x)$. Then there exists at least one root lying between a and b of the polynomial

(a) $f(x)$
(b) $f'(x)$
(c) $f''(x)$
(d) None of these

10. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots.. + \frac{a_{n-1}}{2} + a_n = 0$. Then the function $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots.. + a_n$ has in (0, 1)

(a) At least one zero
(b) At most one zero
(c) Only 3 zeros
(d) Only 2 zeros

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Answer Sheet

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10
b	d	b	a	a	b	d	d	b	a

4.6 Lagrange's Mean Value Theorem

4.6.1 Definition

If a function $f(x)$,

(1) Is continuous in the closed interval [a, b] and

(2) Is differentiable in the open interval (a, b)

Then there is atleast one value $c \in (a,b)$, such that; $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

4.6.2 Analytical Interpretation

First form: Consider the function, $\phi(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}x$

Since, $f(x)$ is continuous in [a, b]

$\therefore$ $\phi(x)$ is also continuous in [a,b]

since, $f'(x)$ exists in (a, b) hence $\phi'(x)$ also exists in (a,b) and $\phi'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$ .....(i)

Clearly, $\phi(x)$ satisfies all the condition of Rolle's theorem

$\therefore$ There is atleast one value of c of x between a and b such that $\phi'(c) = 0$ substituting $x = c$ in (i) we get,

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$ which proves the theorem.

Second form: If we write $b = a+h$ then $a < c < b, c = a + \theta h$ where $0 < \theta < 1$

Thus, the mean value theorem can be stated as follows:

If (i) $f(x)$ is continuous in closed interval [a, a+h]

(ii) $f'(x)$ exists in the open interval $(a, a+h)$ then there exists at least one number $\theta(0 < \theta < 1)$

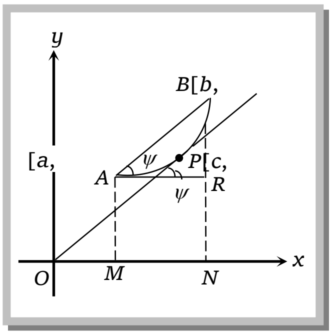
Such that $f(a+h) = f(a) + hf'(a+\theta h)$.

4.6.3 Geometrical Interpretation

Let $f(x)$ be a function defined on [a, b] and let APB be the curve represented by $y = f(x)$. Then co-ordinates of A and B are $(a, f(a))$ and $(b, f(b))$ respectively. Suppose the chord AB makes an angle ψ with the axis of x. Then from the triangle ARB, we have

$$\tan\psi = \frac{BR}{AR} \Rightarrow \tan\psi = \frac{f(b)-f(a)}{b-a}$$

By Lagrange's Mean value theorem, we have, $f'(c) = \frac{f(b)-f(a)}{b-a}$ $\therefore \tan\psi = f'(c)$



$\Rightarrow$ slope of the chord AB = slope of the tangent at $(c, f(c))$

Example: 1 In the mean-value theorem $\frac{f(b)-f(a)}{b-a} = f'(c)$, if $a=0$, $b=\frac{1}{2}$ and $f(x) = x(x-1)(x-2)$, the value of c is **[MP PET 20**

(a) $1-\frac{\sqrt{15}}{6}$ (b) $1+\sqrt{15}$ (c) $1-\frac{\sqrt{21}}{6}$ (d) $1+\sqrt{21}$

Solution: (c) From mean value theorem $f'(c) = \frac{f(b)-f(a)}{b-a}$

$a=0, f(a)=0 \Rightarrow b=\frac{1}{2}, f(b)=\frac{3}{8}$

$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$,

$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1) = c^2 - 3c + 2 + c^2 - 2c + c^2 - c$, $f'(c) = 3c^2 - 6c + 2$

According to mean value theorem

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8}\right)-0}{\left(\frac{1}{2}\right)-0} = \frac{3}{4} \Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$c = \frac{6 \pm \sqrt{36-15}}{2\times 3} = \frac{6\pm\sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}.$$

Example: 2 From mean value theorem $f(b)-f(a) = (b-a)f'(x_1), a < x_1 < b$ if $f(x) = \frac{1}{x}$ then x_1

(a) $\sqrt{ab}$ (b) $\frac{2ab}{a+b}$ (c) $\frac{a+b}{2}$ (d) $\frac{b-a}{b+a}$

Solution: (a) $f'(x_1) = \frac{-1}{x_1^2}$, $\therefore \frac{-1}{x_1^2} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab} \Rightarrow x_1 = \sqrt{ab}$.

Example: 3 The abscissae of the points of the curve $y = x^3$ in the interval [–2, 2], where the slope of the tangent can be obtained by mean value theorem for the interval [– 2, 2] are

(a) $\pm\frac{2}{\sqrt{3}}$ (b) $\pm\frac{\sqrt{3}}{2}$ (c) $\pm\sqrt{3}$ (d) 0

Solution: (a) Given that equation of curve $y = x^3 = f(x)$

So $f(2) = 8$ and $f(-2) = -8$

Now $f'(x) = 3x^2 \Rightarrow f'(x) = \dfrac{f(2) - f(-2)}{2-(-2)} \Rightarrow \dfrac{8-(-8)}{4} = 3x^2; \therefore x = \pm\dfrac{2}{\sqrt{3}}$.

Assignment

Lagrange's Mean Value Theorem

Basic Level

1. If from mean value theorem, $f'(x_1)=\dfrac{f(b)-f(a)}{b-a}$, then **[MP PET 1999]**

(a) $a<x_1\le b$ (b) $a\le x_1<b$ (c) $a<x_1<b$ (d) $a\le x_1\le b$

2. For the function $x+\dfrac{1}{x}, x\in[1,3]$, the value of c for the mean value theorem is **[MP PET 1997]**

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) None of these

3. For the function $f(x)=e^x, a=0, b=1$, the value of c in mean value theorem will be **[DCE 2002]**

(a) log *x* (b) log $(e-1)$ (c) 0 (d) 1

Advance Level

4. If the function $f(x)=x^3-6ax^2+5x$ satisfies the conditions of Lagrange's mean value theorem for the interval [1, 2] and the tangent to the curve $y=f(x)$ at $x=\dfrac{7}{4}$ is parallel to the chord that joins the points of intersection of the curve with the ordinates $x=1$ and $x=2$. Then the value of *a* is

(a) $35/16$ (b) $35/48$ (c) $7/16$ (d) $5/16$

5. If $f(x)=\cos x, 0\le x\le\dfrac{\pi}{2}$, then the real number 'c' of the mean value theorem is **[MP PET 1994]**

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\sin^{-1}\left(\dfrac{2}{\pi}\right)$ (d) $\cos^{-1}\left(\dfrac{2}{\pi}\right)$

6. Let $f(x)$ satisfy all the conditions of mean value theorem in [0, 2]. If $f(0)=0$ and $|f'(x)|\le\dfrac{1}{2}$ for all *x* in [0, 2], then

(a) $f(x)\le 2$ (b) $|f(x)|\le 1$

(c) $f(x)=2x$ (d) $f(x)=3$ for at least one *x* in [0, 2]

7. The function $f(x)=(x-3)^2$ satisfies all the conditions of mean value theorem in [3, 4]. A point on $y=(x-3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1) is

(a) $\left(\frac{7}{2},\frac{1}{2}\right)$ (b) $\left(\frac{7}{2},\frac{1}{4}\right)$ (c) $(1,4)$ (d) (4, 1)

8. Let $f(x)$ and $g(x)$ are defined and differentiable for $x \ge x_0$ and $f(x_0)=g(x_0), f'(x)>g'(x)$ for $x>x_0$, then

(a) $f(x)<g(x)$ for some $x>x_0$

(b) $f(x)=g(x)$ for some $x>x_0$

(c) $f(x)>g(x)$ for all $x>x_0$

(d) None of these

9. Let *f* be differentiable for all *x*. If $f(1)=-2$ and $f'(x)\ge 2$ for all $x\in[1,6]$ then

(a) $f(6)<8$ (b) $f(6)\ge 8$ (c) $f(6)\ge 5$ (d) $f(6)\le 5$

10. The value of *c* in Lagrange's theorem for the function |*x*| in the interval [− 1, 1] is

(a) 0 (b) 1/2 (c) −1/2 (d) Non-existent in the interval

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Answer Sheet

Assignment (Basic and Advance Level)

1	2	3	4	5	6	7	8	9	10
c	b	b	b	c	b	b	c	b	d