

Principle of Mathematical Induction

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The principle of mathematical induction can be stated as

Suppose there is a given statement $P(n)$ involving the natural number n such that

(i) The statement is true for $n = 1$, i.e., $P(1)$ is true, and

(ii) If the statement is true for $n = k$ (where k is some positive integer), then the statement is true for $n = k + 1$, i.e., truth of $P(k)$ implies the truth of $P(k + 1)$.

Then, $P(n)$ is true for all natural numbers n .

- Here, our assumption that the statement is true for $n = k$ is called **inductive hypothesis**.

Example 1:

Prove that $9^{n+1} - 8n - 9$ is divisible by 8.

Solution:

Let the given statement be $P(n)$, i.e.,

$P(n) : 9^{n+1} - 8n - 9$ is divisible by 8

For $n = 1$, $P(1) : 9^{1+1} - 8(1) - 9 = 81 - 8 - 9 = 64$, which is divisible by 8

Thus, $P(n)$ is true for $n = 1$.

Let $P(n)$ be true for $n = k$, i.e., $9^{k+1} - 8k - 9$ is divisible by 8 for some natural number k .

Let $9^{k+1} - 8k - 9 = 8m$, where m is a natural number.

Now, we have to prove that $P(k + 1)$ is true whenever $P(k)$ is true.

$$P(k + 1) = 9^{(k+1)+1} - 8(k + 1) - 9$$

$$= 9^{(k+2)} - 8k - 8 - 9$$

$$= 9^{(k+1)} \cdot 9 - 8k - 8 - 9$$

$$= 9^{(k+1)} (8 + 1) - 8k - 8 - 9$$

$$= 9^{(k+1)} (8 + 1) - 8k - 8 - 9$$

$$\begin{aligned}
&= 8 \cdot 9^{(k+1)} + 9^{(k+1)} - 8k - 8 - 9 \\
&= \{9^{(k+1)} - 8k - 9\} + 8 (9^{(k+1)} - 1) \\
&= 8m + 8 (9^{(k+1)} - 1) \\
&= 8\{m + (9^{(k+1)} - 1)\}
\end{aligned}$$

Thus, $P(k + 1)$ is divisible by 8.

Thus, by the principle of mathematical induction, the given statement is true for every positive integer n .

Example 2:

Prove the following by the principle of mathematical induction.

$2^n > n^2$, where n is a positive integer such that $n > 4$.

Solution:

Let the given statement be $P(n)$, i.e.,

$P(n) : 2^n > n^2$ where $n > 4$

For $n = 5$,

$$2^5 = 32 \text{ and } 5^2 = 25$$

$$\therefore 2^5 > 5^2$$

Thus, $P(n)$ is true for $n = 5$.

Let $P(n)$ be true for $n = k$, i.e.,

$$2^k > k^2 \dots (1)$$

Now, we have to prove that $P(k + 1)$ is true whenever $P(k)$ is true, i.e. we have to prove that $2^{k+1} > (k + 1)^2$.

From equation (1), we get

$$2^k > k^2$$

Multiplying both sides with 2, we obtain

$$2 \times 2^k > 2 \times k^2$$

$$2^{k+1} > 2k^2$$

∴ To prove $2^{k+1} > (k+1)^2$, we only need to prove that $2k^2 > (k+1)^2$.

Let us assume $2k^2 > (k+1)^2$.

$$\Rightarrow 2k^2 > k^2 + 2k + 1$$

$$\Rightarrow k^2 > 2k + 1$$

$$\Rightarrow k^2 - 2k - 1 > 0$$

$$\Rightarrow (k-1)^2 - 2 > 0$$

$$\Rightarrow (k-1)^2 > 2, \text{ which is true as } k > 4$$

Hence, our assumption $2k^2 > (k+1)^2$ is correct and we have $2^{k+1} > (k+1)^2$.

Thus, $P(n)$ is true for $n = k + 1$.

Thus, by the principle of mathematical induction, the given mathematical statement is true for every positive integer n .

Example 3:

Prove the following by the principle of mathematical induction for every positive integer n .

$$2.5 + 4.7 + 6.9 + \dots + 2n(2n+3) = \frac{1}{3}n(n+1)(4n+11)$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n): 2.5 + 4.7 + 6.9 + \dots + 2n(2n+3) = \frac{1}{3}n(n+1)(4n+11)$$

For $n = 1$, $2.5 = \frac{1}{3}(1+1)(4+11) = \frac{1}{3}2.15 = 2.5$, which is true.

Let $P(n)$ be true for $n = k$, i.e.,

$$2.5 + 4.7 + 6.9 + \dots + 2k(2k + 3) = \frac{1}{3}k(k+1)(4k+11) \dots (1)$$

We now have to prove that $P(k + 1)$ is true whenever $P(k)$ is true.

$$2.5 + 4.7 + 6.9 + \dots + 2k(2k + 3) + 2(k + 1)\{2(k + 1) + 3\}$$

$$= \{2.5 + 4.7 + 6.9 + \dots + 2k(2k + 3)\} + 2(k + 1)\{2(k + 1) + 3\}$$

$$= \frac{1}{3}k(k+1)(4k+11) + 2(k + 1)\{2(k + 1) + 3\} \text{ \{from equation (1)\}}$$

$$= (k + 1) \left\{ \frac{1}{3}k(4k + 11) + 2(2k + 5) \right\}$$

$$= \frac{1}{3}(k + 1)\{4k^2 + 23k + 30\}$$

$$= \frac{1}{3}(k + 1)\{(k + 2)(4k + 15)\}$$

$$= \frac{1}{3}(k + 1) \left[\{(k + 1) + 1\} \{4(k + 1) + 11\} \right]$$

Thus, $P(n)$ is true for $n = k + 1$.

Thus, by the principle of mathematical induction, the given statement is true for every positive integer n .