2 Calculus

2.1 Limit

2.1.1 Definition

A number A is said to be limit of a function $f(x)$ at $x = a$ if for any arbitrarily chosen positive integer ϵ, however small but not zero there exist a corresponding number δ greater than zero such that: $|f(x) - A| < \epsilon$ for all values of x for which $0 < |x - a| < \delta$ where $|x - a|$ means the absolute value of $(x - a)$ without any regard to sign.

2.1.2 Right and Left Hand Limits

If x approaches a from the right, that is, from larger value of x than a, the limit of f as defined before is called the right hand limit of $f(x)$ and is written as:

$$\underset{x \to a+0}{Lt} f(x) \text{ or } f(a+0) \text{ or } \underset{x \to a^+}{Lt} f(x)$$

Working rule for finding right hand limit is, put $a + h$ for x in $f(x)$ and make h approach zero.

In short, we have, $$f(a+0) = \underset{h \to 0}{Lt} f(a+h)$$

Similarly if x approaches a from left, that is from smaller values of x than a, the limit of f is called the left hand limit and is written as:

$$\underset{x \to a-0}{Lt} f(x) \text{ or } f(a-0) \text{ or } \underset{x \to a^-}{Lt} f(x)$$

In this case, we have, $$f(a-0) = \underset{h \to 0}{Lt} f(a-h)$$

If both right hand and left hand limit of f, as $x \to$ a exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If however, either or both of these limits do not exist, the limit of f as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

$\therefore$ when $$\underset{x \to a^+}{Lt} f(x) = \underset{x \to a^-}{Lt} f(x)$$

then $$\underset{x \to a}{Lt} f(x) = \underset{x \to a^+}{Lt} f(x) = \underset{x \to a^-}{Lt} f(x)$$

Limit of a function can be any real number, ∞ or $-\infty$. It an sometimes be ∞ or $-\infty$, which are also allowed values for limit of a function.

2.1.3 Various Formulae

These formulae are sometimes useful while taking limits.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots..$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots..$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots...$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots...$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.. |x|<1$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots..\right) |x|<1$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots..$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots..$$

$$\sin h x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots..$$

$$\cos h x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots..$$

Remember: $\log 1 = 0$; $\log e = 1$; $\log \infty = \infty$; $\log 0 = -\infty$

2.1.4 Some Useful Results

1. $\underset{x\to 0}{Lt}\frac{\sin x}{x} = 1$
2. $\underset{x\to 0}{Lt}\cos x = 1$
3. $\underset{x\to 0}{Lt}\frac{\tan x}{x} = 1$
4. $\underset{x\to 0}{Lt}(1+x)^{\frac{1}{x}} = e$
5. $\underset{x\to 0}{Lt}(1+nx)^{\frac{1}{x}} = e^n$
6. $\underset{x\to \infty}{Lt}\left(1+\frac{1}{x}\right)^x = e$
7. $\underset{x\to \infty}{Lt}\left(1+\frac{a}{x}\right)^x = e^a$

2.1.5 Indeterminate Forms

A fraction whose numerator and denominator both tend to zero as $x \to a$ is an example of an indeterminate form written as 0/0. It has no definite values. Other indeterminate forms are: ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^0 (Indeterminate form are not any definite number and hence are not acceptable as limits. To find limit in such cases we use the L'hospital's rule)

2.1.5.1 Indeterminate Form-I $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

Use L'hospital's Rule.

L'Hospital Rule: If $f(x)$ and $\phi(x)$ be two functions of x and if,

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} \phi(x) = 0$$

or if

$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} \phi(x) = \infty,$$

then

$$\underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x \to a}{Lt} \frac{f'(x)}{\phi'(x)}$$

provided, the latter limit exists, finite or infinite.

Working Rule: If the limit of $f(x)/\phi(x)$ as $x \to a$ takes the form 0/0, differentiate the numerator and denominator separately with respect to x and obtain a new function $f'(x)/\phi'(x)$. Now as $x \to a$ if it again takes the form 0/0, differentiate the numerator and denominator again with respect to x and repeat the above process, until the indeterminate form is removed and we get either a real number, ∞ or $-\infty$ as a limit.

Caution: Before applying L'Hospital's rule at any stage, be sure that the form is 0/0. Do not go on applying this rule, if the form is not 0/0.

2.1.5.2 Indeterminate Form-II ($0 \times \infty$)

This form can be easily reduced to the form 0/0 or to the form ∞/∞, and then L'Hospital's rule may be applied.

$$\text{Let } \lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} \phi(x) = \infty.$$

Then we can write

$$\lim_{x \to a} f(x) \cdot \phi(x) = \lim_{x \to a} \frac{f(x)}{1/\phi(x)} \text{ [form 0/0] or } \lim_{x \to a} \frac{\phi(x)}{1/f(x)} \text{ [form } \infty/\infty]$$

Thus $\lim_{x \to a} f(x) \cdot \phi(x)$ is reduced to the form 0/0 or ∞/∞ which can now be evaluated by L' Hospital rule.

2.1.5.3 Indeterminate Form-III (0^0 or 1^∞ or ∞^0)

Suppose $\lim_{x \to a} [f(x)]^{\phi(x)}$ takes any one of these three forms.

Then

$$\text{let } y = \lim_{x \to a} [f(x)]^{\phi(x)}$$

Taking log on both sides, we get

$$\log y = \lim_{x \to a} \phi(x) \cdot \log f(x).$$

Now in any of these above cases log y takes the form $0 \times \infty$ which is changed to the form 0/0 or ∞/∞ then it can be evaluated by previous methods.

2.2 Continuity

2.2.1 Definition

A function $f(x)$ is defined for $x = a$ is said to be continuous at $x = a$ if:

1. $f(a)$ i.e., the value of $f(x)$ at $x = a$ is a definite number and
2. the limit of the function $f(x)$ as $x \to a$ exists and is equal to the value of $f(x)$ at $x = a$.

Note: On comparing the definitions of limit and continuity we find that a function $f(x)$ is continuous at $x = a$ if

$$\underset{x \to a}{\text{Limit}}\ f(x) = f(a)$$

Thus $f(x)$ is continuous at $x = a$ if we have $f(a + 0) = f(a - 0) = f(a)$, otherwise it is discontinuous at $x = a$.

2.2.2 Continuity from Left and Continuity from Right

Let f be a function defined on an open interval I and let a be any point in I. We say that f is continuous from the left at a, if $\underset{x \to a-0}{\text{Limit}}\ f(x)$ exists and is equal to $f(a)$. Similarly f is said to be continuous from the right at a, if $\underset{x \to a+0}{\text{Limit}}\ f(x)$ exists and is equal to $f(a)$.

∴ A function $f(x)$ is continuous at $x = a$, if it is continuous from left as well as continuous from right.

2.2.3 Continuity in an Open Interval

A function f is said to be continuous in open interval (a, b), if it is continuous at each point of open interval.

2.2.4 Continuity in a Closed Interval

Let f be a function defined on the closed interval (a, b) f is said to be continuous on the closed interval $[a, b]$ if it is:

1. continuous from the right at a and
2. continuous from the left at b and
3. continuous on the open interval (a, b).

2.3 Differentiability

Derivative at a point: Let-I denote the open interval (a, b) in R and let $x_0 \in I$. Then a function $f : I \to R$ is said to be differentiable at x_0, if:

$$\underset{h \to 0}{\text{Limit}}\left[\frac{f(x_0 + h) - f(x_0)}{h}\right] \text{ or } \underset{x \to x_0}{\text{Limit}}\left[\frac{f(x) - f(x_0)}{x - x_0}\right]$$

exist (**finitely**) and is denoted by $f'(x_0)$.

2.3.1 Progressive and Regressive Derivatives

The progressive derivative of f (or right derivative of f) at $x = x_0$ is given by

$$\underset{h \to 0}{\text{Limit}}\ \frac{f(x_0 + h) - f(x_0)}{h},\ h > 0 \text{ and is denoted by } Rf'(x_0) \text{ or by } f'(x_0 + 0) \text{ or by } f'(x_0^{+}).$$

The regressive derivative of f (or left derivative of f) at $x = x_0$ is given by

$$\underset{h \to 0}{\text{Limit}}\ \frac{f(x_0 - h) - f(x_0)}{-h},\ h > 0 \text{ and is denoted by } Lf'(x_0) \text{ or by } f'(x_0 - 0) \text{ or by } f'(x_0^{-}).$$

2.3.2 Differentiability in an Open Interval

A function f is said to be differentiable in an open interval (a, b), if it is differentiable at each point of the open interval.

2.3.3 Differentiability in a Closed Interval

A function $f: [a, b] \to R$ is said to be differentiable in closed interval $[a, b]$ if it is

1. differentiable from right a t a [i.e. $R f'(a)$ exists] **and**
2. differentiable from left at b [i.e. $L f'(a)$ exists] **and**
3. differentiable in the open interval (a, b).

2.3.4 Relationship between Differentiability and Continuity

Theorem: If a function is differentiable at any point, then it is necessarily continuous at that point, proof of this theorem follows from definitions of differentiability and continuity.

Note: The converse of this theorem not true.

i.e. Continuity is a necessary but not a sufficient condition for the existence of a finite derivative (differentiability).

i.e. differentiability $\Rightarrow$ continuity

But continuity $\not\Rightarrow$ differentiability

2.4 Mean Value Theorems

2.4.1 Rolle's Theorem

If a function $f(x)$ is such that:

1. $f(x)$ is continuous in the closed interval $a \le x \le b$ **and**
2. $f'(x)$ exists for every point in the open interval $a < x < b$ **and**
3. $f(a) = f(b)$,

   then there exists at least one value of x, say c where $a < c < b$ such that $f'(c) = 0$.

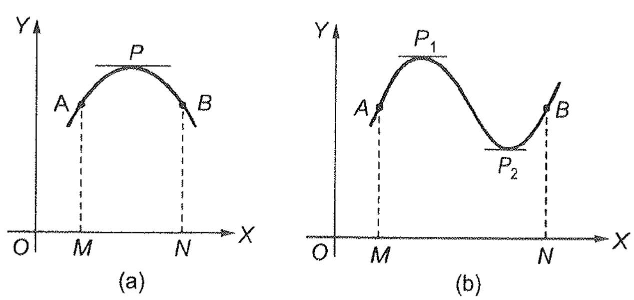
Note: Rolle's theorem will not hold good.

1. If $f(x)$ is discontinuous at some point in the interval $a < x < b$
2. If $f'(x)$ does not exist at some point in the interval $a < x < b$ or
3. If $f(a) \neq f(b)$

2.4.2 Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers a, b, respectively.

Since $f(x)$ is continuous in $[a, b]$, the curve $y = f(x)$ has a tangent at each point between A and B. Also as $f(a) = f(b)$ the ordinates of the points A and B are equal i.e. $MA = NB$ [See Figure (a)].



Then Rolle's theorem asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to x-axis i.e. there exists atleast one real number c in (a, b) such that $f'(c) = 0$. [see figure (a) above]

There may exist more than one point between A and B, the tangents at which are parallel to x-axis [as shown in Figure (b)] i.e. there exists more than one real number c in (a, b) such that $f'(c) = 0$. Rolle's theorem ensures the existence of atleast one real number c in (a, b) such that $f'(c) = 0$.

Remarks:

1. Rolle's theorem fails even if one of the three conditions is not satisfied by the function.
2. The converse of Rolle's theorem is not true, since, $f'(x)$ may be zero at a point in (a, b) without satisfying all the three conditions of Rolle's theorem.

Example 1.

Verify Rolle's theorem for the following functions:

(a) $f(x) = x^2 + x - 6$ in $[-3, 2]$

(b) $f(x) = (x - 1)(x - 2)^2$ in $[1, 2]$

(c) $f(x) = (x^2 - 1)(x - 2)$ in $[-1, 2]$

Solution:

(a) Given $f(x) = x^2 + x - 6$ ... (i)

(i) As $f(x)$ is a polynomial function, it is continuous in $[-3, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(-3, 2)$

(iii) $f(-3) = (-3)^2 - 3 - 6 = 0$, $f(2) = 2^2 + 2 - 6 = 0 \Rightarrow f(-3) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-3, 2)$ such that $f'(x) = 2x + 1$.

Differentiating (i) w.r.t. x, we get $f'(x) = 2x + 1$.

Now $f'(c) = 0 \Rightarrow 2c + 1 = 0 \Rightarrow c = -\frac{1}{2}$.

So there exists $-\frac{1}{2} \in (-3, 2)$ such that $f'\left(-\frac{1}{2}\right) = 0$

Hence, Rolle's theorem is verified.

(b) Given $f(x) = (x - 1)(x - 2)^2$ ... (i)

(i) Since $f(x)$ is a polynomial function, it is continuous is $[1, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(1, 2)$.

(iii) $f(1) = (1 - 1)(1 - 2)^2 = 0$, $f(2) = (2 - 1)(2 - 2)^2 = 0 \Rightarrow f(12) = f(2)$

Thus, all the three conditions of Roll's theorem are satisfied, therefore, there exists atleast one real number c in $(1, 2)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$\begin{aligned} f'(x) &= (x-1)\cdot 2(x-2).\,1 + (x-2)^2.1 \\ &= (x-2)(2x-2+x-2) \\ &= (x-2)(3x-4) \end{aligned}$$

$$\begin{aligned} \text{Now} \quad f'(c) &= 0 \\ \Rightarrow \quad (c-2)(3c-4) &= 0 \\ \Rightarrow \quad c &= 2, 4/3 \end{aligned}$$

But $c \in (1, 2)$, therefore, $c = 4/3$.

So, there exists $(4/3) \in (1, 2)$ such that $f'(4/3) = 0$

Hence, Rolle's theorem is verified.

(c) Given $f(x) = (x^2 - 1)(x - 2)$ ... (i)

(i) Since $f(x)$ is a polynomial function, it is continuous in $[-1, 2]$.

(ii) $f(x)$ being a polynomial function is derivable in $(-1, 2)$.

(iii) $f(-1) = (1 - 1)(1 - 2) = 0$, $f(2) = (4 - 1)(2 - 2) = 0 \Rightarrow f(-1) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-1, 2)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = (x^1 - 1).\ 1 + (x - 2) \cdot 2x = 3x^2 - 4x - 1.$$

Now $$f'(c) = 0 \Rightarrow 3c^2 - 4c - 1 = 0$$

$$\Rightarrow \qquad c = \frac{4 \pm \sqrt{16 - 4.3(-1)}}{2.3} = \frac{2 \pm \sqrt{7}}{3}$$

Also $-1 < \frac{2-\sqrt{7}}{3} < \frac{2+\sqrt{7}}{3} < 2 \Rightarrow \frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ both lie in $(-1, 2)$.

So there exist two real numbers $\frac{2-\sqrt{7}}{3}$ and $\frac{2+\sqrt{7}}{3}$ in $(-1, 2)$ such that

$$f'\left(\frac{2-\sqrt{7}}{3}\right) = 0 \text{ and } f'\left(\frac{2+\sqrt{7}}{3}\right) = 0$$

Hence, Rolle's theorem is verified.

Example 2.

Verify Rolle's theorem for the following functions and find point (or points) where the derivative vanishes:

$$f(x) = \sin x + \cos x \text{ in } \left[0, \frac{\pi}{2}\right]$$

Solution:

Given: $$f(x) = \sin x + \cos x \qquad \text{... (i)}$$

(a) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

(b) $f(x)$ is derivable in $\left[0, \frac{\pi}{2}\right]$ and

(c) $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$,

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Differentiating (i) w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$

Now $$f'(c) = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow c = 1$$

$$\Rightarrow \qquad c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, -\frac{3\pi}{4}, \ldots \text{ but } c \in \left(0, \frac{\pi}{2}\right) \Rightarrow c = \frac{\pi}{4}.$$

So there exists $\frac{\pi}{4}$ in $\left(0, \frac{\pi}{2}\right)$ such that $f'\left(\frac{\pi}{4}\right) = 0$.

Hence, Rolle's theorem is verified and $c = \frac{\pi}{4}$.

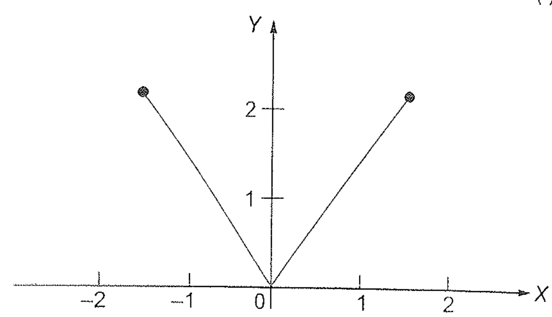
Example 3.

Discuss the applicability of Rolle's theorem for the function $f(x) = |x|$ in $[-2, 2]$.

Solution:

Given: $f(x) = |x|, x \in [-2, 2]$ ... (i)

the graph of $f(x) = |x|$ in $[-2, 2]$ is shown in figure



(a) $f(x)$ is continuous is $[-2, 2]$

(b) Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{x}{|x|}, x \neq 0$$

$\Rightarrow$ the derivative of $f(x)$ does not exist at $x = 0$

$\Rightarrow$ $f(x)$ is not derivable in $(-2, 2)$

Thus, the condition (ii) of Rolle's theorem is not satisfied, therefore, Rolle's theorem is not applicable to the function $f(x) = |x|$ in $[-2, 2]$.

Moreover, $f(-2) = |-2| = 2$ and $f(2) = |2| = 2 \Rightarrow f(-2) = f(2)$, so the condition (iii) of Rolle's theorem is satisfied.

Further, it is clear from the graph that there is not point of the cure $y = |x|$ *i* $(-2, 2)$ at which the tangent is parallel to x-axis.

2.4.3 Lagrange's Mean Value Theorem

If a function $f(x)$ is:

1. Continuous in closed interval $a \leq x \leq b$ and
2. Differentiable in open interval (a, b) i.e., $a < x < b$,

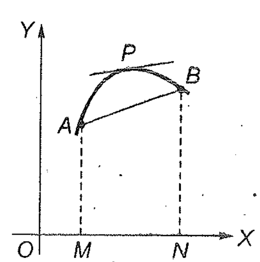
   then there exist at least one value c of x lying in the open interval $a < x < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

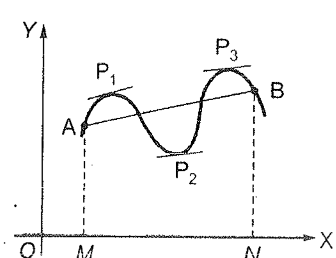
2.4.4 Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers a, b respectively.

Since $f(x)$ is continuous in $[a, b]$, the graph of the curve $y = f(x)$ is continuous from A to B. Again, as $f(x)$ is derivable in (a, b) the curve $y = f(x)$ has a tangent at each point between A and B. Also as $a \neq b$, the slope of the chord AB exists and the slope of the chord $AB = \frac{f(b) - f(a)}{b - a}$.



(a)



(b)

Then Lagrange's Mean Value Theorem asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to the chord AB. There may exist more than one point between A and B the tangents at which are parallel to the chord AB [as shown in Figure (b)]. Lagrange's mean value theorem ensures the existence of atleast one real number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Remarks:

1. Lagrange's mean value theorem fails for the function which does not satisfy even one of the two conditions.

2. The converse of Lagrange's mean value theorem may not be true, for, $f'(c)$ may be equal to $\frac{f(b) - f(a)}{b - a}$ at a point c in (a, b) without satisfying both the conditions of Lagrange's mean value theorem.

Example 1.

Verify Lagrange's mean value theorem for the following functions in the given interval and find 'c' of this theorem.

(a) $f(x) = x^2 + 2x + 3$ in $[4, 6]$ (b) $f(x) = px^2 + qx + r$, $p \neq 0$, in $[a, b]$

Solution:

(a) Given $f(x) = x^2 + 2x + 3$

(i) $f(x)$ being a polynomial function is continuous in $[4, 6]$. ... (i)

(ii) $f(x)$ being a polynomial function is derivable in $(4, 6)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in $(4, 6)$ such that

$$f'(c) = \frac{f(6) - f(4)}{6 - 4}$$

$$f(6) = 6^2 + 2.6 + 3 = 51,\ f(4) = 4^2 + 2.4 + 3 = 27.$$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2x + 2 \Rightarrow f'(c) = 2c + 2$$

$$\therefore \quad f'(c) = \frac{f(6) - f(4)}{6 - 4} \quad 2c + 2 = \frac{51 - 27}{2} \Rightarrow 2c + 2 = 12$$

$$\Rightarrow \quad 2c = 10 \Rightarrow c = 5$$

Thus, there exists $c = 5$ in $(4, 6)$ such that $f'(5) = \frac{f(6) - f(4)}{6 - 4}$.

Hence, Lagrange's mean value theorem is verified and $c = 5$.

(b) Given $f(x) = px^2 + qx + r$, $p \neq 0$

(i) f being a polynomial function is continuous in $[a, b]$

(ii) f being a polynomial function is derivable in (a, b).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in (a, b) such that $f'(x) = \frac{f(b) - f(a)}{b - a}$.

$$f(b) = pb^2 + qb + r,\ f(a) = pa^2 + qa + r.$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = 2px + q \Rightarrow f'(c) = 2pc + q.$$

$$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \qquad 2pc + q = \frac{(pb^2 + qb + r) - (pa^2 + qa + r)}{b - a}$$

$$\Rightarrow \qquad 2pc + q = \frac{p(b^2 + a^2) + q(b - a)}{b - a}$$

$$\Rightarrow \qquad 2pc = p(a + b)$$

$$\Rightarrow \qquad c = \frac{a + b}{2} \text{ and } \frac{a + b}{2} \in (a, b)$$

Thus, there exist $c = \frac{a+b}{2}$ in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Hence Lagrange's mean value theorem is verified and $c = \frac{a+b}{2}$

Example 2.

Find a point on the graph of $y = x^3$ where the tangent is parallel to the chord joining (1, 1) and (3, 27).

Solution:

$$f(x) = x^3 \text{ in the interval } [1, 3]$$

(a) $f(x)$ being a polynomial is continuous in [1, 3].

(b) $f(x)$ being a polynomial is derivable in (1, 3).

Thus, both the conditions of Lagrange's mean value theorem are satisfied by the function $f(x)$ in [1, 3], therefore, there exists atleast one real number c in (1, 3) such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$f(3) = 3^3 = 27 \text{ and } f(1) = 1^3 = 1.$$

Differentiating (1) w.r.t. x, we get

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2.$$

Now $$f'(c) = \frac{f(3) - f(1)}{3 - 1} \Rightarrow 3c^2 = \frac{27 - 1}{3 - 1} \Rightarrow 3c^2 = 13$$

$$\Rightarrow \qquad c^2 = \frac{13}{3} = \frac{39}{9}$$

$$\Rightarrow \qquad c = \pm\frac{\sqrt{39}}{3}$$

But $$c \in (1, 3) \Rightarrow c = \frac{\sqrt{39}}{3}$$

When $$x = \frac{\sqrt{39}}{3}, \text{ from (1) } y = \frac{\sqrt{39}}{3}$$

Hence, there exists a point $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ on the given curve $y = x^3$ where the tangent is parallel to the chord joining the points (1, 1) and (3, 27).

Example 3.

Does the Lagrange's mean value theorem apply to $f(x) = x^{1/3}$, $-1 \le x \le 1$? What conclusions can be drawn?

Solution:

Given, $f(x) = x^{1/3}, x \in [-1, 1]$ ... (i)

(a) $f(x)$ is continuous in $[-1, 1]$

(b) Differentiating (1) w.r.t. x, we get

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}, x \neq 0 \quad \text{... (i)}$$

$\Rightarrow$ The derivative of $f(x)$ does not exist at $x = 0$

$\Rightarrow f(x)$ is not derivable in $(-1, 1)$.

Thus, the condition (ii) of lagrange's mean value theorem is not satisfied by the function $f(x) = x^{1/3}$ in $[-1, 1]$ and hence Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in $[-1, 1]$ and hence Lagrange's mean value theorem is not applicable to the given function $f(x) = x^{1/3}$ in $[-1, 1]$.

Conclusion. However, from (2), $f'(c) = \frac{1}{3c^{2/3}}, c \neq 0$

Also $f(-1) = (-1)^{1/3} = -1$, $f(1) = 1^{1/3} = 1$ (we have taken only real values)

$$\therefore \quad f'(c) = \frac{f(1) - f(-1)}{1-(-1)}$$

$$\Rightarrow \quad \frac{1}{3c^{2/3}} = \frac{1-(-1)}{1-(-1)} = \frac{2}{2} = 1$$

$$\Rightarrow \quad c^{2/3} = \frac{1}{3} \Rightarrow c^2 = \frac{1}{27} \Rightarrow c = \pm\frac{1}{3\sqrt{3}}.$$

As $-1 < -\frac{1}{3\sqrt{3}} < \frac{1}{3\sqrt{3}} < 1 \Rightarrow c = \pm\frac{1}{\sqrt{3}}$ both lie in $(-1, 1)$

Thus, we find that there exist two real numbers $c = \pm\frac{1}{\sqrt{3}}$ in $(-1, 1)$ such that $f'(c) = \frac{f(1) - f(-1)}{1-(-1)}$.

If follows that the converse of Lagrange's mean value theorem may not be true.

2.4.5 Some applications of Lagrange's Mean Value theorem

1. If a function $f(x)$ is
   (a) continuous in $[a, b]$
   (b) derivable in (a, b) and
   (c) $f'(x) > 0$ for all x in (a, b), then $f(x)$ is strictly increasing function in $[a, b]$.

   Proof. Let x_1, x_2 be any two members of $[a, b]$ such that $a \leq x_1 < x_2 \leq b$ then $f(x)$ satisfied both the conditions of Lagrange's mean value theorem in $[x_1, x_2]$, therefore, there exists atleast one real number c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$\Rightarrow \quad (x_2 - x_1)\, f'(c) = f(x_2) - f(x_1)$

But $f'(x) > 0$ for all x in $(a, b) \Rightarrow f'(c) > 0$ for all c in (x_1, x_2). Also $x_1 < x_2$ lie. $x_2 - x_1 > 0$

$\Rightarrow \quad (x_2 - x_1)\, f'(c) > 0$

$\Rightarrow \quad f(x_2) - f(x_1) > 0$

$\Rightarrow f(x_2) > f(x_1)$, for all x_1, x_2 such that $a \leq x_1 < x_2 \leq b$.

Hence, $f(x)$ is strictly increasing in $[a, b]$

2. If a function $f(x)$ is

   (a) continuous in $[a, b]$

   (b) derivable in (a, b)

   (c) $f'(x) < 0$ for all x in (a, b), then $f(x)$ is strictly decreasing function in $[a, b]$.

   (For the proof, proceed as above)

2.4.6 Some Important Deductions from Mean Value Theorems

1. If a function $f(x)$ be such that $f'(x)$ is zero throughout the interval, then $f(x)$ must be constant throughout the interval.
2. If $f(x)$ and $\phi(x)$ be two functions such that $f'(x) = \phi'(x)$ throughout the interval (a, b), then $f(x)$ and $\phi(x)$ differ only by a constant.
3. If $f'(x)$ is:

   (a) continuous in closed interval $[a, b]$

   (b) differentiable in open interval (a, b)

   (c) $f'(x)$ is –ve in $a < x < b$, then $f(x)$ is monotonically decreasing function in the closed interval $[a, b]$ and $f'(x)$ is positive in $a < x < b$, then $f(x)$ is monotonically increasing function in the closed interval $[a, b]$.

2.4.7 Some Standard Results on Continuity and Differentiability of Commonly used Functions

It is important to remember the following facts regarding common functions while checking applicability of Rolle's and Lagrange's mean value theorems:

1. Constant function is differentiable everywhere $[f'(x) = 0, \forall x]$.
2. Any polynomial function is continuous and differentiable everywhere.
3. The exponential function (e^x, a^x etc), $\sin x$, as well as $\cos x$ are also continuous and differentiable everywhere.
4. log function, trigonometric and inverse trigonometric functions are differentiable within their domains.
5. $\tan x$ is discontinuous at $x = \pm \pi/2, \pm 3\pi/2., \ldots$
6. $|x|$ is continuous but not differentiable at $x = 0$.
7. If $f'(x) \to \pm \infty$ as $x \to k$, then that function is not differentiable at $x = k$.
8. Sum, difference, product, quotient and compositions of continuous and differentiable functions are continuous and differentiable.

2.5 Computing the Derivative

Rules of Differentiation:

$$(f + g)' = f' + g' \qquad \text{(Sum rule)}$$

$$(f - g)' = f' - g' \qquad \text{(Difference rule)}$$

$$(f.g)' = fg' + gf' \qquad \text{(Product rule)}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \qquad \text{(Quotient rule)}$$

$$\frac{1}{dx}(f(g(x))) = \frac{df}{dg} \cdot \frac{dg}{dx} \qquad \text{(Chain rule)}$$

Using the above five rules, we can differentiate most of the cases where y is an explicit function of x.

The following is the table of derivatives of commonly occurring functions:

$f(x)$	$f'(x)$
x^n	$n x^{n-1}$
$ln\,x$	$\frac{1}{x}$
$\log_a x$	$\left(\frac{1}{x}\right)\log_a e$
e^x	e^x
a^x	$a^x \log_e a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\sinh x$	$\cosh x$

$f(x)$	$f'(x)$
$\cos h\, x$	$\sin h\, x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\mathrm{cosec}^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\cot^{-1} x$	$\frac{1}{1+x^2}$
$\lvert x\rvert$	$\frac{x}{\lvert x\rvert}\ (x \neq 0)$

Most explicit functions can be differentiated by using above table along with the five rules of differentiation. For more complicates cases, we have to resort to more advanced methods of differentiation as given below:

1. Differentiation by substitution
2. Implicit differentiation
3. Logarithmic differentiation
4. Parametric differentiation

2.5.1 Differentiation by Substitution

There are no hard and fast rules for making suitable substitutions. It is the experience which guides us for the selection of a proper substitution. However, some useful suggestions are given below:

If the function contains an expression of the form

1. $a^2 - x^2$, put $x = a \sin t$ or $x = a \cos t$
2. $a^2 + x^2$, put $x = a \tan t$ or $x = a \cot t$
3. $x^2 - a^2$, put $x = a \sec t$ or $x = a\, \mathrm{cosec}\, t$
4. $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$, put $x = a \cos t$
5. $a \cos x \pm b \sin x$, put $a = r\cos\theta$ and $b = r\sin\theta$, $r > 0$.

Example:

Differentiate the following functions (by suitable substitutions) w.r.t. x.

(a) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(b) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$

(c) $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

(d) $\tan^{-1}\left(\sqrt{1+x^2}+x\right)$

Solution:

(a) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, put $x = \tan\theta$ i.e. $\theta = \tan^{-1} x$, then $\quad y = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin(2\theta)) = 2\theta$

$= 2\tan^{-1}x$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\cdot\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

(b) Let $\quad y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$, put $x = \tan\theta$ i.e. $\theta = \tan^{-1}x$,

then $\quad y = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}+1}{\tan\theta}\right)$

$$= \tan^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right) = \tan^{-1}\left[\frac{\dfrac{1}{\cos\theta}+1}{\dfrac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{2\cos^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}\right]$$

$$= \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)$$

$= \dfrac{\pi}{2}-\dfrac{\theta}{2} = \dfrac{\pi}{2}-\dfrac{1}{2}\tan^{-1}x$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0-\frac{1}{2}\cdot\frac{1}{1+x^2} = -\frac{1}{2(1+x^2)}$$

(c) Let $\quad y = \cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\dfrac{x-\dfrac{1}{x}}{x+\dfrac{1}{x}}\right) = \cos^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$

put $\quad x = \tan\theta$ i.e. $\theta = \tan^{-1}x$,

then $\quad y = \cos^{-1}\left(\dfrac{\tan^2\theta-1}{\tan^2\theta+1}\right) = \cos^{-1}\left(\dfrac{-1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}(\cos(\pi-2\theta))$

$= \pi-2\theta = \pi-2\tan^{-1}x$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0-2\cdot\frac{1}{1+x^2} = -\frac{2}{1+x^2}$$

(d) Let $\quad y = \tan^{-1}\left(\sqrt{1+x^2}+x\right)$

put $\quad x = \cot\theta$ i.e. $\theta = \cot^{-1}x$

then $\quad y = \tan^{-1}\left(\sqrt{1+\cot^2\theta}+\cot\theta\right)$

$= \tan^{-1}(\operatorname{cosec}\theta+\cot\theta)$

$$= \tan^{-1}\left(\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right) = \frac{\pi}{2}-\frac{\theta}{2}$$

$$= \frac{\pi}{2}-\frac{1}{2}\cot^{-1}x\text{, differentiating w.r.t. } x\text{, we get}$$

$$\frac{dy}{dx} = 0-\frac{1}{2}\left(-\frac{1}{1+x^2}\right) = \frac{1}{2(1+x^2)}$$

2.5.2 Implicit Differentiation

If y be a function of x defined by an equation such as

$$y = 7x^4 - 5x^3 + 11x^2 + \sqrt{2}x - 3 \qquad \text{... (i)}$$

y is said to be defined explicitly in terms of x and we write $y = f(x)$ where

$$f(x) = 7x^4 - 5x^3 + 11x^2 + \sqrt{2}x - 3$$

However, if x and y are connected by an equation of the form

$$x^4y^3 - 3x^3y^5 + 7y^3 - 8x^2 + 9 = 0 \qquad \text{... (ii)}$$

i.e. $f(x, y) = 0$, then y cannot be expressed explicitly in terms of x. But, still the value of y depends upon that of x and there may exist one or more functions 'f' connecting y with x so as to satisfy equation (ii) or there may not exist any of the functions satisfying equation (ii).

For example, consider the equations

$$x^2 + y^2 - 25 = 0 \qquad \text{... (iii)}$$

and

$$x^2 + y^2 + 25 = 0 \qquad \text{... (iv)}$$

In equation (ii), y may be expressed explicitly in terms of x, but y is not a function of x. Here we have two functions of x (or two functions of y if y were considered to be independent variable) f_1 and f_2 defined by $f_1(x) = \sqrt{25-x^2}$ and $f_2(x) = -\sqrt{25-x^2}$ which satisfy equation (iii).

In equation (iv), there are no real values of x that can satisfy it.

In cases (ii), (iii) and (iv), we say that y is an implicit function of x (or x is an implicit function of y) and in all such cases, we find the derivative of y with regard to x (or the derivative of x with regard to y) by the process called implication differentiation. Of course, wherever we differentiate implicitly an equation that defines one variable as an implicit function of another variable, we shall assume that the function is differentiable.

Example 1.

Find $\frac{dy}{dx}$ when $x^2 + xy + y^2 = 100$.

Solution:

Given, $x^2 + xy + y^2 = 100$

Keeping in mind that y is a function of x, differentiating both sides w.r.t. x, we get

$$2x + \left(x\cdot\frac{dy}{dx} + y\cdot 1\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x+2y)\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

Example 2.

If $x^{2/3} + y^{2/3} = a^{2/3}$, find $\frac{dy}{dx}$.

Solution:

Given, $\quad x^{2/3} + y^{2/3} = a^{2/3} \quad$ ... (

Differentiating both sides of (i) w.r.t. x, regarding y as a function of x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{x^{1/3}} + \frac{1}{y^{1/3}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}$$

Example 3.

If $\sin^2 y + \cos xy = \pi$, find $\frac{dy}{dx}$

Solution:

Given, $\quad \sin^2 y + \cos xy = \pi$

Differentiating both sides of (i) w.r.t. x, regarding y as function of x, we get

$$2(\sin y)\cdot\cos y\cdot\frac{dy}{dx} - \sin xy\cdot\left(x\frac{dy}{dx} + y\cdot 1\right) = 0$$

$$\Rightarrow \quad (2\sin y\cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \quad (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Example 4.

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \text{to } \infty}}}$, prove that $(1-2y)\frac{dy}{dx} = \sin x$.

Solution:

Given, $\quad y = \sqrt{\cos x + y}$

$$\Rightarrow \quad y^2 = \cos x + y$$

$$\Rightarrow \quad y^2 - y = \cos x$$

differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} - \frac{dy}{dx} = -\sin x$$

$$\Rightarrow \quad (1-2y)\frac{dy}{dx} = \sin x$$

2.5.3 Logarithmic Differentiation

In order to simplify the differentiation of some functions, we first take logarithms and then differentiate. Such a process is called logarithmic differentiation. This is usually done in two types of problems.

1. When the given function is a product of some functions, then the logarithm converts the product into a sum and this facilitates the differentiation.
2. When the variable occurs in the exponent i.e. the given function is of the form $[f(x)]\ \phi(x)$.

   Derivative of u^v where u, v are differentiable functions of x

   Let $\quad y = u^v$, taking logarithm of both sides, we get

   $\log y = v \log u$, differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(v \log u)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\frac{d}{dx}(v \log u) = u^v \frac{d}{dx}(v \log u)$$

Example 1.

Differentiate the following functions w.r.t. x:

(a) x^x

(b) $\cos(x^x)$.

Solution:

(a) Let $\quad y = x^x$,

Taking logarithm of both sides, we get

$$\log y = x \log x,$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$$

(b) Let $\quad y = \cos(x^x)$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\sin(x^x)\cdot\frac{d}{dx}(x^x)$$

Now $\frac{d}{dx}(x^x)$ has been obtained preciously in part (a).

So, $$\frac{dy}{dx} = -\sin(x^x)\cdot x^x(1 + \log x)$$

Example 2.

If $x^y = e^{x-y}$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

Solution:

Given, $\quad x^y = e^{x-y}$, taking logarithm of both sides, we get

$$y \log x = (x - y) \log e = (x - y)\cdot 1 = x - y$$

$$\Rightarrow \quad y + y \log x = x$$

$$\Rightarrow \quad (1 + \log x)\, y = x$$

$$\Rightarrow \qquad y = \frac{x}{1+\log x'} \text{ differentiating w.r.t. } x, \text{ we get}$$

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\cdot\left(0+\frac{1}{x}\right)}{(1+\log x)^2} = \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

2.5.4 Derivatives of Functions in Parametric forms

If x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e. if $x = f(t)$ and $y = g(t)$ then such functions are called parametric functions and the third variable is called the parameter.

In order to find the derivative of a function in parametric form, we use chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

OR

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \left(\text{provide} \frac{dx}{dt} \neq 0\right)$$

Example 1.

If $x = a(t + \sin t)$, $y = a(1 - \cos t)$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$.

Solution:

Given, $\qquad x = a(t + \sin t)$ and $y = a(1 - \cos t)$

Differentiating both w.r.t. t, we get

$$\frac{dx}{dt} = a(1 + \cos t)$$

and $$\frac{dy}{dt} = a(0 - (-\sin t)) = a \sin t.$$

We know that $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2},$$

$$\therefore \qquad \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} = \tan\frac{\pi}{4} = 1.$$

Example 2.

Differentiate $\frac{x^3}{1-x^3}$ w.r.t. x^3.

Solution:

Let $\qquad y = \frac{x^3}{1-x^3}$ and $z = x^3$ so that $\frac{dy}{dz}$ is wanted.

Differentiating both w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1-x^3)\cdot 3x^2 - x^3\cdot(0-3x^2)}{(1-x^3)^2} = \frac{3x^2}{(1-x^3)^2},$$

and $$\frac{dz}{dx} = 3x^2.$$

We know that $$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

$\therefore$ $$\frac{dy}{dz} = \frac{3x^2}{(1-x^3)^2}\times\frac{1}{3x^2} = \frac{1}{(1-x^3)^2},\ x \neq 1$$

2.6 Applications of Derivatives

There are two areas where derivatives are used

1. Increasing and Decreasing Functions
2. Maxima and Minima
   (a) Relative maxima and minima
   (b) Absolute maxima and minima
3. Taylor's and Maclaurin's Series Expansion of Functions
4. Slope determination of line

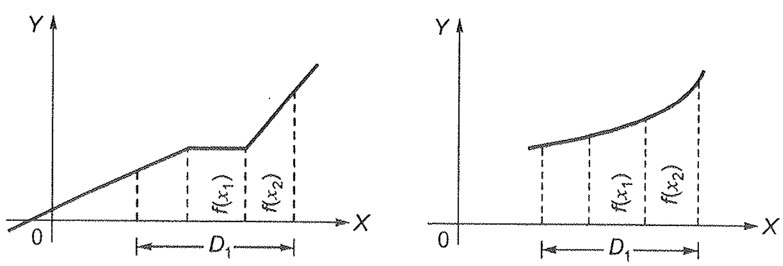
2.6.1 Increasing and Decreasing Functions

Let f be a real valued function defined in an interval D (a subset of R), then f is called an increasing function in an interval D_1(a subset of D) if

for all $$x_1, x_2 \in D_1,$$
$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

and f is called a strict increasing function (or monotonically increasing function) in D_1 if

for all $$x_1, x_2 \in D_1,$$
$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$



Increasing in D_1
(a)

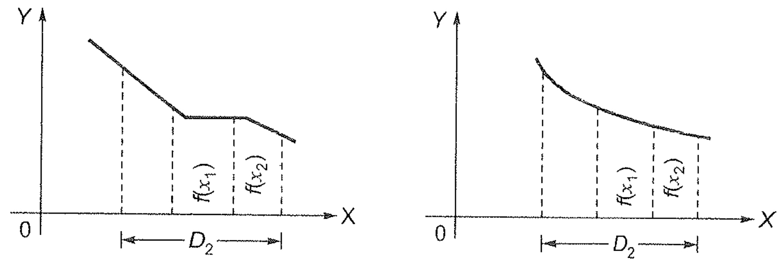
Strict increasing in D_1
(b)

Analogously, f is called a decreasing function in an interval D_2 (a subset of D) if

for all $$x_1, x_2 \in D_2,$$
$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

and f is called a strict decreasing function (or monotonically decreasing function) in D_2 if

for all $$x_1, x_2 \in D_2,$$
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

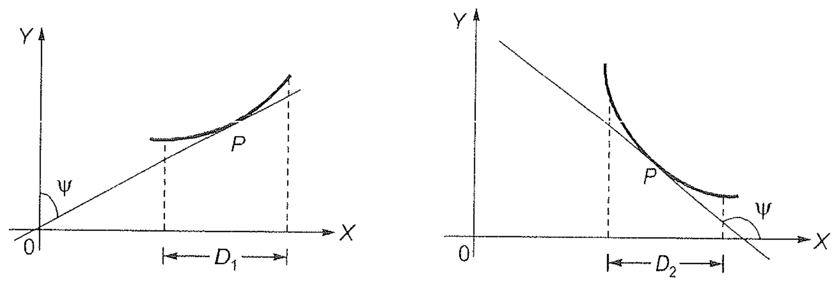


Decreasing in D_2 (a) | Strict decreasing in D_2 (b)

2.6.1.1 Conditions for an Increasing or a Decreasing Function

Now we shall see how to use derivative of a function to determine where it is increasing and where it is decreasing.

We know that the derivative (if it exists) at a point P of a curve represents the slope of the tangent to the curve at P.

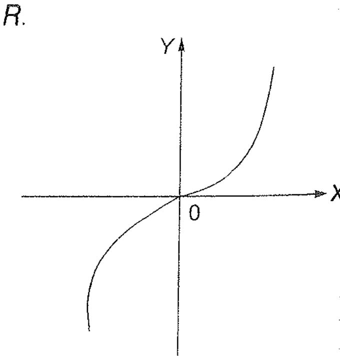


(a) (b)

Intuitively, from above fig. (i) we see that if f is a strict increasing function in D_1 (a subset of D_f), then the tangent to the curve $y = f(x)$ at every point of D_1 makes an acute angle ψ with the positive direction of x-axis, therefore $\tan \psi > 0 \Rightarrow f'(x) > 0$ for all $x \in D_1$.

Analogously, from above figure (ii) we see that if f is a strict decreasing function in D_2 (a subset of D_f), then the tangent to the curve $y = f(x)$ at every point of D_2 makes obtuse angle ψ with the positive direction of x-axis, therefore, $\tan \psi < 0 \Rightarrow f'(x) < 0$ for all $x \in D_2$.

But this intuition may fail, for example, consider the function $f(x) = x^3$, $D_f = R$.



A portion of its graph is shown in figure. It is a strict increasing function. However, here $f'(x) = 3x^2$ and at $x = 0$, $f'(0) = 0$, so the slope of the tangent at $x = 0$ is not positive, it is zero.

In fact, we have:

1. If a function f is increasing in D_1(a subset of D_f), then $f'(x) \geq 0$ for all $x \in D_1$.
2. If a function f is decreasing in D_2 (a subset of D_f), then $f'(x) \leq 0$ for all $x \in D_2$.

Conversely, common sense tells us that a function is increasing when its rate of change (derivative) is positive and decreasing when its rate of change is negative. We state these results as follows:

Theorem 1: If a function f is continuous is $[a, b]$, and derivable in (a, b) and

1. $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing in $[a, b]$
2. $f'(x) > 0$ for all $x \in (a, b)$, then f is strict increasing in $[a, b]$.

Theorem 2: If a function f is continuous in $[a, b]$, and derivable in (a, b) and

1. $f'(x) \leq 0$ for all x in (a, b), then $f(x)$ is decreasing in $[a, b]$.
2. $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is strict decreasing in $[a, b]$.

Remark: The formal proofs of these theorems are based on Lagrange's Mean value Theorem.

Corollary. If a function $f(x)$ is continuous in $[a, b]$, derivable in (a, b) and

1. $f'(x) > 0$ for all x in (a, b) except for a finite number of points where $f'(x) = 0$, then $f(x)$ is strict increasing in $[a, b]$.
2. $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x) = 0$, then $f(x)$ is strict decreasing in $[a, b]$.

Example 1.

Prove that the function $f(x) = ax + b$ is strictly increasing if $a > 0$.

Solution:

Given: $f(x) = ax + b,\ D_f = R.$

Note that f is continuous and differentiable for all $x \in R$.

Differentiating the given function w.r.t. x, we get $f'(x) = a$.

Now the given function is strictly increasing if $f'(x) > 0$ i.e. if $a > 0$.

Hence, the given function is strictly increasing for all $x \in R$ if $a > 0$.

Example 2.

Prove that the function e^{2x} is strictly increasing on R.

Solution:

Let $f(x) = e^{2x},\ D_f = R.$

Differentiating w.r.t. x, we get

$$f'(x) = e^{2x} \cdot 2 > 0 \text{ for all } x \in R.$$

$\Rightarrow f(x)$ is strictly increasing on R.

Example 3.

Prove that $\frac{2}{x} + 5$ is a strictly decreasing function

Solution:

Let $f(x) = \frac{2}{x} + 5,\ D_f = R - [0].$

Dif. it w.r.t. x, we get $f'(x) = 2 \cdot (-1 \cdot x^{-2}) + 0 = -\frac{2}{x^2}$

Since $x^2 > 0$ for all $x \in R,\ x \neq 0$, therefore,

$f'(x) < 0$ for all $x \in R,\ x \neq 0$, i.e., for all $x \in D_f$

$\Rightarrow$ the given function is strictly decreasing.

Example 4.

Prove that the function $f(x) = x^3 - 6x^2 + 15x - 18$ is strictly increasing on R.

Solution:

Given, $f(x) = x^3 - 6x^2 + 15x - 18,\ D_f = R.$

Dif. it w.r.t. we get $f'(x) = 3x^2 - 6.2\,x + 15.1 = 3(x^2 - 4x + 5)$

$= 3[(x-2)^2 + 1] \geq 3 \quad (\because (x-2)^2 \geq 0 \text{ for all } x \in R)$

$\Rightarrow \quad f'(x) > 0$ for all $x \in R$.

$\Rightarrow f(x)$ is strictly increasing function for all $x \in R$.

Example 5.

Find the intervals in which the following functions are strictly increasing or strictly decreasing

(a) $f(x) = 10 - 6x - 2x^2$

(b) $f(x) = x^2 - 12x^2 + 36x + 17$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

Solution:

(a) Given, $f(x) = 10 - 6x - 2x^2$, $D_f = R$.

Differentiating it w.r.t. x, we get

$$f'(x) = 0 - 6.1 - 2.\ 2x = -6 - 4x = -4\left(x + \frac{3}{2}\right).$$

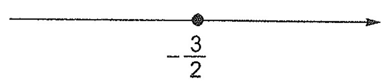
Putting, $f'(x) = 0$, we get $\dfrac{20 \pm \sqrt{400 - 156}}{2} = 0$

$$\Rightarrow \quad x + \frac{3}{2} = 0$$

$$\Rightarrow \quad x = -\frac{3}{2}$$

So there is only one critical point which is $x = -\frac{3}{2}$

Plotting this critical point on the number line we get the following picture

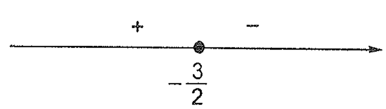


So the critical point divides the real number line into two regions which are $x \in \left(-\infty, -\frac{3}{2}\right)$ and $x \in \left(-\frac{3}{2}, \infty\right)$

Now we find $f'(0) = -6$ which is negative and so the region $x \in \left(-\frac{3}{2}, \infty\right)$ (which contains $x = 0$) is the region where the function is strictly decreasing.

Therefore in the other region i.e. $x \in \left(-\infty, -\frac{3}{2}\right)$ is the region in which the function is strictly increasing.

This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.



(b) Given, $f(x) = x^3 - 12x^2 + 36x + 17$, $D_f = R$.

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12)$$
$$= 3(x - 2)(x - 6).$$

Putting, $f'(x) = 0$ i.e. $3(x - 2)(x - 6) = 0$

$$\Rightarrow \quad (x - 2)(x - 6) = 0$$

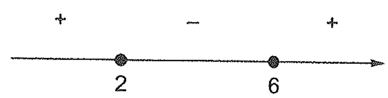
$\Rightarrow$ $x = 2$ or $x = 6$ are the two critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, 2)$ and $x \in (2, 6)$ and $x \in (6, \infty)$.

Now we find $f'(0) = 3(0-2)(0-6) = +36$ which is positive and so in the region $x \in (-\infty, 2)$ (which contains $x = 0$), the function is strictly increasing.

Therefore in the next region i.e. $x \in (2, 6)$, the function is strictly decreasing and in the next region $x \in (6, \infty)$, the function is again strictly increasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.

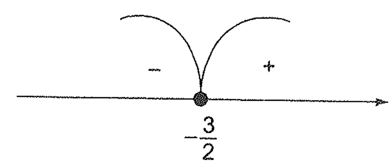


So the final region in which the function strictly increasing is $x \in (-\infty, 2) \cup (6, \infty)$ and the region in which the function is strictly decreasing is $x \in (2, 6)$.

(c) Given, $f(x) = -2x^3 - 9x^2 - 12x + 1,\ D_f = R$

Differentiating w.r.t. x, we get

$$f'(x) = -6x^2 - 18x - 12$$
$$= -6(x^2 + 3x + 2)$$
$$= -6(x+2)(x+1).$$



Putting, $f'(x) = 0$ i.e. $-6(x+2)(x+1) = 0$

$\Rightarrow$ $(x+2)(x+1) = 0$

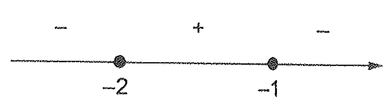
$\Rightarrow$ $x = -2$ and $x = -1$ are the critical points

Plotting these critical points on the number line we get the following picture

So the critical point divides the real number line into three regions which are $x \in (-\infty, -2)$ and $x \in (-2, -1)$ and $x \in (-1, \infty)$.

Now we find $f'(0) = -6(0+2)(0+1) = -12$ which is negative and so in the region $x \in (-1, \infty)$. (which contains $x = 0$), the function is strictly decreasing.

Therefore in the next adjacent region on the left i.e. $x \in (-2, -1)$, the function is strictly increasing and in the next adjacent region on the left $x \in (-\infty, -2)$, the function is again strictly decreasing. This is shown in the following diagram with the sign of $f'(x)$ in each region of the number line.



So the final region in which the function strictly increasing is $x \in (-2, -1)$ and the region in which the function is strictly decreasing is $x \in (-\infty, -2) \cup (-1, \infty)$.

2.6.2 Relative or Local Maxima and Minima (of function of a single independent variable)

Definitions: A function $f(x)$ is said to be a local or relative maximum at $x = a$, if there exist a positive number δ such that $f(a + \delta) < f(a)$ for all values of δ other than zero, in the interval $(-\delta, \delta)$.

A function $f(x)$ is said to be a local or relative minimum at $x = a$, if there exists a positive number δ such that $f(a + \delta) > f(a)$ for all values of δ, other than zero, in the interval $(-\delta, \delta)$.

Maximum and Minimum values of a function are together also called extreme values or turning values and the points at which they are attained are called points of maxima and minima.

The points at which a function has extreme values are called Turning Points.

2.6.2.1 Properties of Relative Maxima and Minima

1. At least one maximum or one minimum must lie between two equal values of a function.
2. Maximum and minimum values must occur alternatively.
3. There may be several maximum or minimum values of same function.
4. A function $y = f(x)$ is maximum at $x = a$, if dy/dx changes sign from +ve to –ve as x passes through a.
5. A function $y = f(x)$ is minimum at $x = a$, if dy/dx changes sign from –ve and +ve as x passes through a.
6. If the sign of dy/dx does not change while x passes through a, then y is neither maximum nor minimum at $x = a$.

2.6.2.2 Conditions for Maximum or Minimum Values

The necessary condition that $f(x)$ should have a maximum or a minimum at $x = a$ is that $f'(a) = 0$.

2.6.2.3 Definition of Stationary Values

A function $f(x)$ is said to be stationary at $x = a$ if $f'(a) = 0$.

Thus for a function $f(x)$ to be a maximum or minimum at $x = a$ it must be stationary at $x = a$.

2.6.2.4 Sufficient Conditions of Maximum or Minimum Values

There is a maximum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ is negative.

Similarly there is a minimum of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$ is positive.

Note: If $f''(a)$ is also equal to zero, then we can show that for a maximum or a minimum of $f(x)$ at $x = a$, we must have $f'''(a) = 0$. Then, if $f^{iv}(a)$ is negative, there will be a maximum at $x = a$ and if $f^{iv}(a)$ is positive there will be minimum at $x = a$.

In general if, $f'(a) = f''(a) = f'''(a) = \dots f^{n-1}(a) = 0$ and $f^n(a) \neq 0$ then n must an even integer for maximum or minimum. Also for a maximum $f^n(a)$ must be negative and for a minimum $f^n(a)$ must be positive.

2.6.2.5 Working rule for Maxima and Minima of *f*(*x*)

1. Find $f'(x)$ and equate to zero.
2. Solve the resulting equation for x. Let its roots be a_1, a_2, ... Then $f(x)$ is stationary at $x = a_1, a_2,$ .... Thus $x = a_1, a_2,$ .... are the only points at which $f(x)$ can be maximum or a minimum.
3. Find $f''(x)$ and substitute in it by terms $x = a_1, a_2,$ ....wherever $f''(x)$ is x we have a maximum and wherever $f''(x)$ is +ve, we have a minimum.
4. If $f''(a_1) = 0$, find $f'''(x)$ put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f^{iv}(x)$ and put $x = a_1$ in it. If $f^{iv}(a_1)$ is –ve, we have maximum at $x = a_1$, if it is positive there is a minimum at $x = a_1$. If $f^{iv}(a_1)$ is zero, we must find $f^{v}(x)$, and so on. Repeat the above process for each root of the equation $f'(x) = 0$.

2.6.3 Working Rules for Finding (Absolute) Maximum and Minimum in Range [*a*, *b*]

If a function f is differentiable in [*a*, *b*] except (possibly) at finitely many points, then to find (absolute) maximum and minimum values adopt the following procedure:

1. Evaluate $f(x)$ at the points where $f'(x) = 0$.
2. Evaluate $f(x)$ at the points where derivative fails to exist.
3. Find $f(a)$ and $f(b)$.

Then the maximum of these values is the absolute maximum of the given function f and the minimum of these values is the absolute minimum of the given function f.

Example 1.

Find the absolute maximum and minimum values of:

(a) $f(x) = 2x^3 - 9x^2 + 12x - 5$ in $[0, 3]$

(b) $f(x) = 12\, x^{4/3} - 6x^{1/3}, x \in [-1, 1]$

Also find points of maxima and minima.

Solution:

(a) Given $f(x) = 2x^3 - 9x^2 + 12x - 5$ ... (i)

It is differentiable for all x in $[0, 3]$, since it is a polynomial

Differentiating (i) w.r.t. x, we get

$$f'(x) = 2 \cdot 3x^2 - 9 \cdot 2x + 12 = 6(x^2 - 3x + 2)$$

Now, $f'(x) = 0$

$\Rightarrow \quad 6(x^2 - 3x + 2) = 0$

$\Rightarrow \quad x^2 - 3x + 2 = 0$

$\Rightarrow \quad (x - 1)(x - 2) = 0$

$\Rightarrow \quad x = 1, 2$

Also 1, 2 both are in $[0, 3]$, therefore 1 and 2 both are stationary points or turning points.

Further, $f(1) = 2 \cdot 1^3 - 9 \cdot 1^2 + 12 \cdot 1 - 5 = 2 - 9 + 12 - 5 = 0$

$f(2) = 2 \cdot 2^3 - 9 \cdot 2^2 + 12 \cdot 2 - 5 = 16 - 36 + 24 - 5 = -1$

$f(0) = -5$

and $f(3) = 2 \cdot 3^3 - 9 \cdot 3^3 + 12 \cdot 3 - 5 = 54 - 81 + 36 - 5 = 4$

Therefore, the absolute maximum value = 4 and the absolute minimum value = – 5. The point of maxima is 3 and the point of minima is 0.

(b) Given, $f(x) = 12x^{4/3} - 6x^{1/3}, x \in [-1, 1]$

Differentiating (i) w.r.t. x, we get

$$f'(x) = 12 \cdot \frac{4}{3} x^{1/3} - 6 \cdot \frac{1}{3} \cdot x^{-2/3} = 16x^{1/3} - \frac{2}{x^{2/3}} = \frac{2(8x - 1)}{x^{2/3}}$$

Now, $f'(x) = 0$

$\Rightarrow \quad \dfrac{2(8x - 1)}{x^{2/3}} = 0$

$\Rightarrow \quad x = \dfrac{1}{8}$

As $\frac{1}{8} \in [-1, 1]$, $\frac{1}{8}$ is a critical point.

Also we note that f is not differentiable at $x = 0$.

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{4/3} - 6\left(\frac{1}{8}\right)^{1/3} = 12\left(\frac{1}{2}\right)^4 - 6.\frac{1}{2}$$

$$= 12 \cdot \frac{1}{16} - 3 = \frac{3}{4} - 3 = -\frac{9}{4}$$

$$f(0) = 12.0 - 6.0 = 0$$

$$f(-1) = 12(-1)^{4/3} - 6(-1)^{1/3} = 12.1 - 6.(-1) = 18$$

$$f(1) = 12.1^{4/3} - 6/1^{1/3} = 12.1 - 6.1 = 6$$

Therefore, the absolute maximum value = 18 and the absolute minimum value = $-\frac{9}{4}$. The point of maxima is –1 and the point of minima is $\frac{1}{8}$.

Example 2.

It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value in the interval [0, 2]. Find the value of a.

Solution:

Let $$f(x) = x^4 - 62x^2 + ax + 9 \quad \text{... (i)}$$

It is differentiable for all x in [0, 2].

Differentiating (i) w.r.t. x, we get

$$f'(x) = 4x^3 - 124x + a$$

$$\therefore \quad f'(1) = 4.1^3 - 124.1 + a = a - 120$$

Given that at $x = 1$, the function (i) has maximum value, therefore, $x = 1$ is a point of maxima

$\Rightarrow$ $x = 1$ is a critical point

$\Rightarrow$ $f'(1) = 0$

$\Rightarrow$ $a - 120 = 0$

$\Rightarrow$ $a = 120$

2.6.4 Taylor's and Maclaurin's Series Expansion of Functions

2.6.4.1 Taylor's Series

If (i) $f(x)$ and its first $(n-1)$ derivatives be continuous in $[a, a + h]$, and (ii) $f^n(x)$ exists for every value of x in $(a, a + h)$, then there is at least one number $\theta(0 < \theta < 1)$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a+\theta h) \quad \text{... (i)}$$

which is called Taylor's theorem with Lagrange's form of remainder, the remainder R_n being $\frac{h^n}{n!}f^n\,(a+\theta h)$.

Consider the function $$\phi(x) = f(x) + (a + h - x)\, f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots + \frac{(a+h-x)^n}{n!}K$$

where K is defined by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}K \quad \text{... (ii)}$$

1. Since $f(x), f'(x), \ldots, f^{n-1}(x)$ are continuous in $[a, a + h]$, therefore $\phi(x)$ is also continuous in $[a, a + h]$,

2. $\phi'(x)$ exists and $= \frac{(a+h-x)^{n-1}}{(n-1)!}[f^n(x) - K]$

3. Also $\phi(a) = \phi(a+h)$ [By (ii)]

Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem, and therefore, there exists at least one number $\theta(0 < \theta < 1)$, such that $\phi'(a + \theta h) = 0$ i.e. $K = f^n(a + \theta h)\ (0 < \theta < 1)$

Substituting this value of K in (2), we get (1).

Cor. 1. Taking $n = 1$ in(1), Taylor's theorem reduces to Lagrange's Mean-value theorem.

Cor. 2. Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^n}{n!}f'^n(0) \qquad \ldots \text{(iii)}$$

which is known as Maclaurin's theorem with Lagrange's form of remainder.

Example

If $f(x) = \log(1 + x)$, $x > 0$, using Taylor's theorem, show that for $0 < \theta < 1$,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

Solution:

Deduce that $\log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

By Maclaurin's theorem with remainder R_3, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \qquad \ldots \text{(i)}$$

Here $\quad f(x) = \log(1 + x), \quad f(0) = 0$

$\therefore \quad f'(x) = \frac{1}{1+x}, \quad f'(0) = 1$

$$f''(x) = \frac{-1}{(1+x)^2}, \quad f''(0) = -1$$

and $\quad f'''(x) = \frac{2}{(1+x)^3}, \quad f'''(\theta x) = \frac{2}{(1+\theta x)^3}$

Substituting in (i), we get $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \qquad \ldots \text{(ii)}$

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

or $\quad (1 + \theta x)^3 > 1$ i.e. $\frac{1}{(1+\theta x)^3} < 1$

$$\therefore \quad x - x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence $\quad \log(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3} \qquad$ [by (ii)]

2.6.4.2 Maclaurin's Series

If $f(x)$ can be expanded as an infinite series, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\infty \qquad \ldots \text{(i)}$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (3) on page 154 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (1).

Example:

Using Maclaurin's series, expand $\tan x$ upto the term containing x^5.

Solution:

Let $f(x) = \tan x \qquad f(0) = 0$

$\therefore \quad f'(x) = \sec^2 x = 1 + \tan^2 x \qquad f'(0) = 1$

$$f''(x) = 2\tan x \sec^2 x = 2\tan x\,(1+\tan^2 x)$$

$$= 2\tan x + 2\tan^3 x \qquad f''(0) = 0$$

$$f'''(0) = 2\sec^2 x + 6\tan^2 x \sec^2 x$$

$$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$$

$$= 2 + 8\tan^2 x + 6\tan^4 x \qquad f'''(0) = 2$$

$$f''''(0) = 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x$$

$$= 16\tan x\,(1+\tan^2 x) + 24\tan^3 x(1+\tan^2 x)$$

$$= 16\tan x + 40\tan^3 x + 24\tan^5 x \qquad f''''(0) = 0$$

$$f'''''(0) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x)$$

$$f'''''(0) = 16$$

and so on.

Substituting the values of $f(0)$, $f'(0)$, etc. in the Maclaurin's series, we get

$$\tan x = 0 + x \times 1 + 0\cdot\frac{x^2}{2!} + \frac{x^3}{3!}\cdot 2 + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 16\cdots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots..$$

2.6.4.3 Expansion by Use of Known Series

When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series

1. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\ldots.$
2. $\sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \ldots\ldots.$
3. $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots\ldots.$
4. $\cosh\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots\ldots.$
5. $\tan\theta = \theta + \frac{\theta^3}{2} + \frac{\theta}{15}\theta^5 + \ldots\ldots\ldots$
6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots$
7. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots$
8. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots$
9. $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots\right)$
10. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$

Example:

Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.

Solution:

We have, $$e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \frac{(\sin x)^4}{4!} + \ldots\ldots$$

$$= 1+\left(x-\frac{x^3}{3!}+......\right)+\frac{1}{2!}\left(x-\frac{x^3}{3!}+......\right)+\frac{1}{3!}\left(x-\frac{x^3}{3!}+......\right)+\frac{1}{4!}(x-...)^4+....$$

$$= 1+\left(x-\frac{x^3}{6}+......\right)+\frac{1}{2}\left(x^2-\frac{x^3}{3}+......\right)+\frac{1}{6}(x^3-...)+\frac{1}{24}(x^4+....)+....$$

$$= 1+x+\frac{x^2}{2}-\frac{x^4}{8}+......$$

Otherwise, let

$$\begin{aligned} f(x) &= e^{\sin x} & f(0) &= 1 \\ \therefore \quad f'(x) &= e^{\sin x}\cos x\, f(x)\cdot\cos x, & f'(0) &= 1 \\ f''(x) &= f'(x)\cos x - f(x)\sin x, & f''(0) &= 1 \\ f'''(x) &= f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x, & f'''(0) &= 0 \\ f''''(x) &= f'''(x)\cos x - 3f'(x)\sin x - 3f'(x)\cos x\, f(x)\sin x, & f''''(0) &= 0 \end{aligned}$$

and so on

substituting the values of $f(0)$, $f'(0)$ etc., in the Maclaurin's series, we obtain

$$e^{\sin x} = 1+x\cdot 1+\frac{x^2}{2!}\cdot 1+\frac{x^3}{3!}\cdot 0+\frac{x^4}{4!}\cdot(-3)+.....$$

$$= 1+x+\frac{x^2}{2}-\frac{x^4}{8}+......$$

2.6.5 Slope Determination of Line

1. This is used to determine slope of straight line in xy plane. For example $y = x + 3$ is a line its slope is given by $\frac{dy}{dx} = 1$.

2. If two lines are perpendicular then product of their slopes is -1.
   For example let m_1 be the slope of first line and m_2 is the slope of second line. If both lines are perpendicular then

   $$m_1 \cdot m_2 = -1$$

3. The derivatives are also used to find slope of tangent on any curve.
   For example $\quad y = f(x)$ is a curve in x-y plane

   $$\frac{dy}{dx} = f'(x)\Big|_{(x_0,\, y_0)}$$ is the slope the tangent at point (x_0, y_0)

2.7 Partial Derivatives

2.7.1 Definition of Partial Derivative

If a derivative of a function of several independent variables be found with respect to any one of them, keeping the others as constants, it is said to be a partial derivative. The operation of finding the partial derivative of a function of more than one independent variables is called Partial Differentiation.

The symbols $\partial/\partial x$, $\partial/\partial y$ etc., are used to denote such differentiations and the expressions $\partial u/\partial x$, $\partial u/\partial y$ etc., are respectively called partial differential coefficients of u with respect to x and y.

If $u = f(x, y, z)$ the partial differential coefficient of u with respect to x i.e., $\partial u/\partial x$ is obtained by differentiating u with respect to x keeping y and z as constants.

2.7.2 Second order partial differential coefficients

If $u = f(x, y)$ then $\partial u/\partial x$ or f_x and $\partial u/\partial y$ or f_y are themselves function of x and y and can be again differentiated partially.

We call $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)$ as second order partial derivatives of u and these are respectively denoted by $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y \partial x}$.

Note: If $u = f(x, y)$ and its partial derivatives are continuous, the order of differentiation is immaterial i.e.,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

2.7.3 Homogenous Functions

An expression in which every term is of the same degree is called homogenous function. Thus, $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n$ is a homogenous function of x and y of degree n. This can also be written as,

$$x^n\left\{a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right\}$$

or $x^n f\left(\frac{y}{x}\right)$, where $f\left(\frac{y}{x}\right)$ is some function of $\frac{y}{x}$.

Note: To test whether a given function $f(x, y)$ is homogenous or not we put tx for x and ty for y in it.

If we get $f(tx, ty) = t^n f(x, y)$ the function $f(x, y)$ is homogenous of degree n otherwise $f(x, y)$ is not a homogenous function.

Note: If u is a homogenous function of x and y of degree n then $\partial u/\partial x$ and $\partial u/\partial y$ are also homogenous function of x and y each being of degree $(n-1)$.

2.7.4 Euler's Theorem on homogenous functions

If u is a homogenous function of x and y of degree n, then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Note: Euler's theorem can be extended to a homogenous function of any number of variables. Thus if $f(x_1, x_2, \ldots x_n)$ be a homogenous function of $x_1, x_2, \ldots x_n$ of degree n then, $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_n\frac{\partial f}{\partial x_n} = nf$

Example:

Show that $u = x^3 + y^3 + 3xy^2$ is a homogenous function of degree 3.

Solution:

Now, $\frac{\partial u}{\partial x} = 3x^2 + 3y^2$ and

$$\frac{\partial u}{\partial x} = 3y^2 + 6xy$$

Now,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = x(3x^2 + 3y^2) + y(3y^2 + 6xy)$$
$$= 3(x^3 + y^3 + 3xy^2)$$
$$= 3u$$

So, Euler's theorem says that u is a homogenous function of degree 3.

2.8 Total Derivatives

If
$$u = f(x, y), \text{ where } x = \phi_1(t) \text{ and } y = \phi_2(t),$$

then,
$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Here $\frac{du}{dt}$ is called the total differential coefficient of u with respect to t while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u.

In the same way if $u = f(x, y, z)$ where x, y, z are all functions of some variable t, when

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

This result can be extended to any number of variables.

Corollary 1: If u be a function of x and y, where y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

Corollary 2: If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

and
$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}$$

Corollary 3: If x and y are connected by an equation of the form $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{\partial f / \partial x}{\partial f / \partial y}$$

2.9 Maxima and Minima (of Function of Two Independent Variables)

2.9.1 Definitions

Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively.

Then, $f(a, b)$ is said to be maximum and a minimum value of $f(x, y)$ according as $f(a + h, b + k)$ is less or greater than $f(a, b)$ for all sufficiently small independent values of h and k, positive or negative, provided both of them are not equal to zero.

2.9.2 Necessary Conditions

The necessary conditions that $f(x, y)$ should have a maximum or minimum at $x = a$, $y = b$ is that

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}} = 0 \text{ and } \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}} = 0$$

2.9.3 Sufficient Condition for Maxima or Minima

Let $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}}$; $s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a\\y=b}}$; $t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$

Case 1: $f(x, y)$ will have a maximum or a minimum at $x = a$, $y = b$, if **rt > s2**. Further, $f(x, y)$ is maximum or minimum according as r in negative or positive.

Case 2: $f(x, y)$ will have neither maximum or minimum at $x = a$, $y = b$ if $rt < s^2$. i.e. $x = a$, $y = b$ is a saddle point.

Case 3: If $rt = s^2$ this case is doubtful case and further advanced investigation is needed to determine whether $f(x, y)$ is a maximum or minimum at $x = a$, $y = b$ or not. For gate problems case 3 will not apply. Check only case 1 or case 2.

2.10 Theorems of Integral Calculus

1. The integral of the product of a constant and a function is equal to be product of the constant and the integral of function.

   Thus if λ is constant, then $\int \lambda f(x) dx = \lambda \int f(x) dx$.

2. The integral of a sum of or difference of a finite number of functions is equal to sum or difference of integrals. Symbolically

   $$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)] dx = \int f_1(x) dx \pm \int f_2(x) dx \pm \int f_3(x) dx \pm \ldots \pm \int f_n(x) dx$$

2.10.1 Fundamental Formulae

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1}$
2. $\int \frac{1}{x} dx = \log x$
3. $\int \sin x \, dx = \cos x$
4. $\int \cos x \, dx = \sin x$
5. $\int \sec^2 x \, dx = \tan x$
6. $\int \text{cosec}^2 x \, dx = -\cot x$
7. $\int \sec x \tan x = \sec x$
8. $\int \text{cosec}\, x \cot x = -\text{cosec}\, x$
9. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$
10. $\int \frac{1}{1+x^2} dx = \tan^{-1} x$
11. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$
12. $\int \cos hx = \sin hx$
13. $\int \sin hx \, dx = \cos hx$

2.10.2 Useful Trigonometric Identities

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	–1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

1. $\sin(-x) = -\sin x$
2. $\cos(-x) = \cos x$
3. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
4. $\sin(x - y) = \sin x \cos y - \cos x \sin y$
5. $\cos(x + y) = \cos x \cos y - \sin x \sin y$
6. $\cos(x - y) = \cos x \cos y + \sin x \sin y$

7. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

8. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

9. (i) $\sin\left(\frac{\pi}{2} + x\right) = \cos x$(ii) $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

(iii) $\sin(\pi - x) = \sin x$(iv)$\cos(\pi - x) = -\cos x$

(v) $\sin(\pi + x) = -\sin x$(vi)$\cos(\pi + x) = -\cos x$

(vii) $\sin(2\pi - x) = -\sin x$(viii) $\cos(2\pi - x) = \cos x$

10. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

11. $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

12. $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$

13. $\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$

14. $\cot(x + y) = \frac{\cot x \cot y + 1}{\cot y + \cot x}$

15. $\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

16. $\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}$

17. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

18. $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

19. $\sin^2 x = 1 - \cos^2 x$

20. $\cos^2 x = 1 - \sin^2 x$

21. $e^{it} = \cos t + i \sin t$

2.10.3 Methods of Integration

There are various methods of integration by which we can reduce the given integral to one of the known standard integrals. There are four principal methods of integration.

1. **Integration by substitution:** A change in the variable of integration often reduces an integral to one of fundamental integrals.

   Let $I = \int f(x)dx$, then by differentiation w.r.t to x we have $\frac{dI}{dx} = f(x)$. Now put,

$$x = \phi(t), \text{ so that } \frac{dx}{dt} = \phi'(t)$$

Then, $\frac{dI}{dt} = \frac{dI}{dx} \cdot \frac{dx}{dt} = f(x) \cdot \phi'(t) = f\{\phi(t) \cdot \phi'(t)\}$ for $x = \phi(t)$

This gives $$I = \int f\{\phi(t) \cdot \phi'(t)\} dt$$

Rule to Remember:

To evaluate $\int f\{\phi(x) \cdot \phi'(x)\} dx$

Put $\phi(x) = t$

and $\phi'(x) dx = dt$

where $\phi'(x)$ is the differential coefficient of $\phi(x)$ with respect to x.

Three Forms of Integrals:

(a) $$\int \frac{f'(x)}{f(x)} dx = \log f(x)$$

Put $f(x) = t$ differentiating we get $f'(x) \cdot dx = dt$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log t = \log f(x)$$

Thus the integral of a fraction whose numerator is the exact derivative of its denominator is equal to the logarithmic of its denominator.

Example:

$$\int \frac{4x^3}{1+x^4} dx = \log(1+x^4) \qquad \text{... (i)}$$

Because, if we put $(1+x^4) = t$

$\Rightarrow$ $4x^3\, dx = dt$

(i) reduces to $\Rightarrow \int \frac{dt}{t} \Rightarrow \log t \Rightarrow \log(1+x^4)$.

Some Important Formulae Based on the Above Form:

(i) $$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{(-\sin x)}{\cos x} dx$$

$$= -\log \cos x$$
$$= \log(\cos x)^{-1}$$
$$= \log \sec x$$

(ii) $\int \cot x\, dx = \log \sin x$

(iii) $\int \sec x = \log(\sec x + \tan x)$

(iv) $\int \operatorname{cosec} x = \log\left(\tan \frac{x}{2}\right)$

(b) $\int [f(x)^n f'(x) dx = \frac{[f(x)]^{n+1}}{(n+1)}$ when $n \neq 1$: If the integrand consists of the product of a constant power of a function $f(x)$ and the derivative $f'(x)$ of $f(x)$, to obtain the integral we increase the index by unity and then divide by increased index. This is known as power formula.

Formulae:

(i) $\int f'(ax+b)dx = \dfrac{f(ax+b)}{a}$

(ii) $\int \dfrac{1}{\sqrt{a^2+x^2}}\,dx = \sin h^{-1}\left(\dfrac{x}{a}\right) = \log\left[x+\sqrt{x^2+a^2}\right]$

(iii) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$

(iv) $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \cos h^{-1}\left(\dfrac{x}{a}\right) = \log\left[x+\sqrt{x^2-a^2}\right]$

(v) $\int \sqrt{x^2+a^2}\,dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\sin h^{-1}\left(\dfrac{x}{a}\right)$

or $\dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\log\left\{x+\sqrt{x^2+a^2}\right\}$

(vi) $\int \sqrt{a^2-x^2}\,dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$

2. Integral of the product of two functions

Integration by parts: Let u and v be two functions of x. Then we have from differential calculus.

$$\frac{d}{dx}(uv) = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx} \qquad \text{... (i)}$$

Integrating both sides of (1) with respect to x, we have

$$uv = \int u\cdot\frac{dv}{dx}dx + \int v\cdot\frac{du}{dx}dx$$

$$\Rightarrow \quad \int u\frac{dv}{dx}dx = uv - \int v\cdot\frac{du}{dx}\cdot dx \qquad \text{... (ii)}$$

i.e. $\int udv = uv - \int v\cdot du$

This can also be written as $\int uvdx = u\int vdx - \int[du\int vdx]dx$

The choice of which function will be u and which function will be dv is very important in solving by integration by parts.

The ILATE method helps to decide this.

ILATE stands for

I : Inverse trigonometric functions ($\sin^{-1}x$, $\cos^{-1}x$ etc)

L : Logarithmic functions ($\log x$, $ln\,x$ etc.)

A : Algebraic functions (x^2, x^3+x^2+2, etc.)

T : Trigonometric functions ($\sin x$, $\cos x$ etc.)

E : Exponential function (e^x, a^x etc.)

whichever of the two functions comes first in ILATE, get designated as u and other function gets designated as dv.

Formulae Based Upon Above Method:

(a) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(b) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$

Integration by Partial Fractions:

(a) $$I = \int \frac{1}{x^2 - a^2}\,;\ (x > a)$$

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\left\{\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right\}$$

$$= \frac{1}{2a}\{\log(x-a) - \log(x+a)\} = \frac{1}{2a}\log\frac{x-a}{x+a}$$

Thus $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\frac{x-a}{x+a},\ x > a$

(b) $$I = \int \frac{1}{a^2 - x^2}\,dx\ (x < a)$$

In this case $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\frac{a+x}{a-x},\ x < a$

The following is a summary of some of the integrals derived so far by using the three methods of integration.

(a) $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(b) $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\frac{a+x}{a-x}$

(c) $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\frac{x-a}{x+a}$

(d) $\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sin h^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2 + a^2}\right]$

(e) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$

(f) $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cos h^{-1}\left(\frac{x}{a}\right) = \log\left[x + \sqrt{x^2 - a^2}\right]$

(g) $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$

(h) $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right)$

(i) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right)$

A few other useful integration formulae:

(a) $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\frac{n+1}{2}}{2\Gamma\left[\frac{m+n+2}{2}\right]}$$

where $\Gamma(x)$ is called the gamma function which satisfies the following properties

$$\Gamma(n+1) = n\Gamma n$$

$$\Gamma(n+1) = n! \qquad \text{if } n \text{ is a positive integer}$$

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(b) Walle's formula

$$\int_0^{\pi/2} \sin^n x = \int_0^{\pi/2} \cos^n x = \begin{cases} \frac{(n-1)(n-3)(n-5)}{(n)(n-2)(n-4)} \cdots \frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{(n-1)(n-3)(n-5)}{(n)(n-2)(n-4)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & \text{when } n \text{ is even} \end{cases}$$

2.11 Definite Integrals

If $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral of $f(x)$ between the limit of a and b. $b \to$ upper limit; $a \to$ lower limit.

2.11.1 Fundamental Properties of Definite Integrals

1. We have $\int_a^b f(x)dx = \int_a^b f(t)\,dt$ i.e., the value of a definite integral does not change with the change of variable of integration provided the limits of integration remain the same.

   Let $\int f(x)dx = F(x)$ and $\int f(t)dt = F(t)$

   Now $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

   $\int_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$

2. $\int_a^b f(x)dt = -\int_b^a f(x)dt$. Interchanging the limits of a definite integral does not change in the absolute value but change the sign of integrals.

3. We have $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

   Note 1: This property also holds true even if the point c is exterior to the interval (a, b).
   Note 2: In place of one additional point c, we can take several points. Thus several points.

   Thus, $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \ldots + \int_{c_n}^b f(x)dx$

4. (a) We have $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

   (b) We have $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

Proof: Let $I = \int_0^a f(x)dx$

Put $x = a - t \Rightarrow dx = -dt$ where $x = 0$, $t = a$ and when $x = a$, $t = 0$

$$\Rightarrow \quad I = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)dt = \int_0^a f(a-x)dx$$

5. $\int_{-a}^{+a} f(x)dx = 0$ or $2\int_0^a f(x)dx$ according as $f(x)$ is an odd or even function of x.

Odd and Even function

(a) An odd function of x if $f(-x) = -f(x)$

(b) An even function of x if $f(-x) = f(x)$.

6. $$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \quad \text{if } f(2a-x) = f(x)$$

and $$\int_0^{2a} f(x)dx = 0, \quad \text{if } f(2a-x) = -f(x)$$

Corollary: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

7. $$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$$

if $f(x) = f(x+a)$ [periodic function with period a]

8. $$\frac{d}{dt}\int_{\phi(t)}^{\Psi(t)} f(x)\,dx = f[\Psi(t)]\,\Psi'(t) - f[\phi(t)]\,(p)$$

Example 1.

Evaluate the following definite integrals:

(a) $\int_{-5}^{5} |x+2|dx$ (b) $\int_1^4 (|x| + |x-3|)dx$

Solution:

(a) Since for $-5 \le x \le -2$, $x + 2 \le 0$

$\Rightarrow \quad |x+2| = -(x+2)$

and for $-2 \le x \le 5$, $x + 2 \ge 0$

$\Rightarrow \quad |x+2| = x+2,$

$$\therefore \int_{-5}^{5} |x+2|\,dx = \int_{-5}^{-2} |x+2|dx + \int_{-2}^{5} |x+2|dx \quad \text{(Property 3)}$$

$$= \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5}(x+2)dx = \left[-\frac{x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= (-2+4) - \left(-\frac{25}{2} + 10\right) + \left(\frac{25}{2} + 10\right) - (2-4) = 29.$$

(b) Since for $1 \le x \le 3$, $x \ge 0$, $x - 3 \le 0 \Rightarrow |x| = x$, $|x-3| = -(x-3)$

Also for $3 \le x \le 4$, $x \le 0$, $x - 3 \ge 0 \Rightarrow |x| = x$, $|x-3| = x-3$.

$$\therefore \int_1^4 (|x| + |x-3|)dx = \int_1^3 (|x| + |x-3|)dx + \int_3^4 (|x| + |x-3|)dx \quad \text{(Property 3)}$$

$$= \int_1^3 (x-(x-3))dx + \int_2^4 (x+x-3)dx$$

$$= \int_1^3 3dx + \int_3^4 (2x-3)dx$$

$$= 3[x]_1^3 + \left[2\cdot\frac{x^2}{2} - 3x\right]_3^4$$

$$= 3(3-1) + (16-12) - (9-9)$$

$$= 16 + 4 - 0 = 10.$$

Example 2.

Evaluate the following definite integrals:

(a) $\int_{-1}^{2} f(x)dx$ where $f(x) = \begin{cases} 2x+1, & x \le 1 \\ x-5, & x > 1 \end{cases}$ (b) $\int_{-1}^{1} \frac{|x|}{x} dx$ (c) $\int_0^1 [3x]dx$

Solution:

(a) First note that the given function is discontinuous at $x = 1$.

$$\therefore \int_{-1}^{2} f(x)dx = \int_{-1}^{1} f(x)dx + \int_1^2 f(x)dx \qquad \text{(Property 3)}$$

$$= \int_{-1}^{1} (2x+1)dx + \int_1^2 (x-5)dx$$

$$= \left[x^2+x\right]_{-1}^{1} + \left[\frac{x^2}{2} - 5x\right]_1^{-2}$$

$$= (1+1)-(1-1)+(2-10)-\left(\frac{1}{2}-5\right) = 2-0-8+\frac{9}{2} = -\frac{3}{2}$$

(b) First note that $\frac{|x|}{x}$ is discontinuous at $x = 0$.

$$\therefore \int_{-1}^{1} \frac{|x|}{x} dx = \int_{-1}^{0} \frac{|x|}{x} dx + \int_0^1 \frac{|x|}{x} dx = \int_{-1}^{0} \frac{-x}{x} dx + \int_0^1 \frac{x}{x} dx$$

($\because -1 \le x \le 0 \Rightarrow |x| = -x$ and $0 \le x \le 1 \Rightarrow |x| = x$)

$$= \int_{-1}^{0} -1dx + \int_0^1 1dx = [-x]_{-1}^{0} + [x]_0^1$$

$$= -(0-(-1)) + (1-0) = -1 + 1 = 0.$$

(c) First note that $[3x]$ is discontinuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$,

$$\therefore \int_0^1 [3x]\, dx = \int_0^{1/3} [3x]\, dx + \int_{1/3}^{2/3} [3x]dx + \int_{2/3}^{1} [3x]dx$$

$$= \int_0^{1/3} 0\, dx + \int_{1/3}^{2/3} 1\, dx + \int_{2/3}^{1} 2dx = 0 + [x]_{1/3}^{2/3} + 2[x]_{2/3}^{1}$$

$$= \left(\frac{2}{3}-\frac{1}{3}\right)+2\left(1-\frac{2}{3}\right) = \frac{1}{3}+\frac{2}{3}=1$$

Example 3.

By using properties of definite integral, evaluate the following:

(a) $\int_{-\pi/2}^{\pi/2} \sin^4 x\, dx$ (b) $\int_{-\pi/4}^{\pi/2} x^3 \sin^4 x\, dx$ (c) $\int_0^{2\pi} |\cos x|\, dx$

Solution:

(a) Let $f(x) = \sin^4 x \Rightarrow f(-x) = \sin^4(-x) = (-\sin x)^4 = \sin^4 x = f(x)$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \sin^4 x\, dx = 2\int_0^{\pi/2} \sin^4 x\, dx = 2\int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(1-2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}\int_0^{\pi/2}(3-4\cos 2x+\cos 4x)\,dx$$

$$= \frac{1}{4}\left[3x-4.\frac{\sin 2x}{2}+\frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(3\frac{\pi}{2}-2\sin\pi+\frac{1}{4}\sin 2\pi\right)-\left(0-2\sin 0+\frac{1}{4}\sin 0\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{3\pi}{2}-0+0\right)-(0-0+0)\right] = \frac{3\pi}{8}$$

(b) Let $f(x) = x^3 \sin^4 x \Rightarrow f(-x) = (-x)^3 \sin^4(-x) = -x^3 \sin^4 x = -f(x)$

$\Rightarrow f(x)$ is an odd function; therefore, by property 5,

$$\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\, dx = 0$$

(c) Let $f(x) = |\cos x| \Rightarrow f(2\pi - x) = |\cos(2\pi - x)| = |\cos x| = f(x)$, therefore, by property 6,

$$\int_0^{2\pi} |\cos x|\, dx = 2\int_0^{\pi} |\cos x|\, dx$$

Again, $f(\pi - x) = |\cos(\pi - x)| = |-\cos x| = |\cos x| = f(x)$, therefore, by property 6,

$$\int_0^{\pi} |\cos x|\, dx = 2\int_0^{\pi/2} |\cos x|\, dx$$

$\therefore$ From (i) and (ii), we get

$$\int_0^{2\pi} |\cos x|\, dx = 2.2\int_0^{2\pi} |\cos x|\, dx = 4\int_0^{\pi/2} \cos x\, dx$$

$$\left(\because \text{ for } 0 \le x \le \frac{\pi}{2}, \cos x \ge 0 \Rightarrow |\cos x| = \cos x\right)$$

$$= 4[\sin x]_0^{\pi/2} = 4\left(\sin\frac{\pi}{2} - \sin 0\right) = 4(1-0) = 4.$$

Example 4.

Evaluate the following $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$.

Solution:

Let $$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \quad \text{... (i)}$$

Then, by using property 4b, we get

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

Example 5.

Evaluate the following definite integrals:

(a) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

(b) $\int_0^{\pi/2} \sin 2x \log(\tan x) dx$

Solution:

(a) Let $$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \text{... (i)}$$

Then, by using property 4b, we get

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx = \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right)^{-1} dx = \int_0^1 -1.\log\left(\frac{1-x}{x}\right) dx = -\int_0^1 \log\left(\frac{1-x}{x}\right) dx$$

$$= -I$$

$$\Rightarrow \quad 2I = 0$$

$$\Rightarrow \quad I = 0$$

(b) Let $$I = \int_0^{\pi/2} \sin 2x \log(\tan x) dx \quad \text{... (i)}$$

Then, by using property 4b, we get

Let $$I = \int_0^{\pi/2} \sin\left(2\left(\frac{\pi}{2}-x\right)\right)\log\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx$$

$$= \int_0^{\pi/2} \sin(\pi-2x)\log(\cot x)dx = \int_0^{\pi/2} \sin 2x \log\left((\tan x)^{-1}\right)dx$$

$$= \int_0^{\pi/2} \sin 2x(-1)\log(\tan x)dx = \int_0^{\pi/2} \sin 2x \log(\tan x)dx$$

$$= -I \qquad \text{[using (i)]}$$

$\Rightarrow$ $$2I = 0$$

$\Rightarrow$ $$I = 0$$

Example 6.

Evaluate the following definite integrals $\int_0^{\pi} \log(1+\cos x)dx$.

Solution:

$$I = \int_0^{\pi} \log(1+\cos x)dx \qquad \ldots \text{(i)}$$

Then, by using property 4b, we get

$$I = \int_0^{\pi} \log(1+\cos(\pi-x))dx = \int_0^{\pi} \log(1-\cos x)dx \qquad \ldots \text{(ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi} (\log(1+\cos x)+\log(1-\cos x))dx = \int_0^{\pi} \log\left(1-\cos^2 x\right)dx$$

$$= \int_0^{\pi} \log\left(\sin^2 x\right)dx = 2\int_0^{\pi} \log \sin x\, dx$$

$\Rightarrow$ $$I = \int_0^{\pi} \log \sin x\, dx$$

Let $f(x) = \log \sin x \Rightarrow f(\pi - x) = \log(\sin(\pi - x)) = \log \sin x = f(x)$, therefore, by using property 6, we get

$$I = 2\int_0^{\pi/2} \log \sin x\, dx = 2\left(-\frac{\pi}{2}\log 2\right) = -\pi \log 2.$$

2.12 Applications of Integration

We study three areas where integration is applied

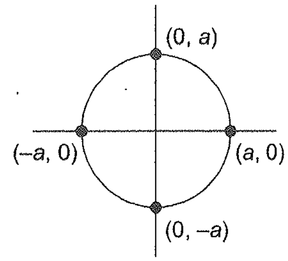
1. Areas of curves
2. Length of curves
3. Volumes of revolution

2.12.1 Preliminary : Curve Tracing

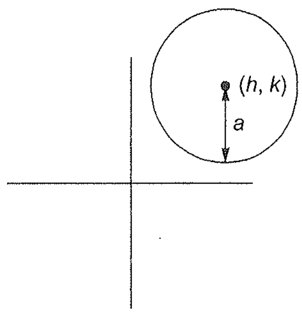
In order to find area under curves, as well as for evaluating double and triple integrals, it is used to know how to trace some common curves from their equations.

Circle : Cartesian Form:

1. $x^2 + y^2 = a^2$: Circle with centre (0, 0) and radius a.



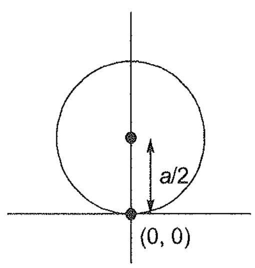
2. $(x - h)^2 + (y - k)^2 = a^2$: Circle with centre (h, k) and radius a.



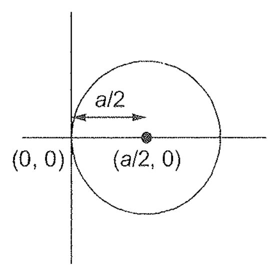
Polar Form:

1. $r = a$: Circle with centre (0, 0) and radius a.

2. $r = a \sin \theta$: Circle with centre $\left(0, \frac{a}{2}\right)$ and radius $\frac{a}{2}$.

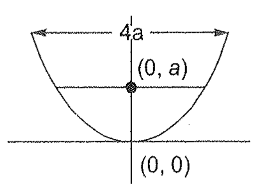


3. $r = a \cos \theta$: Circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

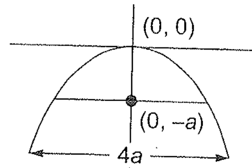


Parabola:

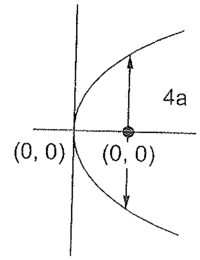
1. $x^2 = 4ay$: Parabola with vertex at (0, 0) and focus at (0, a) and latus rectum = $4a$.



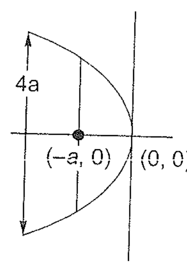
2. $x^2 = -4ay$: Parabola with vertex at (0, 0) and focus at (0, $-a$) and latus rectim = $4a$.



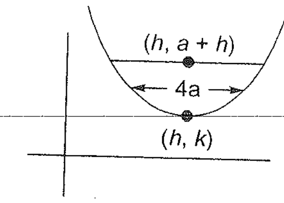
3. $y^2 = 4ax$: Parabola with vertex at (0, 0) and focus at (a, 0) and latus rectum = $4a$.



4. $y^2 = -4ax$: Parabola with vertex at (0, 0) and focus at ($-a$, 0) and latus rectum = $4a$.

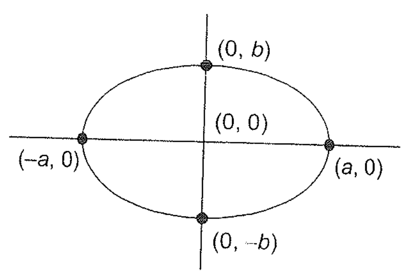


5. $(x - h)^2 = 4a(y - k)$: Parabola with centre at (h, k) focus at ($0 + h$, $a + k$) and latus rectum = $4a$.

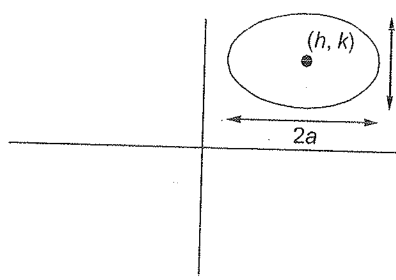


Ellipse:

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$: Ellipse with centre at (0, 0) and major axis = $2a$ and minor axis = $2b$.

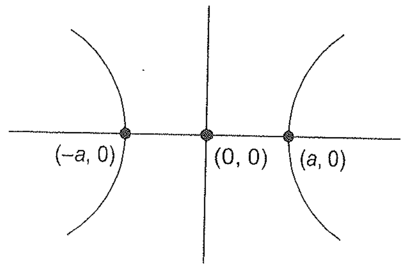


2. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$: Ellipse with centre at (h, k) and major axis = $2a$ and minor axis = $2b$.

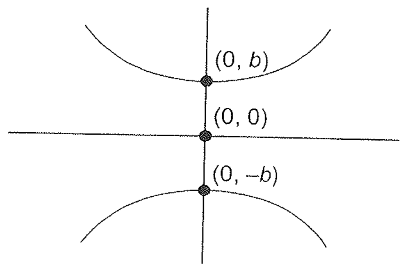


Hyperbola:

1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$: Hyperbola with vertex at $(a, 0)$ and $(-a, 0)$ and centre at $(0, 0)$.



2. $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$: Hyperbola with vertex at $(0, b)$ and $(0, -b)$ and centre at $(0, 0)$.

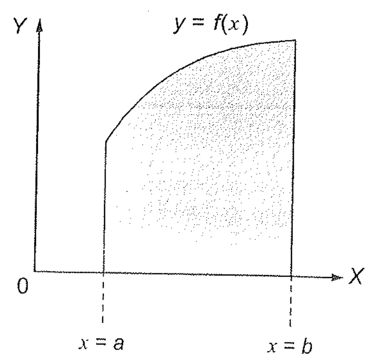


2.12.2 Areas of Cartesian Curves

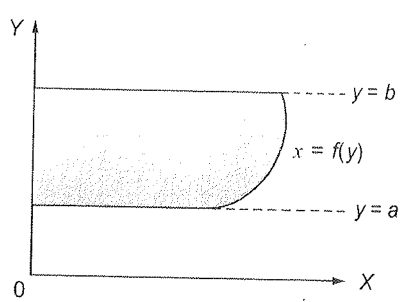
Theorem:

1. Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$\int_a^b y\,dx = \int_a^b f(x)\,dx$$



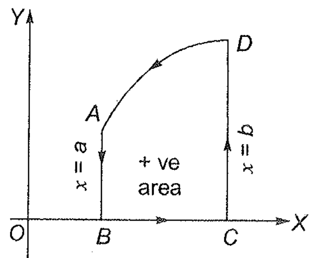
2. Interchanging x and y in the above formula, we see that the area bounded by the curve $x = f(y)$, the x-axis and the abscissa $y = a$, $y = b$ is $\int_a^b x dy = \int_a^b f(y) dy$ as shown in figure below.



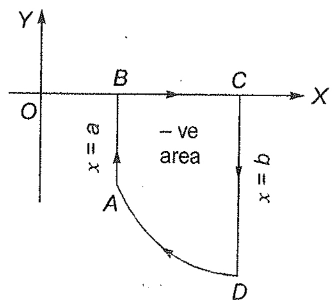
Note. 1 : The area bounded by a curve, the x-axis and two ordinates is called the **area under the curve.**

The process of finding the area of plane curves is often called **quadrature.**

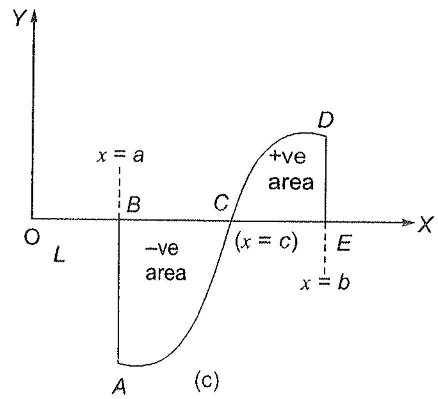
Note. 2 : Sign of an area. An area whose boundary is described in the anti-clockwise direction (i.e. lies above x-axis) is considered positive (Fig. a) and an area whose boundary is described in the clockwise direction (i.e. lies below x-axis) is taken as negative (Fig. b).



(a)



(b)



(c)

In Fig. (c) above, the area given by $\int_a^b y\,dx$ will not consist of the sum of the area $ABC\left(=\int_a^c y\,dx\right)$ and the area $CDE\left(=\int_c^b y\,dx\right)$ but their difference.

Thus to find the total area in such cases the numerical value of the area of each portion must be evaluated separately by taking modulus and their results added afterwards.

Example:

Find the area of the segment cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$.

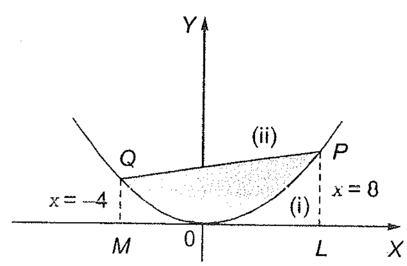
Solution:

Given parabola is $\qquad x^2 = 8y \qquad \ldots (i)$

and the straight line is

$$x - 2y + 8 = 0$$

$$\Rightarrow \quad y = \frac{x+8}{2} \qquad \ldots (ii)$$



Substituting the value of y from (ii) in (i), we get

$$x^2 = 4(x+8)$$

or $\quad x^2 - 4x - 32 = 0$

or $\quad (x-8)(x+4) = 0$

$\therefore \quad x = 8, -4$

Thus (i) and (ii) intersect at P and Q where $x = 8$ and $x = -4$.

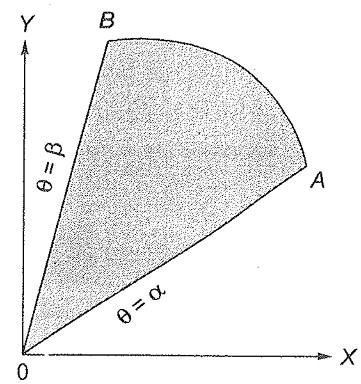
$\therefore$ Required area POQ (i.e. dotted area) = [area bounded by st. line (ii) and x-axis from $x = -4$ to $x = 8$] – [area bounded by parabola (i) and x-axis from $x = -4$ to $x = 8$]

$$= \int_{-4}^{8} y\,dx, \text{ from (ii)} - \int_{-4}^{8} y\,dx, \text{ from (i)}$$

$$= \int_{-4}^{8} \frac{x+8}{2}dx - \int_{-4}^{8} \frac{x^2}{8}dx = \frac{1}{2}\left|\frac{x^2}{2} + 8x\right|_{-4}^{8} - \frac{1}{8}\left|\frac{x^3}{3}\right|_{-4}^{8}$$

$$= \frac{1}{2}\{(32+64)-(-24)\} - \frac{1}{24}(512+64) = 36.$$

2.12.3 Areas of Polar Curves



Theorem: Area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha, \theta = \beta$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

Example:

Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.

Solution:

The equations of the circles are

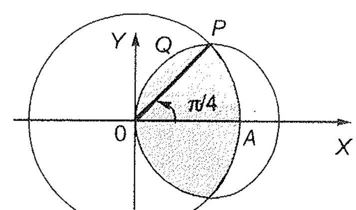
$$r = a\sqrt{2} \text{ and} \quad \ldots \text{(i)}$$
$$r = 2a\cos\theta \quad \ldots \text{(ii)}$$

(i) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$.

(ii) represents a circle symmetrical about OX, with centre at $(a, 0)$ and radius a.

The circles are shown in Fig. below. At their point of intersection P, eliminating r from (i) and (ii),

$$a\sqrt{2} = 2a\cos\theta \text{ i.e., } \cos\theta = \frac{1}{\sqrt{2}}$$



or $\theta = \pi/4$

$\therefore$ Required area $= 2 \times$ area $OAPQ$ (by symmetry)

$$= 2(\text{area } OAP + \text{area } OPQ)$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2 d\theta, \text{ for (i)} + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2 d\theta, \text{ for (ii)}\right]$$

$$= \int_0^{\pi/4}\left(a\sqrt{2}\right)^2 d\theta + \int_{\pi/4}^{\pi/2}(2a\cos\theta)^2 d\theta$$

$$= 2a^2 |\theta|_0^{\pi/4} + 4a^2\int_{\pi/4}^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta$$

$$= 2a^2(\pi/4 - 0) + 2a^2\left|\theta + \frac{\sin 2\theta}{2}\right|_{\pi/4}^{\pi/2}$$

$$= \frac{\pi a^2}{2} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = a^2(\pi - 1).$$

2.12.4 Derivative of arc Length d

Theorem: For the curve $y = f(x)$, we have

$$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

Proof: Let $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve AB (Figure below). Let arc $AP = s$, arc $PQ = \delta s$.

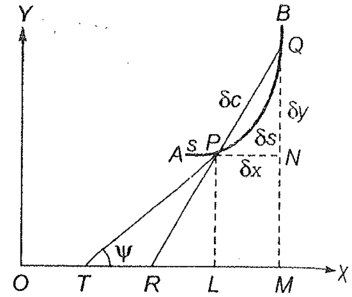
Draw PL, QM ⊥s on the x-axis and $PN \perp QM$.

$\therefore$ From the *rt.* triangle PNQ,

$$PQ^2 = PN^2 + NQ^2$$

i.e. $$\delta c^2 = \delta x^2 + \delta y^2$$

or $$\left(\frac{\delta c}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$



$\therefore$ $$\left(\frac{\delta s}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta x}\right)^2 = \left(\frac{\delta s}{\delta c}\right)^2\left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]$$

Taking limits as $Q \to P$ (i.e. $\delta c \to 0$),

$$\left(\frac{ds}{dx}\right)^2 = 1.\left[1+\left(\frac{dy}{dx}\right)^2\right] \qquad \left[\text{Since, } \underset{x\to 0}{\text{Lt}}\frac{\delta s}{\delta c} = 1\right]$$

If s increases with x as in Figure above, dy/dx is positive.

Thus $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$, taking positive sign before the radical ... (i)

Cor. 1. If the equation of the curve is $x = f(y)$, then

$\therefore$ $$\frac{ds}{dy} = \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]} \qquad \text{... (ii)}$$

Cor. 2. If the equation of the curve is in parametric form $x = f(t)$, $y = \phi(t)$, then

$$\frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{dt} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)^2\right]}$$

$\therefore$ $$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]} \qquad \text{... (iii)}$$

2.12.5 Lengths of Curves

Theorem: The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is

$$s = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$$

The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$, is

$$\int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}dy$$

The length of the arc of the curve $x = f(t)$, $y = f(t)$ between the points where $t = a$ and $t = b$, is

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}dt$$

The length of the arc of the curve $r = f(\theta)$, between the points where $\theta = \alpha$ and $\theta = \beta$, is

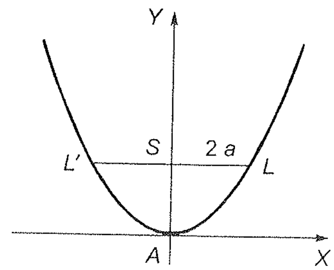
$$\int_{\alpha}^{\beta} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta$$

Example:

Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.

Solution:

Let A be the vertex and L an extremity of the latus-rectum so that at A, $x = 0$ and at L, $x = 2a$, as shown in figure.



Now, $y = x^2/4a$

so that $\frac{dy}{dx} = \frac{1}{4a}.2x = \frac{x}{2a}$

$$\therefore \quad \text{arc } AL = \int_0^{2a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]} dx = \int_0^{2a} \sqrt{\left[1+\left(\frac{x}{2a}\right)^2\right]} dx$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{[(2a)^2 + x^2]} dx = \frac{1}{2a}\left[\frac{x\sqrt{[(2a)^2 + x^2]}}{2} + \frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a}$$

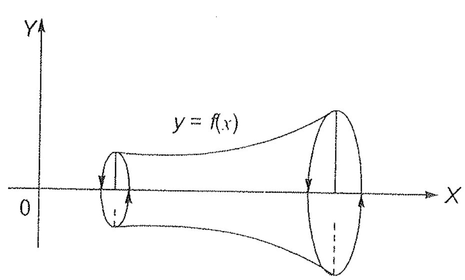
$$= \frac{1}{2a}\left[\frac{2a\sqrt{(8a)^2}}{2} + 2a^2 \sinh^{-1}1\right]$$

$$= a\left[\sqrt{2} + \sinh^{-1}1\right] = a\left[\sqrt{2} + \log(1+\sqrt{2})\right] \quad \left[\because \sinh^{-1}x = \log[x + \sqrt{(1+x^2)}]\right]$$

2.12.6 Volumes of Revolution

1. **Revolution about x-axis:** The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b \pi y^2 dx$.

   Let AB to the curve $y = f(x)$ between the ordinates LA ($x = a$) and MB($x = b$).



Example:

Find the volume of a sphere of radius a.

Solution:

Let the sphere be generated by the revolution of the semicircle ABC, of radius a about its diameter CA, (Figure)

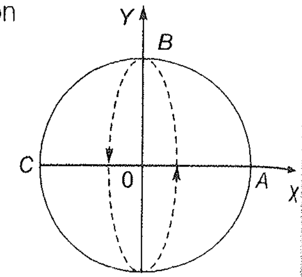
Taking CA as the x-axis and its midpoint O as the origin, the equation of the circle ABC is

$$x^2 + y^2 = a^2$$

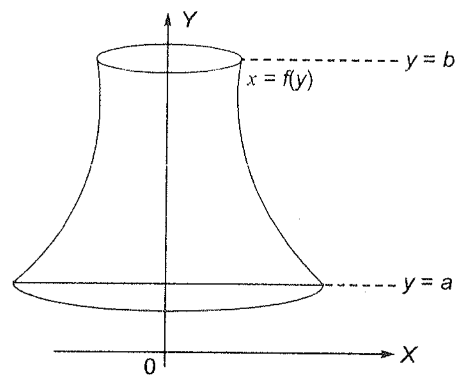
$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

$$= 2\int_0^a \pi y^2 dx = 2\pi\int_0^b (a^2 - x^2)dx$$

$$= 2\pi\left|a^2 x - \frac{x^3}{3}\right|_0^a = 2\pi\left|a^3 - \frac{a^3}{3} - (0-0)\right| = \frac{4}{3}\pi a^3$$



2. **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the volume of the solid generated by the revolution, about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissa $y = a$, $y = b$ is $\int_a^b \pi x^2 dy$.



Example:

Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum.

Solution:

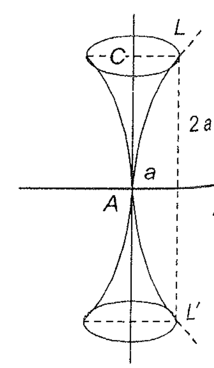
Given parabola is $x = y^2/4a$.

Let A be the vertex and L one extremity of the latus-rectum. For the arc AL, y varies from 0 to 2a (Figure)

$\therefore$ Required volume = 2 (volume generated by the revolution about the y-axis of the area ALC)

$$= 2\int_0^{2a} \pi x^2 dy = 2\pi\int_0^{2a} \frac{y^4}{16a^2}.dy$$

$$= \frac{\pi}{8a^2}\left|\frac{y^5}{5}\right|_0^{2a} = \frac{\pi}{40a^2}(32a^5 - 0) = \frac{4\pi a^3}{5}$$



2.13 Multiple Integrals and Their Applications

1. Double integrals
2. Change of order of integration
3. Double integrals in polar coordinates
4. Areas enclosed by plane curves
5. Triple integrals

2.13.1 Double Integrals

The definite integral $\int_a^b f(x)dx$ is defined as the limit of the sum

$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots + f(x_n)\delta x_n$,

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n-elementary areas $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_r, y_r) be any point within the r^{th} elementary area δA_r. Consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots + f(x_n, y_n)\delta A_n, \text{ i.e. } \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the double integral of $f(x, y)$ over the region R and is written as $\iint_R f(x,y)dA$

Thus
$$\iint_R f(x, y)dA = \lim_{\substack{n\to\infty \\ \delta A \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \quad \ldots \text{(i)}$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purposes of evaluation, (i) is expressed as the repeated integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dxdy$. Its value is found as follows:

1. When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x, y)$ is first integrated w.r.t. y (keeping x fixed) between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2 i.e.

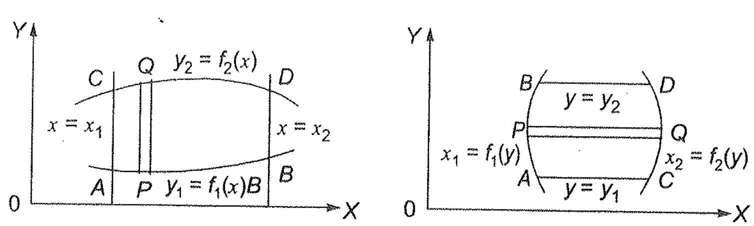
$$I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)dy}\, dx$$

where integrations carried from the inner to the outer rectangle.

Fig. (a) below illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

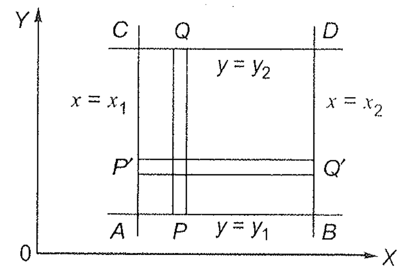
Thus the whole region of integration is the area $ABDC$.



(a) (b)

2. When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. between the limits y_1, y_2, i.e.

$I_2 = \int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\, dy$ which is geometrically illustrated by Fig. (b).



Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

3. When both pairs of limits are constants, the region of integration is the rectangle $ABDC$ (Fig.)

   In I_1, we integrated along the vertical strip PQ and then slide it from AC to BD.

   In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it fro AB to CD.

   Here obviously $I_1 = I_2$. Thus for constant limits, it hardly mattery whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

Example:

Evaluate $\int_0^5 \int_0^{x^2} x\left(x + y^2\right) dx\, dy$.

Solution:

$$I = \int_0^5 dx \int_0^{x^2} \left(x^2 + xy^2\right) dy = \int_0^5 \left[x^2 y + x.\frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^5 \left[x^2.x^2 + x.\frac{x^6}{3}\right] dx = \int_0^5 \left(x^4 + \frac{x^7}{3}\right) dx = \left|\frac{x^5}{5} + \frac{x^8}{24}\right|_0^5$$

$$= \frac{5^5}{5} + \frac{5^8}{24} \cong 16901.04$$

2.13.2 Change of order of Integration

In a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketh of the region of integration.

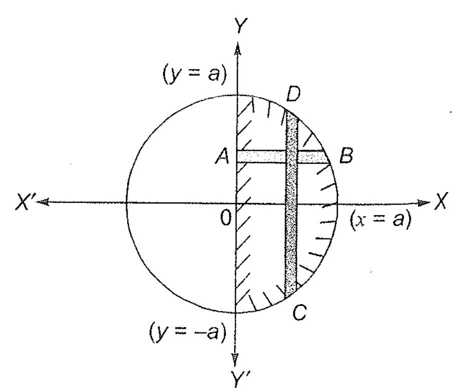
The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example: 1

Change the order of integration in the integral,

$$I = \int_{-a}^{a} \int_0^{\sqrt{(a^2 - y^2)}} f(x,y)\,dx\,dy.$$

Solution:



The elementary strip AB from $x = 0$ to $x = \sqrt{a^2 - y^2}$ (corresponding to the circle $x^2 + y^2 = a^2$), can be slided up from $y = -a$ to $y = a$ and integration is carried out. This shaded semicircular area is, therefore, the region of integration (Figure).

This corresponds to the given integral

$$I = \int_{-a}^{a}\int_{0}^{\sqrt{(a^2-y^2)}} f(x,y)dxdy.$$

The order of integration can be changed, if we first integrate with respect to y along a vertical strip CD (going from $y = -\sqrt{a^2-x^2}$ to $y = \sqrt{a^2-x^2}$), and then integrate with respect to x as x goes from $x = 0$ to $x = a$. (i.e. slide the strip CD from left to right from $x = 0$ to $x = a$)

This will result in the integral,

$$I = \int_{0}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)dy$$

or

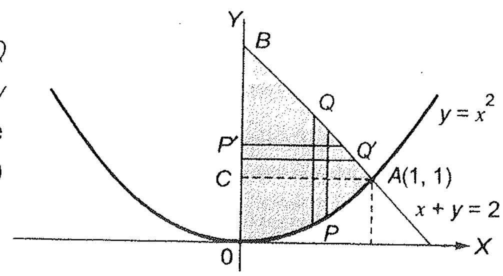
$$= \int_{0}^{a}\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)dy\, dx$$

Example: 2

Change the order of integration in $I = \int_{0}^{1}\int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate the same.

Solution:

Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Figure.



On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), i.e. the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_{0}^{1} dy \int_{0}^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

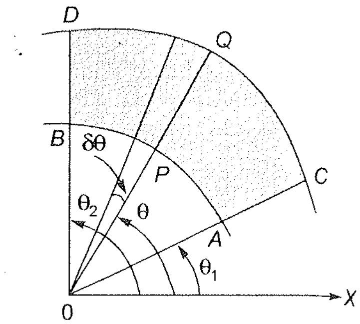
$$I_2 = \int_{1}^{2} dy \int_{0}^{2-y} xy\, dx$$

$$I = \int_{0}^{1} dy \int_{0}^{\sqrt{y}} xy\, dx + \int_{1}^{2} dy \int_{0}^{2-y} xy\, dx$$

$$= \int_{0}^{1} dy \left|\frac{x^2}{2}.y\right|_{0}^{\sqrt{y}} + \int_{1}^{2} dy \left|\frac{x^2}{2}.y\right|_{0}^{2-y}$$

$$= \frac{1}{2}\int_{0}^{1} y^2 dy + \frac{1}{2}\int_{1}^{2} y(2-y)^2 dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

2.13.3 Double Integrals in Polar Coordinates



Illustrates the process geometrically.

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness δθ.

Then $\int_{r_1}^{r_2} f(r,\theta)dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example:

Calculate $\iint r^3 dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

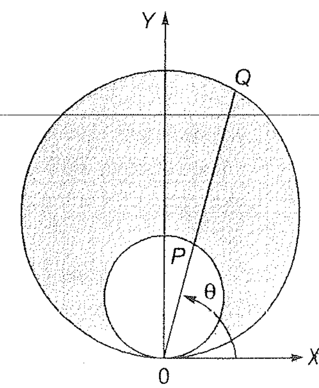
Solution:

Given circles $\quad r = 2\sin\theta \quad$ ... (i)

and $\quad r = 4\sin\theta \quad$ ... (ii)

are shown in Figure below. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is



$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr$$

$$= \int_0^{\pi} d\theta \left| \frac{r^4}{4} \right|_{2\sin\theta}^{4\sin\theta}$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta$$

$$= 60 \times 2\int_0^{\pi/2} \sin^4\theta\, d\theta$$

using reduction formula,

$$\int_0^{\pi/2} \sin^4\theta\, d\theta = \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots}\cdot\left(\frac{\pi}{2}\right)$$

Here $n = 4$ [using walle's formula with n is even]

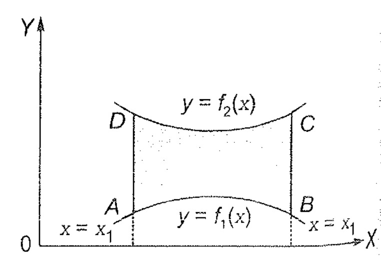
So, $$\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\times 1}{4\times 2}\left(\frac{\pi}{2}\right)$$

So the required integral, $$I = 120\times\frac{3\times 1}{4\times 2}\left(\frac{\pi}{2}\right) = 22.5\pi\,.$$

2.13.4 Area Enclosed by Plane Curves

The area enclosed by curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ is shown in figure and is given by the double integral

$$\int_{y_2}^{y_1}\int_{f_1(y)}^{f_2(y)} dx\, dy\,.$$



Example:

Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Solution:

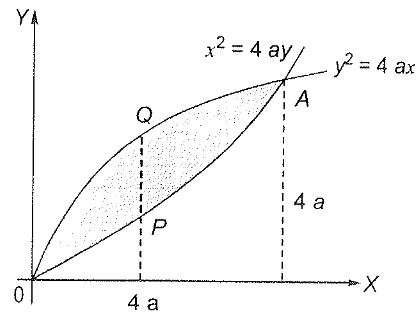
The equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at $O(0, 0)$ and $A(4a, 4a)$. As such for the shaded area between these parabolas (Fig. below) x varies from 0 to $4a$ and y varies from P to Q i.e. from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the required area

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dy\, dx$$

$$= \int_0^{4a}(2\sqrt{(ax)} - x^2/4a)dx$$

$$= \left| 2\sqrt{a}\cdot\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a}$$

$$= \frac{32}{2}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$



2.13.5 Triple Integrals

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the r^{th} sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r,$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the triple integral of $f(x, y, z)$ over the region V and is denoted by

$$\iiint f(x,y,z)dV$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dx\, dy\, dz.$$

If x_1, x_2 are constants; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows:

First $f(x, y, z)$ is integrated w.r.t. z between the limits, z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus
$$I = \boxed{\int_{x_1}^{x_2}\boxed{\int_{y_1(x)}^{y_2(x)}\boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz}\,dy}\,dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example: 1

Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x + y + z)dx\, dy\, dz$.

Solution:

Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1}\int_{0}^{z}\left|xy+\frac{y^2}{2}+yz\right|_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[(x+z)(2z)+\frac{1}{2}4xz\right]dx\,dz = 2\int_{-1}^{1}\left|\frac{x^2z}{2}+z^2x+\frac{x^2}{2}z\right|dz$$

$$= 2\int_{-1}^{1}\left(\frac{z^2}{2}+z^3+\frac{z^3}{2}\right)dz = 4\left|\frac{z^4}{4}\right|_{-1}^{1} = 0$$

Example: 2

Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz$.

Solution:

We have, $$I = \int_0^1 x\left[\int_0^{\sqrt{(1-x^2)}} y\left\{\int_0^{\sqrt{(1-x^2-y^2)}} z\,dz\right\}dy\right]dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y\cdot\left|\frac{z^2}{2}\right|_0^{\sqrt{(1-x^2-y^2)}} dy\right\}dx = \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y\cdot\frac{1}{2}(1-x^2-y^2)dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left|(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right|_0^{\sqrt{(1-x^2)}} dx = \frac{1}{8}\int_0^1[(1-x^2)^2\cdot 2x-(1-x^2)^2\cdot x]dx$$

$$= \frac{1}{8}\int_0^1(x-2x^3+x^5)dx = \frac{1}{8}\left|\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right|_0^1 = \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{48}$$

2.14 Vectors

2.14.1 Introduction

This chapter deals with vectors and vector functions in 3-space and extends the differential calculus to these vector functions. Forces, velocities and various other quantities are vectors. This makes the algebra and calculus of these vector functions the natural instrument for the engineer and physicist in solid mechanics, fluid flow, heat flow, electrostatics, and so on. The engineer must understand these fields as the basis of the design and construction of system, or robots. In three dimensions (as opposed to higher dimensions), geometrical ideas become influential, enriching the theory, and many geometrical quantities (tangents and normal, for example) can be given by vectors.

We first explain the basic algebraic operations with vectors in 3-space. Vector differential calculus begins next with a discussion of vector functions, which represent vector fields and have various physical and geometrical applications. Then the basic concepts of differential calculus are extended to vector functions in a simple and natural fashion. Vector functions are useful in studying curves and their applications as paths of moving bodies in mechanics.

We finally discuss three physically and geometrically important concepts related to scalar and vector fields, namely, the gradient, divergence, and curl. Integral theorems involving these concepts follow in vector integral calculus.

2.14.2 Basic Definitions

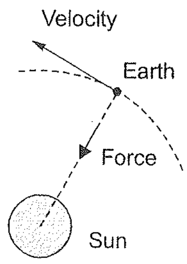
In geometry and physics and its engineering applications we use two kinds of quantities, scalars and vectors. A scalar is a quantity that is determined by its magnitude, its number of units measured on a suitable scale. For instance, length, temperature, and voltage are scalars.

A vector is a quantity that is determined by both its magnitude and its direction; thus it is an arrow or directed line segment. For instance, a force is a vector, and so is a velocity, giving the speed and direction of motion (Figure below). We denote vectors by lower case bold face letters *a*, *b*, *v* etc.

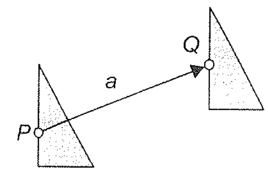
A vector (arrow) has a tail, called its initial point, and a tip, called its terminal point. For instance, in Figure, the triangle is translated (displaced without rotation); the initial point P of the vector **a** is the original position of a point and the terminal point *Q* is its position after the translation.

The length (or magnitude) of a vector a (length of the arrow) is also called the norm (or Euclidean norm) of ***a*** and is denoted by |***a***|.

A vector of length 1 is called a unit vector.



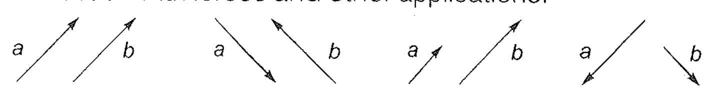
Force and Velocity
(i)



Translation
(ii)

2.14.3 Equality of Vectors

By definition, two vectors ***a*** and ***b*** are equal, written, ***a*** = ***b***, if they have the same length and the same direction (Figure below). Hence a vector can be arbitrarily translated, that is, its initial point can be chosen arbitrarily. This definition is practical in connection with forces and other applications.



Equal vectors, a = b	Vectors having the same length but different direction	Vectors having the same direction but different length	Vectors having different length and different direction

Vectors

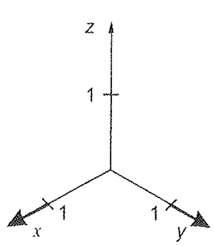
2.14.4 Components of a Vector

We choose an *xyz* Cartesian coordinate system in space, that is, a usual rectangular coordinate system with the same scale of measurement on the three mutually perpendicular coordinate axes. Then if a given vector ***a*** has initial point P: (x_1, y_1, z_1) and terminal point *Q:* $(x_2\ y_2, z_2)$ the three numbers,

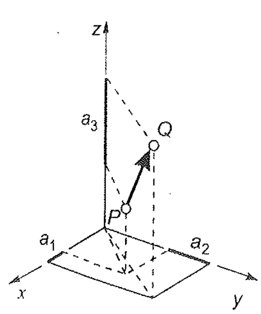
1. $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$; are called the components of the vector a with respect to that coordinate system, and we write simply $a = [a_1, a_2, a_3]$.

   Length in Terms of Components: By definition, the length |*a*| of a vector a is the distance between its initial point *P* and terminal point *Q*. From the Pythagorean theorem, and figure (ii) below we see that

2. $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$



Cartesian coordinate
(i)



Components of a vector
(ii)

Example:

Components and length of a vector.

The vector a with initial point **P**: (4, 0, 2) and terminal point **Q**: (6, – 1, 2) has the components $a_1 = 6 - 4 = 2$, $a_2 = -1 - 0 = -1$, $a_3 = 2 - 2 = 0$.

Solution:

Hence, $\mathbf{a} = [2, -1, 0]$.

$$|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

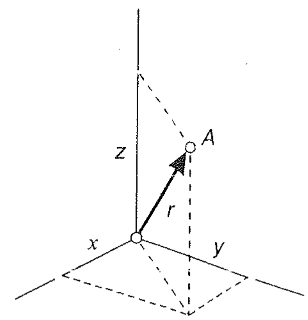
If w choose (–1, 5, 8) as the initial point of a, the corresponding terminal point is (1, 4, 8).

If we choose the origin (0, 0, 0) as the initial point of a, the corresponding terminal point is (2, –1, 0); i.e. its coordinates equal the components of a, if origin is closer as initial point. This suggests that we can determine each point in space by a vector, as follows:

2.14.5 Position Vector

A Cartesian coordinate system being given, the position vector r of a point A; (x, y, z) is the vector with the vector with the origin (0, 0, 0) as the initial point and A as the terminal point. Thus, $r = [x, y, z]$.

Furthermore, if we translate a vector $\mathbf{a}$, with initial point P and terminal point Q, then corresponding coordinates of P and Q change by the same amount, so that the components of the vector remain unchanged. This proves



Position vector r of a point A: (x, y, z)

2.14.5.1 Vectors as Ordered Triples of Real Numbers

Theorem: A fixed Cartesian coordinate system being given, each vector is uniquely determined by its ordered triple of corresponding components. Conversely, to each ordered triple of real numbers $(a_1, a_2\ a_3)$ there corresponds precisely one vector $\mathbf{a} = [a_1, a_2, a_3]$. In particular, the ordered triple (0, 0, 0) corresponds to the zero vector "0", which has length 0 and no direction.

Hence a vector equation $\mathbf{a} = \mathbf{b}$ is equivalent to the three equations $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$ for the components.

We see that from our "geometrical" definition of vectors as arrows we have arrived at an "algebraic" characterization by above Theorem. We could have started from the latter and reversed our process. This shows that the two approaches (i.e. "geometrical" and "algebraic" approaches) are equivalent.

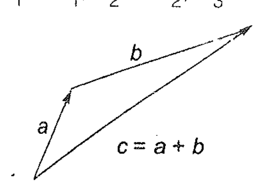
2.14.6 Vector Addition, Scalar Multiplication

Applications have suggested algebraic calculations with vectors that are practically useful and almost as simple as calculations with numbers.

2.14.6.1 Definition: 1

Addition of Vectors: The sum $\mathbf{a} + \mathbf{b}$ of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is obtained by adding.

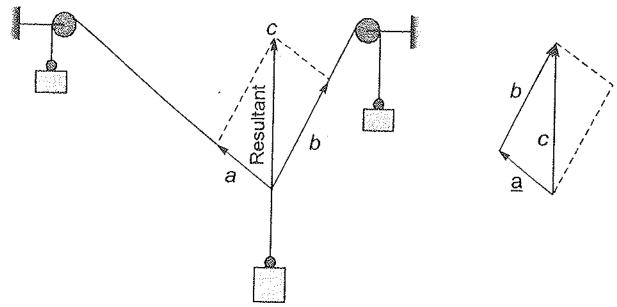
$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$



Vector addition

Geometrically, place the vectors as in Fig. above (the initial point of b at the terminal point of a): then $\mathbf{a} + \mathbf{b}$ is the vector drawn from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$.

Figure shows that for forces, this addition is the parallelogram law by which we obtain the resultant of two forces in mechanics.



Resultant of two forces (parallgram law)

Figure illustrates (for the plane) that the "algebraic" way and the "geometric" way of vector addition amount to the same thing.

Basic properties of Vector addition follow immediately from the familiar laws for real numbers

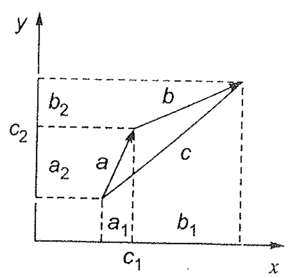
(a) $\dot{a}+\dot{b}=\dot{b}+\dot{a}$ (commutativity)

(b) $(\dot{u}+\dot{v})+\dot{w}=\dot{u}+(\dot{v}+\dot{w})$ (associativity)

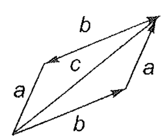
(c) $\dot{a}+0=0+\dot{a}=\dot{a}$

(d) $\dot{a}+(-\dot{a})=0$

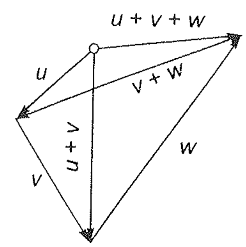
where $-a$ denotes the vector having the length $|a|$ and the direction opposite to that of a.



Vector addition



Commutativity of vector addition



Associativity of vector addition

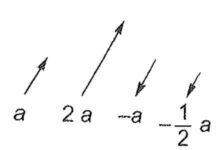
In property (b) above, instead of $u + (v + w)$ or $(u + v) + w$, we may simply write $u + v + w$ without brackets, and similarly for sums of more than three vectors. Also instead of $a + a$ we also write $2a$, and so on. This (and the notation $-a$ before) suggests that we define the second algebraic operation for vectors, namely, the multiplication of vectors by a scalar as follows.

2.14.6.2 Definition: 2

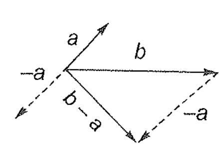
Scalar Multiplication (Multiplication by a Number): The product $c\dot{a}$ of any vector $a = [a_1, a_2, a_3]$ and any scalar c (real number c) is the vector obtained by multiplying each component of a by c,

$$ca = [ca_1, ca_2, ca_3]$$

Geometrically, if $a \neq 0$, then ca with $c > 0$ has the direction of $\dot{a}$ and with $c < 0$ the direction opposite to a. In any case, the length of ca is $|ca| = |c||a|$, and $ca = 0$ if $a = 0$ or $c = 0$ (or both).



Scalar multiplication [multiplication of vector by scalars (numbers)]



Difference of vectors

Example:

Vector Addition and Multiplication by Scalars.

With respect to a given coordinate system, let

$$\dot{a} = [4, 0, 1] \text{ and } \dot{b} = \left[2, -5, \tfrac{1}{3}\right]$$

Solution:

Then.

$$-a = [-4, 0, -1], 7a = [28, 0, 7]. \; a + b = \left[6, -5, \frac{4}{3}\right], \text{ and}$$

$$2(a-b) = 2\left[2, -5, \tfrac{2}{3}\right] = \left[4, 10, \tfrac{4}{3}\right] = 2\dot{a} - 2\dot{b}.$$

2.14.7 Unit Vectors

Any vector whose length is 1 is a unit vector $\dot{i}$, $\dot{j}$ and $\dot{k}$ are example of special unit vectors, which are along x, y and z coordinate axes.

$$|i| = |j| = |k| = 1$$

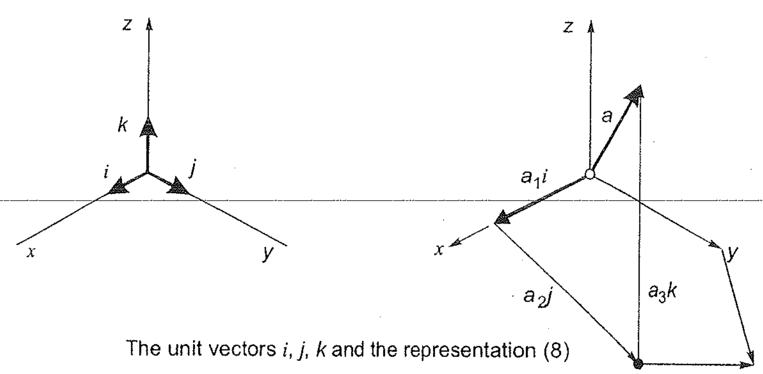
$$u = \cos\theta\, i + \sin\theta j$$

gives every unit vector in the plane.

2.14.7.1 Representation of Vectors in Terms of *i, j,* and *k*

$$\dot{a} = [a_1, a_2, a_3] = a_1 i + a_2 j + a_3 k.$$

In this representation, i, j, k are the unit vectors in the positive directions of the axes of a Cartesian coordinate system.



The unit vectors *i, j, k* and the representation (8)

$i = [1, 0. 0]$ $\quad j = [0, 1, 0]$, $\quad k = [0, 0, 1]$

and the right side of $a = a_1 i + a_2 j + a_3 k$ is a sum of three vectors parallel to the three axes.

Example:

i, j, k Notation for Vectors:

Solution:

In previous example where $a = [4, 0, 1]$ and $b = \left[2, -5, \frac{1}{3}\right]$,

we have $a = 4i + k$, $b = 2i - 5j + \frac{1}{3}k$, and so on, in i, j, k notation.

2.14.8 Length and Direction of Vectors

Any vector ***a*** may be written as a product of its length and direction as follows:

$$\dot{a} = |\dot{a}|\left(\frac{\dot{a}}{|\dot{a}|}\right)$$

here $|\dot{a}|$ is the length of vector and $\frac{\dot{a}}{|\dot{a}|}$ is a unit vector in direction of a

Example 1.

Express $3i - 4j$ as a product of length and direction:

$$v = 3i - 4j$$

Solution:

$$\text{length of } v = |v| = \sqrt{3^2 + 4^2}$$

The unit vector in direction of $v = \dfrac{v}{|v|} = \dfrac{3i - 4j}{5} = \dfrac{3}{5}i - \dfrac{4}{5}j$

$\therefore$ $$v = 3i - 4j = 5\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

Note that $$\left|\frac{3}{5}i - \frac{4}{5}j\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$$

Since, $\dfrac{3}{5}i - \dfrac{4}{5}j$ is a unit vector.

Example 2.

Find a unit vector in direction of $4i + 6j$.

Solution:

The required vector is $\dfrac{v}{|v|} = \dfrac{4i + 6j}{\sqrt{4^2 + 6^2}} = \dfrac{4}{\sqrt{52}}i + \dfrac{6}{\sqrt{52}}j$

Example 3.

Find unit vector, tangent and normal to the curve

$$y = \frac{x^3}{2} + \frac{1}{2} \text{ at pt}(1,1)$$

Solution:

Unit vector tangent to curve:

$$y' = \left[\frac{3x^2}{2}\right]_{(1,1)} = \frac{3 \times 1^2}{2} = \frac{3}{2}$$

Any vector with slope of $\dfrac{3}{2}$ can be written as

$$v = k(2i + 3j)$$

$$|v| = k\sqrt{2^2 + 3^2} = \sqrt{13}\,A$$

A unit vector in direction of v is

$$u = \frac{v}{|v|} = \frac{k(2i + 3j)}{\sqrt{13}\,k} = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{3}}j$$

Note also that $$-u = \frac{-2}{\sqrt{13}}i - \frac{3}{\sqrt{13}}j$$

is another unit vector tangent to the curve, but in opposite direction to u.

Unit vector normal to curve:

$$u = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j$$

Any vector normal to $ai + bj$ is of the form of $bi - aj$, since product of this slopes is

$$\left(\frac{b}{a}\right)\left(-\frac{a}{b}\right) = -1$$

So a vector normal to $u = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j$ is $n = \frac{3}{\sqrt{13}}i - \frac{2}{\sqrt{13}}j$

Note that $-n = \frac{3}{\sqrt{13}}i + \frac{2}{\sqrt{13}}j$ is another unit vector normal to the curve, but in opposite direction to n.

2.14.9 Inner Product (Dot Product)

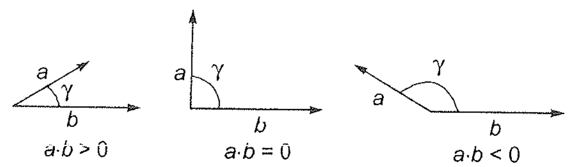
We shall now define a multiplication of two vectors that gives a scalar as the product and is suggested by various applications.

Definition. Inner Product (Dot Product) of Vectors

The inner product or dot product $\mathbf{a} \cdot \mathbf{b}$ (read "$\mathbf{a}$ dot $\mathbf{b}$") of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the product of their lengths times the cosine of their angle, see Fig. below.

1. $\dot{a} \cdot \dot{b} = |\dot{a}|\,|\dot{b}| \cos \gamma$

   The angle γ, $0 \leqq \gamma \leqq \pi$, between a and b is measured when the vectors have their initial points coinciding, as in Fig. below.



Angle between vectors and value of inner product

In components, $a = [a_1, a_2, a_3]$, $b = [b_1, b_2, b_3]$, and

2. $\dot{a} \cdot \dot{b} = a_1b_1 + a_2b_2 + a_3b_3$

   can be derived from (i).

   Since the cosine in (i) may be positive, zero, or negative, so may be the inner product. The case that the inner product is zero is of great practical interest and suggests the following concept.

   A vector a is called orthogonal to a vector b if $a \cdot b = 0$. Then b is also orthogonal to a and we call these vectors orthogonal vectors. Clearly, the zero vector is orthogonal to every vector. For nonzero vectors we have $\dot{a} \cdot \dot{b} = 0$ if and only if $\cos \gamma = 0$; thus $\gamma = \pi/2 (90°)$. This proves the following important theorem.

   Theorem: 1 (Orthogonality)

   The inner product of two nonzero vectors is zero if and only if these vectors are perpendicular.

   Length and Angle in Terms of Inner Product: Equation (i) above with $\dot{b} = \dot{a}$ gives $a \cdot a = |a|^2$.

3. $|a| = \sqrt{a \cdot a}$

   From (i) and (iii) we obtain for the angle γ between two nonzero vectors

4. $\cos \gamma = \frac{a \cdot b}{|a||b|} = \frac{a \cdot b}{\sqrt{a \cdot a}\sqrt{b \cdot b}}$

Example:

Find the inner product and the lengths of $\dot{a} = [1, 2, 0]$ and $\dot{b} = [3, -2, 1]$ as well as the angle between these vectors.

Solution:

$$a \cdot b = 1.3 + 3.(-2) + 0.1 = -1$$

$$|a| = \sqrt{a \cdot a} = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

$$|b| = \sqrt{b \cdot b} = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

$$\gamma = \text{arc cos}\frac{a \cdot b}{|a||b|} = \text{arc cos}(-0.11952)$$

$$= 1.69061 \doteq 96.865°$$

The given vectors make an obtuse angle between them solves and notice that the inner product has come out negative because of this.

General Properties of Inner Products: From the definition we see that the inner product has the following properties. For any vectors a, b, c and scalars q_1, q_2.

(a) $[q_1 a + q_2 b] \cdot \dot{c} = q_1 a \cdot \dot{c} + q_2 b \cdot \dot{c}$

(Linearity)

(b) $a \cdot b = b \cdot a$ (Symmetry)

(c) $a \cdot a \geqq 0$ (Positive-definiteness)

(d) $a \cdot a = 0$ if and only if $a = 0$ (Positive-definiteness)

Hence dot multiplication is commutative and is distributive with respect to vector addition; in fact, from above (a) with $q_1 = 1$ and $q_2 = 1$ we have

5. $(\dot{a} + \dot{b}) \cdot \dot{c} = \dot{a} \cdot \dot{c} + \dot{b}\dot{c}$ (Distributivity)

Furthermore, from $a \cdot b = |\dot{a}||\dot{b}| \cos\gamma$ and $|\cos\gamma| \leq |$, So

6. $|a \cdot b| \leqq |a|\,|b|$ (Schwarz inequality)

7. $|a + b| \leqq |a| + |b|$ (Triangle inequality)

A simple direct calculation with inner products shows that

8. $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$ (Parallelogram equality)

Equations (6) – (8) play a basic role in so-called Hilbert spaces (abstract inner product spaces), which form the basis of quantum mechanics.

Derivation of $\dot{a} \cdot \dot{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ from $a \cdot b = |\dot{a}||\dot{b}| \cos\gamma$

Let $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$.

Since i, j and k are unit vectors, we have from (3) $i \cdot i = |i|^2 = 1$, $j \cdot j = |j|^2 = 1$ and $k \cdot k = |k|^2 = 1$.

Since i, j, k are or orthogonal to each other (The coordinate axes being perpendicular to each other), we get from theorem, $i \cdot j = 0$, $j \cdot k = 0$, $k \cdot i = 0$.

Now, $a \cdot b = (a_1 i + a_2 j + a_3 k) \cdot (b_1 i + b_2 j + b_3 k)$

using distributive property, we first have a sum of nine inner products.

$$a \cdot b = a_1 b_1 i \cdot i + a_1 b_2 i \cdot j + \ldots + a_3 b_3 k \cdot k$$

Since six of these products are zero, we obtain $a.b = a_1 b_1 + a_2 b_2 + a_3 b_3$

Applications of Inner Products: Typical applications of inner products are shown in the following examples.

Example:

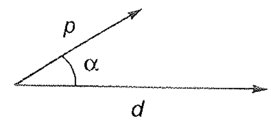
Work done by a force as inner product.

Solution:

Consider a body on which a constant force p acts. Let the body be given a displacement d. Then the work done by p in the displacement is defined as

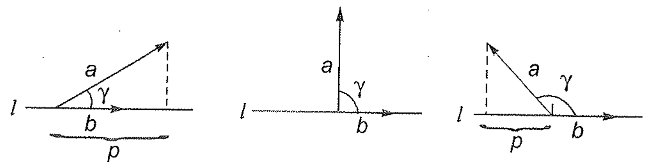
$$W = |p||d|\cos\alpha = p \cdot d$$

that is, magnitude $|p|$ of the force times length $|d|$ of the displacement times the cosine of the angle a between p and d. If $\alpha < 90°$, as in Fig. below then $W > 0$. If p and d are orthogonal, then the work done is zero. If $\alpha > 90°$, then $W < 0$, which means that in the displacement one has to do work against the force.



Work done by a force

Vector Projection: $Proj_b^a$ is the vector projection of a on another vector b.



$$p = Proj_b^a$$

$$= \text{(Scalar component of a in direction of } b) \times \text{(a unit vector in direction of } b)$$

$$Proj_b^a = (|a|\cos\gamma)\left(\frac{b}{|b|}\right)$$

$$= \left(\frac{a \cdot b}{|b|}\right)\left(\frac{b}{|b|}\right) = \left(\frac{a \cdot b}{b.b}\right).b$$

Typical application of projection is finding component of n force in a given direction as is often required in mechanics.

Example:

Vector projection of a on another vector b.

Find the vector projection of a vector $a = 2i - 3j$ or $b = 3i + 4j$.

Solution:

$$Proj_b^a = \left(\frac{a \cdot b}{b \cdot b}\right)b = \left(\frac{2.3 - 3.4}{3.3 + 4.4}\right)(3i - 4j) = \frac{-6}{25}(3i - 4j) = \frac{-18}{25}i - \frac{24}{25}j$$

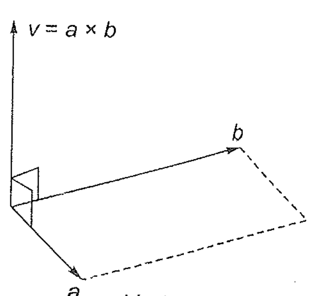
2.14.10 Vector Product (Cross Product)

The dot product is a scalar. We shall see that some applications, for instance, in connection with rotations, require a product of two vector which is again a vector. This is called vector product of two vectors or the cross product.

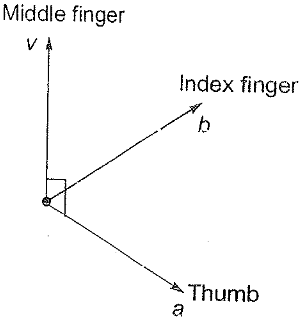
Definition. Vector product (Cross product)

The **vector product (cross product)** $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is a vector.

$v = a \times b = |a|\,|b| \sin\gamma\, \hat{n}$ such that $\dot{a} \cdot \dot{b}$ and $\hat{n}$ from a right handed system, with $\hat{n}$ being a unit normal vector perpendicular to plane of a and b.



Vector product



If a and b have the same or opposite direction or if one of these vectors is the zero vector, then $v = \mathbf{a} \times \mathbf{b} = 0$. In any other case, $v = a \times b$ has the length.

1. $|\dot{v}| = |\dot{a}|\,|\dot{b}| \sin\gamma$

This is the area of the parallelogram in Figure above with $\dot{a}$ and $\dot{b}$ as adjacent sides. (γ is the angle between a and b). The direction of $v = a \times b$ is perpendicular to both a and b and such that a, b, v, in this order, form a right-handed triple as shown in figure above.

In components, $v = [v_1, v_2, v_3] = a \times b$ is

2. $v_1 = a_2b_3 - a_3b_2, \quad v_2 = a_3b_1 - a_1b_3, \quad v_3 = a_1b_2 - a_2b_1$

i.e. If a is in direction of (right hand) thumb, b is in direction of index figure, then $v = a \times b$ will be a vector in direction of the middle figure.

In terms of determinants:

$$v_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad v_2 = \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \qquad v_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Hence $v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k$ is the expansion of the symbolical third-order determinant

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

by the first row. (We call it "symbolical" because the first row consists of vectors rather than numbers.)

2.14.10.1 Finding a Unit Vector Perpendicular to two Given Vectors a and b

A unit vector perpendicular to two given vectors a and b is given by

$$n = \frac{a \times b}{|a||b|\sin\gamma} = \frac{a \times b}{|a \times b|}$$

Example 1.

With respect to a right-handed Cartesian coordinate system, let $a = [4, 0, -1]$ and $b = [-2, 1, 3]$.

Solution:

$$a \times b = \begin{vmatrix} i & j & k \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{vmatrix} = i - 10j + 4k = [1, -10, 4]$$

Example 2.

Find a unit vector perpendicular to both $a = 3i + j + 2k$ and $b = 2i - 2j + 4k$.

Solution:

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8i - 8j - 8k$$

A unit vector perpendicular to both a and b is

$$n = \frac{a \times b}{|a \times b|} = \frac{8i - 8j - 8k}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(i - j - k)$$

There are 2 unit vectors perpendicular to both a and b. They are $\pm n = \pm\frac{1}{\sqrt{3}}(i - j - k)$

Example 3.

The vectors from origin to the points A and B are $\vec{a} = \hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$ respectively. Find the area of

(a) the triangle OAB

(b) the parallelogram formed by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as adjacent sides.

Solution:

Given $\overrightarrow{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\overrightarrow{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$,

$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (12-2)\hat{i} - (-6-4)\hat{j} + (3+12)\hat{k} = 10\hat{i} + 10\hat{j} + 15\hat{k}$$

$$\Rightarrow \quad \left|\vec{a} \times \vec{b}\right| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$$

(a) area of $\Delta OAB = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2} \cdot 5\sqrt{17}$ sq. units $= \frac{5}{2}\sqrt{17}$ sq.units.

(b) Area of parallelogram formed by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as adjacent sides

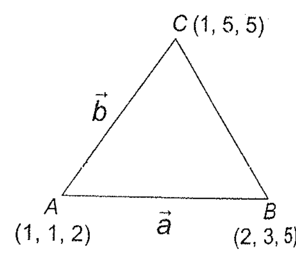
$$= \left|\vec{a} \times \vec{b}\right| = 5\sqrt{17} \text{ sq.units}$$

Example. 4.

Using vectors, find the area of the triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$

Solution:

Let the vectors $\vec{a}$ and $\vec{b}$ represents the sides AB and AC of ΔABC, then



$$\vec{a} = \overrightarrow{AB} = \text{P.V. of } B - \text{P.V. of } A$$

$$= (2\hat{i} + 3\hat{j} + 5\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k})$$

$$= \hat{i} + 2\hat{j} + 3\hat{k}$$

and

$$\vec{b} = \overrightarrow{AC} = \text{P.V. of } C - \text{P.V. of } A$$

$$= (\hat{i} + 5\hat{j} + 5\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k}) = 4\hat{j} + 3\hat{k}$$

No

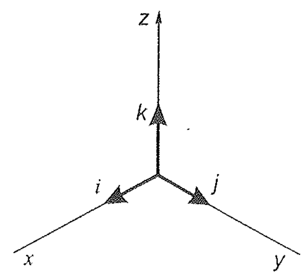
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = (6-12)\hat{i} - (3-0)\hat{j} + (4-0)\hat{k}$$

$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\Rightarrow \quad \left|\vec{a} \times \vec{b}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{61}$$

$\therefore$ The area of $\Delta ABC = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{61}$

2.14.10.2 Vector Products of the Standard Basis Vectors

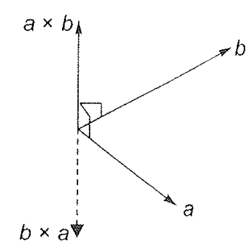


Since i, j, k are orthogonal (mutually perpendicular) unit vectors, the definition of vector product gives some useful formulas for simplifying vector products: in right-handed coordinates these are

$i \times j = k \qquad j \times k = i, \qquad k \times i = j$

$j \times i = -k \qquad k \times j = -i, \qquad i \times k = -j.$

2.14.10.3 General Properties of Vector Products



Vector Product has the property that for every scalar l,

$$(la) \times b = l(a \times b) = a \times (lb).$$

It is distributive with respect to vector addition, that is,

$$\dot{a} \times (\dot{b} + \dot{c}) = (\dot{a} \times \dot{b}) + (\dot{a} \times \dot{c}),$$

$$(\dot{a} + \dot{b}) \times \dot{c} = (\dot{a} + \dot{c}) + (\dot{b} \times \dot{c})$$

It is **not commutative** but **anti-commutative**, that is,

$$b \times a = -(a \times b)$$

It is not associative, that is,

$$a \times (b \times c) \neq (a \times b) \times c \qquad \text{(in general)}$$

so that the parentheses cannot be omitted.

2.14.11 Scalar Triple Product

The scalar triple product or mixed triple product of three vectors

$a = [a_1, a_2, a_3], \qquad b = [b_1, b_2, b_3], \qquad c = [c_1, c_2, c_3]$

is denoted by $(a\ b\ c)$ and is defined by $(a\ b\ c)$ and is defined by $(a\ b\ c) = a.(b \times c)$

We can write this as a third-order determinant. For this we set $b \times c = v = [v_1, v_2, v_3]$. Then from the dot product in components we obtain

$$a \cdot (b \times c) = a \cdot v = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \left(- \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} \right) + a_3 \begin{vmatrix} b_1 & b_2 \\ c_2 & c_2 \end{vmatrix}$$

The expression on the right is the expansion of a third-order determinant by its first row. Thus

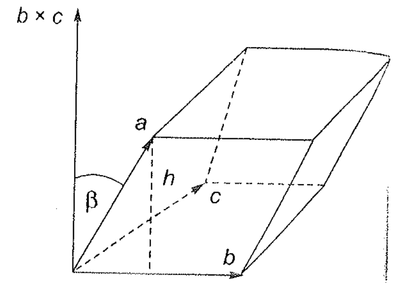
$$[a\ b\ c] = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric Interpretation of Scalar Triple Products

The absolute value of the scalar triple product is the volume of the parallelepiped with a, b, c as edge vectors (Figure, $|a \cdot (b \times c)| = |a||b \times c|\cos\beta$ where $|a||\cos\beta|$ is the height h and, by (1), the base, the parallelogram with sides b and c, has area $|b \times c|$. Naturally, if vectors a, b and c are coplanar, then this volume is zer

$a \cdot (b \times c) = 0$, if a, b and c are coplanar.

we also have for any scalar k.



Geometrical interpretation of a scalar triple product

$$[ka\ b\ c] = k[a\ b\ c]$$

because the multiplication of a row of a determinant by k multiples the value of the determinant by k. Furthermore, we prove that

$$a \cdot (b \times c) = (a \times b) \cdot c$$

Proof: LHS of above $= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\text{RHS of above} = (a \times b) \cdot c = c \cdot (a \times b) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

By properties of determinants it can be seen that the LHS and RHS determinants are indeed both equal.

So, $a \cdot (b \times c) = (a \times b) \cdot c$

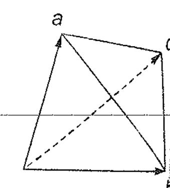
In fact $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$

i.e. the value of triple product depends upon the cycle order of the vectors, but is independent of the position of dot and cross. However if the order is non-cycle, then value changes.

i.e. $a \cdot (b \times c) \neq b \cdot (a \times c)$

Example:

A tetrahedron is determined by three edge vectors a, b, c as indicated in Fig. Find its volume if with respect to right-handed Cartesian coordinates, $a = [2, 0, 3]$, $b = [0, 6, 2]$, $c = [3, 3, 0]$.



Tetrahedron

Solution:

The volume V of the parallelopiped with these vectors as edge vectors is the absolute value of the scalar triple product.

$$[a\ b\ c] = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 6 & 2 \\ 3 & 3 & 0 \end{vmatrix} = 2\begin{vmatrix} 6 & 2 \\ 3 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 6 \\ 3 & 3 \end{vmatrix} = -12 - 54 = -66$$

That is, $V = 66$. The minus sign indicates that a, b, c, in this order, form a left-handed triple. The volume of the tetrahedron is $\frac{1}{6}$ of that of the parallelepiped, hence 11.

Testing Linear Independence of 3 Vectors using Scalar Triple Product:

Linear independence of three vectors can be tested by scalar triple products, as follows. We call a given set of vectors $a_{(1)}, \ldots, a_{(m)}$ linearly independent if the only scalar $c_1, \ldots, c_m$ for which the vector equation.

$$c_1 a_{(1)} + c_2 a_{(2)} + \ldots\ldots + c_m a_{(m)} = 0$$

is satisfied are $c_1 = 0$, $c_2 = 0$, ..., $c_m = 0$. otherwise, that is, if that equation also holds for an m-tuple of scalars not all zero, we call that set of vectors linearly dependent.

Now three vectors, if we let their initial point coincide, form a linearly independent set if and only if they do not lie in the same plane (or on the same line). i.e. These vectors are linearly independent, if and only if they are not co-planar. The interpretation of a scalar triple product as a volume thus gives the following criterion.

Theorem: 1 (Linear Independence of Three Vectors)

Three vectors form a linearly independent set if and only if their scalar triple product is not zero.

The scalar triple product is the most important "repeated product." Other repeated product exist, but are used only occasionally.

2.14.12 Vector Triple Product

If a, b and c are three vectors then the vector triple product is written as $a \times (b \times c)$

It can be proved that $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

Example:

Let $a = i + j - k, \quad b = i - j + k; \quad c = i - j - k$

Find the vector $a \times (b \times c)$

Solution:

Since,
$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$
$$a \cdot c = 1 - 1 + 1 = 1$$
$$a \cdot b = 1 - 1 - 1 = -1$$
So,
$$a \times (b \times c) = 1 \cdot b - (-1) \cdot c = b + c$$
$$= (i - j + k) + (i - j - k) = 2i - 2j$$

2.14.13 Vector and Scalar Functions and Fields. Derivatives

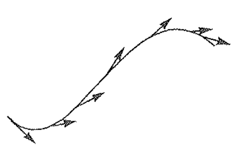
This is the beginning of vector calculus, which involves two kinds of functions, vector functions, whose values are vectors.

$$v = v(P) = [v_1(P), v_2(P), v_3(P)]$$

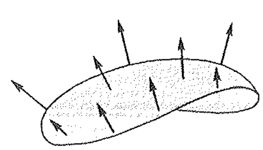
depending on the points P in space, and scalar functions, whose values are scalars

$$f = f(P)$$

depending on P. In applications, the domain of definition for such a function is a region of space or a surface in space or a curve in space. We say that a vector function defines a vector field in that region (or on that surface or curve). Examples are shown in figures. Similarly, a scalar function defines a scalar field in a region or on a surface or a curve. Examples, are the temperature field in a body (scalar function) and the pressure field of the air in the earth's atmosphere. Vector (vector function) and scalar functions may also depends on time t or on further parameters.



Field of tangent vectors of a curve



Field of normal vectors of a surface

Comment on Notation. If we introduce Cartesian coordinates x, y, z, then instead of $v(P)$ and $f(P)$ we can also write

$$v(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$$

and $f(x, y, \ )$, but we keep in mind that a vector or scalar field that has a geometrical or physical meaning should depend only on the points P where it is defined but not on the particular choice of Cartesian coordinates.

Example: 1

Scalar function (Euclidean distance in space).

Solution:

The distance $f(P)$ of any point P from a fixed point P_0 in space is a scalar function whose domain of definition is the whole space. $f(P)$ defines a scalar field in space. If we introduce a Cartesian coordinate system and P_0 has the coordinates x_0, y_0, z_0 then f is given by the well-known formula

$$f(P) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

where x, y, z are the coordinates of P. If we replace the given Cartesian coordinate system by another such system, then the values of the coordinates of P and P_0 will in general change, but $f(p)$ will have the same value as before. Hence $f(P)$ is a scalar function. The direction cosines of the line through P and P_0 are not scalars because their values will depend on the choice of the coordinate system.

Example: 2

Vector field (Velocity field).

Solution:

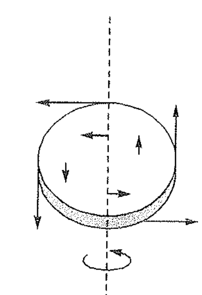
At any instant the velocity vectors $v(P)$ of a rotating body B constitute a vector field, the so-called velocity field of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotation, then

$$v(x, y, z) = w \times r = w \times [x, y, z] = w \times (xi + yj + zk)$$

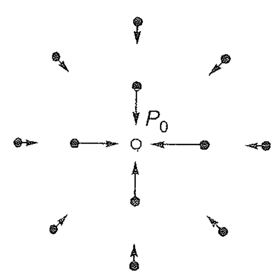
where x, y, z are the coordinates of any point P of B at the instant under consideration. If the coordinates are such that the z-axis is the ais of rotation and w points in the positive z-direction, then $w = \omega k$ and

$$v = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega(-yi + xj) = \omega[-y, x, 0]$$

An example of a rotating body and the corresponding velocity field are shown in Figure below. Also shown is another example of vector field, the gravitational field.



Velocity field of a rotating body



Gravitational field

Vector Calculus: We show next that the basic concepts of calculus, such as convergence, continuity, and differentiability, can be defined for vector functions in a simple and natural way. Most important here is the derivative.

Convergence: An infinite sequence of vectors $a_{(n)}$, $n = 1, 2, \ldots\ldots$, is said to **converge** if there is a vector a such that

$$\lim_{n\to\infty} \left|\dot{a}_{(n)} - \dot{a}\right| = 0$$

a is called the limit vector of that sequence, and we write

$$\lim_{n\to\infty} \dot{a}_{(n)} = \dot{a}$$

Cartesian coordinates being given, this sequence of vectors converges to a if and only if the three sequences of components of the vectors converge to the corresponding components of a.

Similarly, a vector function $v(t)$ of a real variable t is said to have the limit l as t approaches t_0, if $v(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t \to t_0} |v(t) - l| = 0$$

Then we write, $$\lim_{t \to t_0} v(t) = l$$

Continuity: A vector function $v(t)$ is said to be continuous at $t = t_0$ if it is defined in some neighborhood of t_0 and

$$\lim_{t \to t_0} v(t) = v(t_0)$$

If we introduce a Cartesian coordinate system, we may write

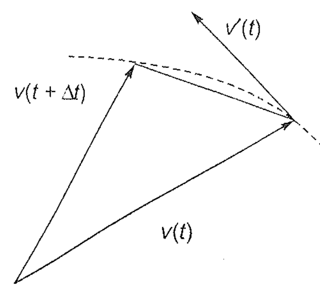
$$v(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)i + v_2(t)j + v_3(t)k.$$

Then v(t) is continuous at t_0 if and only if its three components are continuous at t_0. We now state the most important of these definitions.

2.14.13.1 Derivative of a Vector Function

A vector function $v(t)$ is said to be differentiable at a point t if the following limit exists:

$$v'(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$



Derivative of a vector function

exists. The vector $v'(t)$ is called the derivative of $v(t)$. See Figure above (the curve in this figure is the locus of the heads of the arrows representing v for values of the independent variable in some interval containing t and $t + \Delta t$).

In terms of components with respect to a given Cartesian coordinate system $v(t)$ is differentiable at a point t if and only if its three components are differentiable at t, and then the derivative $v'(t)$ is obtained by differentiating each component separately.

$$v'(t) = [v_1'(t),\ v_2'(t),\ v_3'(t)]$$

It follows that the familiar rules of differentiation yield corresponding rules for differentiating vector functions, for example,

$$(cv)' = cv' \qquad \text{(c constant)}$$
$$(u + v)' = u' + v' \text{ and in particular.}$$
$$(u \cdot v)' = u' \cdot v + u \cdot v'$$
$$(u \times v)' = u' \times v + u \times v'$$
$$[u\ v\ w]' = [u'\ v\ w] + [u\ v'\ w] + [u\ v\ w']$$

The order of the vectors must be carefully observed because cross multiplication is not commutative.

Example:

Derivative of a vector function of constant length.

Solution:

Let $v(t)$ b a vector function whose length is constant, say, $|v(t)| = c$. Then $|v|^2 = v \cdot v = c^2$, and $(v \cdot v)' = v'v + vv' = 2v \cdot v' = 0$, by differentiation. The yields the following result. The derivative of a vector function $v(t)$ of constant length is either the zero vector or is perpendicular to $v(t)$

2.14.13.2 Partial Derivatives of a Vector Function

From our present discussion we see that partial differentiation of vector functions depending on two or more variables can be introduced as follows. Suppose that the components of a vector function

$$v = [v_1, v_2, v_3] = v_1 i + v_2 j + v_3 k$$

are differentiable functions of n variables $t_1, \ldots, t_n$. Then the partial derivative of v with respect to t_l is denoted by $\partial v/\partial t_l$ and is defined as the vector function

$$\frac{\partial v}{\partial t_l} = \frac{\partial v_1}{\partial t_l} i + \frac{\partial v_2}{\partial t_l} j + \frac{\partial v_3}{\partial t_l} k$$

Similarly, $$\frac{\partial^2 v}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m} i + \frac{\partial^2 v_2}{\partial t_l \partial t_m} j + \frac{\partial^2 v_3}{\partial t_l \partial t_m} k \text{ and so on.}$$

Example:

Let $$r(t_1, t_2) = a \cos t_1 i + a \sin t_1 j + t_2 k.$$

Solution:

Then $$\frac{\partial r}{\partial t_1} = -a \sin t_1 i + a \cos t_1 j$$

$$\frac{\partial r}{\partial t_2} = k$$

Various physical and geometrical applications of derivatives of vector functions will be discussed in the next sections.

2.14.14 Gradient of a Scalar Field

We shall see that some of the vector fields in applications-(not all of them) can be obtained from scalar fields. This is a considerable advantage because scalar fields can be handled more easily. The relation between the two types of fields is accomplished by the "gradient". Hence the gradient is of great practical importance.

Definition of Gradient: The gradient grad f of a given scalar function $f(x, y, z)$ is the vector function defined by

1. $\text{grad } f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

   Here we must assume that f is differentiable. It has become popular, particularly with physicists and engineers, to introduce the differential operator.

2. $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

   (read nabla or del) and to write

   $$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

   For instance, if $f(x, y, z) = 2x + yz - 3y^2$, then grad $f = \nabla f = 2\boldsymbol{i} + (z - 6y)\boldsymbol{j} + y\boldsymbol{k}$.

   We show later that grad f is a vector; that is, although it is defined in terms of components, it has a length and direction that is independent of the particular choice of Cartesian coordinates. But first we explore how the gradient is related to the rate of change of f in various directions. In the directions of the three coordinate axes, this rate is given by the partial derivatives, as we know from calculus. The idea of extending this to arbitrary directions seems natural and leads to the concept of directional derivative.

2.14.15 Directional Derivative

The rate of change of f at any point P in any fixed direction given by a vector b is defined as in calculus. We denote it by $\nabla_b f$ or df/ds, call it the directional derivative of f at P in the direction of b, and define it by figure.

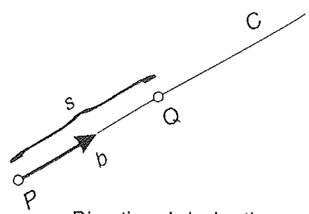
3. $\nabla_b f = \dfrac{df}{ds} = \lim_{s \to 0} \dfrac{f(Q) - f(P)}{s}$ (s = distance between P and Q)

where Q is a variable point on the ray C in the direction of b as in Fig. below.

The next idea is to use Cartesian xyz-coordinates and for b a unit vector. Then the ray C is given by

4. $r(s) = x(s)i + y(s)j + z(s)k = p_0 + sb \qquad (s \geqq 0, |b| = 1)$

(p_0 the position vector of P). Equation (3) now shows that $D_b f = df/ds$ is the derivative of the function $f(x(s), y(s), z(s))$ with respect to the arc length s of C. Hence, assuming that f has continuous partial derivatives and applying the chain rule. We obtain



Directional derivative

5. $D_b f = \dfrac{df}{ds} = \dfrac{\partial f}{\partial x}x' + \dfrac{\partial f}{\partial y}y' + \dfrac{\partial f}{\partial z}z'$

where primes denote derivatives with respect to s (which are taken at $s = 0$). But here, $r' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} = b$ by (4). Hence (5) is simply the inner product of b and grad f [see (2), Sec. 8.2],

6. $D_b f = \dfrac{df}{ds} = b \cdot \text{grad} f \qquad (|b| = 1)$

Attention! In general, if the direction is given by a vector a of any length, then

$D_b f = \dfrac{df}{ds} = \dfrac{1}{|a|} a \cdot \text{grad} f$ (where $\dfrac{a}{|a|}$ is a unit vector is direction of a)

Example:

Gradient. Directional Derivative

Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point P: (2, 1, 3) in the direction of the vector $a = i - 2k$.

Solution:

We obtain grad $f = 4xi + 6yj + 2zk$, and at P: (2, 1, 3), grad $f = 8i + 6j + 6k$

$$D_a f = \frac{a}{|a|} \cdot \text{grad} f$$

$$= \frac{1}{\sqrt{5}}(i - 2k).(8i + 6j + 6k)$$

$$= \frac{1.8 - 2.6}{\sqrt{5}} = -\frac{4}{\sqrt{5}} \approx -1.789$$

The minus sign indicates that f decreases at P in the direction of a.

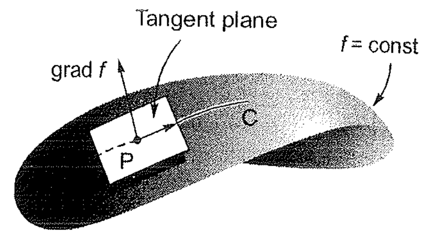
2.14.16 Gradient Characterizes Maximum Increase

Theorem. 1 (Gradient, Maximum Increase)

Let $f(P) = f(x, y, z)$ be a scalar function having continuous first partial derivatives. Then grad f exists and its length and direction are independent of the particular choice of Cartesian coordinates in space. If at a point P the gradient of f is not the zero vector, it has the direction of maximum increase of f at P. Proof. From (6) and the definition of inner product we have

7. $D_b f = |b|\,|\text{grad } f| \cos\gamma = |\text{grad } f| \cos\gamma$

where γ is the angle between b and grad f. Now f is a scalar function. Hence its value at a point P depends on P but not on the particular choice of coordinates. The same holds for the arc length s of the ray C (see hence also for $D_b f$. Now (7) shows that $D_b f$ is maximum when $\cos\gamma = 1$, i.e. $\gamma = 0$, and the $D_b f = |\text{grad } f|$. It follows that the length and direction of grad f are independent of the coordinates.



Gradient as surface normal vector

Since $\gamma = 0$ if and only if b has the direction of grad f, the latter is the direction of maximum increase of f at P, provided grad $f \neq 0$ at P.

Gradient as Surface Normal Vector: Another basic use of the gradient results in connection with surfaces S in space given by

8. $$f(x, y, z) = c = \text{const.}$$

as follows. We recall that a curve C in space can be given by

9. $$r(t) = x(t)i + y(t)j + z(t)k$$

Now if we want C to lie on S, its components must satisfy (8); thus

10. $$f(x(t), y(t), z(t)) = c$$

A tangent vector of C is

$$r'(t) = x'(t)i + y'(t)j + z'(t)k.$$

If C lies on S, this vector is tangent to S. At a fixed point P on S, these tangent vectors of all curves on S through P will generally form a plane, called the tangent plane of S at P (Figure above). Its normal (the straight line through P and perpendicular to the tangent plane) s called the surface normal of S at P. A vector parallel to it is called a surface normal vector of S at P. Now if we differentiate (10) with respect to t, we get by the chain rule.

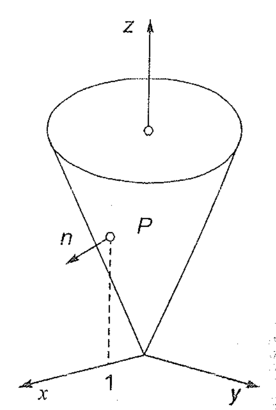
11. $$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = (\text{grad } f)r' = 0$$

This means orthogonality of grad f and all the vectors r' in the tangent plane. This result in shown pictorially in the figure above, where grad f is shown as normal to tangent plane of vectors r'. So, we have the theorem 2 given below.

Theorem. 2 (Gradient as Surface Normal Vector)

Let f be a differentiable scalar function that represents a surface S: $f(x, y, z) = c = \text{const}$. Then if the gradient of f at a point P of S is not the zero vector, it is a normal vector of S at P.

Comment. The surfaces given by (8) with various values of c are called the level surfaces of the scalar function f.



Cone and unit normal vector n

Example:

Gradient as Surface Normal Vector

Find a unit normal vector n of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point P: (1, 0. 2).

Solution:

The cone is the level surface $f = 0$ $f(x, y, z) = 4(x^2 + y^2) - z^2$. thus grad $f = 8xi + 8yj - 2zk$ and at $P(1, 0, 2)$, grad $f = 8i - 4k$

Hence, by Theorem 2, grad f is a normal vector of the cone at point P.

Now a unit normal vector at point P will be,

$$n = \frac{1}{|\text{grad } f|}\text{grad } f = \frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}k$$

and the other one is $-n$.

2.14.17 Vector Fields that are Gradients of a Scalar Field ("Potential")

Some vector fields have the advantage that they can be obtained from scalar fields, which can be handled more easily. Such a vector field is given by a vector function $v(P)$, which is obtained as the gradient of a scalar function, say, $v(P) = \text{grad } f(P)$. The function $f(P)$ is called a potential function or a potential of $v(P)$. Such a v(P) and the corresponding vector field are called conservative because in such a vector fields, energy is conserved; that is, no energy is lost (or gained) in displacing a body (or a charge in the case of an electrical field) from a point P to another point in the field and back to P.

2.14.18 Divergence of a Vector Field

Vector calculus owes much of its importance in engineering and physics to the gradient, divergence, and curl. Having discussed the gradient, we turn next to the divergence. The curl follows in next section.

Let $v(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of v. Then the function

1. $$\text{div } v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of v or the divergence of the vector field defined by v. Another common notation for the divergence of v is $\nabla . v$,

$$\text{div } v = \nabla . v$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).(v_1 i + v_2 j + v_3 k) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

with the understanding that the "product" $(\partial/\partial x)v_1$ in the dot product means the partial derivative $\partial v_1/\partial x$, etc. This is a convenient notation, but nothing more. Note that $\nabla . v$ means the scalar div v, whereas, ∇f means the vector grad f.

Example:

If $$v = 3xzi + 2xyj - yz^2k,$$

then $$\text{div } v = 3z + 2x - 2yz$$

We shall see below that the divergence has an important physical meaning. Clearly the values of a function that characterize a physical or geometrical property must be independent of the particular choice of coordinates; that is, those values must be invariant with respect to coordinate transformations.

Theorem. 1 (Invariance of The Divergence)

1. The values of div v depend only on the points in space (and, of course, on v) but not on the particular choice of the coordinates.

Now, let us turn to the more immediate practical task of getting a feel for the significance of the divergence.

If $f(x, y, z)$ is a twice differentiable scalar function, then

$$\text{grad } f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

and $\quad \text{div}(\text{grad } f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

2. The expression on the right is the Laplacian of f. Thus

$$\text{div}(\text{grad } f) = \nabla^2 f.$$

Example 1.

Gravitational force

The gravitational force p, is the gradient of the scalar function $f(x, y, z) = c/r$, which satisfies Laplace's equation $\nabla^2 f = 0$. According to (3), this means that div $p = 0$ ($r > 0$)

The following example, taken from hydrodynamics, shows the physical significance of the divergence of a vector field (and more will be added in next section when the so-called divergence theorem of Gauss will be available).

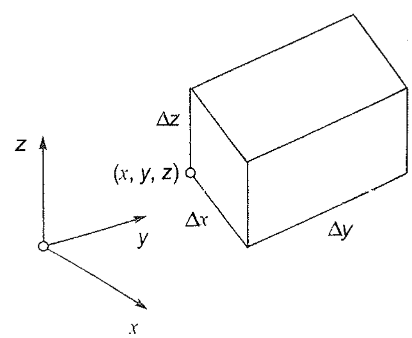
Example 2.

1. **Motion of a compressible fluid, Physical meaning of the divergence**

We consider the motion of a fluid in a region R having no sources or sinks in R, that is, no points at which fluid is produced or disappears. The concept of fluid state is meant to cover also gases and vapors. Fluids in the restricted sense, or liquids (water or oil, for instance), have very small compressibility, which can be neglected in many problems. Gasses and vapors have large compressibility; that is, their density r (= mass per unit volume) depends on the coordinates x, y, z in space (and may depend on time t). We assume that our fluid is compressible.

We consider the flow through a small rectangular box W of dimensions Δx, Δy, Δz with edges, parallel to the coordinate axes (Fig. below). W has the volume $\Delta V = \Delta x\,\Delta y\,\Delta z$. Let $v = [v_1, v_2\; v_3] = v_1 i + v_2 j_+ v_3 k$ be the velocity vector of the motion. We set

2. $u = \rho v = [u_1, u_2, u_3] = u_1 i + u_2 j + u_3 k$ and assume that u and v are continuously differentiable vector functions of x, y, z, and t (that is, they have first partial derivatives, which are continuous). Let us calculate the change in the mass included in W by considering the flux across the boundary, that is, the total loss of mass leaving W per unit time. Consider the flow through the left face of W, whose area is $\Delta x\,\Delta z$. The components, v_1 and v_3 of v are parallel to that face and contribute nothing to this flow. Hence the mass of fluid entering through that face during a short time interval Δt is given approximately by



Physical interpretation of the divergence

$$(\rho v_2)_y \Delta x\,\Delta z\,\Delta t = (u_2)_y \Delta x\,\Delta z\,\Delta t,$$

where the subscript y indicates that this expression refers to the left face. The mass of fluid leaving the box W through the opposite face during the same time interval is approximately $(u_2)_{y+\delta y}\,\Delta x\,\Delta z\,\Delta t$, where the subscript $y + \Delta y$ indicates that this expression refers to the right face (which is not visible in Fig. above figure). The difference

$$\Delta u_2\,\Delta x\,\Delta z\,\Delta t = \frac{\Delta u_2}{\Delta_y}\Delta V\,\Delta t \quad \left[\Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y\right]$$

is the approximate loss of mass. Two similar expressions are obtained by considering the other two pairs of parallel faces of W. If we add these three expression, we find that the total loss of mass in W during the time interval Δt is approximately.

$$\left(\frac{\Delta u_1}{\Delta x}+\frac{\Delta u_2}{\Delta y}+\frac{\Delta u_3}{\Delta z}\right)\Delta V\,\Delta t$$

where, $\Delta u_1 = (u_1)_{x+\Delta x} - (u_1)_x$

and $\Delta u_3 = (u_3)_{z+\Delta z} - (u_3)_z$

This loss of mass in W is cause by the time rate of change of the density and is thus equal to

$$-\frac{\Delta\rho}{\Delta t}\Delta V\,\Delta t$$

If we equate both expressions, divide the resulting equation by $\Delta V \Delta t$, we get

$$\frac{\Delta u_1}{\Delta x}+\frac{\Delta u_2}{\Delta y}+\frac{\Delta u_3}{\Delta z} = -\frac{\Delta P}{\Delta t}$$

Now we let Δx, Δy, Δz and Δt approach zero and get,

$$\text{div } u = div(\rho v) = -\frac{\partial\rho}{\partial t}$$

3. i.e. $$\frac{\partial\rho}{\partial t}+div\,(\rho v) = 0$$

This important relation is called the condition for the conservation of mass or the continuity equation of a compressible fluid flow.

If the flow is steady, that is, independent of time, then $\frac{\partial\rho}{\partial t} = 0$ and the continuity equation is

4. $$\text{div }(\rho v) = 0$$

If the density r is constant, so that the fluid is incompressible, then equation (6) becomes

5. $$\text{div } v = 0$$

This relation is known as the condition of incompressibility. It expresses the fact that the balance of outflow and inflow for a given volume element is zero at any time. Clearly, the assumption that the flow has no source or sinks in R is essential to our argument.

From this discussion you should conclude and remember that, roughly speaking, the divergence measures outflow minus inflow.

If v denotes the velocity of fluid in a medium and if div(v) = 0, then the fluid is said to be **incompressible**. In electromagnetic theory, if div(v) = 0, then the vector field v is said to be **solenoidal**.

2.14.19 Curl of a Vector Field

Gradient, divergence, and curl are basic in connection with fields. We now define and discuss the curl.

Let x, y, z be right-handed Cartesian coordinates, and let

$$v(x, y, z) = v_1 i + v_2 j + v_3 k$$

be a differentiable vector function. Then the function

$$\text{curl } v = \nabla\times v \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\text{curl } v = \left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)i+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)j+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)k$$

is called the curl of the vector function v or the curl of the vector field defined by v.

Instead of curl v, the notation rot v is also used, (since one application of curl is to signify rotation of a rigid body)

Example 1.

With respect to right-handed Cartesian coordinates, let

$$v = yzi + 3zxj + zk.$$

Then (1) gives

$$\text{curl } v = \nabla \times v$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$$

$$= -3xi + yj + (3z - z)k = -3xi + yj + 2zk.$$

The curl plays an important role in many applications. Let us illustrate this with a typical basic example. (We shall say more about the role and nature of the curl in next section).

Example 2.

Rotation of a rigid body. Relation to the curl

1. Rotation of a rigid body B about a fixed axis in space can be described by a vector w of magnitude ω in the direction of the axis of rotation, where $\omega(> 0)$ is the angular speed of the rotation, and ω is directed so that the rotation appears clockwise if we look in the direction of w. The velocity field of the rotation can be represented in the form

$$v = w \times r$$

where r is the position vector of a moving point with respect to a Cartesian coordinate system having the origin on the axis of rotation. Let us choose right-handed Cartesian coordinates such that

$$w = \omega k \text{ and } r = xi + yj + zk$$

that is, the axis of rotation is the z-axis. Then

$$v = w \times r = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega yi + \omega xj$$

and therefore, $\text{curl v} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega k$,

since $w = \omega k$,

2. $\text{curl } v = 2w.$

Hence, in the case of a rotation of a rigid body, the curl of the velocity field has the direction of the axis of rotation, and its magnitude equals twice the angular speed ω of the rotation.

Note that out result does not depend on the particular choice of the Cartesian coordinate system in space.

For any twice continuously differentiable scalar function f,

3. $\text{curl (grad } f) = 0,$

as can easily be verified by direct calculation, as shown below:

$$\text{grad } f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\text{curl(grad } f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2}{\partial y \partial z} f\right) - j\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)$$

$$= 0i - 0j + 0k = 0$$

Hence if a vector function is the gradient of a scalar function, its curl is the zero vector. Since the curl characterizes the rotation in a field, we also say more briefly that gradient fields describing a motion are irrotational. (If such a field occurs in some other connection, not as a velocity field, it is usually called conservative;

If curl $v = 0$, then v is solid to be an irrotational field.

Example:

The gravitational field has curl $p = 0$. The field in the rotation of rigid body example this section is not irrotational since we saw that curl $v = 2w \neq 0$. A similar velocity field is obtained by stirring coffee in a cup.

Other than (3), another key formula for any twice continuously differentiable scalar function is

4. $$\text{div (curl } v) = 0$$

It is plausible because of the interpretation of the curl as a rotation and the divergence as a flux. A proof of (4) follows readily from the definitions of curl and div; the six terms cancel in pairs.

Let $v = v_1 i + v_2 j + v_3 k$

$$\text{curl } v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - j\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + k\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\text{div(curl } v) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$$= 0$$

The curl is defined in terms of coordinates, but if it is supposed to have a physical or geometrical significance, it should not depend on the choice of these coordinates. This is true, as follows.

Theorem. 1 (Invariance of The Curl)

The length and direction of curl v are independent of the particular choice of Cartesian coordinate systems in space.

2.14.19.1 Important Repeated Operations by Nable Operator (∇)

1. div grad $f = \nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

2. curl grad $f = \nabla \times \nabla f = 0$
3. div curl $f = \nabla \cdot (\nabla \times F) = 0$
4. curl curl f = grad div $F - \nabla^2 F = \nabla(\nabla \cdot F) - \nabla^2 F$
5. grad div f = curl curl $F + \nabla^2 F = \nabla \times \nabla \times F + \nabla^2 \times F$

2.14.20 Vector Integral Calculus: Integral Theorems

2.14.20.1 Line Integral

The concept of a line integral is a simple and natural generalization of a definite integral

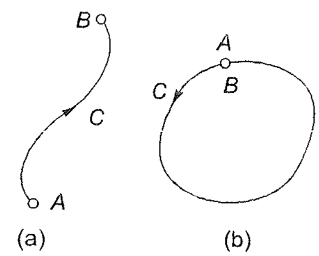
1. $\int_a^b f(x)\,dx$ known from calculus. In (1) we integrate the integrand $f(x)$ from $x = a$ along the x-axis to $x = b$.

In a line integral we shall integrate a given function, called the integrand, along a curve C in space (or in the plane). Hence curve integral would be a better term, but line integral is standard.

We represent the curve C by a parametric representation.

2. $r(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k \quad (a \leqq t \leqq b)$

We call C the path of integrating. A: $r(a)$ its initial point, and B: $r(b)$ its terminal point. C is now oriented. The direction from A to B, in which t increases, is called the positive direction on C. We can indicate the direction by an arrow (as in above Figure (a)). The points A and B may coincide (as in above figure (b)). Then C is called a closed path.



Oriented curve

We call C a smooth curve if C has a unique tangent at each of its points whose direction varies continuously as we move along C.

Technically : C has a representation (2) such that $r(t)$ is differentiable and the derivative $r'(t) = dx/dt$ is continuous and different from the zero vector at every point of C.

2.14.20.2 Definition and Evaluation of Line Integrals

A line integral of a vector function $F(r)$ over a curve C is defined by

3. $$\int_C F(r) \cdot dr = \int_a^b F(r(t)) \cdot \frac{dr}{dt} dt$$

In terms of components, with $dr = [dx, dy, dz]$ and $= d/dt$, formula (3) becomes

3′. $$\int_C F(r) \cdot dr = \int_C (F_1 i + F_2 j + F_3 k).(dx\,i + dy\,j + dz\,k)$$

$$= \int_C F_1 dx + F_2 dy + F_3 dz = \int_a^b (F_1 x' + F_2 y' + F_3 z')$$

If the path of integrating C in (3) is a closed curve, then instead of

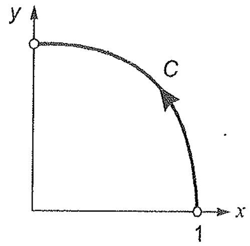
$$\int_C \text{ we also write } \oint_C .$$

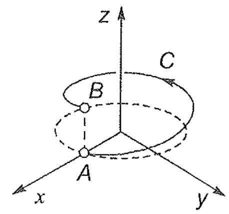
We see that the integral in (3) on the right is a definite integral of a function of t taken over the interval $a \leqq t \leqq b$ on the t-axis in the positive direction (the direction of increasing t). This definite integral exists for continuous F and piecewise smooth C, because this makes $F.r'$ piecewise continuous.

Example 1.

Find the value of the line integral (3) when $F(r) = [-y, -xy] = -yx - xyj$ and C is the circular arc as from A to B shown in figure titled Example below:

Solution:





We may represent C by

$$r(t) = [\cos t, \sin t] = \cos t\,i + \sin t\,j \quad (0 \le t \le \pi/2)$$

Thus $x(t) = \cos t,\ y(t) = \sin t$, so that

$$F(r(t)) = -y(t)i - x(t)y(t)j = [-\sin t, -\cos t \sin t] = -\sin t\,i - \cos t \sin t\,j$$

By differentiation, $r'(t) = -\sin t\,i + \cos t\,j,$

So by (3)

$$\int_C F(r).dr = \int_a^b F(r(t)) \cdot \frac{dr}{dt}dt = \int_0^{\pi/2}(-\sin t\,i - \cos t \sin t\,j).(-\sin t\,i + \cos t\,j)dt$$

$$= \int_0^{\pi/2}(\sin^2 t - \cos^2 t \sin t)dt$$

$$= \int_0^{\pi/2}\sin^2 t\,dt - \int_0^{\pi/2}\cos^2 t \sin t\,dt$$

$$= \int_0^{\pi/2}\left(\frac{1-\cos^2 t}{2}\right)dt + \int^0 u^2\,dt \quad (\text{where } u = \cos t)$$

$$= \left(\frac{\pi}{4} - 0\right) - \left(\frac{1}{3}\right) \simeq 0.4521$$

Example 2.

Line integral in space.

Evaluation of line integrals in space is practically the same as it is in the plane. To see this, find the value of (3) when $\int F(r)dr = [z, x, y] = zi + xj + yk$ and C is the helix (Figure above titled Example 2) $r(t) = [\cos t, \sin t, 3t]$ where $0 \le t \le 2\pi$.

Solution:

We have $x(t) = \cos t,\ y(t) = \sin t,\ z(t) = 3t$.

Thus $F(r) = zi + xj + yk = 3ti + \cos t\,j + \sin t\,k$

$$\int F(r(t)dr = \int F(r(t)).r'(t)dt$$

$$= \int_0^{2\pi}(3ti + \cos tj + \sin tk).(-\sin ti + \cos tj + 3k)dt$$

$$= \int_0^{2\pi}(-3t\sin t + \cos^2 t + 3\sin t)dt$$

$$= 6\pi + \pi + 0 = 7\pi$$

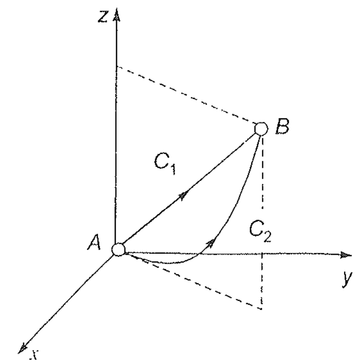
$$\approx 21.99.$$

1. **Choice of representation:** Does the value of a line integral with given F and C depend on the particular choice of a representation of C? The answer is no; see theorem 1 below.
2. **Choice of path:** Does this value change if we integrate from the old point A to the old point B but along another path. The answer is yes, in general; see example 3.

Example 3.

Dependence of a line integral on path (same endpoints)

Evaluate the line integral (3) with $F(r) = [5z, xy, x^2z] = 5zi + xyj + x^2zk$ along two different paths with the same initial point A: (0, 0, 0) and the same terminal point B: (1, 1, 1), namely.(Fig. below titled example 3)



(a) C_1 : the straight-line segment $r_1(t) = [t, t, t] = ti + tj + tk,\ 0 \leqq t \leqq 1$, and

(b) C_2 : the parabolic are $r_2(t) = [t, t, t^2] = ti + tj + t^2k,\ 0 \leqq t \leqq 1$.

Solution:

(a) By substituting r_1 into F we obtain $F(r_1(t)) = [5t, t^2, t^3] = 5ti + t^2j + t^3k$. We also need $r'_1 = [1,1,1] = i + j + k$.

Hence the integral over C_1 is

$$\int_{C_1} F(r).dr = \int_0^1 F(r_1(t)) \cdot r_1'dt = \int_0^1 (5ti + t^2j + t^2k).(i + j + k)dt$$

$$= \int_0^1 (5t + t^2 + t^3)dt = \frac{5}{2} + \frac{1}{3} + \frac{1}{4} = \frac{31}{12}$$

(b) Similarly, by substituting r_2 into F and calculating r'_2 we obtain for the integral over the path C_2.

$$\int_{C_2} F(r) \cdot dr = \int_0^1 F(r_2(t)).r_2'(t)dt = \int_0^1 (5t^2 + t^2 + 2t^5)dt = \frac{5}{3} + \frac{1}{3} + \frac{2}{6} = \frac{28}{12}.$$

The two results are different, although the endpoints are the same. This shows that the value of a line integral (3) will in general depend not only on F and on the endpoints A, B of the path but also on the path along which we integrate from A to B.

Can we find conditions that guarantee independence? This is a basic question in connection with physical applications. The answer is yes, as we show in next section.

2.14.20.3 General Properties of the Line Integral (3)

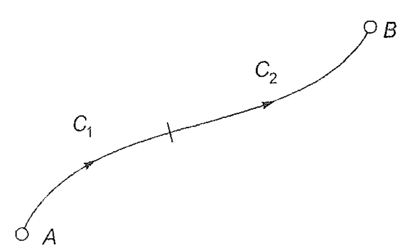
From familiar properties of integrals in calculate we obtain corresponding formulas for line integrals.

$$\int_C kF \cdot dr = k\int_C F \cdot dr \qquad (k \text{ constant})$$

$$\int_C (F + G) \cdot dr = \int_C F \cdot dr + \int_C G \cdot dr$$

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$$

where in third formula above the path C is subdivided into two arcs C_1 and C_2, that have the same orientation as C (Fig. below). In (second formula above) the orientation of C is the same in both integrals. If the sense of integration along C is reversed, the value of the integral is multiplied by −1.



2.14.20.4 Line Integrals Independent of Path

1. $$\int_C F(r) \cdot dr = \int_C (F_1dx + F_2dy + F_3dz)$$

as before. In (1) we integrate from a point A to a point B over a path C. The value of such an integral generally depends not only on A and B, but also on the path C along which we integrate. This was shown in example 3 of the last section. It raises the question of conditions for independence of path, so that we get the same value in integrating from A to B along any path C. This is of great practical importance. For instance, in mechanics, independence of path may mean that we have to do the same amount of work regardless of the path to the mountain top, be it short and steep or long and gentle, or that we gain back the work done in extending an elastic spring when we release it. Not all forces are of this type - think of swimming in a big whirlpool.

We define a line integral (1) to be independent of path in a domain D in space if for every pair of endpoints A, B in D the integral (1) has the same value for all path in D that begin at A and end at B. A very practical criterion for path independence is the following.

Theorem. 1 (Independence of Path)

A line integral (1) with continuous F_1, F_2, F_3 in a domain D in space is independent of path in D if and only if $F = [F_1, F_2, F_3]$ is the gradient of some function f in D.

2. $$F = \text{grad } f;$$

in components,

2′. $$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

Example 1.

Independence of path. Show that the integral

$$\int_C F \cdot dr = \int_C (2xdx + 2ydy + 4zdz)$$

is independent of path in any domain in space and find its value if C has the initial point A: (0, 0, 0) and terminal point B: (2, 2, 2).

Solution:

By inspection we find that

$$F = [2x, 2y, 4z] = 2xi + 2yj + 4zk = \text{grad } f,$$

where $$f = x^2 + y^2 + 2z^2.$$

(If F is more complicated, proceed by integration, as in Example 2, below.) Theorem 1 now implies independence of path. To find the value of the integral, we can choose the convenient straight path

$$C: \quad r(t) = [t, t, t] = t(i + j + k), \quad 0 \le t \le 2.$$

and get r', $= i + j + k$; thus $F \cdot r' = 2t + 2t + 4t = 8t$ and from this

$$\int_C (2xdx + 2ydy + 4zdz) = \int_0^2 F \cdot r'dt = dt \int_0^2 8t\, dt = 16$$

Proof of Theorem 1:

1. Let (2) hold for some function f in D. Let C be any path in D from any point A to any point B, given by

$$r(t) = x(t)i + y(t)j + z(t)k, \quad 0 \leqq t \leqq b$$

by chain rule, we get,

$$\int_A^B (F_1dx + F_2dy + F_3dz) = \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt$$

$$= \int_a^b \frac{df}{dt}dt = f[x(t), y(t), z(t)]\Big|_{t=0}^{t=b} = f(B) - f(A)$$

This shows that the value of the integral is simply the difference of the values of f at the two end points of C and is, therefore, independent of the path C.

2. The converse proof of this theorem, that independence of path implies that F is gradient of some function f, is more complicated and not given here.

The above example 1 can, now be solved more easily as

$$\int_C F\,dr = f(B) - f(A) = f(2, 2, 2) - f(0, 0,0\,0)$$
$$= (2^2 + 2^2 + 2.2^2) - (0^2 + 0^2 + 2.0^2) = 16$$

An easy way of solving this problem follows from proof of theory 1, shown below:
The last formula in part (a) of the proof,

$$\int_A^B (F_1dx + F_2dy + F_3dz) = f(B) - f(A) \quad [F = \text{grad } f]$$

is the analog of the usual formula for definite integrals in calculus.

$$\int_a^b g(x)dx = G(x)\Big|_a^b = G(b) - G(a) \; [G'(x) = g(x)].$$

3. **Potential theory** relates to our present discussion, if we remember, that f is called a potential of F = grad f. Thus the integral (1) is independent of path in D if and only if F is the gradient of a potential in D.

Example 2.

Independence of path. Determination of a potential
Evaluate the integral

$$I = \int_C (3x^2dx + 2yzdy + y^2dz)$$

from A: (0, 1, 2) to B: (1, – 1, 7) by showing that F has a potential and applying line integral formula.

Solution:

If F has a potential f, we should have

$f_x = F_1 = 3x^2$, $\quad f_y = F_2 = 2yz$, $\quad f_z = F_z = y^2$

We show that we can satisfy these conditions. By integration and differentiation.

$f = x^3 + g(y, z)$, $\Rightarrow$ $f_y = g_y = 2yz$, $\Rightarrow$ $g = y^2z + h(z)$

$f = x^3 + g(y, z)$ $\Rightarrow$ $f_z = g_z = y^2 + h'$,

Now from first step we know that, $f_z = y^2$, $\Rightarrow$ $g = y^2z + 0 = y^2z$

$\therefore\; y^2 + h^1 = y^2$ $\Rightarrow$ $h' = 0$, $\Rightarrow$ h = constant = 0 (say)

This gives $f(x, y, z) = x^3 + y^2z$ and the required integral $I = f(B) - f(A)$

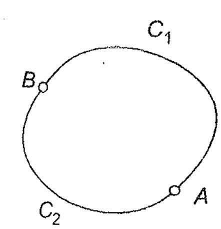
$$I = f(1, -1, 7) - f(0, 1, 2) = (1 + 7) - (0 + 2) = 6$$

Theorem. 2 (Independence of path)

The integral (1) is independent of path in a domain D if and only if its value around every closed path in D is zero.

Proof: If we have independence of path, integration from A to B along C_1 and along C_2 in Fig. 205 gives the same value. Now C_1 and C_2 together make up a closed curve C, and if we integrate from A along C_1 to B as before, but then in the opposite sense along C_2 back to A (so that this integral is multiplied by –1), the sum of the two integrals is zero, but this is the integral around the closed curve C.
Conversely, assume that the integral around any closed path C in D is zero.

Given any points A and B and any two curves C_1 and C_2 from A to B in D, we see that C_1 with the orientation reversed and C_1 together form a closed path C. By assumption, the integral over C is zero.
Hence the integrals over C_1 and C_2. Both taken from A to B, must be equal. This proves the theorem.



Proof of Theorem 2

Work. Conservative and Nonconservative (Dissipative) Physical Systems: Recall from the last section that in mechanics, the integral $\int_C F(r) \cdot dr$ represents the work done by a force F in the displacement of a body along C. Then theorem 2 states that work is independent of path if and only if it is zero for displacement around any closed path. Furthermore, Theorem 1 tells us that this happens if and only if F is the gradient of a potential. In this case, F and the vector field defined by F are called conservative, because in this case mechanical energy is conserved, that is, no work is done in the displacement from a point A and back to A. Similarly for the displacement of an electrical charge (an electron, for instance) in an electrostatic field.

Physically, the kinetic energy of a body can be interpreted as the ability of the body to do work by virtue of its motion, and if the body moves in a conservative field of force, after the completion of a round-trip the body will return to its initial position with the same kinetic energy it had originally. For instance, the gravitational force is conservative; if we throw a ball vertically up, it will (if we assume air resistance to be negligible) return to our hand with the same kinetic energy it had when it left our hand.

Friction, air resistance, and water resistance always act against the direction of motion, tending to diminish the total mechanical energy of a system (usually converting it into heat or mechanical energy of the surrounding medium, or both), and if in the motion of a body, these forces are so large that they can no longer be neglected, then the resultant F of the forces acting on the body is no longer conservative. Quite generally, a physical system is called conservative, if all the forces acting in it are conservative; otherwise it is called nonconservative or dissipative.

Exactness and Independence of Path: Theorem 1 relates path independence of the line integral (1) to the gradient and theorem 2 to integration around closed curves. A third idea and theorem 3, below) relate path independence to the exactness of the differential form

4. $$F_1dx + F_2dy + F_3dz$$

under the integral sign in (1). This form (4) is called exact in a domain D in space if it is the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

of a differentiable function $f(x, y, z)$ everywhere in D. That is, if we have

$$F_1dx + F_2dy + F_3dz = df$$

Comparing these two formulas, we see that the form (4) is exact if and only if there is a differentiable function $f(x, y, z)$ in D such that everywhere in D,

5. $$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

In vectorial form these three equation (5′) can be written

5′. $$F = \text{grad } f.$$

Hence, by Theorem 1, the integral (1) is independent of path in D if and only if the differential form (4) has continuous components F_1, F_2, F_3 and is exact in D.

This is practically important because there is a useful exactness criterion involving the following concept.

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

For example, the interior of a sphere or a cube, the interior of a sphere with finitely many points removed, and the domain between two concentric spheres are simply connected, while the interior of a torus (a doughnut) and the interior of a cube with one space diagonal removed are not simply connected.

The criterion for path independence based on exactness is then as follows.

Theorem. 3 (Criterion for exactness and independence of path)

Let F_1, F_2, F_3 in the line integral,

$$\int_C F(r)\cdot dr = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

be continuous and have continuous first partial derivatives in a domain D in space. Then:

(a) If this integral is independent of path in D—and thus the differential form under the integral sign is exact,

6. $$\text{curl } F = 0$$

in components therefore condition of exactness follows from curl $F = 0$, which gives,

since $$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{curl } F = i\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$$

6′. $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

(b) If (6′) holds in D and D is simply connected, then the integral is independent of path in D.

Proof:

(a) If the line integral is independent of path in D, then F = grad f by (2) and

curl F = curl (grad f) = 0 So, that (6) holds.

(b) The proof of the converse requires "Stokes's theorem" and is omittcs here.

Comment For a line integral in the plane

$$\int_C F(r)\cdot dr = \int_C (F_1 dx + F_2 dy),$$

curl F has just one component and (6′) reduces to the single relation 6″.

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Example:

Exactness and independence of path. Determination of a potential

Using (6′), show that the differential form under the integral sign of

$$t = \int_C \left[2xyz^2 dx + x^2 z^2 + z\cos yz)dy + (2x^2 yz + y\cos yz)dz\right]$$

is exact, so that we have independence of path in any domain, and find the value of I from A: (0, 0, 1) to B: (1, π/4, 2).

Solution:

Exactness follows from (6′), which gives

$$(F_3)_y = 2x^2z + \cos yz - yz \sin yz = (F_2)_z$$
$$(F_1)_z = 4xyz = (F_3)_x$$
$$(F_2)_x = 2xz^2 = (F_1)_y.$$

To find f, we integrate F_2 (which is "long," so that we save work) and then differentiate to compare with F_1 and F_3.

$$f_x = F_1 = 2xyz^2$$
$$f_y = F_2 = (x^2z^2 + z\cos yz)$$
$$f_z = F_3 = 2x^2yz + y\cos yz)$$
$$f = \int F_2 dy = \int (x^2z^2 + z\cos yz) = x^2z^2y + \sin yz + g(x, z)$$
$$f_x = 2xz^2y + g_x = f_1 = 2xyz^2, \quad g_x = 0, \quad g = h(z),$$
$$f_z = 2x^2zy + y\cos yz + h' = F_3 = 2x^2zy + y\cos yz, \; h' = 0$$

so that, taking $h = 0$, we have

$$f(x, y, z) = x^2yz^2 + \sin yz.$$

From this and (3) we get, $I = f(B) - f(A)$

$$= f(1, \pi/4, 2) - f(0, 0, 1) = \pi + \sin\frac{1}{2}\pi - 0 = \pi + 1$$

The assumption in Theorem 3 that D be simply connected is essential and cannot be omitted.

2.14.21 Green's Theorem in the Plane

Double integrals over a plane region may be transformed into line integrals over the boundary of the region and conversely. This is of practical interest because it may help to make the evaluation of an integral easier. It also helps in the theory whenever one wants to switch from one kind of integral to the other. The transformation can be done by the following theorem.

Theorem. 1 (Green's Theorem in The Plane)

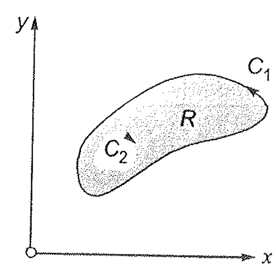
(Transformation between double integrals and line integrals)

Let R be a closed bounded region (see Sec. 9.3) in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R. Then.

1. $$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1dx + F_2dy)$$

here we integrate along the entire boundary C or R such that R is on the left as we advance in the direction of integration (See Figure).

Region R whose boundary is C consists of two parts: C_1 is traversed counterclockwise, while C_2 is traversed clockwise, so that R is on left as we advance.



Comment. Formula (1) can be written in vectorial form

1′. $$\iint_R (\text{curl } F) \cdot \hat{k}\, dxdy = \oint_C F \cdot dr \qquad (F = [F_1, F_2] = F_1 i + F_2 j)$$

This follows from the fact that the third component of curl F is $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

Example:

Verification of Green's theorem in the plane.

Green's theorem in the plane will be quite important in our further work. Before proving it, let us get used to it by verifying it for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$

Solution:

In (1) on the left we get

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint_R [(2y+2)-(2y-7)]dxdy = 9\iint_R dxdy$$

$$= 9\pi \quad \text{(since the circular disk } R \text{ has area } \pi).$$

On the right in (1) we represent C (oriented counterclockwise!) by

$$r(t) = [\cos t, \sin t]$$

Then $$r'(t) = [-\sin t, \cos t].$$

On C we thus obtain

$$F_1 = \sin^2 t - 7 \sin t,$$

$$F_2 = 2 \cos t \sin t + 2 \cos t.$$

Hence the integral in (1) on the right becomes

$$\int_C (F_1 x' + F_2 y')dt = \int_C^{2\pi} [(\sin^2 t - 7\sin t)(-\sin t) + 2(\cos t \sin t + \cos t)(\cos t)]dt$$

$$= 0 + 7\pi + 0 + 2\pi = 9\pi.$$

This verifies Green's theorem in the plane.

2.14.22Triple Integrals : Divergence Theorem of Gauss

In this section we first discuss triple integrals. Then we obtain the first "big" integral theorem, which transforms surface integrals into triple integrals. It is called **Gauss's divergence theorem** because it involves the divergence of a vector function.

The triple integral is a generalization of the double integral. For defining this integral we consider a function $f(x, y, z)$ defined in a bounded closed region T in space. We subdivide this three-dimensional region T by planes parallel to the three coordinate planes. Then those boxes of subdivision (rectangular parallelopiped) that lie entirely inside T are numbered 1 to n. In each such box we choose an arbitrary point, Say, (x_k, y_k, z_k) in box k, and form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k$$

where DV_k is the volume of box k. This we do for larger and larger positive integers n arbitrarily but so that the maximum length of all the edges of those n boxes approaches zero as n approaches infinity. This gives a sequence of real numbers J_{n_1}, J_{n_2}, ... We assume that $f(x, y, z)$ is continuous in a domain containing T and T is bounded by finitely many smooth surfaces (see Sec. 9.5). Then it can be shown (See Ref. [5] in Appendix 1) that the sequence converges to a limit that is independent of the choice of subdivisions and corresponding points (x_k, y_k, z_k). This limit is called the triple integral of $f(x, y, z)$ over the region T and is denoted by

$$\iiint_T f(x, y, z)dx\,dy\,dz \quad \text{or} \quad \iiint_T f(x, y, z)dV$$

Triple integrals can be evaluated by three successive integrations. This is similar to the evaluation of double integrals by two successive integrations.

2.14.22.1 Divergence Theorem of Gauss

Triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely. This is of practical interest because one of the two kinds of integral is often simpler than the other.

It also helps in establishing fundamental equations in fluid flow, heat conduction, etc., as we shall see. The transformation is done by the divergence theorem, which involves the divergence of a vector function $F = [F_1, F_2, F_3] = F_1 i + F_2 j + F_3 k$,

1. $$\text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{(Sec. 8.10)}$$

Theorem. 1 (Divergence Theorem of Gauss)

Transformation between volume integrals and surface integrals

Let T be a closed[11] bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $F(x, y, z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then.

2. $$\iiint_T \text{div } F\, dV = \iint_S F \cdot n\, dA$$

where n is the outer unit normal vector of S (pointing to the outside of S, as in Fig. 231).

Formula (2) in Components. using (1) and $n = [\cos\alpha, \cos\beta, \cos\gamma]$, we can write (2)

3*. $$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA$$

since, $$\iint_S F.n dA = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

equation 2 may also be written as,

3. $$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1\, dy\, dz + F_2 dz\, dx + F_3 dx\, dy)$$

Example:

Evaluation of a surface integral by the divergence theorem

Before we prove the divergence theorem, let us show a typical application. By transforming to a triple integral, evaluate

$$I = \iint_S (x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy).$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$.

Solution:

In (3) we now have

$$F_1 = x^3,\ F_2 = x^2 y,\ F_3 = x^2 z$$

Hence, $$\text{div } F = 3x^2 + x^2 + x^2 = 5x^2$$

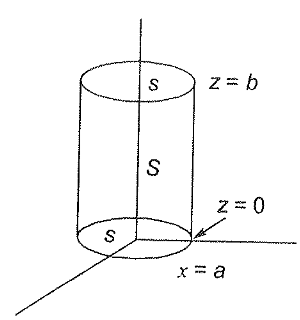
Introducing polar coordinates r, θ defined by $x = r\cos\theta$, $y = r\sin\theta$ (thus, cylindrical coordinates r, θ, z), we have $dx\, dy\, dz = r\, dr\, d\theta\, dz$, and we obtain

$$I = \iiint_T 5x^2 dx\, dy\, dz$$

$$= 5\int_{z=0}^{b}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2 \cos^2\theta\, r\, dr\, d\theta\, dz$$

$$= 5b\int_0^a\int_0^{2\pi} r^3 \cos^2\theta\, dr\, d\theta$$

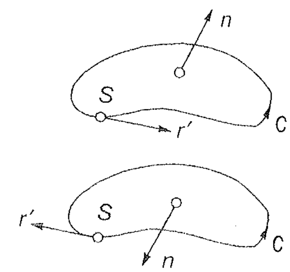
$$= 5b\frac{a^4}{4}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{5}{4}\pi b a^4$$



2.14.22.2 Stokes's Theorem

Having seen the great usefulness of Gauss's theorem, we now turn to the second "big" theorem in this chapter, Stokes's theorem, which transforms line integrals into surface integrals and conversely. Hence this theorem generalizes Green's theorem. It involves the curl,

1. $$\text{curl } F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



Stoke's theorem

Theorem. 2 (Stokes' Theorem)

Transformation between surface integrals and line integrals

Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. Let $F(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

2. $$\iint_S (\text{curl } F)\cdot n\,dA = \oint_C F\cdot r'(s)\,ds$$

where n is a unit normal vector of S and, depending on n, the integration around C is taken in the sense shown in Figure above. Furthermore, $r' = dr/ds$ is the unit tangent vector and s the arc length of C. Formula 2 can be written in terms of components:

3. $$\iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right] du\,dv$$

$$= \oint_{\bar{C}} (F_1 dx + F_2 dy + F_3 dz)$$

where R is the region with boundary curve $\bar{C}$ in the uv-plane corresponding to S represented by $r(u, v)$, and $\mathbf{N} = [N_1, N_2, N_3] = \mathbf{r}_u \times \mathbf{r}_v$.

Example 1.

Verification of Stokes's theorem

Before proving Stokes's theorem, let us get used to it by verifying it for $F = [y, z, x]\; yi + zj + xk$ and S the paraboloid.

$$z = f(x, y) = 1 - (x^2 + y^2),\ z \geq 0.$$

Solution:

The curve C is the circle $r(s) = [\cos s, \sin s, 0] = \cos s i + \sin s j$. It has the unit tangent vector $r'(s) = [-\sin s, \cos s, 0] = -\sin s i + \cos j$. Consequently, the line integral in (2) on the right is simply

$$\oint_C F\cdot dr = \int_0^{2\pi} [(\sin s)(-\sin s) + 0 + 0]ds = -\pi$$

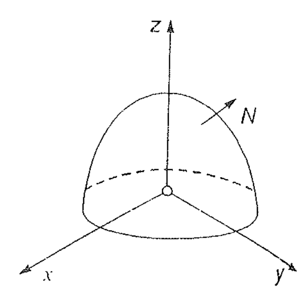
On the other hand, in (2) on the left we need (verify this)

$$\text{curl } F = [-1, -1, -1]$$

and

$$N = \text{grad }(z - f(x, y)) = [2x, 2y, 1]$$

so that $(\text{curl } F).N = -2x - 2y - 1$. From (3) in previous section we get



Surface S in Example 1

$$\iint_S (\text{curl } F)\cdot n\,dA = \iint_R (-2x - 2y - 1)dx\,dy$$

$$= \iint_{\bar{R}} (-2r\cos\theta - 2r\sin\theta - 1)\, r\, dr\, d\theta$$

where $x = r\cos\theta$, $y = r\sin\theta$, and $dx\, dy = r\, dr\, d\theta$. Now the projection R of S in the xy-plane is given in polar coordinates by $\bar{R}: r \leq 1, 0 \leq \theta 2\pi$. The integration of the cosine and sine terms over θ from 0 to 2π gives zero. The remaining term $-1.\ r$ has integral $(-1/2)\, 2\pi = -\pi$, in agreement with the previous result. Note well that N is an upper normal vector of S, and $r(s)$ orients C counterclockwise, as required in Stokes's theorem.

Example 2.

Green's theorem in the plane as a special case of Stokes's theorem

Let $F = [F_1, F_2] = F_1 i + F_2 j$ be a vector function that is continuously differentiable in a domain in the xy-plane containing a simply connected bounded closed region S whose boundary C is a piecewise smooth simple closed curve. Then, according to (1),

$$(\text{curl } F)\cdot a = (\text{curl } F)\cdot k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Solution:

Hence the formula in stoke's theorem now takes the form

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C (F_1 dx + F_2 dy)$$

This shows that Green's theorem in the plane is a special case of Stokes's theorem.

Previous GATE and ESE Questions

Q.1 $\underset{x \to 0}{\text{Lt}} \dfrac{\sin^2 x}{x}$ is equal to

(a) 0 (b) ∞

(c) 1 (d) – 1

[ME, GATE-2003, 1 mark]

Q.2 If P, Q and R are three points having coordinates (3, –2, –1), (1, 3, 4), (2, 1, –2) in XYZ space, then the distance from point P to plane OQR (O being the origin of the coordinate system) is given by

(a) 3 (b) 5

(c) 7 (d) 9

[CE, GATE-2003, 1 mark]

Q.3 The value of the function $f(x) = \lim\limits_{x \to 0} \dfrac{x^3 + x^2}{2x^3 - 7x^2}$ is

(a) 0 (b) $-\dfrac{1}{7}$

(c) $\dfrac{1}{7}$ (d) ∞

[CE, GATE-2004, 1 mark]

Q.4 If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then dy/dx will be equal to

(a) $\sin\left(\dfrac{\theta}{2}\right)$ (b) $\cos\left(\dfrac{\theta}{2}\right)$

(c) $\tan\left(\dfrac{\theta}{2}\right)$ (d) $\cot\left(\dfrac{\theta}{2}\right)$

[ME, GATE-2004, 1 mark]

Q.5 The function $f(x) = 2x^3 - 3x^2 - 36x + 2$ has its maxima at

(a) $x = -2$ only (b) $x = 0$ only

(c) $x = 3$ only (d) both $x = -2$ and $x = 3$

[CE, GATE-2004, 2 marks]

Q.6 The volume of an object expressed in spherical co-ordinates is given by

$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\phi \, dr\, d\phi\, d\theta$. The value of the intergal is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{2\pi}{3}$ (d) $\dfrac{\pi}{4}$

[ME, GATE-2004, 2 marks]

Q.7 The angle between two unit-magnitude coplanar vectors P(0.866, 0.500, 0) and Q(0.259, 0.966, 0) will be

(a) 0° (b) 30°

(c) 45° (d) 60°

[ME, GATE-2004, 1 mark]

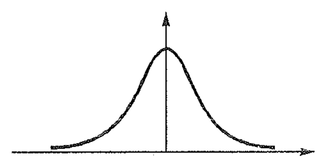
Q.8 A rail engine accelerates from its stationary position for 8 seconds and travels a distance of 280 m. According to the Mean Value Theorem, the speedometer at a certain time during acceleration must read exactly

(a) 0 (b) 8 kmph

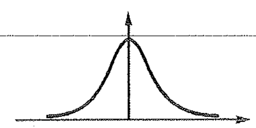
(c) 75 kmph (d) 126 kmph

[CE, GATE-2005, 2 marks]

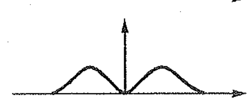
Q.9 The derivative of the symmetric function drawn in given figure will look like



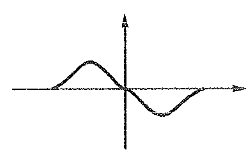
(a)



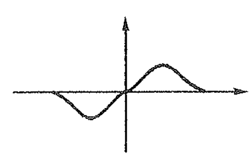
(b)



(c)



(d)



[EC, GATE-2005, 2 marks]

Q.10 The right circular cone of largest volume that can be enclosed by a sphere of 1 m radius has a height of

(a) 1/3 m (b) 2/3 m

(c) $\dfrac{2\sqrt{2}}{3}$ m (d) 4/3 m

[ME, GATE-2005, 2 marks]

Q.11 For the function $f(x) = x^2 e^{-x}$, the maximum occurs when x is equal to

(a) 2 (b) 1
(c) 0 (d) –1

[EE, GATE-2005, 2 marks]

Q.12 If $S = \int_1^{\infty} x^{-3}\, dx$, then S has the value

(a) –1/3 (b) 1/4
(c) 1/2 (d) 1

[EE, GATE-2005, 1 mark]

Q.13 Changing the order of the integration in the double integral $I = \int_0^8 \int_{x/4}^2 f(x, y)\,dy\,dx$ leads to $I = \int_r^s \int_p^q f(x, y)\,dx\,dy$. What is q?

(a) $4y$ (b) $16y^2$
(c) x (d) 8

[ME, GATE-2005, 1 mark]

Q.14 By a change of variable $x(u, v) = uv$, $y(u, v) = v/u$ is double integral, the integrand $f(x, y)$ changes to $f(uv, v/u)\,\phi(u, v)$. Then, $\phi(u, v)$ is

(a) $2\,u/v$ (b) $2\,uv$
(c) v^2 (d) 1

[ME, GATE-2005, 2 marks]

Q.15 For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, magnitude of the gradient at the point (1, 3) is

(a) $\sqrt{\frac{13}{9}}$ (b) $\sqrt{\frac{9}{2}}$
(c) $\sqrt{5}$ (d) $\frac{9}{2}$

[EE, GATE-2005, 2 marks]

Q.16 The line integral $\int \vec{V}.d\vec{r}$ of the vector $\vec{V}.(\vec{r}) = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$ from the origin to the point $P(1, 1, 1)$

(a) is 1
(b) is zero
(c) is –1
(d) cannot be determined without specifying path

[ME, GATE-2005, 2 marks]

Q.17 Value of the integral $\oint_c (xydy - y^2dx)$, where, c is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$ will be (Use Green's theorem to change the line integral into double integral)

(a) $\frac{1}{2}$ (b) 1
(c) $\frac{3}{2}$ (d) $\frac{5}{3}$

[CE, GATE-2005, 2 marks]

Q.18 Stokes theorem connects

(a) a line integral and a surface integral
(b) a surface integral and a volume integral
(c) a line integral and a volume integral
(d) gradient of a function and its surface integral

[ME, GATE-2005, 1 mark]

Q.19 If $f(x) = \frac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$, then $\lim_{x \to 3} f(x)$ will be

(a) – 1/3 (b) 5/18
(c) 0 (d) 2/5

[ME, GATE-2006, 2 marks]

Q.20 As x increased from $-\infty$ to ∞, the function $f(x) = \frac{e^x}{1+e^x}$

(a) monotonically increases
(b) monotonically decreases
(c) increases to a maximum value and then decreases
(d) decreases to a minimum value and then increases

[EC, GATE-2006, 2 marks]

Q.21 Assuming $i = \sqrt{-1}$ and t is a real number, $\int_0^{\pi/3} e^{it}dt$ is

(a) $\frac{\sqrt{3}}{2} + i\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2} - i\frac{1}{2}$
(c) $\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$ (d) $\frac{1}{2} + i\left(1 - \frac{\sqrt{3}}{2}\right)$

[ME, GATE-2006, 2 marks]

Q.22 The integral $\int_0^{\pi} \sin^3\theta\, d\theta$ is given by

(a) 1/2 (b) 2/3
(c) 4/3 (d) 8/3

[EC, GATE-2006, 2 marks]

Q.23 What is the area common to the circles $r = a$ and $r = 2a\cos\theta$?

(a) $0.524\, a^2$ (b) $0.614\, a^2$
(c) $1.047\, a^2$ (d) $1.228\, a^2$

[CE, GATE-2006, 2 marks]

Q.24 The expression $V = \int_0^H \pi R^2 (1 - h/H)^2\, dh$ for the volume of a cone is equal to

(a) $\int_0^R \pi R^2 (1 - h/H)^2\, dr$
(b) $\int_0^R \pi R^2 (1 - h/H)^2\, dh$
(c) $\int_0^H 2\pi r H (1 - r/R)\, dh$
(d) $\int_0^R \pi r H \left(1 - \frac{r}{R}\right)^2 dr$

[EE, GATE-2006, 2 marks]

Q.25 A surface $S(x, y) = 2x + 5y - 3$ is integrated once over a path consisting of the points that satisfy $(x + 1)^2 + (y - 1)^2 = \sqrt{2}$. The integral evaluates to

(a) $17\sqrt{2}$ (b) $\frac{17}{\sqrt{2}}$
(c) $\frac{\sqrt{2}}{17}$ (d) 0

[EE, GATE-2006, 2 marks]

Q.26 The directional derivative of

$$f(x, y, z) = 2x^2 + 3y^2 + z^2$$

at the point $P(2, 1, 3)$ in the direction of the vector $a = i - 2k$ is

(a) −2.785 (b) −2.145
(c) −1.789 (d) 1.000

[CE, GATE-2006, 2 marks]

Q.27 Equation of the line normal to function $f(x) = (x - 8)^{2/3} + 1$ at $P(0, 5)$ is

(a) $y = 3x - 5$ (b) $y = 3x + 5$
(c) $3y = x + 15$ (d) $3y = x - 15$

[ME, GATE-2006, 2 marks]

Q.28 $\nabla \times \nabla \times P$, where P is a vector is equal to

(a) $P \times \nabla \times P - \nabla^2 P$ (b) $\nabla^2 P + \nabla(\nabla \times P)$
(c) $\nabla^2 P + \nabla \times P$ (d) $\nabla(\nabla \cdot P) - \nabla^2 P$

[EC, GATE-2006, 1 mark]

Q.29 $\iint (\nabla \times P) \cdot ds$, where P is a vector, is equal to

(a) $\oint P \cdot dl$ (b) $\oint \nabla \times \nabla \times P \cdot dl$
(c) $\oint \nabla \times P \cdot dl$ (d) $\iiint \nabla \cdot P\, dv$

[EC, GATE-2006, 1 mark]

Q.30 $\lim_{x \to 0} \frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3} =$

(a) 0 (b) 1/6
(c) 1/3 (d) 1

[ME, GATE-2007, 2 marks]

Q.31 $\lim_{\theta \to 0} \frac{\sin(\theta/2)}{\theta}$ is

(a) 0.5 (b) 1
(c) 2 (d) not defined

[EC, GATE-2007, 1 mark]

Q.32 The minimum value of function $y = x^2$ in the interval [1, 5] is

(a) 0 (b) 1
(c) 25 (d) undefined

[ME, GATE-2007, 1 mark]

Q.33 Which one of the following functions is strictly bounded?

(a) $1/x^2$ (b) e^x
(c) x^2 (d) e^{-x^2}

[EC, GATE-2007, 1 mark]

Q.34 Consider the function $f(x) = x^2 - x - 2$. The maximum value of $f(x)$ in the closed interval [−4, 4] is

(a) 18 (b) 10
(c) −2.25 (d) indeterminate

[EC, GATE-2007, 2 marks]

Q.35 For the function e^{-x}, the linear approximation around $x = 2$ is

(a) $(3-x)e^{-2}$

(b) $1-x$

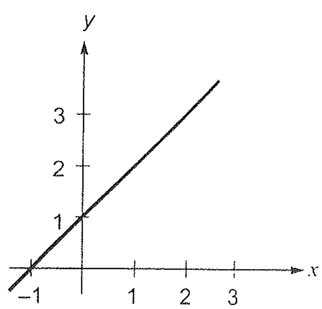
(c) $[3+2\sqrt{2}-(1+\sqrt{2})x]e^{-2}$

(d) e^{-2}

[EC, GATE-2007, 1 mark]

Q.36 The following plot shows a function y which varies linearly with x. The value of the integral

$$I = \int_1^2 y\, dx \text{ is}$$



(a) 1.0 (b) 2.5

(c) 4.0 (d) 5.0

[EC, GATE-2007, 1 mark]

Q.37 The area of a triangle formed by the tips of vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is

(a) $\frac{1}{2}(\bar{a}-\bar{b})\cdot(\bar{a}-\bar{c})$ (b) $\frac{1}{2}\left|(\bar{a}-\bar{b})\times(\bar{a}-\bar{c})\right|$

(c) $\frac{1}{2}\left|\bar{a}\times\bar{b}\times\bar{c}\right|$ (d) $\frac{1}{2}(\bar{a}\times\bar{b})\cdot\bar{c}$

[ME, GATE-2007, 2 marks]

Q.38 Let x and y be two vectors in a 3 dimensional space and $<x, y>$ denote their dot product. Then the determinant $\det\begin{bmatrix} <x,x> & <x,y> \\ <y,x> & <y,y> \end{bmatrix}$.

(a) is zero when x and y are linearly independent

(b) is positive when x and y are linearly independent

(c) is non-zero for all non-zero x and y

(d) is zero only when either x or y is zero

[EE, GATE-2007, 2 marks]

Q.39 A velocity vector is given as $\vec{V} = 5xy\vec{i} + 2y^2\vec{j} + 3yz^2\vec{k}$. The divergence of this velocity vector at (1, 1, 1) is

(a) 9 (b) 10

(c) 14 (d) 15

[CE, GATE-2007, 2 marks]

Q.40 Potential function ϕ is given as $\phi = x^2 - y^2$. What will be the stream function (ψ) with the condition $\psi = 0$ at $x = y = 0$?

(a) $2xy$ (b) $x^2 + y^2$

(c) $x^2 - y^2$ (d) $2x^2y^2$

[CE, GATE-2007, 2 marks]

Q.41 The Value of $\lim_{x \to 8} \frac{x^{1/3} - 2}{(x-8)}$

(a) $\frac{1}{16}$ (b) $\frac{1}{12}$

(c) $\frac{1}{8}$ (d) $\frac{1}{4}$

[ME, GATE-2008, 1 mark]

Q.42 $\lim_{x \to \infty} \frac{x - \sin x}{x + \cos x}$ equals

(a) 1 (b) –1

(c) ∞ (d) $-\infty$

[CS, GATE-2008, 1 mark]

Q.43 Consider function $f(x) = (x^2 - 4)^2$ where x is a real number. Then the function has

(a) only one minimum (b) only two minima

(c) three minima (d) three maxima

[EE, GATE-2008, 2 marks]

Q.44 A point on a curve is said to be an extremum if it is a local minimum or a local maximum. The number of distinct exterma for the curve $3x^4 - 16x^3 - 24x^2 + 37$ is

(a) 0 (b) 1

(c) 2 (d) 3

[CS, GATE-2008, 2 marks]

Q.45 In the Taylor series expansion of e^x about $x = 2$, the coefficient of $(x-2)^4$ is

(a) 1/4! (b) $2^4/4!$

(c) $e^2/4!$ (d) $e^4/4!$

[ME, GATE-2008, 1 mark]

Q.46 Which of the following functions would have only odd powers of x in its Taylor series expansion about the point $x = 0$?

(a) $\sin(x^3)$ (b) $\sin(x^2)$

(c) $\cos(x^3)$ (d) $\cos(x^2)$

[EC, GATE-2008, 1 mark]

Q.47 In the Taylor series expansion of $\exp(x) + \sin(x)$ about the point $x = \pi$, the coefficient of $(x-\pi)^2$ is

(a) $\exp(\pi)$ (b) $0.5\exp(\pi)$

(c) $\exp(\pi) + 1$ (d) $\exp(\pi) - 1$

[EC, GATE-2008, 2 marks]

Q.48 Let $f = y^x$. What is $\frac{\partial^2 f}{\partial x \partial y}$ at $x = 2, y = 1$?

(a) 0 (b) ln 2

(c) 1 (d) $\frac{1}{\ln 2}$

[ME, GATE-2008, 2 marks]

Q.49 Which of the following integrals is unbounded ?

(a) $\int_0^{\pi/4} \tan x\, dx$ (b) $\int_0^{\infty} \frac{1}{x^2+1} dx$

(c) $\int_0^{\infty} x e^{-x} dx$ (d) $\int_0^{1} \frac{1}{1-x} dx$

[ME, GATE-2008, 2 marks]

Q.50 The length of the curve $y = \frac{2}{3}x^{3/2}$ between $x = 0$ and $x = 1$ is

(a) 0.27 (b) 0.67

(c) 1 (d) 1.22

[ME, GATE-2008, 2 marks]

Q.51 The value of the integral of the function $g(x,y) = 4x^3 + 10y^4$ along the straight line segment from the point (0, 0) to the point (1, 2) in the x - y plane is

(a) 33 (b) 35

(c) 40 (d) 56

[EC, GATE-2008, 2 marks]

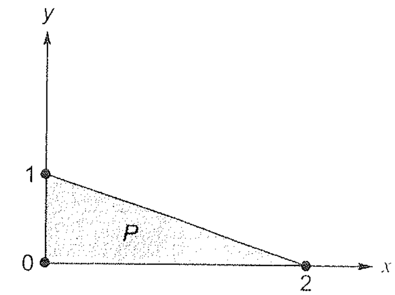
Q.52 The value of $\int_0^3 \int_0^x (6 - x - y)\, dx\, dy$ is

(a) 13.5 (b) 27.0

(c) 40.5 (d) 54.0

[CE, GATE-2008, 2 marks]

Q.53 Consider the shaded triangular region P shown in the figure. What is $\iint_P xy\,dx\,dy$?



(a) $\frac{1}{6}$ (b) $\frac{2}{9}$

(c) $\frac{7}{16}$ (d) 1

[ME, GATE-2008, 2 marks]

Q.54 The inner (dot) product of two non zero vectors $\vec{P}$ and $\vec{Q}$ is zero. The angle (degrees) between the two vectors is

(a) 0 (b) 30

(c) 90 (d) 120

[CE, GATE-2008, 2 marks]

Q.55 The divergence of the vector field $(x - y)\hat{i} + (y - x)\hat{j} + (x + y + z)\hat{k}$ is

(a) 0 (b) 1

(c) 2 (d) 3

[ME, GATE-2008, 1 mark]

Q.56 The directional derivative of the scalar function $f(x, y, z) = x^2 + 2y^2 + z$ at the point $P = (1,1,2)$ in the direction of the vector $\vec{a} = 3\hat{i} - 4\hat{j}$ is

(a) -4 (b) -2

(c) -1 (d) 1

[ME, GATE-2008, 2 marks]

Q.57 Consider points P and Q in the x-y plane, with $P = (1,0)$ and $Q = (0,1)$. The line integral $2\int_P^Q (x\,dx + y\,dy)$ along the semicircle with the line segment PQ as its diameter

(a) is -1

(b) is 0

(c) is 1

(d) depends on the direction (clockwise or anti-clockwise) of the semicircle

[EC, GATE-2008, 2 marks]

Q.58 The distance between the origin and the point nearest to it on the surface $z^2 = 1 + xy$ is

(a) 1 (b) $\frac{\sqrt{3}}{2}$

(c) $\sqrt{3}$ (d) 2

[ME, GATE-2009, 2 marks]

Q.59 A cubic polynomial with real coefficients

(a) can possibly have no extrema and no zero crossings

(b) may have up to three extrema and upto 2 zero crossings

(c) cannot have more than two extrema and more than three zero crossings

(d) will always have an equal number of extrema and zero crossings

[EE, GATE-2009, 2 marks]

Q.60 The Taylor series expansion of $\frac{\sin x}{x-\pi}$ at $x = \pi$ is given by

(a) $1+\frac{(x-\pi)^2}{3!}+\ldots$ (b) $-1-\frac{(x-\pi)^2}{3!}+\ldots$

(c) $1-\frac{(x-\pi)^2}{3!}+\ldots$ (d) $-1+\frac{(x-\pi)^2}{3!}+\ldots$

[EC, GATE-2009, 2 marks]

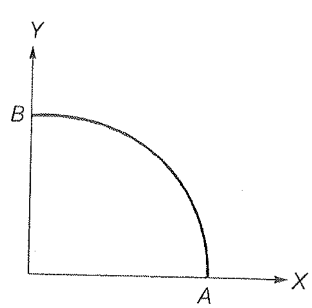
Q.61 $\int_0^{\pi/4} \frac{(1-\tan x)}{(1+\tan x)} dx$ evaluates to

(a) 0 (b) 1

(c) ln 2 (d) 1/2 ln 2

[CS, GATE-2009, 2 marks]

Q.62 A path AB in the form of one quarter of a circle of unit radius is shown in the figure. Integration of $(x + y)^2$ on path AB traversed in a counter-clockwise sense is



(a) $\frac{\pi}{2}-1$ (b) $\frac{\pi}{2}+1$

(c) $\frac{\pi}{2}$ (d) 1

[ME, GATE-2009, 2 marks]

Q.63 The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is

(a) $\frac{16}{3}$ (b) 8

(c) $\frac{32}{3}$ (d) 16

[ME, GATE-2009, 2 marks]

Q.64 $f(x, y)$ is a continuous function defined over $(x, y) \in [0, 1] \times [0, 1]$. Given the two constraints, $x > y^2$ and $y > x^2$, the volume under $f(x, y)$ is

(a) $\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} f(x, y)dx\,dy$

(b) $\int_{y=x^2}^{y=1} \int_{x=y^2}^{x=1} f(x, y)dx\,dy$

(c) $\int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x,y)dx\,dy$

(d) $\int_{y=0}^{y=\sqrt{x}} \int_{x=0}^{x=\sqrt{y}} f(x, y)dx\,dy$

[EE, GATE-2009, 2 marks]

Q.65 For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the gradient at the point $P(1, 2, -1)$ is

(a) $2\vec{i}+6\vec{j}+4\vec{k}$ (b) $2\vec{i}+12\vec{j}-4\vec{k}$

(c) $2\vec{i}+12\vec{j}+4\vec{k}$ (d) $\sqrt{56}$

[CE, GATE-2009, 1 mark]

Q.66 For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the directional derivative at the point $P(1, 2, -1)$ in the direction of a vector $\vec{i}-\vec{j}+2\vec{k}$ is

(a) -18 (b) $-3\sqrt{6}$

(c) $3\sqrt{6}$ (d) 18

[CE, GATE-2009, 2 marks]

Q.67 The divergence of the vector field $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at a point (1,1,1) is equal to

(a) 7 (b) 4
(c) 3 (d) 0

[ME, GATE-2009, 1 mark]

Q.68 The $\lim_{x \to 0} \dfrac{\sin\left[\frac{2}{3}x\right]}{x}$ is

(a) 2/3 (b) 1
(c) 3/2 (d) ∞

[CE, GATE-2010, 1 mark]

Q.69 What is the value of $\lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{2n}$?

(a) 0 (b) e^{-2}
(c) $e^{-1/2}$ (d) 1

[CS, GATE-2010, 1 mark]

Q.70 The function $y = |2 - 3x|$

(a) is continuous $\forall\, x \in R$ and differentiable $\forall\, x \in R$
(b) is continuous $\forall\, x \in R$ and differentiable $\forall\, x \in R$ except at $x = 3/2$
(c) is continuous $\forall\, x \in R$ and differentiable $\forall\, x \in R$ except at $x = 2/3$
(d) is continuous $\forall x \in R$ except $x = 3$ and differentiable $\forall x \in R$

[ME, GATE-2010, 1 mark]

Q.71 Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$. The optimal value of $f(x, y)$

(a) is a minimum equal to 10/3
(b) is a maximum equal to 10/3
(c) is a minimum equal to 8/3
(d) is a maximum equal to 8/3

[CE, GATE-2010, 2 marks]

Q.72 At $t = 0$, the function $f(t) = \dfrac{\sin t}{t}$ has

(a) a minimum
(b) a discontinuity
(c) a point of inflection
(d) a maximum

[EE, GATE-2010, 2 marks]

Q.73 If $e^y = x^{\frac{1}{x}}$, then y has a

(a) maximum at $x = e$
(b) minimum at $x = e$
(c) maximum at $x = e^{-1}$
(d) minimum at $x = e^{-1}$

[EC, GATE-2010, 2 marks]

Q.74 The value of the integral $\int_{-\infty}^{\infty} \dfrac{dx}{1+x^2}$ is

(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) π

[ME, GATE-2010, 1 mark]

Q.75 The value of the quantity P, where $P = \int_0^1 xe^x dx$, is equal to

(a) 0 (b) 1
(c) e (d) $1/e$

[EE, GATE-2010, 1 mark]

Q.76 A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h(x^2/L^2)$, where x is the horizontal coordinate and y is the vertical coordinate with the origin at the centre of the cable. The expression for the total length of the cable is

(a) $\int_0^L \sqrt{1+64\dfrac{h^2x^2}{L^4}}dx$ (b) $2\int_0^{L/2} \sqrt{1+64\dfrac{h^3x^2}{L^4}}dx$

(c) $\int_0^{L/2} \sqrt{1+64\dfrac{h^2x^2}{L^4}}dx$ (d) $2\int_0^{L/2} \sqrt{1+64\dfrac{h^2x^2}{L^4}}dx$

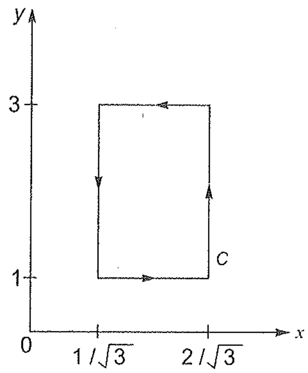
[CE, GATE-2010, 2 marks]

Q.77 The parabolic arc $y = \sqrt{x}$, $1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is

(a) $\pi/4$ (b) $\pi/2$
(c) $3\pi/4$ (d) $3\pi/2$

[ME, GATE-2010, 1 mark]

Q.78 If $\vec{A} = xy\hat{a}_x + x^2\hat{a}_y$, $\oint_C \vec{A}\cdot d\vec{l}$ over the path shown in the figure is



(a) 0 (b) $\frac{2}{\sqrt{3}}$

(c) 1 (d) $2\sqrt{3}$

[EC, GATE-2010, 2 marks]

Q.79 Velocity vector of a flow field is given as $\vec{V} = 2xy\hat{i} - x^2z\hat{j}$. The vorticity vector at (1, 1, 1) is

(a) $4\hat{i} - \hat{j}$ (b) $4\hat{i} - \hat{k}$

(c) $\hat{i} - 4\hat{j}$ (d) $\hat{i} - 4\hat{k}$

[ME, GATE-2010, 2 marks]

Q.80 Divergence of the three-dimensional radial vector field $\vec{r}$ is

(a) 3 (b) 1/r

(c) $\hat{i} + \hat{j} + \hat{k}$ (d) $3(\hat{i} + \hat{j} + \hat{k})$

[EE, GATE-2010, 1 mark]

Q.81 What is $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ equal to?

(a) θ (b) $\sin\theta$

(c) 0 (d) 1

[ME, GATE-2011, 1 mark]

Q.82 What should be the value of λ such that the function defined below is continuous at $x = \pi/2$?

$$f(x) = \begin{cases} \dfrac{\lambda \cos x}{\dfrac{\pi}{2} - x} & \text{if } x \neq \pi/2 \\ 1 & \text{if } x = \pi/2 \end{cases}$$

(a) 0 (b) $2/\pi$

(c) 1 (d) $\pi/2$

[CE, GATE-2011, 2 marks]

Q.83 The function $f(x) = 2x - x^2 + 3$ has

(a) a maxima at $x = 1$ and a minima at $x = 5$

(b) a maxima at $x = 1$ and a minima at $x = -5$

(c) only a maxima at $x = 1$

(d) only a minima at $x = 1$

[EE, GATE-2011, 2 marks]

Q.84 A series expansion for the function $\sin\theta$ is

(a) $1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$

(b) $\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

(c) $1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots$

(d) $\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$

[ME, GATE-2011, 1 mark]

Q.85 Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} dx$?

(a) 0 (b) 2

(c) $-i$ (d) i

[CS, GATE-2011, 2 marks]

Q.86 What is the value of the definite integral, $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$?

(a) 0 (b) $a/2$

(c) a (d) $2a$

[CE, GATE-2011, 2 marks]

Q.87 If $f(x)$ is an even function and a is a positive real number, then $\int_{-a}^{a} f(x)dx$ equals

(a) 0 (b) a

(c) $2a$ (d) $2\int_0^a f(x)dx$

[ME, GATE-2011, 1 marks]

Q.88 If $\vec{a}$ and $\vec{b}$ are two arbitrary vectors with magnitudes a and b, respectively, $|\vec{a} \times \vec{b}|^2$ will be equal to

(a) $a^2b^2 - (\vec{a}\cdot\vec{b})^2$ (b) $ab - \vec{a}\cdot\vec{b}$

(c) $a^2b^2 + (\vec{a}\cdot\vec{b})^2$ (d) $ab + \vec{a}\cdot\vec{b}$

[CE, GATE-2011, 2 marks]

Q.89 The two vectors [1, 1, 1] and [1, a, a^2], where

$$a = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right), \text{ are}$$

(a) orthonormal (b) orthogonal
(c) parallel (d) collinear

[EE, GATE-2011, 2 marks]

Q.90 $\lim_{x \to 0}\left(\frac{1-\cos x}{x^2}\right)$ is

(a) 1/4 (b) 1/2
(c) 1 (d) 2

[ME, GATE-2012, 1 mark]

Q.91 Consider the function $f(x) = |x|$ in the interval $-1 < x \le 1$. At the point $x = 0$, $f(x)$ is

(a) continuous and differentiable
(b) noncontinuous and differentiable
(c) continuous and non-differentiable
(d) neither continuous nor differentiable

[ME, GATE-2012, 1 mark]

Q.92 At $x = 0$, the function $f(x) = x^3 + 1$ has

(a) a maximum value (b) a minimum value
(c) a singularity (d) a point of inflection

[ME, GATE-2012, 1 mark]

Q.93 The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6] is

(a) 21 (b) 25
(c) 41 (d) 46

[EC, EE, IN, GATE-2012, 2 marks]

Q.94 Consider the function $f(x) = \sin(x)$ in the interval $x \in [\pi/4, 7\pi/4]$. The number and location(s) of the local minima of this function are

(a) One, at $\pi/2$
(b) One, at $3\pi/2$
(c) Two, at $\pi/2$ and $3\pi/2$
(d) Two, at $\pi/4$ and $3\pi/2$

[CS, GATE-2012, 1 mark]

Q.95 The infinite series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ corresponds to

(a) $\sec x$ (b) e^x
(c) $\cos x$ (d) $1 + \sin^2 x$

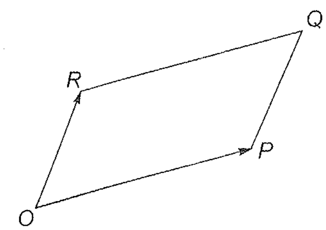
[CE, GATE-2012, 1 mark]

Q.96 The area enclosed between the straight line $y = x$ and the parabola $y = x^2$ in the x - y plane is

(a) 1/6 (b) 1/4
(c) 1/3 (d) 1/2

[ME, GATE-2012, 1 mark]

Q.97 For the parallelogram $OPQR$ shown in the sketch, $\overline{OP} = a\hat{i} + b\hat{j}$ and $\overline{OR} = c\hat{i} + d\hat{j}$. The area of the parallelogram is



(a) $ad - bc$ (b) $ac + bd$
(c) $ad + bc$ (d) $ab - cd$

[CE, GATE-2012, 2 marks]

Q.98 For the spherical surface $x^2 + y^2 + z^2 = 1$, the unit outward normal vector at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is given by

(a) $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ (b) $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$
(c) $\hat{k}$ (d) $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

[ME, GATE-2012, 1 mark]

Q.99 The direction of vector A is radially outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla \cdot A = 0$ is

(a) −2 (b) 2
(c) 1 (d) 0

[IN, GATE-2012, 2 marks]

Q.100 Which one of the following functions is continuous at $x = 3$?

(a) $f(x) = \begin{cases} 2, & \text{if } x = 3 \\ x - 1, & \text{if } x > 3 \\ \frac{x+3}{3}, & \text{if } x < 3 \end{cases}$

(b) $f(x) = \begin{cases} 4, & \text{if } x = 3 \\ 8 - x & \text{if } x \ne 3 \end{cases}$

(c) $f(x) = \begin{cases} x+3, & \text{if} \quad x \le 3 \\ x-4 & \text{if} \quad x > 3 \end{cases}$

(d) $f(x) = \dfrac{1}{x^3 - 27}$, if $x \ne 3$

[CS, GATE-2013, 1 Mark]

Q.101 A function $y = 5x^2 + 10x$ is defined over an open interval $x = (1, 2)$. At least at one point in this interval, $\dfrac{dy}{dx}$ is exactly

(a) 20 (b) 25
(c) 30 (d) 35

[EE, GATE-2013, 2 Marks]

Q.102 A polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x - a_0$ with all coefficients positive has

(a) no real roots
(b) no negative real root
(c) odd number of real roots
(d) at least one positive and one negative real root

[EC, GATE-2013, 1 Mark]

Q.103 The value of $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$ is

(a) 0 (b) $\dfrac{1}{15}$
(c) 1 (d) $\dfrac{8}{3}$

[CE, GATE-2013, 2 Mark]

Q.104 The value of the definite integral $\int_1^e \sqrt{x} \ln(x) dx$ is

(a) $\frac{4}{9}\sqrt{e^3} + \frac{2}{9}$ (b) $\frac{2}{9}\sqrt{e^3} - \frac{4}{9}$
(c) $\frac{2}{9}\sqrt{e^3} + \frac{4}{9}$ (d) $\frac{4}{9}\sqrt{e^3} - \frac{2}{9}$

[ME, GATE-2013, 2 Marks]

Q.105 The curl of the gradient of the scalar field defined by $V = 2x^2y + 3y^2z + 4z^2x$ is

(a) $4xy\, \mathbf{a}_x + 6yz\, \mathbf{a}_y + 8zx\, \mathbf{a}_z$
(b) $4\mathbf{a}_x + 6\mathbf{a}_y + 8\mathbf{a}_z$
(c) $(4xy + 4z^2)\, \mathbf{a}_x + (2x^2 + 6yz)\, \mathbf{a}_y + (3y^2 + 8zx)\, \mathbf{a}_z$
(d) 0

[EE, GATE-2013, 1 Mark]

Q.106 The divergence of the vector field $\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is

(a) 0 (b) 1/3
(c) 1 (d) 3

[EC, GATE-2013, 1 Mark]

Q.107 Function f is known at the following points

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0
$f(x)$	0	0.09	0.36	0.81	1.44	2.25	3.24	4.41	5.76	7.29	9.00

The value of $\int_0^3 f(x)dx$ computed using the continuous at $x = 3$?

(a) 8.983 (b) 9.003
(c) 9.017 (d) 9.045

[CS, GATE-2013, 1 Mark]

Q.108 For a vector E, which one of the following statements is **NOT TRUE**?

(a) If $\nabla \cdot E = 0$, E is called solenoidal.
(b) If $\nabla \times E = 0$, E is called conservative.
(c) If $\nabla \times E = 0$, E is called irrotational.
(d) If $\nabla \cdot E = 0$, E is called irrotational.

[IN, GATE-2013 : 1 mark]

Q.109 Given a vector field $\vec{F} = y^2x\hat{a}_x - yz\hat{a}_y - x^2\hat{a}_z$, the line integral $\int \vec{F} \cdot \vec{dl}$ evaluated along a segment on the x-axis from $x = 1$ to $x = 2$ is

(a) –2.33 (b) 0
(c) 2.33 (d) 7

[EE, GATE-2013, 1 Mark]

Q.110 The following surface integral is to be evaluated over a sphere for the given steady velocity vector field $F = xi + yj + zk$ defined with respect to a Cartesian coordinate system having i, j and k as unit base vectors.

$$\iint_S \frac{1}{4}(F \cdot n) dA$$

where S is the sphere, $x^2 + y^2 + z^3 = 1$ and n is the outward unit normal vector to the sphere. The value of the surface integral is

(a) π (b) 2π
(c) $3\pi/4$ (d) 4π

[ME, GATE-2013, 2 Marks]

Q.111 Consider a vector field $\vec{A}(\vec{r})$. The closed loop line integral $\oint \vec{A} \cdot \vec{dl}$ can be expressed as

(a) $\oiint (\nabla \times \vec{A}) \cdot \vec{ds}$ over the closed surface bounded by the loop

(b) $\oiiint (\nabla \cdot \vec{A}) dv$ over the closed volume bounded by the top

(c) $\iiint (\nabla \cdot \vec{A}) dv$ over the open volume bounded by the loop

(d) $\iint (\nabla \times \vec{A}) \cdot \vec{ds}$ over the open surface bounded by the loop

[EC, GATE-2013, 1 Mark]

Q.112 $\lim_{x \to \infty} \left(\frac{x + \sin x}{x} \right)$ equal to

(a) $-\infty$ (b) 0
(c) 1 (d) ∞

[CE, GATE-2014 : 1 Mark]

Q.113 The expression $\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha}$ is equal to

(a) $\log x$ (b) 0
(c) $x \log x$ (d) ∞

[CE, GATE-2014 : 2 Marks]

Q.114 $\lim_{x \to 0} \frac{x - \sin x}{1 - \cos x}$

(a) 0 (b) 1
(c) 3 (d) not defined

[ME, GATE-2014 : 1 Mark]

Q.115 $\lim_{x \to 0} \left(\frac{e^{2x} - 1}{\sin(4x)} \right)$ is equal to

(a) 0 (b) 0.5
(c) 1 (d) 2

[ME, GATE-2014 : 1 Mark]

Q.116 The value of $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ is

(a) $ln 2$ (b) 1.0
(c) e (d) ∞

[EC, GATE-2014 : 1 Mark]

Q.117 The integrating factor for differential equation $\frac{dP}{dt} + k_2 P = k_1 L_o e^{-k_1 t}$ is

(a) $e^{-k_1 t}$ (b) $e^{-k_2 t}$
(c) $e^{k_1 t}$ (d) $e^{k_2 t}$

[CE, GATE-2014 : 1 Mark]

Q.118 If a function is continuous at a point,

(a) the limit of the function may not exist at the point.
(b) the function must be derivable at the point.
(c) the limit of the function at the point tends to infinity.
(d) the limit must exist at the point and the value of limit should be same as the value of the function at that point.

[ME, GATE-2014 : 1 Mark]

Q.119 A function $f(x)$ is continuous in the interval [0, 2]. It is known that $f(0) = f(2) = -1$ and $f(1) = 1$. Which one of the following statements must be true?

(a) There exists a y in the interval (0, 1) such that $f(y) = f(y + 1)$
(b) For every y in the interval (0, 1), $f(y) = f(2 - y)$
(c) The maximum value of the function in the interval (0, 2) is 1
(d) There exists a y in the interval (0, 1) such that $f(y) = -f(2 - y)$

[CS, GATE-2014 : 2 Marks]

Q.120 Let the function

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

where $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $f'(\theta)$ denote the derivative of f with respect to θ. Which of the following statements is/are TRUE?

(I) There exists $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) = 0$.

(II) There exists $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) \neq 0$.

(a) I only (b) II only
(c) Both I and II (d) Neither I nor II

[CS, GATE-2014 : 1 Mark]

Q.121 The function $f(x) = x \sin x$ satisfies the following equation: $f''(x) + f(x) + t\cos x = 0$. The value of t is ________.

[CS, GATE-2014 : 2 Marks]

Q.122 If $y = f(x)$ is the solution of $\frac{d^2y}{dx^2} = 0$, with the boundary conditions $y = 5$ at $x = 0$, and $\frac{dy}{dx} = 2$ at $x = 10$, $f(15) =$ ______.

[EC, GATE-2014 : 2 Marks]

Q.123 For a right angled triangle, if the sum of the lengths of the hypotenuse and a side is kept constant, in order to have maximum area of the triangle, the angle between the hypotenuse and the side is

(a) 12° (b) 36°
(c) 60° (d) 45°

[EC, GATE-2014 : 2 Marks]

Q.124 If $z = xy\ln(xy)$, then

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$ (b) $y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$

(c) $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$ (d) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

[EC, GATE-2014 : 1 Mark]

Q.125 Let $f(x) = x\,e^{-x}$. The maximum value of the function in the interval $(0, \infty)$ is

(a) e^{-1} (b) e
(c) $1 - e^{-1}$ (d) $1 + e^{-1}$

[EE, GATE-2014 : 1 Mark]

Q.126 Minimum of the real valued function $f(x) = (x-1)^{2/3}$ occurs at x equal to

(a) $-\infty$ (b) 0
(c) 1 (d) ∞

[EE, GATE-2014 : 1 Mark]

Q.127 The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval $[-3, 3]$ is

(a) 20 (b) 28
(c) 16 (d) 32

[EE, GATE-2014 : 2 Marks]

Q.128 For $0 \le t < \infty$, the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

(a) $t = \log_e 4$ (b) $t = \log_e 2$
(c) $t = 0$ (d) $t = \log_e 8$

[EC, GATE-2014 : 1 Mark]

Q.129 The maximum value of the function $f(x) = \ln(1 + x) - x$ (where $x > -1$) occur at $x =$ ______.

[EC, GATE-2014 : 1 Mark]

Q.130 The maximum value of $f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval $0 \le x \le 3$ is

[EC, GATE-2014 : 2 Marks]

Q.131 The value of the inegral $\int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^2 + \cos(x-1)}dx$ is

(a) 3 (b) 0
(c) −1 (d) −2

[ME, GATE-2014 : 1 Mark]

Q.132 If $\int_0^{2\pi} |x\sin x|\,dx = k\pi$, then the value of k is equal to ________.

[CS, GATE-2014 : 1 Mark]

Q.133 The value of the integral given below is

(a)
(a) -2π (b) π
(c) $-\pi$ (d) 2π

[CS, GATE-2014 : 2 Marks]

Q.134 The line integral of function $F = yzi$, in the counter clockwise direction, along the circle $x^2 + y^2 = 1$ at $z = 1$ is

(a) -2π (b) $-\pi$
(c) π (d) 2π

[EE, GATE-2014 : 2 Marks]

Q.135 The value of the integral $\int_0^2 \int_0^x e^{x+y}\,dy\,dx$

(a) $\frac{1}{2}(e-1)$ (b) $\frac{1}{2}(e^2-1)^2$

(c) $\frac{1}{2}(e^2-e)$ (d) $\frac{1}{2}\left(e-\frac{1}{e}\right)^2$

[ME, GATE-2014 : 2 Marks]

Q.136 To evaluate the double integral $\int_0^8 \left(\int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2}\right) dx\right) dy$, we make the substitution $= \left(\frac{2x-y}{2}\right)$ and $v = \frac{y}{2}$. The integral will reduce to

(a) $\int_0^4 \left(\int_0^2 2\,u\,du\right)dv$ (b) $\int_0^4 \left(\int_0^1 2\,u\,du\right)dv$

(c) $\int_0^4 \left(\int_0^1 u\,du\right)dv$ (d) $\int_0^4 \left(\int_0^2 u\,du\right)dv$

[EE, GATE-2014 : 2 Marks]

Q.137 Which one of the following describes the relationship among the three vectors, $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+4\hat{k}$?

(a) The vectors are mutually perpendicular

(b) The vectors are linearly dependent

(c) The vectors are linearly independent

(d) The vectors are unit vectors

[ME, GATE-2014 : 1 Mark]

Q.138 Curl of vector $\vec{F} = x^2z^2\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$ is

(a) $(4yz^3 + 2xy^2)\hat{i} + 2x^2z\hat{j} - 2y^2z\hat{k}$

(b) $(4yz^3 + 2xy^2)\hat{i} - 2x^2z\hat{j} - 2y^2z\hat{k}$

(c) $2xz^2\hat{i} - 4xyz\hat{j} + 6y^2z^2\hat{k}$

(d) $2xz^2\hat{i} + 4xyz\hat{j} + 6y^2z^2\hat{k}$

[ME, GATE-2014 : 1 Mark]

Q.139 Divergence of the vector field $x^2z\hat{i} + xy\hat{j} - yz^2\hat{k}$ at (1, –1, 1) is

(a) 0 (b) 3

(c) 5 (d) 6

[ME, GATE-2014 : 1 Mark]

Q.140 Let $\nabla\cdot(f\vec{v}) = x^2y + y^2z + z^2x$, where f and v are scalar and vector fields respectively. If $\vec{v} = y\vec{i} + z\vec{j} + x\vec{k}$, then $\vec{v}\cdot\nabla f$ is

(a) $x^2y + y^2z + z^2x$ (b) $2xy + 2yz + 2zx$

(c) $x + y + z$ (d) 0

[EE, GATE-2014 : 1 Mark]

Q.141 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to

(a) e^{-2} (b) e

(c) 1 (d) e^2

[CE, GATE-2015 : 1 Mark]

Q.142 The value of $\lim_{x\to 0}\frac{1-\cos(x^2)}{2x^4}$ is

(a) 0 (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) undefined

[ME, GATE-2015 : 1 Mark]

Q.143 The value of $\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + \cos x}\right)$ is ______.

[ME, GATE-2015 : 1 Mark]

Q.144 $\lim_{x\to\infty} x^{1/x}$ is

(a) ∞ (b) 0

(c) 1 (d) Not defined

[CS, GATE-2015 : 1 Mark]

Q.145 The value of $\lim_{x\to\infty}(1+x^2)^{e^{-x}}$ is

(a) 0 (b) 1/2

(c) 1 (d) ∞

[CS, GATE-2015 : 1 Mark]

Q.146 Let $f(x) = x^{-(1/3)}$ and A denote the area of the region bounded by $f(x)$ and the X-axis, when x varies from –1 to 1. Which of the following statements is/are True?

1. f is continuous in [–1, 1]
2. f is not bounded in [–1, 1]
3. A is nonzero and finite

(a) 2 only (b) 3 only

(c) 2 and 3 only (d) 1, 2 and 3

[CS, GATE-2015 : 2 Marks]

Q.147 A function $f(x) = 1 - x^2 + x^3$ is defined in the closed interval [–1, 1]. The value of x in the open interval (–1, 1) for which the mean value theorem is satisfied, is

(a) –1/2 (b) –1/3

(c) 1/3 (d) 1/2

[EC, GATE-2015 : 1 Mark]

Q.148 While minimizing the function $f(x)$, necessary and sufficient conditions for a point x_0 to be a minima are

(a) $f'(x_0) > 0$ and $f''(x_0) = 0$

(b) $f'(x_0) < 0$ and $f''(x_0) = 0$

(c) $f'(x_0) = 0$ and $f''(x_0) < 0$

(d) $f'(x_0) = 0$ and $f''(x_0) > 0$

[CE, GATE-2015 : 1 Mark]

Q.149 At $x = 0$, the function $f(x) = |x|$ has

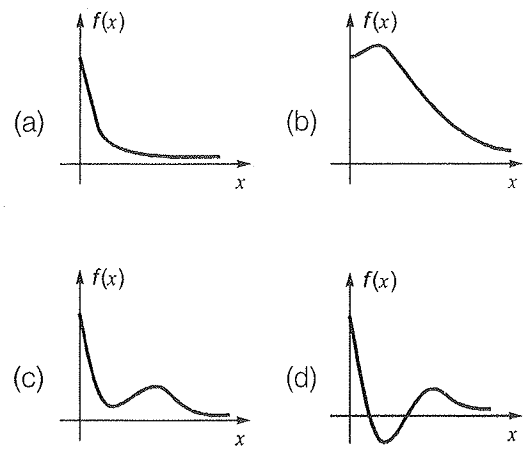
(a) a minimum
(b) a maximum
(c) a point of inflection
(d) neither a maximum nor minimum

[ME, GATE-2015 : 1 Mark]

Q.150 If the sum of the diagonal elements of a 2×2 symmetric matrix is -6, then the maximum possible value of determinant of the matrix is _____.

[EE, GATE-2015 : 1 Mark]

Q.151 Which one of the following graphs describes the function $f(x) = e^{-x}(x^2 + x + 1)$?



[EC, GATE-2015 : 2 Marks]

Q.152 The maximum area (in square unit) of a rectangle whose vertices lies on the ellipse $x^2 + 4y^2 = 1$ is_______.

[EC, GATE-2015 : 2 Marks]

Q.153 The contour on the x - y plane, where the partial derivative of $x^2 + y^2$ with respect to y is equal to the partial derivative of $6y + 4x$ with respect to x, is

(a) $y = 2$
(b) $x = 2$
(c) $x = y = 4$
(d) $x - y = 0$

[EC, GATE-2015 : 1 Mark]

Q.154 If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 25$ where $a \neq b$ then $\int_1^2 f(x)dx$ is

(a) $\frac{1}{a^2 - b^2}\left[a(\ln 2 - 25) + \frac{47b}{2}\right]$

(b) $\frac{1}{a^2 - b^2}\left[a(2\ln 2 - 25) - \frac{47b}{2}\right]$

(c) $\frac{1}{a^2 - b^2}\left[a(2\ln 2 - 25) + \frac{47b}{2}\right]$

(d) $\frac{1}{a^2 - b^2}\left[a(\ln 2 - 25) - \frac{47b}{2}\right]$

[CS, GATE-2015 : 2 Marks]

Q.155 Consider an ant crawling along the curve $(x - 2)^2 + y^2 = 4$, where x and y are in meters. The ant starts at the point (4, 0) and moves counter-clockwise with a speed of 1.57 meters per second. The time taken by the ant to reach the point (2, 2) is (in seconds) _______.

[ME, GATE-2015 : 2 Marks]

Q.156 Consider a spatial curve in three-dimensional space given in parametric form by

$$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = \frac{2}{\pi}t,\ 0 \le t \le \frac{\pi}{2}$$

The length of the curve is ________.

[ME, GATE-2015 : 2 Marks]

Q.157 The volume enclosed by the surface $f(x, y) = e^x$ over the triangle bounded by the lines $x = y$; $x = 0$; $y = 1$ in the xy plane is _______.

[EE, GATE-2015 : 2 Marks]

Q.158 The double integral $\int_0^a \int_0^y f(x, y)\, dx\, dy$ is equivalent to

(a) $\int_0^x \int_0^y f(x, y)\, dx\, dy$
(b) $\int_0^a \int_x^y f(x, y)\, dx\, dy$
(c) $\int_0^a \int_x^a f(x, y)\, dy\, dx$
(d) $\int_0^a \int_0^a f(x, y)\, dx\, dy$

[IN, GATE-2015 : 1 Mark]

Q.159 The directional derivative of the field $u(x, y, z) = x^2 - 3yz$ in the direction of the vector $(\bar{i} + \bar{j} - 2\bar{k})$ at point (2, − 1, 4) is _________.

[CE, GATE-2015 : 2 Marks]

Q.160 Curl of vector $V(x, y, z) = 2x^2 i + 3z^2 j + y^3 k$ at $x = y = z = 1$ is

(a) $-3i$ (b) $3i$

(c) $3i - 4j$ (d) $3i - 6k$

[ME, GATE-2015 : 1 Mark]

Q.161 Let ϕ be an arbitrary smooth real valued scalar function and V be an arbitrary smooth vector valued function in a three-dimensional space. Which one of the following is an identity?

(a) $\text{Curl}(\phi\vec{V}) = \nabla(\phi \text{Div} \vec{V})$

(b) $\text{Div} \vec{V} = 0$

(c) $\text{Div Curl} \vec{V} = 0$

(d) $\text{Div}\left(\phi\vec{V}\right) = \phi \text{ Div} \vec{V}$

[ME, GATE-2015 : 1 Mark]

Q.162 The magnitude of the directional derivative of the function $f(x, y) = x^2 + 3y^2$ in a direction normal to the circle $x^2 + y^2 = 2$, at the point (1, 1), is

(a) $4\sqrt{2}$ (b) $5\sqrt{2}$

(c) $7\sqrt{2}$ (d) $9\sqrt{2}$

[IN, GATE-2015 : 1 Mark]

Q.163 The value of $\int_c [(3x - 8y^2)dx + (4y - 6xy)dy]$, (where C is the boundary of the region boundary by $x = 0$, $y = 0$ and $x + y = 1$) is ________.

[ME, GATE-2015 : 2 Marks]

Q.164 The surface integral $\iint_S \frac{1}{\pi}(9xi - 3yj)\cdot n dS$ over the sphere given by $x^2 + y^2 + z^2 = 9$ is _____.

[ME, GATE-2015 : 2 Marks]

Q.165 $\lim_{x\to 4} \frac{\sin(x-4)}{x-4} =$ _____ .

[CS, GATE-2016 : 1 Mark]

Q.166 $\lim_{x\to 4} \frac{\sin(x-4)}{x-4} =$ _____ .

[CS, GATE-2016 : 1 Mark]

Q.167 A scalar potential φ has the following gradient : $\nabla\varphi = yz\hat{i} + xz\hat{j} + xy\hat{k}$. Consider the integral $\int_C \nabla\varphi . d\vec{r}$ on the curve $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The curve C is parameterized as follows :

$$\begin{cases} x = t \\ y = t \quad \text{and } 1 \le t \le 3 \\ z = 3t^2 \end{cases}$$

The value of the integral is____

[ME, GATE-2016 : 2 Marks]

Q.168 $\lim_{n\to\infty} \left(\sqrt{n^2 + n} - \sqrt{n^2 + 1}\right)$ is__________.

[IN, GATE-2016 : 1 Mark]

Q.169 $\underset{x\to 0}{\text{Lt}} \frac{\log_e(1+4x)}{e^{3x} - 1}$ is equal to

(a) 0 (b) $\frac{1}{12}$

(c) $\frac{4}{3}$ (d) 1

[ME, GATE-2016 : 1 Mark]

Q.170 $\lim_{x\to\infty} \sqrt{x^2 + x - 1} - x$ is

(a) 0 (b) ∞

(c) 1/2 (d) $-\infty$

[ME, GATE-2016 : 2 Marks]

Q.171 What is the value of $\lim_{\substack{x\to 0 \\ y\to 0}} \frac{xy}{x^2 + y^2}$?

(a) 1 (b) –1

(c) 0 (d) Limit does not exit

[CE, GATE-2016 : 1 Mark]

Q.172 Given the following statements about a function $f : R \to R$, select the right option:

P: If $f(x)$ is continuous at $x = x_0$, then it is differential at $x = x_0$.

Q: If $f(x)$ is continuous at $x = x_0$, then it may not be differentiable at $x = x_0$.

R: If $f(x)$ is differentiable at $x = x_0$, then it is also continuous at $x = x_0$.

(a) P is true, Q is false, R is false

(b) P is false, Q is true, R is true

(c) P is false, Q is true, R is false

(d) P is true, Q is false, R is true

[EC, GATE-2016 : 1 Mark]

Q.173 The values of x for which the function

$$f(x) = \frac{x^2 - 3x - 4}{x^2 + 3x - 4}$$

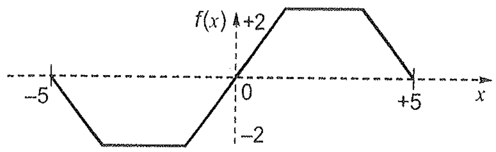
is NOT continuous are

(a) 4 and −1 (b) 4 and 1

(c) −4 and 1 (d) −4 and −1

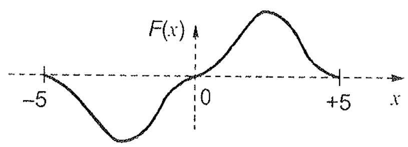
[ME, GATE-2016 : 1 Mark]

Q.174 Consider the plot $f(x)$ versus x as shown below.

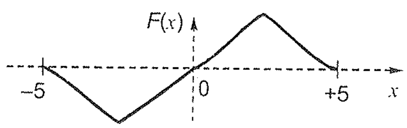


Suppose $F(x) = \int_{-5}^{x} f(y)dy$. Which one of the following is a graph of $F(x)$?

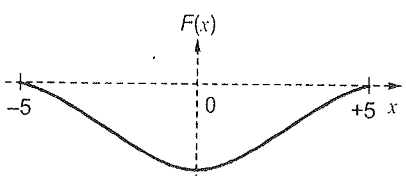
(a)



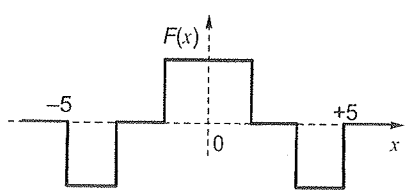
(b)



(c)



(d)



[EC, GATE-2016 : 1 Mark]

Q.175 Let $f(x)$ be a polynomial and $g(x) = f'(x)$ be its derivative. If the degree of $(f(x) + f(-x))$ is 10, then the degree of $(g(x) - g(-x))$ is ____.

[CS, GATE-2016 : 1 Mark]

Q.176 As x varies from −1 to +3, which one of the following describes the behaviour of the function $f(x) = x^3 - 3x^2 + 1$?

(a) $f(x)$ increases monotonically.

(b) $f(x)$ increases, then decreases and increases again.

(c) $f(x)$ decreases, then increases and decreases again.

(d) $f(x)$ increases and then decreases

[EC, GATE-2016 : 1 Mark]

Q.177 Let f: [−1, 1] → R, where $f(x) = 2x^3 - x^4 - 10$. The minimum value of $f(x)$ is______.

[IN, GATE-2016 : 2 Marks]

Q.178 The maximum value attained by the function $f(x) = x(x-1)(x-2)$ in the interval [1, 2] is_____.

[EE, GATE-2016 : 1 Mark]

Q.179 The optimum value of the function $f(x) = x^2 - 4x + 2$ is

(a) 2 (maximum) (b) 2 (minimum)

(c) −2 (maximum) (d) −2 (minimum)

[CE, GATE-2016 : 1 Mark]

Q.180 The quadratic approximation of $f(x) = x^3 - 3x^2 - 5$ a the point $x = 0$ is

(a) $3x^2 - 6x - 5$ (b) $-3x^2 - 5$

(c) $-3x^2 + 6x - 5$ (d) $3x^2 - 5$

[CE, GATE-2016 : 2 Marks]

Q.181 The angle of intersection of the curves $x^2 = 4y$ and $y^2 = 4x$ at point (0, 0) is

(a) 0° (b) 30°

(c) 45° (d) 90°

[CE, GATE-2016 : 2 Marks]

Q.182 How many distinct values of x satisfy the equation $\sin(x) = x/2$, where x is in radians?

(a) 1 (b) 2

(c) 3 (d) 4 or more

[EC, GATE-2016 : 1 Mark]

Q.183 The value of the line integral $\oint_C \bar{F} \cdot \bar{r}' ds$, where C is a circle of radius $\frac{4}{\sqrt{\pi}}$ units is______

Here, $\bar{F}(x, y) = y\hat{i} + 2x\hat{j}$ and $\bar{r}'$ is the UNIT tangent vector on the curve C at an arc length s from a reference point on the curve $\hat{i}$ and $\hat{j}$ are the basis vectors in the x–y Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction.

[ME, GATE-2016 : 2 Marks]

Q.184 A straight line of the form $y = mx + c$ passes through the origin and the point $(x, y) = (2, 6)$. The value of m is_______.

[IN, GATE-2016 : 1 Mark]

Q.185 The integral $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$ is equal to _____.

[EC, GATE-2016 : 1 Mark]

Q.186 The value of $\int_0^{\infty} \frac{1}{1+x^2}dx + \int_0^{\infty} \frac{\sin x}{x}dx$ is

(a) $\frac{\pi}{2}$ (b) π

(c) $\frac{3\pi}{2}$ (d) 1

[CE, GATE-2016 : 2 Marks]

Q.187 A triangle in the xy - plane is bounded by the straight lines $2x = 3y$, $y = 0$ and $x = 3$. The volume above the triangle and under the plane $x + y + z = 6$ is ________.

[EC, GATE-2016 : 2 Marks]

Q.188 The area between the parabola $x^2 = 8y$ and the straight line $y = 8$ is ______.

[CE, GATE-2016 : 2 Marks]

Q.189 The integral $\frac{1}{2\pi}\iint_D (x+y+10)dx,dy$, where D denotes the disc: $x^2 + y^2 \le 4$, evaluates to ____.

[EC, GATE-2016 : 2 Marks]

Q.190 Suppose C is the closed curve defined as the circle $x^2 + y^2 = 1$ with C oriented anti clockwise. The value of $\oint(xy^2dx + x^2ydy)$ over the curve C equals ________.

[EC, GATE-2016 : 2 Marks]

Q.191 The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is

(a) $\frac{59}{6}$ (b) $\frac{9}{2}$

(c) $\frac{10}{3}$ (d) $\frac{7}{6}$

[CE, GATE-2016 : 2 Marks]

Q.192 The region specified by $\{(\rho, \phi, z): 3 \le \rho \le 5, \frac{\pi}{8} \le \varphi \le \frac{\pi}{4}, 3 \le z \le 4.5\}$ in cylindrical coordinates has volume of _____.

[EC, GATE-2016 : 2 Marks]

Q.193 Consider the time-varying vector $I = \hat{x}15\cos(\omega t) + \hat{y}5\sin(\omega t)$ in Cartesian coordinates, where $\omega > 0$ is a constant. When the vector magnitude $|I|$ is at its minimum value, the angle θ that I makes with the x axis (in degrees, such that $0 \le \theta \le 180$) is _____.

[EC, GATE-2016 : 1 Mark]

Q.194 The vector that is NOT perpendicular to the vectors $(i + j + k)$ and $(i + 2j + 3k)$ is_____.

(a) $(i - 2j + k)$ (b) $(-i + 2j - k)$

(c) $(0i + 0j + 0k)$ (d) $(4i + 3j + 5k)$

[IN, GATE-2016 : 1 Mark]

Q.195 Which one of the following is a property of the solutions to the Laplace equation: $\nabla^2 f = 0$?

(a) The solutions have neither maxima nor minima anywhere except at the boundaries.

(b) The solutions are not separable in the coordinates.

(c) The solutions are not continuous.

(d) The solutions are not dependent on the boundary conditions.

[EC, GATE-2016 : 1 Mark]

Q.196 The value of the line integral

$$\int_C (2xy^2dx + 2x^2ydy + dz)$$

along a path joining the origin (0, 0, 0) and the point (1, 1, 1) is

(a) 0 (b) 2

(c) 4 (d) 6

[EE, GATE-2016 : 1 Mark]

Q.197 The line integral of the vector field $F = 5xz\hat{i} + (3x^2 + 2y)\hat{j} + x^2z\hat{k}$ along a path from (0, 0, 0) to (1, 1, 1) parameterized by (t, t^2, t) is ____.

[EE, GATE-2016 : 2 Marks]

Q.198 Let x be a continuous variable defined over the interval $(-\infty, \infty)$, and $f(x) = e^{-x-e^{-x}}$. The integral $g(x) = \int f(x)dx$ is equal to

(a) $e^{e^{-x}}$ (b) $e^{-e^{-x}}$

(c) $e^{-e^{x}}$ (d) e^{-x}

[CE, GATE-2017 : 1 Mark]

Q.199 The divergence of the vector $-yi + xj$ is ____________.

[ME, GATE-2017 : 1 Mark]

Q.200 The surface integral $\iint_S \vec{F}\cdot\hat{n}\,\overrightarrow{dS}$ over the surface S of the sphere $x^2 + y^2 + z^2 = 9$, where $\vec{F} = (x + y)i + (x + z)j + (y + z)k$ and $\hat{n}$ is the unit outward surface normal, yields _______.

[ME, GATE-2017 : 2 Marks]

Q.201 The value of $\lim_{x\to 0}\frac{x^3 - \sin(x)}{x}$ is

(a) 0 (b) 3
(c) 1 (d) – 1

[ME, GATE-2017 : 1 Mark]

Q.202 A parametric curve defined by $x = \cos\left(\frac{\pi u}{2}\right), y = \sin\left(\frac{\pi u}{2}\right)$ in the range $0 \le u \le 1$ is rotated about the x-axis by 360 degrees. Area of the surface generated is

(a) $\frac{\pi}{2}$ (b) π
(c) 2π (d) 4π

[ME, GATE-2017 : 2 Marks]

Q.203 For the vector $\vec{V} = 2yz\hat{i} + 3xz\hat{j} + 4xy\hat{k}$, the value of $\nabla\cdot\left(\nabla\times\vec{V}\right)$ is _________.

[ME, GATE-2017 : 2 Marks]

Q.204 If V is a non-zero vector of dimension 3×1, then the matrix $A = VV^T$ has rank = _______

[IN, GATE-2017 : 1 Mark]

Q.205 The angle between two vectors $X_1 = [2\ 6\ 14]^T$ and $X_2 = [-12\ 8\ 16]^T$ in radian is _______.

[IN, GATE-2017 : 2 Marks]

Q.206 Let x and y be integers satisfying the following equations.

$2x^2 + y^2 = 34$

$x + 2y = 11$

The value of $(x + y)$ is __________.

[EE, GATE-2017 : 1 Mark]

Q.207 Let $y^2 - 2y + 1 = x$ and $\sqrt{x} + y = 5$. The value of $x + \sqrt{y}$ equals______.

(Give the answer up to three decimal places).

[EE, GATE-2017 : 1 Mark]

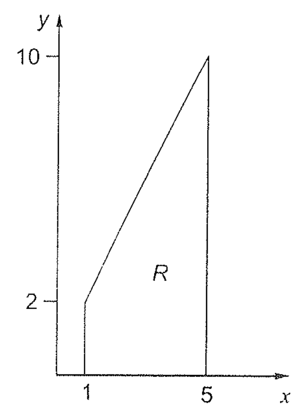
Q.208 Let $g(x) = \begin{cases} -x, & x \le 1 \\ x+1, & x \ge 1 \end{cases}$ and $f(x) = \begin{cases} 1-x, & x \le 0 \\ x^2, & x > 0 \end{cases}$.

Consider the composition of f and g i.e. (fog) (x) = $f(g(x))$. The number of discontinuities in (fog) (x) present in the interval $(-\infty, 0)$ is:

(a) 0 (b) 1
(c) 2 (d) 4

[EE, GATE-2017 : 2 Marks]

Q.209 Let $I = \iint_R xy^2\,dx\,dy$, where R is the region shown in the figure and $c = 6 \times 10^{-4}$. The value of I equals ________ . (Give the answer up to two decimal places.)



[EE, GATE-2017 : 1 Mark]

Q.210 A function $f(x)$ is defined as $f(x) = \begin{cases} e^x & x < 1 \\ \ln x + ax^2 + bx, & x \ge 1 \end{cases}$, where $x \in R$ which one of the following statements is TRUE?

(a) $f(x)$ is NOT differentiable at $x = 1$ for any values of a and b.

(b) $f(x)$ is differentiable at $x = 1$ for the unique values of a and b.

(c) $f(x)$ is differentiable at $x = 1$ for all the values of a and b such that $a + b = e$.

(d) $f(x)$ is differentiable at $x = 1$ for all values of a and b.

[EE, GATE-2017 : 2 Marks]

Q.211 The smaller angle (in degrees) between the planes $x + y + z = 1$ and $2x - y + 2z = 0$ is _____.

[EC, GATE-2017 : 1 Mark]

Q.212 The minimum value of the function $f(x) = \frac{1}{3}x(x^2-3)$ in the interval $-100 \le x \le 100$ occurs at x = ______.

[EC, GATE-2017 : 2 Marks]

Q.213 The values of the integrals

$$\int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}dy\right)dx \quad \text{and} \quad \int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}dx\right)dy$$

are

(a) same and equal to 0.5
(b) same and equal to – 0.5
(c) 0.5 and – 0.5 respectively
(d) – 0.5 and 0.5 respectively

[EC, GATE-2017 : 2 Marks]

Q.214 If the vector function

$$\vec{F} = \hat{a}_x(3y - k_1 z) + \hat{a}_y(k_2 x - 2z) - \hat{a}_z(k_3 y + z)$$

is irrotational, then the values of the constants k_1, k_2 and k_3, respectively, are

(a) 0.3, – 2.5, 0.5
(b) 0.0, 3.0, 2.0
(c) 0.3, 0.33, 0.5
(d) 4.0, 3.0, 2.0

[EC, GATE-2017 : 2 Marks]

Q.215 Let $I = \int_C (2zdx + 2ydy + 2xdz)$ where x, y, z are real, and let C be the straight line segment from point A: (0, 2, 1) to point B: (4, 1, –1). The value of I is ______

[EC, GATE-2017 : 2 Marks]

Q.216 Let $f(x) = e^{x+x^2}$ for real x. From among the following, choose the Taylor series approximation of $f(x)$ around $x = 0$, which includes all powers of x less than or equal to 3,

(a) $1 + x + x^2 + x^3$
(b) $1 + x + \frac{3}{2}x^2 + x^3$
(c) $1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$
(d) $1 + x + 3x^2 + 7x^3$

[EC, GATE-2017 : 2 Marks]

Q.217 A three dimensional region R of finite volume is described by

$$x^2 + y^2 \le z^3 \; ; \; 0 \le z \le 1,$$

where x, y, z are real. The volume of R (up to two decimal places) is ________.

[EC, GATE-2017 : 2 Marks]

Q.218 If $f(x) = R\sin\left(\frac{\pi x}{2}\right) + S, f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)dx = \frac{2R}{\pi}$, then the constants R and S are, respectively

(a) $\frac{2}{\pi}$ and $\frac{16}{\pi}$
(b) $\frac{2}{\pi}$ and 0
(c) $\frac{4}{\pi}$ and 0
(d) $\frac{4}{\pi}$ and $\frac{16}{\pi}$

[CS, GATE-2017 : 1 Mark]

Q.219 The value of $\lim_{x\to 1}\frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

(a) is 0
(b) is –1
(c) is 1
(d) does not exist

[CS, GATE-2017 : 2 Marks]

Q.220 Let $w = f(x, y)$, where x and y are functions of t. Then, according to the chain rule, $\frac{dw}{dt}$ is equal

(a) $\frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dt}{dt}$
(b) $\frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$
(c) $\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$
(d) $\frac{dw}{dx}\frac{\partial x}{\partial t} + \frac{dw}{dy}\frac{\partial y}{\partial t}$

[CE, GATE-2017 : 1 Mark]

Q.221 The divergence of the vector field $V = x^2 i + 2y^3 j + z^4 k$ at $x = 1$, $y = 2$, $z = 3$ is ________.

[CE, GATE-2017 : 1 Mark]

Q.222 The tangent to the curve represented by $y = x\ln x$ is required to have 45° inclination with the x-axis. The coordinates of the tangent point would be

(a) (1, 0)
(b) (0, 1)
(c) (1, 1)
(d) $(\sqrt{2}, \sqrt{2})$

[CE, GATE-2017 : 2 Marks]

Q.223 Consider the following definite integral:

$$I = \int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx$$

The value of the integral is

(a) $\frac{\pi^3}{24}$ (b) $\frac{\pi^3}{12}$

(c) $\frac{\pi^3}{48}$ (d) $\frac{\pi^3}{64}$

[CE, GATE-2017 : 2 Marks]

Q.224 Two cars *P* and *Q* are moving in a racing track continuously for two hours. Assume that no other vehicles are using the track during this time. The expressions relating the distance travelled *d* (in km) and time t (in hours) for both the vehicles are given as

$P: d = 60t$

$Q: d = 60t^2$

Within the first one hour, the maximum space headway would be

(a) 15 km at 30 minutes

(b) 15 km at 15 minutes

(c) 30 km at 30 minutes

(d) 30 km at 15 minutes

[CE, GATE-2017 : 2 Marks]

Q.225 $\lim_{x \to \infty}\left(\frac{\tan x}{x^2 - x}\right)$ is equal to ______.

[CE, GATE-2017 : 1 Mark]

Q.226 The minimum value of the function $f(x) = \left(\frac{x^3}{3}\right) - x$ occurs at

(a) $x = 1$ (b) $x = -1$

(b) $x = 0$ (d) $x = \frac{1}{\sqrt{3}}$

[ESE Prelims-2017]

Q.227 The value of the integral $\int_0^{2\pi}\left(\frac{3}{9 + \sin^2\theta}\right) d\theta$ is

(a) $\frac{2\pi}{\sqrt{10}}$ (b) $2\sqrt{10}\,\pi$

(c) $\sqrt{10}\,\pi$ (d) 2π

[ESE Prelims-2017]

Q.228 Which of the following statements are correct regarding dot product of vectors?

1. Dot product is less than or equal to the product of magnitudes of two vectors.
2. When two vectors are perpendicular to each other, then their dot product is non-zero.
3. Dot product of two vectors is positive or negative depending whether the angle between the vectors is less than or greater than $\pi/2$.
4. Dot product is equal to the product of one vector and the projection of the vector on the first one.

Select the correct answer using the codes given below:

(a) 1, 2 and 3 only (b) 1, 3 and 4 only

(c) 1, 2 and 4 only (d) 2, 3 and 4 only

[EE, ESE-2017]

Q.229 At the point $x = 0$, the function $f(x) = x^3$ has

(a) local maximum

(b) local minimum

(c) both local maximum and minimum

(d) neither local maximum nor local minimum

[CE, GATE-2018 : 1 Mark]

Q.230 The value of the integral $\int_0^{\pi} x\cos^2 x\,dx$ is

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{4}$

(c) $\frac{\pi^2}{2}$ (d) p^2

[CE, GATE-2018 : 2 Marks]

Q.231 The solution of the equation $x\frac{dy}{dx} + y = 0$ passing through the point (1, 1) is

(a) x (b) x^2

(c) x^{-1} (d) x^{-2}

[CE, GATE-2018 : 1 Mark]

Q.232 The value (up to two decimal places) of a line integral $\int_C \vec{F}(\vec{r}) \cdot \vec{dr}$, for $\vec{F}(\vec{r}) = x^2\hat{i} + y^2\hat{j}$ along C which is a straight line joining (0, 0) to (1, 1) is ___.

[CE, GATE-2018 : 2 Marks]

Q.233 According to the Mean Value Theorem, for a continuous function $f(x)$ in the interval $[a, b]$, there exists a value ξ in this interval such that $\int_n^b f(x)dx =$

(a) $f(\xi)(b-a)$ (b) $f(b)(\xi-a)$
(c) $f(a)(b-\xi)$ (d) 0

[ME, GATE-2018 : 1 Mark]

Q.234 The value of integral

$$\oiint_S \vec{r}\cdot\vec{n}ds$$

over the closed surface S bounding a volume, where $\vec{r} = x\hat{i} + y\hat{i} + z\hat{k}$ is the position vector and $\vec{n}$ is the normal to the surface S, is

(a) V (b) 2 V
(c) 3 V (d) 4 V

[ME, GATE-2018 : 2 Marks]

Q.235 The divergence of the vector field $\vec{u} = e^x\left(\cos y\hat{i} + \sin y\hat{j}\right)$ is

(a) 0 (b) $e^x \cos y + e^x \sin y$
(c) $2e^x \cos y$ (d) $2e^x \sin y$

[ME, GATE-2018 : 1 Mark]

Q.236 For a position vector $r = x\hat{i} + y\hat{j} + z\hat{k}$ the norm of the vector can be defined as $|\vec{r}| = \sqrt{x^2+y^2+z^2}$. Given a function $\phi = \ln|\vec{r}|$, its gradient $\nabla\phi$ is

(a) $\vec{r}$ (b) $\dfrac{\vec{r}}{|\vec{r}|}$

(c) $\dfrac{\vec{r}}{\vec{r}\cdot\vec{r}}$ (d) $\dfrac{\vec{r}}{|\vec{r}|^3}$

[ME, GATE-2018 : 2 Marks]

Q.237 Taylor series expansion of $f(x) = \int_0^x e^{-\left(\frac{t^2}{2}\right)}dt$ around $x = 0$ has the form

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots$$

The coefficient a_2 (correct to two decimal places) is equal to ______.

[EC, GATE-2018 : 1 Mark]

Q.238 Let $f(x, y) = \dfrac{ax^2 + by^2}{xy}$, where a and b are constants. If $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y}$ at $x = 1$ and $y = 2$, then the relation between a and b is

(a) $a = \dfrac{b}{4}$ (b) $a = \dfrac{b}{2}$

(c) $a = 2b$ (d) $a = 4b$

[EC, GATE-2018 : 1 Mark]

Q.239 The value of the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^2 + zx^2$ at the point $(2, -1, 1)$ in the direction of the vector $p = i + 2j + 2k$ is

(a) 1 (b) 0.95
(c) 0.93 (d) 0.9

[EE, GATE-2018 : 1 Mark]

Q.240 Let f be a real valued function of a real variable defined as $f(x) = x - [x]$, where $[x]$ denotes the largest integer less than or equal to x. The value of $\int_{0.25}^{1.25} f(x)\,dx$ is ______ (up to 2 decimal places).

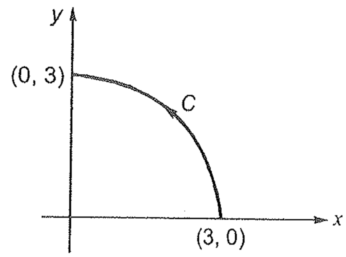
[EE, GATE-2018 : 1 Mark]

Q.241 Let f be a real-valued function of a real variable defined as $f(x) = x^2$ for $x \geq 0$, and $f(x) = -x^2$ for $x < 0$. Which one of the following statements is true?

(a) $f(x)$ is discontinuous at $x = 0$.
(b) $f(x)$ is continuous but not differentiable at $x = 0$.
(c) $f(x)$ is differentiable but its first derivative is not continuous at $x = 0$.
(d) $f(x)$ is differentiable but its first derivative is not differentiable at $x = 0$.

[EE, GATE-2018 : 1 Mark]

Q.242 As shown in the figure, C is the arc from the point $(3, 0)$ to the point $(0, 3)$ on the circle $x^2 + y^2 = 9$. The value of the integral $\int_C (y^2 + 2yx)\,dx + (2xy + x^2)\,dy$ is ______ (upto 2 decimal places).



[EE, GATE-2018 : 2 Marks]

Q.243 Let $f(x) = 3x^3 - 7x^2 + 5x + 6$. The maximum value of $f(x)$ over the interval [0, 2] is ______ (upto 1 decimal place).

[EE, GATE-2018 : 2 Marks]

Q.244 Consider the following equations

$$\frac{\partial V(x,y)}{\partial x} = px^2 + y^2 + 2xy$$

$$\frac{\partial V(x,y)}{\partial y} = x^2 + qy^2 + 2xy$$

where p and q are constants. $V(x, y)$ that satisfies the above equations is

(a) $p\frac{x^3}{3} + q\frac{y^3}{3} + 2xy + 6$

(b) $p\frac{x^3}{3} + q\frac{y^3}{3} + 5$

(c) $p\frac{x^3}{3} + q\frac{y^3}{3} + x^2y + xy^2 + xy$

(d) $p\frac{x^3}{3} + q\frac{y^3}{3} + x^2y + xy^2$

[IN, GATE-2018 : 2 Marks]

Q.245 Given $\vec{F} = (x^2 - 2y)\,\vec{i} - 4yz\,\vec{j} + 4xz^2\vec{k}$, the value of the line integral $\int_c \vec{F}\cdot d\vec{l}$ along the straight line c from (0, 0, 0) to (1, 1, 1) is

(a) $\frac{3}{16}$ (b) 0

(c) $\frac{-5}{12}$ (d) −1

[IN, GATE-2018 : 2 Marks]

Q.246 The value of $\int_0^{\pi/4} x\cos(x^2)\,dx$ correct to three decimal places (assuming that $\pi = 3.14$) is ________.

[CS, GATE-2018 : 1 Mark]

Q.247 If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$, what is the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$?

(a) 0 (b) 1

(c) u (d) eu

[ESE Prelims-2018]

■■■■

Answers Calculus

1. (a)	2. (a)	3. (b)	4. (c)	5. (a)	6. (a)	7. (c)	8. (d)	9. (c)
10. (d)	11. (a)	12. (c)	13. (a)	14. (a)	15. (c)	16. (a)	17. (c)	18. (a)
19. (b)	20. (a)	21. (a)	22. (c)	23. (d)	24. (d)	25. (d)	26. (c)	27. (b)
28. (d)	29. (a)	30. (b)	31. (a)	32. (b)	33. (d)	34. (a)	35. (a)	36. (b)
37. (b)	38. (b)	39. (d)	40. (a)	41. (b)	42. (a)	43. (b)	44. (d)	45. (c)
46. (a)	47. (b)	48. (c)	49. (d)	50. (d)	51. (a)	52. (a)	53. (a)	54. (c)
55. (d)	56. (b)	57. (b)	58. (a)	59. (c)	60. (b)	61. (d)	62. (b)	63. (a)
64. (a)	65. (b)	66. (b)	67. (c)	68. (a)	69. (b)	70. (c)	71. (a)	72. (d)
73. (a)	74. (d)	75. (b)	76. (d)	77. (d)	78. (c)	79. (d)	80. (a)	81. (d)
82. (c)	83. (c)	84. (b)	85. (d)	86. (b)	87. (d)	88. (a)	89. (b)	90. (b)
91. (c)	92. (d)	93. (c)	94. (b)	95. (b)	96. (a)	97. (a)	98. (a)	99. (a)
100. (a)	101. (b)	102. (d)	103. (b)	104. (c)	105. (d)	106. (d)	107. (d)	108. (d)
109. (b)	110. (a)	111. (d)	112. (c)	113. (a)	114. (a)	115. (b)	116. (c)	117. (d)
118. (d)	119. (a)	120. (c)	121. (–2)	122. (35)	123. (c)	124. (c)	125. (a)	126. (c)
127. (b)	128. (a)	129. (0)	130. (6)	131. (b)	132. (4)	133. (a)	134. (b)	135. (b)
136. (b)	137. (b)	138. (a)	139. (c)	140. (a)	141. (d)	142. (c)	143. (0)	144. (c)
145. (c)	146. (c)	147. (b)	148. (d)	149. (a)	150. (9)	151. (b)	152. (4)	153. (a)
154. (a)	155. (2)	156. (1.86)	157. (0.72)	158. (c)	159. (–5.7)	160. (a)	161. (c)	162. (a)
163. (1.67)	164. (216)	165. (1)	166. (1)	167. (726)	168. (0.5)	169. (c)	170. (c)	171. (d)
172. (b)	173. (c)	174. (c)	175. (9)	176. (b)	177. (–13)	178. (0)	179. (d)	180. (b)
181. (d)	182. (c)	183. (16)	184. (3)	185. (2)	186. (b)	187. (10)	188. (85.3)	189. (20)
190. (0)	191. (b)	192. (4.7)	193. (90)	194. (d)	195. (a)	196. (b)	197. (4.41)	198. (b)
199. (0)	200. (226)	201. (d)	202. (c)	203. (0)	204. (1)	205. (0.72)	206. (7)	207. (5.73)
208. (a)	209. (0.99)	210. (a)	211. (54.7)	212. (–100)	213. (c)	214. (b)	215. (–11)	216. (c)
217. (0.78)	218. (c)	219. (c)	220. (c)	221. (134)	222. (a)	223. (a)	224. (a)	225. (–1)
226. (a)	227. (a)	228. (b)	229. (d)	230. (b)	231. (c)	232. (0.66)	233. (a)	234. (c)
235. (c)	236. (c)	237. (0)	238. (d)	239. (a)	240. (0.5)	241. (d)	242. (0)	243. (12)
244. (d)	245. (d)	246. (0.289)	247. (b)					

1. (a)

$$\underset{x\to 0}{\text{Lt}}\frac{\sin^2 x}{x} = \underset{x\to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right)^2 x = 1\times 0 = 0$$

2. (a)

Solution by Coordinate Geometry:

This problem can be done through coordinate geometry formula or through vectors.

Given, $P(3, -2, -1)$

$Q(1, 3, 4)$

$R(2, 1, -2)$

$O(0, 0, 0)$

Equation of plane OQR is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

i.e. $2x - 2y + z = 0$

Now $\perp$ distance of (x_1, y_1, z_1) from $ax + by + cz + d = 0$ is given by

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}\right|$$

Therefore, $\perp$ distance of $(3, -2, -1)$ from plane $2x - 2y + z = 0$ is given by

$$\left|\frac{2\times 3 - 2\times(-2) + (-1)}{\sqrt{2^2+(-2)^2+1^2}}\right| = 3$$

(b)

$$f(x) = \lim_{x\to 0}\left[\frac{x^3+x^2}{2x^3-7x^2}\right]$$

Since this has $\frac{0}{0}$ form, limit can be found by repeated application of L' Hospitals rule.

$$f(x) = \lim_{x\to 0}\left[\frac{3x^2+2x}{6x^2-14x}\right]$$

$$= \lim_{x\to 0}\left[\frac{6x+2}{12x-14}\right]$$

$$= \left[\frac{6\times 0+2}{12\times 0-14}\right] = -\frac{1}{7}$$

4. (c)

Given, $x = a(\theta + \sin\theta),\ y = a(1-\cos\theta)$

$$\therefore \quad \frac{dx}{d\theta} = a(1+\cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{2a\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{a\times 2\cos^2\left(\frac{\theta}{2}\right)} = \tan(\theta/2)$$

5. (a)

Putting

$$f'(x) = 6x^2 - 6x - 36 = 0$$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow \quad x = 3 \text{ or } -2$

Now $\quad f''(x) = 12x - 6$

and $\quad f''(3) = 30 > 0$ (minima)

and $\quad f''(-2) = -30 < 0$ (maxima)

Hence maxima is at $x = -2$ only.

6. (a)

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2\sin\phi\cdot dr\cdot d\phi\cdot d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{r^3}{3}\right]_0^1\sin\phi\ d\phi\ d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}[-\cos\phi]_0^{\pi/3}d\theta$$

$$= \frac{1}{3}\times\frac{1}{2}\times\int_0^{2\pi}d\theta = \frac{1}{3}\times\frac{1}{2}\times 2\pi = \frac{\pi}{3}$$

7. (c)

$$\vec{P} = 0.866\hat{i} + 0.500\hat{j} + 0\hat{k}$$

$$\vec{Q} = 0.259\hat{i} + 0.966\hat{j} + 0\hat{k}$$

$$\therefore \quad \vec{P}.\vec{Q} = |\vec{P}||\vec{Q}|\cos\theta$$

Here, $\quad |P| = |Q| = 1 \quad$ (unit magnitude)

So, $(0.866\hat{i} + 0.5\hat{j} + 0\hat{k})\cdot(0.259\hat{i} + 0.966\hat{j} + 0\hat{k})$

$$= \sqrt{(0.866)^2+(0.5)^2}\times\sqrt{(0.259)^2+(0.966)^2}\cdot\cos\theta$$

$$\therefore \quad \cos\theta = \frac{0.866 \times 0.259 + 0.5 \times 0.966}{\sqrt{1} \times \sqrt{1}}$$

$$= 0.707$$

$$\therefore \quad \theta = 45^\circ$$

8. (d)

Since the position of rail engine $S(t)$ is continuous and differentiable function, according to Lagranges mean value theorem

At t where $0 \le t \le 8$ such that

$$S'(t) = v(t) = \frac{S(8) - S(0)}{8 - 0} = \frac{(280 - 0)}{(8 - 0)} \text{ m/sec}$$

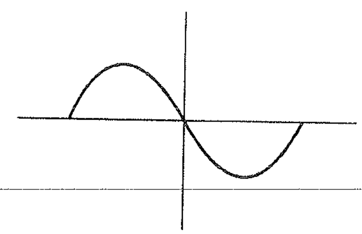
$$= \frac{280}{8} \text{ m/sec}$$

$$= \frac{280}{8} \times \frac{3600}{1000} \text{ kmph} = 126 \text{ kmph}$$

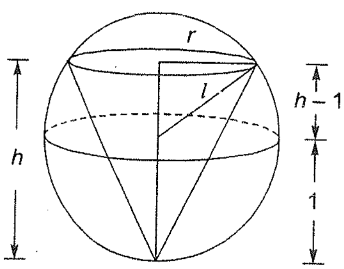
where $v(t)$ is the velocity of the rail engine.

9. (c)

Given function has negative slope in +ve half and +ve slope in –ve half. So its differentiation curve is satisfied by (c).



10. (d)



$$r^2 + (h - 1)^2 = 1^2$$

$$r^2 + h^2 - 2h + 1 = 1$$

$$r^2 = 2h - h^2$$

Volume of the cone, $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{\pi}{3}(2h - h^2)h = \frac{\pi}{3}(2h^2 - h^3)$$

$$\frac{dV}{dh} = \frac{\pi}{3}(4h - 3h^2)$$

$$\frac{dV}{dh} = 0 \quad \text{for minima and maxima}$$

$$4h - 3h^2 = 0$$

$$h(4 - 3h) = 0$$

$$h = \frac{4}{3}, 0$$

$$V'' = \frac{\pi}{3}(4 - 6h)$$

$$h = 0\ ;\ V'' = \frac{4\pi}{3} > 0 \text{ minima}$$

$$h = \frac{4}{3}\ ;\ V'' = -\frac{4\pi}{3} < 0 \text{ maxima}$$

$\therefore$ Volume is maximum when $x = \frac{4}{3}$

11. (a)

$$f(x) = x^2 e^{-x}$$

$$f'(x) = x^2(-e^{-x}) + e^{-x} \times 2x$$

$$= e^{-x}(2x - x^2)$$

Putting $f'(x) = 0$

$$e^{-x}(2x - x^2) = 0$$

$$e^{-x}x(2 - x) = 0$$

$x = 0$ or $x = 2$ are the stationary points.

Now, $f''(x) = e^{-x}(2 - 2x) + (2x - x^2)(-e^{-x})$

$$= e^{-x}(2 - 2x - (2x - x^2))$$

$$= e^{-x}(x^2 - 4x + 2)$$

at $x = 0, f''(0) = e^{-0}(0 - 0 + 2) = 2$

Since $f''(x) = 2$ is > 0 at $x = 0$ we have a minima.

Now at $x = 2 f''(2) = e^{-2}(2^2 - 4 \times 2 + 2)$

$$= e^{-2}(4 - 8 + 2) = -2\, e^{-2} < 0$$

$\therefore$ at $x = 2$ we have a maxima.

12. (c)

$$S = \int_1^\infty x^{-3} dx$$

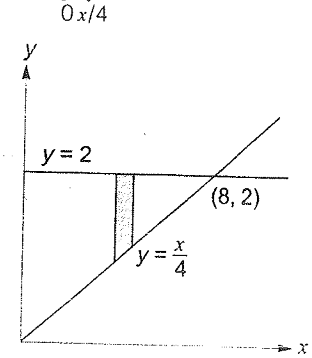
$$= \left[\frac{x^{-2}}{-2}\right]_1^\infty = -\left[\frac{1}{2x^2}\right]_1^\infty$$

$$= -\left[\frac{1}{\infty} - \frac{1}{2}\right] = \frac{1}{2}$$

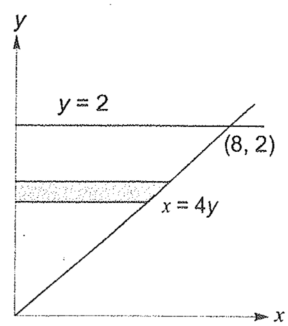
13. (a)

When $\quad I = \int_0^8 \int_{x/4}^2 f(x \cdot y)\, dy\, dx$

i.e.



Now,



$$I = \int_0^2 \int_0^{4y} f(x,y)\,dx\,dy$$

$\therefore \quad q = 4y$

14. (a)

$$\frac{\partial x}{\partial u} = v \quad \frac{\partial x}{\partial v} = u$$

and $\quad \dfrac{\partial y}{\partial u} = -\dfrac{v}{u^2}$

$$\frac{\partial y}{\partial v} = \frac{1}{u}$$

and $\quad \phi(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} v & u \\ -\dfrac{v}{u^2} & \dfrac{1}{u} \end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}$$

15. (c)

$$u = \frac{x^2}{2} + \frac{y^2}{3}$$

$$\text{grad } u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} = xi + \frac{2}{3}yj$$

$$\text{At}(1,3), \text{grad } u = (1)j + \left(\frac{2}{3}\cdot 3\right)j = i + 2j$$

$$|\text{grad } u| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

16. (a)

$$f_x = 2xyz,\ f_y = x^2z,\ f_z = x^2y$$

By integraning, we get

$$f = \text{Potential function of } \vec{V} = x^2yz$$

$\therefore$ line integral of the vector function from point A (0, 0, 0) to the point B(1, 1, 1) is

$$= f(B) - f(A)$$

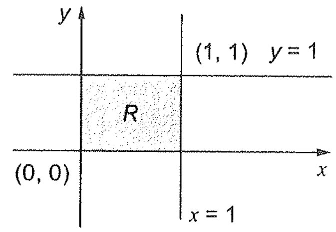
$$= (x^2yz)_{1,1,1} - (x^2yz)_{(0,0,0)}$$

$$= 1 - 0 = 1$$

17. (c)

Green's Theorem is

$$\oint_c \phi\,dx + \psi\,dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$$



Here $\quad I = \oint_c (xydy - y^2dx)$

$$= \oint_c (-y^2)dx + (xy)dy$$

$$\therefore \quad \phi = -y^2,\ \psi = xy$$

$$\frac{\partial \psi}{\partial x} = y,\ \frac{\partial \phi}{\partial y} = -2y$$

Substituting in Green's theorem, we get,

$$I = \int_{y=0}^{1} \int_{x=0}^{1} [y - (-2y)]\,dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} 3y\,dx\,dy$$

$$= \int_{y=0}^{1} [3xy]_{x=0}^{1}\,dy = \int_{y=0}^{1} 3ydy = \frac{3}{2}$$

18. (a)

A line integral and a surface integral is related by Stoke's theorem.

19. (b)

$$\underset{x \to 3}{\text{Lt}} f(x) = \underset{x \to 3}{\text{Lt}} \left(\frac{2x^2 - 7x + 3}{5x^2 - 12x - 9}\right)$$

Here this is of the form of $\left(\dfrac{0}{0}\right)$

So, applying L-Hospital's rule

$$\underset{x \to 3}{\text{Lt}} \left(\frac{4x - 7}{10x - 12}\right) = \frac{5}{18}$$

20. (a)

$$f'(x) = \frac{e^x(1+e^x) - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

since e^x is +ve for all values of x, $f'(x)$ is +ve for all values of x and hence $f(x)$ monotonically increases.

21. (a)

$$I = \int_0^{\pi/3} e^{it}\,dt = \left[\frac{e^{it}}{i}\right]_0^{\pi/3}$$

$$= \left[\frac{\cos t + i\sin t}{i}\right]_0^{\pi/3} = \frac{1}{i}\left[\frac{1}{2} + i\frac{\sqrt{3}}{2} - 1\right]$$

$$= \left[-\frac{1}{2i} + \frac{\sqrt{3}}{2}\right] = \left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right]$$

22. (c)

$$I = \int_0^{\pi} \sin^3\theta \cdot d\theta$$

$$\int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta\, d\theta,$$

Let $\cos\theta = t$

$-\sin\theta\, d\theta = dt,$

at $\theta = 0,\ t = \cos 0 = 1$

at $\theta = \pi,\ t = \cos\pi = -1$

So, $$I = -\int_{-1}^{+1}\left(1-t^2\right)dt$$

$$= \left|t - \frac{t^3}{3}\right|_{-1}^{1} = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)$$

$$I = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

23. (d)

Area common to circles,

$r = a$

and $r = 2a\cos\theta$ is $1.228a^2$

24. (d)

We consider options (a) and (d) only, because these contains variable r, as variable of integration. By integrating (d), we get

$1/3\,\pi\, a^2H$, which is volume of a cone.

25. (d)

$x + 1 = \sqrt{2}\cos\theta$; $y - 1 = \sqrt{2}\sin\theta$

$x = \sqrt{2}\cos\theta - 1$; $y = \sqrt{2}\sin\theta + 1$

$$= \int_0^{2\pi}(2\sqrt{2}\cos\theta - 2 + 5\sqrt{2}\sin\theta + 5 - 3)d\theta$$

$$= \int_0^{2\pi}(2\sqrt{2}\cos\theta + 5\sqrt{2}\sin\theta)d\theta$$

$$= 2\sqrt{2}(\sin\theta)\Big|_0^{2\pi} + 5\sqrt{2}(-\cos\theta)\Big|_0^{2\pi}$$

$$= 2\sqrt{2}\,(\sin 2\pi - \sin 0) - 5\sqrt{2}(\cos 2\pi - \cos 0)$$

$$= 2\sqrt{2}(0-0) - 5\sqrt{2}(1-1) = 0$$

26. (c)

$f = 2x^2 + 3y^2 + z^2$, $P(2, 1, 3)$, $a = i - 2k$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 4xi + 6yj + 2zk$$

at $P(2, 1, 3)\nabla f = 4 \times 2 \times i + 6 \times 1 \times j + 2 \times 3 \times k$

$= 8i + 6j + 6k$

directional derivative of f in direction of vector

$a = i - 2k$ is

nothing but the component of gradf in the direction of vector a and is given by $\frac{a}{|a|} \cdot \text{grad } f$

$$= \left(\frac{i - 2k}{\sqrt{1^2 + (-2)^2}}\right) \cdot (8i + 6j + 6k)$$

$$= \frac{1}{\sqrt{5}}(1.8 + 0.6 + (-2)6)$$

$$= \frac{-4}{\sqrt{5}} = -1.789$$

27. (b)

Given $f(x) = (x-8)^{2/3} + 1$

$$f'(x) = \frac{2}{3}(x-8)^{-1/3}$$

Slope of tangent at point (0, 5)

$$m = \frac{2}{3}(0-8)^{-1/3} = -\frac{1}{3}$$

Slope of normal at point (0, 5)

$$m_1 = -\frac{1}{m} = 3$$

Equation of normal at point (0, 5)

$y - 5 = 3(x - 0)$

$\Rightarrow$ $y = 3x + 5$

28. (d)

From property of vector triple product.

$A \times (B \times C) = (A \cdot C)\,B - (A \cdot B)C$

and putting, $A = \nabla$, $B = \nabla$ & $C = P$

We get, $\nabla \times \nabla \times P = (\nabla \cdot P)\nabla - (\nabla \cdot \nabla)P$

$= \nabla(\nabla \cdot P) - \nabla^2 P$

29. (a)

$$\iint (\nabla \times P) \cdot ds = \oint P \cdot dl \quad \text{(stokes Theorem)}$$

30. (b)

$$\underset{x \to 0}{\text{Lt}} \frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3}$$

This is of the form of $\left(\frac{0}{0}\right)$

Applying L' Hospital rule

$$\underset{x \to 0}{\text{Lt}} \frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3} \left(\frac{0}{0}\right)$$

$$\underset{x \to 0}{\text{Lt}} \frac{e^x - 1}{6x}$$

$$\underset{x \to 0}{\text{Lt}} \frac{e^x}{6} = \frac{1}{6}$$

31. (a)

$$\lim_{\theta \to 0} \frac{\frac{1}{2} \times \sin\left(\frac{\theta}{2}\right)}{\theta \times \frac{1}{2}} = \frac{1}{2} \lim_{\theta \to 0} \frac{\sin \theta/2}{\theta/2} = \frac{1}{2} = 0.5$$

32. (b)

Given, $y = x^2$

$\Rightarrow \quad \dfrac{dy}{dx} = 2x = 0$ at $x = 0$

$\Rightarrow \quad \left.\dfrac{d^2y}{dx^2}\right]_{x=0} = 2$ which is +ve

so we have a local minima at $x = 0$

at $x = 0, \; y = 0$

but since $x = 0 \notin [1, 5]$

it is not a candidate for minima or maxima in that range

At the end point $x = 1$

$y = 1$

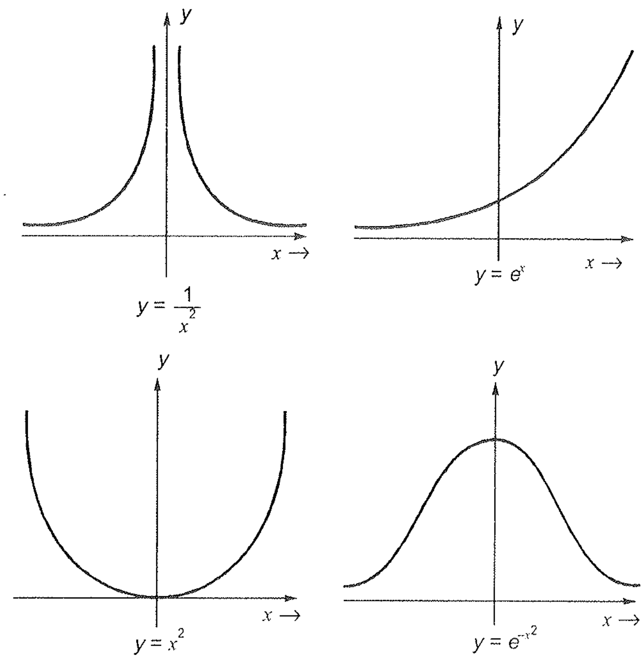
at second end point $x = 5$

$y = 25$

So, absolute minimum value of function in [1, 5] is 1.

33. (d)

From the graphs below, we can see that only e^{-x^2} is strictly bounded



34. (a)

$$f(x) = x^2 - x - 2 = (x + 1)(x - 2)$$

$$f'(x) = 2x - 1 = 0$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$f''(x) = 2$$

$$f''\left(\frac{1}{2}\right) = 2 \quad > 0$$

So at $x = \dfrac{1}{2}$,

we have a local minima so this is not a candidate for maxima in range [–4, 4].

Now $f(-4) = 18$

$f(+4) = 10$

so maximum value in range [–4, 4] is 18.

35. (a)

The Taylor's series expansion of $f(x)$ allowed $x = 2$ is

$$f(x) = f(2) + (x - 2) f'(2) + \frac{(x-2)^2}{2!} f''(2) \ldots$$

For linear approximation we take only the first two terms and get

$$f(x) = f(2) + (x - 2) f'(2)$$

Here, $f(x) = e^{-x}$ and $f'(x) = -e^{-x}$

$\therefore \quad f(x) = e^{-2} + (x - 2)(-e^{-2}) = (3 - x) e^{-2}$

36. (b)

Equation of line with slope 1 and y- intercept of 1 is,

$$y = x + 1$$

$$I = \int_1^2 y\,dx = \int_1^2 (x+1)\,dx$$

$$= \left.\frac{(x+1)^2}{2}\right|_1^2 = \frac{1}{2}(9-4) = 2.5$$

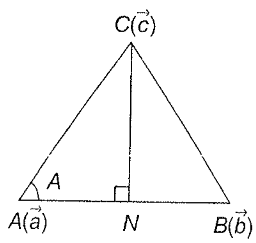
37. (b)

From C, draw $CN \perp AB$. From right-angled ΔCAN,

$$\sin A = \frac{|CN|}{|AC|} \Rightarrow |CN| = |AC| \sin A.$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|AB| \times |CN|$$

$$= \frac{1}{2}|AB| \cdot |AC| \sin A = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$



From above figure, $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$.

$$\text{So, Area of } \Delta ABC = \frac{1}{2}|(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})|$$

$$= \frac{1}{2}|(\vec{a}-\vec{b}) \times (\vec{a}-\vec{c})|$$

Choice (b) is correct.

38. (b)

Let $D = \begin{vmatrix} x \cdot x & x \cdot y \\ y \cdot x & y \cdot y \end{vmatrix}$

Let $x = x_1 i + x_2 j$

$y = y_1 i + y_2 j$

$x \cdot x = x_1^2 + x_2^2$

$y \cdot y = y_1^2 + y_2^2$

$x \cdot y = x_1 x_2 + y_1 y_2$

$\therefore \quad D = \begin{vmatrix} x_1^2 + x_2^2 & x_1 x_2 + y_1 y_2 \\ x_1 x_2 + y_1 y_2 & y_1^2 + y_2^2 \end{vmatrix}$

$= (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1 x_2 + y_1 y_2)^2$

$= x_2^2 y_1^2 + x_1^2 y_2^2 - 2x_1 y_1 x_2 y_2$

$= (x_2 y_1 - x_1 y_2)^2$

Now, $D = 0$

$x_2 y_1 - x_1 y_2 = 0$

$\Rightarrow \quad \dfrac{x_1}{x_2} = \dfrac{y_1}{y_2}$

$\Rightarrow$ Vector $x_1 i + x_2 j$ and $y_1 i + y_2 j$ are linearly dependent.

$\therefore$ Linear dependence $\Rightarrow D = 0$

So, Linear independence $\Rightarrow D \neq 0$

i.e. is negetive or positive.

However, [notice that here since $D = (x_2 y_1 - x_1 y_2)^2$, it cannot be negative].

So, Linear independence $\Rightarrow D$ is positive.

39. (d)

$\vec{V} = 5xyi + 2y^2 j + 3yz^2 k = v_1 i + v_2 j + v_3 k$

$\text{div}(\vec{V}) = \dfrac{dv_1}{dx} + \dfrac{dv_2}{dy} + \dfrac{dv_3}{dz} = 5y + 4y + 6yz$

at (1, 1, 1) $\text{div}(\vec{V}) = 5.1 + 4.1 + 6.1.1 = 15$

40. (a)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$2x = \frac{\partial \psi}{\partial y}$$

$$\psi = 2x + y + c$$

$$\psi|_{(0,0)} = 0 + c = 0$$

and $\psi = 2xy$

41. (b)

$(x - 8) = h$ (say)

$\Rightarrow \quad x = 8 + h$

$$\therefore \lim_{h \to 0} \frac{(8+h)^{1/3} - 2}{h}$$

Above form in the $\left(\frac{0}{0}\right)$ by putting the value $h = 0$

Applying L' Hospital rule

$$\lim_{h \to 0} \frac{\frac{1}{3}(8+h)^{\left(\frac{1}{3}-1\right)}}{1} = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{12}$$

42. (a)

$$\lim_{x \to \infty} \frac{x - \sin x}{x + \cos x} = \lim_{x \to \infty} \frac{1 - \sin x / x}{1 + \cos x / x}$$

$$= \frac{\lim_{x \to \infty} (1 - \sin x/x)}{\lim_{x \to \infty} (1 + \cos x/x)}$$

$$= \frac{1 - \lim_{x \to \infty} \frac{\sin x}{x}}{1 + \lim_{x \to \infty} \frac{\cos x}{x}} = \frac{1-0}{1+0} = 1$$

43. (b)

$$f(x) = (x^2-4)^2$$
$$f'(x) = 2(x^2-4)\times 2x = 4x(x^2-4) = 0$$

$x = 0, x = 2$ and $x = -2$ are the stationary pts.

$$f''(x) = 4[x(2x)+(x^2-4)\times 1]$$
$$= 4[2x^2+x^2-4] = 4[3x^2-4]$$
$$= 12x^2-16$$
$$f''(0) = -16 < 0 \quad (\text{so maxima at } x = 0)$$
$$f''(2) = (12)2^2-16 = 32 > 0 \quad (\text{so minima at } x = 2)$$
$$f''(-2) = 12(-2)^2-16 = 32 > 0 \quad (\text{so minima at } x = -2)$$

$\therefore$ There is only one maxima and only two minima for this function.

44. (d)

$$y = 3x^4-16x^3-24x^2+37$$
$$\frac{dy}{dx} = 12x^3-48x^2-48x = 0$$
$$x(12x^2-48x-48) = 0$$
$$x = 0$$
or $12x^2-48x-48 = 0$
$$x^2-4x-4 = 0$$
$$x = \frac{4\pm\sqrt{16+16}}{2}$$
$$= \frac{4\pm\sqrt{32}}{2} = \frac{4\pm 4\sqrt{2}}{2} = 1\pm\sqrt{2}$$
$$\frac{d^2y}{dx^2} = 36x^2-96x-48$$

Now at $x = 0$

$$\frac{d^2y}{dx^2} = -48 \neq 0$$

at $1\pm\sqrt{2}$ also $\frac{d^2y}{dx^2} \neq 0$

$\therefore$ There are 3 extrema in this function.

45. (c)

$f(x)$ in the neighbourhood of a is,

$$f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$$

where, $b_n = \frac{f^n(a)}{n!}$

$$f^{4\prime}(x) = e^x;\ f^{4\prime}(2) = e^2$$

$\therefore$Coefficient of $(x-2)^4 = b_4 = \frac{f^4(2)}{4!} = \frac{e^2}{4!}$

46. (a)

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \ldots$$
$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots$$

From this, $\sin x^2 = x^2 - \frac{x^6}{\lfloor 3} + \frac{x^{10}}{\lfloor 5} - \frac{x^{14}}{\lfloor 7}$

$$\cos x^2 = 1 - \frac{x^4}{\lfloor 2} + \frac{x^8}{\lfloor 4} - \frac{x^{12}}{\lfloor 6}$$

So, $\sin x^2$ and $\cos x^2$ have only even powers of x

Similarly, $\sin x^3 = x^3 - \frac{x^9}{\lfloor 3} + \frac{x^{15}}{\lfloor 5} \ldots$

$$\cos x^3 = 1 - \frac{x^6}{\lfloor 2} + \frac{x^{12}}{\lfloor 4} \ldots$$

So, only $\sin(x^3)$ has all odd powers of x.

$\therefore$ correct choice is (a).

47. (b)

$$f(x) = e^x + \sin x$$

We wish to expand about $x = \pi$

Taylor's series expansion about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)\ldots$$

Now about $x = \pi$

$$f(x) = f(\pi) + (x-\pi)f'(\pi) + \frac{(x-\pi)^2}{2!}f''(\pi) + \ldots$$

The coefficient of $(x-\pi)^2$ is $\frac{f''(\pi)}{2!}$

Here $f(x) = e^x + \sin x$
$$f'(x) = e^x + \cos x$$
$$f''(x) = e^x - \sin x$$
$$f''(\pi) = e^\pi - \sin\pi = e^\pi - 0 = e^\pi$$

The coefficient of $(x-\pi)^2$ is therefore

$$\frac{e^\pi}{2!} = 0.5\exp(\pi)$$

48. (c)

$$f = y^x$$

Treating x as constant, we get

$$\frac{\partial t}{\partial y} = xy^{x-1}$$

Now we treat y as a constant and get,

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(y^{x-1}x\right) = y^{x-1} + xy^{x-1}\ln y$$

whose value at $x = 2$ and $y = 1$ is $= 1^{(2-1)}(1 + 2\,ln 1) = 1$.

49. (d)

Choice (a) $\int_0^{\frac{\pi}{4}} \tan x\,dx = \log\sqrt{2}$

Choice (b) $\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$

Choice (c) $\int_0^{\infty} x\,e^{-x}dx$

Integrating by parts, taking $u = x$ and $dv = e^{-x}\,dx$ we get $du = dx$ and $v = -e^{-x}$

So, $\int x e^{-x} dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} - e^{-x}$

$= -e^{-x}(x+1)$

Now $\int_0^{\infty} x\,e^{-x}dx = [-e^{-x}(x+1)]_0^{\infty} = 1$

Choice (d) $\int_0^1 \frac{1}{1-x}dx = \ln 0 - \ln 1 = -\infty - 0 = -\infty$

Since, only (d) is unbounded, (d) is the answer.

50. (d)

$$y = \frac{2}{3}x^{3/2}$$

$$\frac{dy}{dx} = x^{1/2}$$

length of the curve is given by

$$\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_{x=0}^{x=1} = 1.22$$

51. (a)

Equation of straight line from point (0,0) to (1,2) is

$$y - 0 = \frac{(2-0)}{(1-0)}(x-0)$$

or $\quad y = 2x$

$$g(x, y) = 4x^3 + 10y^4$$
$$= 4x^3 + 10(2x)^4 = 4x^3 + 160x^4$$

$$\int_0^1 \left(4x^3 + 160x^4\right)dx = \left(\frac{4x^4}{4} + \frac{160x^5}{5}\right)\Bigg|_0^1$$

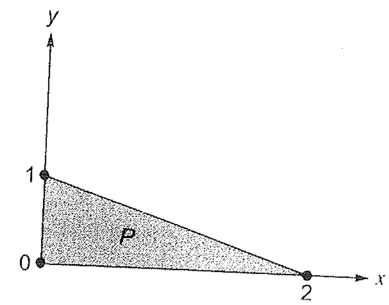
$$= 1 + 32 = 33$$

52. (a)

$$\int_0^3\int_0^x (6-x-y)dx\,dy = \int_0^3 \left[(6-x)y - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^3 \left[(6-x)x - \frac{x^2}{2}\right]dx = 13.5$$

53. (a)



The equation of the straight line with x - intercept = 2 and y-intercept = 1 is

$$\frac{x}{2} + \frac{y}{1} = 1$$

$$\Rightarrow \quad y = 1 - \frac{x}{2}$$

$$\Rightarrow \quad x = 2 - 2y$$

$$\int_0^1 \int_0^{(2-2y)} (xy\,dx)dy = \int_0^1 \left[\left(\frac{yx^2}{2}\right)\Bigg|_0^{2-2y}\right]dy$$

$$= \int_0^1 \frac{y}{2}(2-2y)^2\,dy$$

$$= \int_0^1 2y(1-y)^2\,dy = \frac{1}{6}$$

Alternatively, we may also write this integral as

$\int_0^2 \int_0^{\frac{2-x}{2}} (xy\,dy)dx$ which is also $= \frac{1}{6}$

54. (c)

$$\vec{P}\cdot\vec{Q} = 0$$

$$\vec{P}\cdot\vec{Q} = |P|\;|Q|\cos\theta$$

if $\quad \vec{P}\cdot\vec{Q} = 0$

$\Rightarrow |P|\;|Q|\cos\theta = 0$

Since, P and Q are non-zero vectors

$\Rightarrow \quad \cos\theta = 0$

$\Rightarrow \quad \theta = 90°$

55. (d)

$\text{div}\{(x-y)\hat{i} + (y-x)\hat{j} + (x+y+z)\hat{k}\}$

$$= \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-x) + \frac{\partial}{\partial z}(x+y+z) = 3$$

56. (b)

$$\text{grad } f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = 2xi + 4yj + k$$

at point $P(1, 1, 2)$, grad $f = 2i + 4j + k$

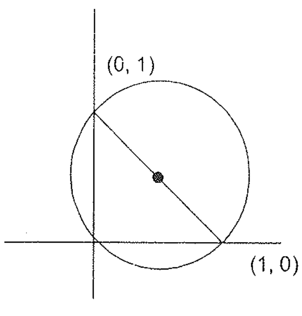
Now directional derivative of f at $P(1, 1, 2)$ in direction of vector $a = 3i - 4j$ is given by

$$\frac{a}{|a|}\text{ grad } f = \left(\frac{3i-4j}{\sqrt{25}}\right)\cdot(2i+4j+k)$$

$$= \frac{1}{5}(3\times 2 - 4\times 4 + 0) = -2$$

57. (b)

Taking $f(x, y) = xy$, we can show that, $xdx + ydy$, is exact. So, the value of the integral is independent of path



$$= 2\int_P^Q (x\,dx + y\,dx)$$

$$= 2\int_1^0 x\,dx + 2\int_0^1 y\,dy$$

$$= 2\left[\left.\frac{x^2}{2}\right|_1^0 + \left.\frac{y^2}{2}\right|_0^1\right] = 0$$

or Integral $= f(Q) - f(P)$

$$= [xy]_{(0,1)} - [xy]_{(1,0)} = 0 - 0 = 0$$

58. (a)

Let the point be (x, y, z) on surface $z^2 = 1 + xy$

Distance from origin $= l$

$$= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$l = \sqrt{x^2 + y^2 + 1 + xy}$$

[since $z^2 = 1 + xy$ is given]

This distance is shortest when l is minimum we need to find minima of $x^2 + y^2 + 1 + xy$

Let $u = x^2 + y^2 + 1 + xy$

$$\frac{\partial u}{\partial x} = 2x + y$$

$$\frac{\partial u}{\partial y} = 2y + x$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

$\Rightarrow$ $2x + y = 0$ and $2y + x = 0$

Solving simultaneously, we get

$x = 0$ and $y = 0$

is the only solution and so (0, 0) is the only stationary point.

Now, $r = \frac{\partial^2 u}{\partial x^2} = 2$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2$$

Since $rt = 2 \times 2 = 4 > s^2 = 1$

We have case 1, i.e. either a maximum or minimum exists at (0, 0)

Now, since $r = 2 > 0$, so it is a minima at (0, 0).

Now at $x = 0$, $y = 0$,

$$z = \sqrt{1+xy} = \sqrt{1+0} = 1$$

So, the point nearest to the origin on surface

$z^2 = 1 + xy$ is (0, 0, 1)

The distance $l = \sqrt{0^2 + 0^2 + 1^2} = 1$

So, correct answer is choice (a).

59. (c)

An n^{th} degree polynomial bends exactly $n - 1$ times and therefore can have a maximum of $n - 1$ extremas. Also an n^{th} degree polynomial has at most n roots (zero crossings). So a cubic polynomial (degree 3) cannot have more than 2 extrema and cannot have more than 3 zero crossings.

60. (b)

Let, $x - \pi = t$

$x = \pi + t$

$$f(t) = \frac{-\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)}{t}$$

$$f(t) = -\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots\right)$$

$$f(t) = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \cdots$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + \cdots$$

61. (d)

Since, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore \quad I = \int_0^{\pi/4} \frac{1-\tan x}{1+\tan x}\,dx$$

$$= \int_0^{\pi/4} \frac{1-\tan\left(\frac{\pi}{4}-x\right)}{1+\tan\left(\frac{\pi}{4}-x\right)}\,dx$$

Since, $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$

$$\therefore \quad I = \int_0^{\pi/4} \frac{1-\left[\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]}{1+\left[\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]}\,dx$$

$$= \int_0^{\pi/4} \frac{1-\left[\dfrac{1-\tan x}{1+\tan x}\right]}{1+\left[\dfrac{1-\tan x}{1+\tan x}\right]}\,dx$$

$$= \int_0^{\pi/4} \frac{(1+\tan x)-(1-\tan x)}{(1+\tan x)+(1-\tan x)}\,dx$$

$$= \int_0^{\pi/4} \frac{2\tan x}{2}\,dx = \int_0^{\pi/4} \tan x\,dx$$

$$= [\log(\sec x)]_0^{\pi/4}$$

$$= \ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec 0)$$

$$= \ln(\sqrt{2}) - \ln(1) = \ln(2^{1/2}) - 0$$

$$= \frac{1}{2}\ln 2$$

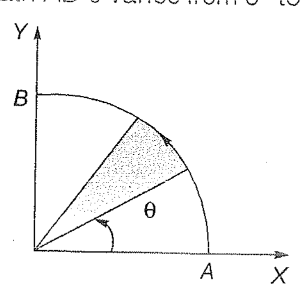
62. (b)

Path $AB : x^2 + y^2 = 1$

$x = \cos\theta$

$y = \sin\theta$

Along path AB θ varies from 0° to 90° [0 to $\pi/2$]



$$\int_{\text{Path } AB} (x+y)^2\,(r\,d\theta) = \int_0^{\pi/2} (\cos\theta + \sin\theta)^2\, 1\cdot d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} (1+\sin 2\theta)\,d\theta$$

$$= \theta + \left.\frac{(-\cos 2\theta)}{2}\right|_0^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{1}{2}\left[\cos 2\frac{\pi}{2} - \cos 0\right]$$

$$= \frac{\pi}{2} - \frac{1}{2}[-1-1] = \frac{\pi}{2} + 1$$

63. (a)

Curve 1 : $y^2 = 4x$

Curve 2 : $x^2 = 4y$

Intersection points of curve 1 and 2

$$y^2 = 4x = 4\sqrt{4y} = 8\sqrt{y}$$

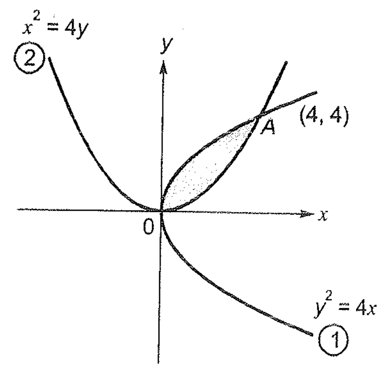
$$y^4 = 8 \times 8\,y \Rightarrow y(y^3 - 64) = 0$$

Solution $y = 4$ and $y = 0$

then $x = 4$ $x = 0$

Therefore intersection point are $A(4, 4)$ and $0(0, 0)$

The area enclosed between curves 1 and 2 are given by



$$\text{Area} = \int_{x_1}^{x_2} y_1\,dx - \int_{x_1}^{x_2} y_2\,dx$$

$$= -\int_0^4 \frac{x^2}{4}\,dx$$

$$= 2\left.\frac{x^{3/2}}{3/2}\right|_0^4 - \left.\frac{x^3}{3\times 4}\right|_0^4$$

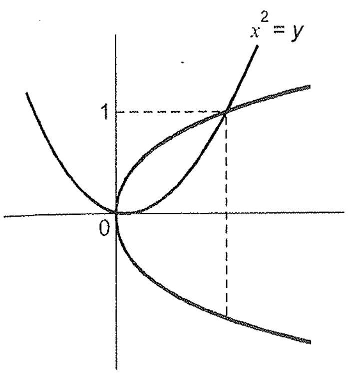
$$= \frac{4}{3}(4)^{3/2} - \frac{(4)^3}{3\times 4} = \frac{16}{3}$$

Alternately, the same answer could have been obtained by taking a double integral as follows:

$$\text{Required Area} = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\,dy$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \frac{16}{3}$$

64. (a)



$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} f(x, y)\,dx\,dy$$

65. (b)

$$f = x^2 + 3y^2 + 2z^2$$

$$\Delta f = \text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2x) + j(6y) + k(4z)$$

The gradient at $P(1, 2, -1)$ is

$$= i(2 \times 1) + j(6 \times 2) + k(4 \times -1)$$

$$= 2i + 12j - 4k$$

66. (b)

$$\Delta f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

here $\quad f = x^2 + 3y^2 + 2z^2$

$\therefore \quad \Delta f = i(2x) + j(6y) + k(4z)$

at $p(1, 2, -1)\ \Delta f = i(2 \times 1) + j(6 \times 2) + k(4 \times -1)$

$= 2i + 12j - 4k$

The directional derivative in direction of vector a $= i - j + 2k$ is given by

$$\frac{a}{|a|} \cdot \text{grad } f = \frac{i - j + 2k}{\sqrt{1^2 + (-1)^2 + 2^2}} \cdot (2i + 12j - 4k)$$

$$= \frac{1}{\sqrt{6}}(1.2 + (-1) \cdot 12 + 2(-4))$$

$$= -\frac{18}{\sqrt{6}} = -3\sqrt{6}$$

67. (c)

Vector field,

$$\vec{f} = 3xz\,\hat{i} + 2xy\,\hat{j} - yz^2\,\hat{k}$$

$$= v_1 i + v_2 j + v_3 k$$

Divergence of vector field

$$\text{Div}(f) = \nabla \cdot f = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \frac{\partial}{\partial x}[3xz] + \frac{\partial}{\partial y}[2xy] + \frac{\partial}{\partial z}\left[-2yz^2\right]$$

$$= 3z + 2x - 2zy$$

$$\text{Div}(f)\big|_{(1,1,1)} = 3(1) + 2(1) - 2(1)(1) = 3$$

68. (a)

$$\lim_{x \to 0} \frac{\sin\left(\frac{2}{3}x\right)}{x} = \lim_{\frac{2}{3}x \to 0} \frac{\sin\left(\frac{2}{3}x\right)}{\frac{2}{3}x} \cdot \frac{2}{3}$$

$$= (1)\left(\frac{2}{3}\right) = \frac{2}{3}$$

69. (b)

$$\lim_{n \to \infty}\left(1 - \frac{1}{n}\right)^{2n} = \lim_{n \to \infty}\left[\left(1 - \frac{1}{n}\right)^{n}\right]^{2}$$

$$= \left[\lim_{n \to \infty}\left(1 - \frac{1}{n}\right)^{-n}\right]^{-2}$$

$$= e^{-2}$$

70. (c)

$$y = |2 - 3x| = 2 - 3x \quad 2 - 3x \geq 0$$

$$= 3x - 2 \quad 2 - 3x < 0$$

Therefore, $\quad y = 2 - 3x \quad x \leq \frac{2}{3}$

$$= 3x - 2 \quad x > \frac{2}{3}$$

Since $2 - 3x$ and $3x - 2$ are polynomials, these are continuous at all points. The only concern is at $x = \frac{2}{3}$

Left limit at $x = \frac{2}{3}$ is $2 - 3 \times \frac{2}{3} = 0$.

Right limit at $x = \frac{2}{3}$ is $3 \times \frac{2}{3} - 2 = 0$.

$$f\left(\frac{2}{3}\right) = 2 - 3 \times \frac{2}{3} = 0$$

Since, Left limit = Right limit = $f\left(\frac{2}{3}\right)$,

Function is continuous at $\frac{2}{3}$.

y is therefore continuous $\forall\, x \in R$

Now since $2 - 3x$ and $3x - 2$ are polynomials, they are differentiable.

only concern is at $x = \frac{2}{3}$.

Now, at $x = \frac{2}{3}$, LD = Left derivative $= -3$

RD = Right derivative $= +3$

LD $\neq$ RD

$\therefore$ The function y is not differentiable at $x = \frac{2}{3}$

So, we can say that y is differentiable $\forall x \in R$, except at $x = \frac{2}{3}$.

71. (a)

$$f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$$

$$\frac{\partial f}{\partial x} = 8x - 8$$

$$\frac{\partial f}{\partial y} = 12y - 4$$

Putting, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$8x - 8 = 0 \text{ and } 12y - 4 = 0$$

Given, $x = 1$ and $y = \frac{1}{3}$

$\left(1, \frac{1}{3}\right)$ is the only stationary point.

$$r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8$$

$$s = \left[\frac{\partial^2 f}{\partial x dy}\right]_{\left(1, \frac{1}{3}\right)} = 0$$

$$t = \left[\frac{\partial^2 f}{\partial y^2}\right]_{\left(1, \frac{1}{3}\right)} = 12$$

Since, $rt = 8 \times 12 = 96$

$s^2 = 0$

Since, $rt > s^2$,

we have either a maxima or minima at $\left(1, \frac{1}{3}\right)$

also since, $r = \left[\frac{\partial f}{\partial x^2}\right]_{\left(1, \frac{1}{3}\right)} = 8 > 0$, the point $\left(1, \frac{1}{3}\right)$ is a point of minima.

The minimum value is

$$f\left(1, \frac{1}{3}\right) = 4 \times 1^2 + 6 \times \frac{1}{3^2} - 8 \times 1 - 4 \times \frac{1}{3} + 8$$

$$= \frac{10}{3}$$

So the optimal value of $f(x, y)$ is a minimum equal to $\frac{10}{3}$.

72. (d)

$$f(t) = \frac{\sin t}{t}$$

$$f(t) = \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots}{t}$$

$$f(t) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots$$

$$f'(t) = -\frac{2t}{3!} + \frac{4t^3}{5!} - \dots$$

$$f''(t) = -\frac{2t}{3!} + \frac{4t^3}{5!} - \dots$$

At $t = 0$, $f'(t) = 0$, $f''(t) < 0$

$\therefore$ $f(t)$ attains maxima.

73. (a)

$$e^y = x^{1/x}$$

Taking log on both sides,

$$y = \frac{1}{x} \log x$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{x} + \log x \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}(1 - \log x)$$

putting $\frac{dy}{dx} = 0$

$$\frac{1}{x^2}(1 - \log x) = 0$$

$\Rightarrow$ $\log x = 1$

$\Rightarrow$ $x = e$ is a stationary point

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} \times \left(-\frac{1}{x}\right) + (1 - \log x) \times \left(-\frac{2}{x^3}\right)$$

$$= -\frac{1}{x^3}[1 + 2(1 - \log x)] = -\frac{1}{x^3}(3 - 2\log x)$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=e} = -\frac{1}{e^3}(3 - 2\log e) = -\frac{1}{e^3} < 0$$

So, at $x = e$, we have a maximum.

74. (d)

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{-\infty}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} - \left[\frac{-\pi}{2}\right] = \pi$$

75. (b)

$$P = \int_0^1 x\, e^x\, dx\ dx$$

Integrating by parts:

Let $u = x,$

$dv = e^x\, dx$

$du = dx,$

$v = \int e^x\, dx = e^x$

Now, $\int u\, dv = uv - \int v\, du$

$\therefore \int x\, e^x\, dx = x\, e^x - \int e^x\, dx$

$= x\, e^x - e^x + c$

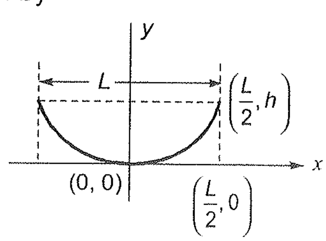
$$\int_0^1 x\, e^x\, dx = [x\, e^x - e^x]_0^1$$

$$= (1 \cdot e^1 - e^1) - (0 \cdot e^0 - e^0)$$

$$= 0 - (-1) = 1$$

76. (d)

Length of curve $y = f(x)$ between $x = a$ and $x = b$ is given by



$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

here, $y = 4h\frac{x^2}{L^2}$ ... (i)

$$\frac{dy}{dx} = 8h\frac{x}{L^2}$$

since, $y = 0$ at $x = 0$

and $y = h$ at $x = \frac{L}{2}$

(As can be seen from equation (i), by substituting $x = 0$ and $x = L/2$)

$$\therefore \frac{1}{2}(\text{Length of cable}) = \int_0^{L/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{L/2} \sqrt{1+\left(\frac{8hx}{L^2}\right)^2}\, dx$$

$$\text{Length of cable} = 2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}\, dx$$

77. (d)

The volume of a solid generated by revolution about the x-axis, of the area bounded by curve $y = f(x)$, the x-axis and the ordinates $x = a$, $y = b$ is

$$\text{Volume} = \int_a^b \pi y^2 dx$$

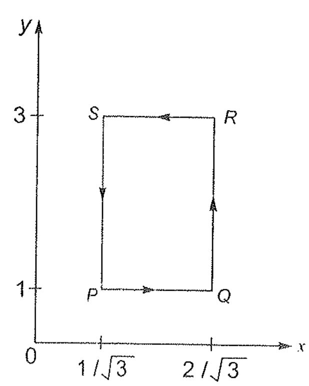
Here, $a = 1$, $b = 2$ and $y = \sqrt{x} \Rightarrow y^2 = x$

$$\therefore \quad \text{Volume} = \int_1^2 \pi \cdot x \cdot dx$$

$$= \pi \cdot \left[\frac{x^2}{2}\right]_1^2 = \frac{\pi}{2}\left[x^2\right]_1^2$$

$$= \frac{\pi}{2}\left[2^2 - 1^2\right] = \frac{3}{2}\pi$$

78. (c)



$$\vec{A} = xy\hat{a}_x + x^2\hat{a}_y$$

$$\vec{l} = x\hat{a}_x + y\hat{a}_y$$

$$\vec{dl} = dx\hat{a}_x + dy\hat{a}_y$$

$$\vec{A}\cdot\vec{dl} = xy\, dx + x^2\, dy$$

P – Q: $y = 1$, $dy = 0$

$$\int_P^Q \vec{A}\cdot\vec{dl} = \int_{1/\sqrt{3}}^{2/\sqrt{3}} x\, dx = \frac{x^2}{2}\Big|_{1/\sqrt{3}}^{2/\sqrt{3}} = \frac{1}{2}$$

$Q-R$: $x = \frac{2}{\sqrt{3}}$, $dx = 0$

$$\int_Q^R \vec{A}\cdot\vec{dl} = \int_1^3 \left(\frac{2}{\sqrt{3}}\right)^2 dy = \frac{4}{3}(3-1) = \frac{8}{3}$$

$R-S$: $y = 3$, $dy = 0$

$$\int_R^S \vec{A}\cdot\vec{dl} = \int_{2/\sqrt{3}}^{1/\sqrt{3}} 3x\,dx$$

$$= \frac{3}{2}x^2\Big|_{2/\sqrt{3}}^{1/\sqrt{3}} = \frac{3}{2}\left(\frac{1}{3}-\frac{4}{3}\right) = \frac{-3}{2}$$

S – P: $x = \frac{1}{\sqrt{3}}$, $dx = 0$

$$\int_S^P \vec{A}\cdot\vec{dl} = \int_3^1 \left(\frac{1}{\sqrt{3}}\right)^2 dy = \frac{1}{3}(1-3) = \frac{-2}{3}$$

So,

$$\oint_C \vec{A}\cdot\vec{dl} = \int_P^Q \vec{A}\cdot\vec{dl} + \int_Q^R \vec{A}\cdot\vec{dl} + \int_R^S \vec{A}\cdot\vec{dl} + \int_S^P \vec{A}\cdot\vec{dl}$$

$$= \frac{1}{2} + \frac{8}{3} - \frac{3}{2} - \frac{2}{3} = 1$$

79. (d)

Velocity vector = $\vec{V} = 2xy\hat{i} - x^2z\hat{j}$

The vorticity vector = curl (velocity vector)

$$= \text{curl }(\vec{V})$$

$$= \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -x^2z & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-x^2z)\right]\hat{i}$$

$$-\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2xy)\right]\hat{j}$$

$$+\left[\frac{\partial}{\partial x}(-x^2z) - \frac{\partial}{\partial y}(2xy)\right]\hat{k}$$

$$= x^2\hat{i} + [-2xz - 2x]\hat{k}$$

at (1, 1, 1), by substituting $x = 1$, $y = 1$ and $z = 1$, we get,vorticity vector = $\hat{i} - 4\hat{k}$

80. (a)

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{div } \vec{r} = \nabla . \vec{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\ \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

81. (d)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

82. (c)

If $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}} \frac{\lambda\cos x}{\frac{\pi}{2} - x} = f\left(\frac{\pi}{2}\right) = 1 \quad \text{...(i)}$$

Since the limit is in form of $\frac{0}{0}$, we can use L' hospitals rule on LHS of equation (i) and get

$$\lim_{x\to\frac{\pi}{2}} \frac{-\lambda\sin x}{-1} = 1$$

$$\Rightarrow \quad \lambda\sin\frac{\pi}{2} = 1$$

$$\Rightarrow \quad \lambda = 1$$

83. (c)

$$f(x) = 2x - x^2 + 3$$

$$f'(x) = 2 - 2x = 0$$

$\Rightarrow$ $x = 1$ is the stationary point

$$f''(x) = -2$$

$$\Rightarrow \quad f''(1) = -2 < 0$$

So at $x = 1$ we have a relative maxima.

84. (b)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^2}{5!}\ldots$$

85. (d)

$$\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}dx = \int_0^{\pi/2}\frac{e^{ix}}{e^{-ix}}dx = \int_0^{\pi/2} e^{2ix}\,dx$$

$$= \left[\frac{e^{2ix}}{2i}\right]_0^{\pi/2} = \frac{1}{2i}\left[e^{i\pi} - e^{o}\right]$$

$$= \frac{1}{2i}[-1-1] \quad \left(\text{since } e^{i\pi} = -1\right)$$

$$= \frac{-2}{2i} = \frac{-1}{i} = i$$

86. (b)

Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ ...(i)

Since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx$ ...(ii)

(i) + (ii) $\Rightarrow 2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$

$\Rightarrow \quad 2I = \int_0^a dx$

$\Rightarrow \quad 2I = a$

$\Rightarrow \quad I = a/2$

87. (d)

If $f(x)$ is even function then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

89. (b)

Given [1, 1, 1] and [1, a, a^2]

hence $a = \omega = -\frac{1}{2}+j\frac{\sqrt{3}}{2}$

$a^2 = \omega^2$

So the vectors we

$u = [1, 1, 1]$

and $v = [1, \omega, \omega^2]$

Now $u \cdot v = 1 \cdot 1 + 1 \cdot \omega + 1 \cdot \omega^2$

$= 1 + \omega + \omega^2 = 0$

So u & v are orthogonal.

90. (b)

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \frac{1-\cos 0}{0^2} = \frac{0}{0}$$

So use L' Hospitals rule

$$\lim_{x\to 0}\frac{(1-\cos x)}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{0}{0}$$

So use L'Hospitals rule again

$$= \lim_{x\to 0}\left[\frac{\cos x}{2}\right] = \frac{1}{2}$$

91. (c)

$|x| = x \qquad x \geq 0$

$\quad = -x \qquad x < 0$

at $x = 0$ left limit = 0

Right limit = −0 = 0

$f(0) = 0$

Since left limit = Right limit = $f(0)$

So $|x|$ is continuous at $x = 0$

Now, LD = Left derivative (at $x = 0$) = −1

RD = Right derivative (at $x = 0$) = +1

LD ≠ RD

So $|x|$ is not differentiable at $x = 0$

So $|x|$ is continuous and non-differentiable at $x = 0$

92. (d)

$f(x) = x^3 + 1$

Put $f'(x) = 0$

$\Rightarrow \quad 3x^2 = 0$

$\Rightarrow \quad x = 0$ is the only critical point

at this critical point

$f''(x) = 6x$

$f''(0) = 6 \times 0 = 0$

Now $f'''(x) = 6$ and

so $f'''(0) = 6$ which is non zero.

Since the first non zero derivative value occurs at the third derivative which is an odd derivative, this function has a point of inflection at $x = 0$.

93. (c)

We need absolute maximum of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6]

First find local maximum if any by putting $f'(x) = 0$.

i.e. $f'(x) = 3x^2 - 18x + 24 = 0$

i.e. $x^2 - 6x + 8 = 0$

$x = 2, 4$

Now $f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -6 < 0$

(So $x = 2$ is a point of local maximum)

and $f''(4) = 24 - 18 = +6 > 0$

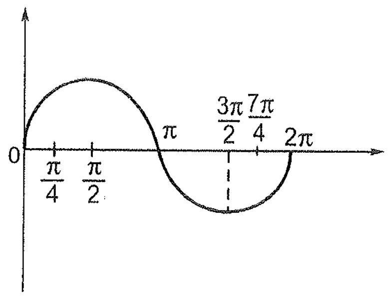
(So $x = 4$ is a point of local minimum)

Now tabulate the values of f at end point of interval and at local maximum point, to find absolute maximum in given range, as shown below:

x	$f(x)$
1	21
2	25
6	41

Clearly the absolute maxima is at $x = 6$ and absolute maximum value is 41.

94. (b)



From the plot of $\sin x$ given above, we can easily see that in the range $[\pi/4, 7\pi/4]$, there is only one local minima, at $3\pi/2$.

95. (b)

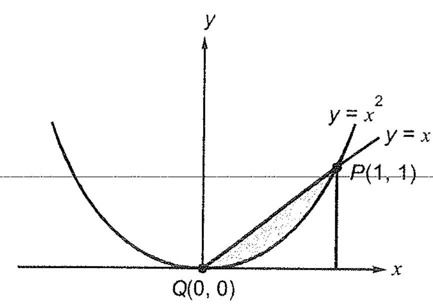
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

(By McLaurin's series expansion)

96. (a)

The area enclosed is shown below as shaded:

The coordinates of point P and Q is obtained by solving



$y = x$

and $y = x^2$ simultaneously,

i.e. $x = x^2$

$\Rightarrow \quad x(x-1) = 0$

$\Rightarrow \quad x = 0, \quad x = 1$

Now, $x = 0 \Rightarrow y = 0$ which is pt $Q(0, 0)$

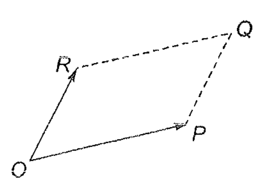
and $x = 1 \Rightarrow y = 1^2 = 1$ which is pt $P(1, 1)$

So required area is

$$= \int_0^1 x\,dx - \int_0^1 x^2 dx$$

$$\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

97. (a)



The area of parallelogram $OPQR$ in figure shown above, is the magnitude of the vector product

$$= \left|\overrightarrow{OP} \times \overrightarrow{OR}\right|$$

$$\overrightarrow{OP} = a\hat{i} + b\hat{j}$$

$$\overrightarrow{OR} = e\hat{i} + d\hat{j}$$

$$\overrightarrow{OP} \times \overrightarrow{OR} = \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (ad - bc)\hat{k}$$

$$\left|\overrightarrow{OP} \times \overrightarrow{OR}\right| = \sqrt{0^2 + 0^2 + (ad - bc)^2} = ad - bc$$

98. (a)

$x^2 + y^2 + z^2 = 1$

$$f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\hat{i} + 2y\,j + 2z\,\hat{k}$$

at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$$\text{grad } f = \frac{2}{\sqrt{2}}\hat{i} + \frac{2}{\sqrt{2}}\hat{j} + 2 \times 0 \times \hat{k}$$

$$= \sqrt{2}\,i + \sqrt{2}\,j + 0\hat{k}$$

$$|\text{grad } f| = \sqrt{2+2} = \sqrt{4} = 2$$

The unit outward normal vector at point P is

$$n = \frac{1}{|\text{grad } f|}(\text{grad } f)_{\text{at } P}$$

$$= \frac{1}{2}\left(\sqrt{2}\,i + \sqrt{2}\,j\right) = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

99. (a)

$$|A| = kr^n$$

$$\Rightarrow \quad A = k\,r^n \frac{\bar{r}}{r}$$

$$\nabla \cdot A = \nabla \cdot (kr^{n-1}\bar{r}) = 0$$

We have, $\nabla \cdot (\phi A) = (\nabla\phi) \cdot A + \phi(\nabla \cdot A)$

$$K\left[\nabla(r^{n-1}) \cdot \bar{r} + r^{n-1}(\nabla \cdot \bar{r})\right] = 0$$

$$K\left[(n-1)r^{n-2}\frac{\bar{r}}{r} \cdot \bar{r} + 3r^{n-1}\right] = 0$$

$$(n-1)\,r^{n-3}\,r^2 + 3r^{n-1} = 0$$

$$[(n-1) + 3]r^{n-1} = 0$$

$$n = -2$$

100. (a)

$$f(x)=\begin{cases} 2, & \text{if } x=3 \\ x-1, & \text{if } x>3 \\ \dfrac{x+3}{3}, & \text{if } x<3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+3}{3} = 2$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x-1 = 2$$

Also, $f(3) = 2$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

So it is continuous at $x = 3$

option (a) is correct.

101. (b)

$$\frac{dy}{dx} = 10x + 10$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 20$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 30$$

$\because$ x is defined open interval $x = (1, 2)$

$\therefore$ $1 < x < 2$

$\therefore$ $20 < \dfrac{dy}{dx} < 30$

102. (d)

Using R-H criterion

x^4	a_4	a_2	$-a_o$
x^3	a_3	a_1	
x^2	A		
x^1	a_1		
x^0	$-a_0$		

Where $A = \dfrac{a_3 a_2 - a_1 a_4}{a_3}$

So, from the above table it is clear that there is atleast one sign change in the first column. So, at least one positive and one negative real root.

103. (b)

Let $3\theta = t$

$$3 \times d\theta = dt$$

$$d\theta = \frac{dt}{3}$$

$$\theta = \frac{\pi}{6} \qquad t = \frac{\pi}{2}$$

$$\theta = 0 \qquad t = 0$$

$$I = \int_0^{\pi/2} \cos^4 t \cdot \sin^3 2t \cdot \frac{dt}{3}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^4 t \cdot (2\sin t \cos t)^3 \cdot dt$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^4 t \cdot \sin^3 t \cdot \cos^3 t dt$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^7 t \sin^3 t dt$$

$$= \frac{8}{3}\left[\frac{6\cdot 4\cdot 2\cdot 2}{10\cdot 8\cdot 6\cdot 4\cdot 2}\right] = \frac{1}{15}$$

104. (c)

$$I = \int_1^e \sqrt{x} \ln x \, dx$$

$$u = \ln x \quad ; \quad dv = \sqrt{x} dx$$

$$du = \frac{1}{x}dx \quad ; \quad dv = \int \sqrt{x}\, dx = \frac{x^{3/2}}{\frac{3}{2}}$$

$$\int u dv = uv - \int v du$$

$$\int_1^e \ln x \cdot \sqrt{x}\, dx = \left(\ln x \cdot \frac{x^{3/2}}{3/2}\right)\Bigg|_1^e - \int_1^e \frac{x^{3/2}}{3/2} \cdot \frac{1}{x} dx$$

$$= \left(\frac{2}{3} e^{3/2} - 0\right) - \frac{2}{3}\int_1^e x^{1/2} dx$$

$$= \frac{2}{3} e^{3/2} - \frac{2}{3}\left(\frac{x^{3/2}}{3/2}\right)\Bigg|_1^e$$

$$= \frac{2}{3} e^{3/2} - \frac{4}{9}(e^{3/2} - 1)$$

$$= \frac{2}{9} e^{3/2} + \frac{4}{9}$$

105. (d)

Curl of gradient of a scalar field is always zero.

$$\nabla \times \nabla V = 0$$

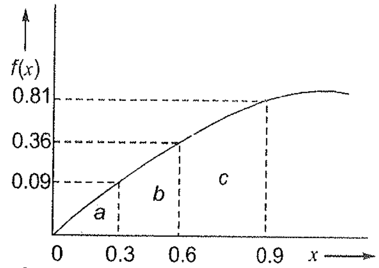
106. (d)

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$$

$$\nabla \cdot \vec{A} = 3$$

107. (d)



Area of region a is

$$= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 0.09 \times 0.3$$

Area of region b is

$$= \frac{1}{2} \times \text{height} \times (\text{base1} + \text{base2})$$

$$= \frac{1}{2} \times 0.3 \times (0.09 + 0.36)$$

Area of region c is

$$= \frac{1}{2} \times \text{height} \times (\text{base 2} + \text{base 3})$$

$$= \frac{1}{2} \times 0.3 \times (0.36 + 0.81)$$

$$\int_0^3 f(x)dx = \frac{1}{2}(0.3) \times (0.09) + \frac{1}{2}(0.3) \times (0.09 + 0.36) + \cdots + \frac{1}{2}(0.3) \times (7.29 + 9.0) = 9.045$$

option (d) is correct.

108. (d)

Option (d) is not true as irrotational vector has cross product as zero. Thus for vector to be irrotational $\nabla \times E = 0$

109. (b)

To find: $\int \vec{F} \cdot \vec{dl}$ along a segment on the x-axis from $x = 1$ to $x = 2$.

i.e. $y = 0, \quad z = 0, \quad dy = 0$ and $dz = 0$

$$\int \vec{F} \cdot \vec{dl} = \int (y^2 \times \hat{a}_x - yz\hat{a}_y - x^2\hat{a}_z) \cdot (\hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz)$$

$$= \int y^2 x dx - yzdy - x^2 dz$$

Putting, $y = 0$, $z = 0$, $dy = 0$ and $dz = 0$

We get,

$$\int \vec{F} \cdot \vec{dl} = 0$$

110. (a)

$$\frac{1}{4} \iiint_0 (\nabla \cdot F) dV$$

$$= \frac{1}{4} \times 3 \times \iiint dV = \frac{3}{4} \times \frac{4}{3} \pi (1)^3 = \pi$$

111. (d)

According to Stoke's theorem

$$\oint_c \vec{A} \cdot \vec{dl} = \iint_s (\nabla \times \vec{A}) \cdot \vec{ds}$$

112. (c)

$$\lim_{x \to \infty} \left(\frac{x + \sin x}{x} \right) = \frac{1 + \frac{\sin x}{x}}{1} = \frac{1 + 0}{1} = 1$$

Since, $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

113. (a)

$$\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} \quad \left[\frac{0}{0} \text{form} \right]$$

Use L-Hospital Rule (Note: Differentiate numerator and denominator w.r.t. α keeping x as constant.)

$$\Rightarrow \lim_{\alpha \to 0} \frac{x^\alpha \ln x}{1} = \log x$$

114. (a)

$$\operatorname{Lt}_{x \to 0} \frac{x - \sin x}{1 - \cos x}$$

Applying L' Hospital's rule

$$\operatorname{Lt}_{x \to 0} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(1 - \cos x)} = \operatorname{Lt}_{x \to 0} \frac{(1 - \cos x)}{\sin x}$$

(It is still of $\frac{0}{0}$ form)

Again applying L' Hospital's rule

$$\operatorname{Lt}_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)} = \operatorname{Lt}_{x \to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

115. (b)

$\operatorname{Lt}_{x \to 0} \frac{(e^{2x} - 1)}{\sin 4x}$, it is of $\left(\frac{0}{0} \right)$ from

Applying L' Hospital's rule,

$$\operatorname{Lt}_{x \to 0} \frac{2e^{2x}}{4\cos 4x} = \frac{2 \times 1}{4 \times 1} = \frac{1}{2}$$

116. (c)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty}\frac{1}{x}\cdot x} = e^1 = e$$

117. (d)

$$IF = e^{\int k_2 dt} = e^{k_2 t}$$

118. (d)

$f(x)$ is continuous at any point

if $\underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to a^+}{\text{Lt}} f(x) = f(a)$

119. (a)

(a) As $y \in (0, 1)$; $f(y)$ varies from –1 to 1 similarly $f(y + 1)$ varies from +1 to –1

$\therefore$ Let, $g(x) = f(y) - f(y+1)$; $y \in (0, 1)$

we get, $g(x) = 0$ for some value of x

i.e. $f(y) = f(y+1)$ for some $y \in (0, 1)$

(b) $f(y) = f(2-y)$ only at $y = 0$ and $y = 1$

$\therefore$ In (0, 1) we cannot say $f(y) = f(2-y)$

(c) We cannot conclude that the maximum value of $f(y)$ is 1 in (0, 2)

(d) As $y \in (0, 1)$; $f(y)$ varies from –1 to 1 and $-f(2-y)$ varies from 1 to –1

$\therefore$ Let $g(x) = f(y) + f(2-y)$; $y \in (0, 1)$

$\therefore$ $g(x) = 0$ for same value of x

i.e. $f(y) = -f(2-y)$ for some $y \in (0, 1)$

But the difference between y and $2 - y$ should be less than the length of the interval 2 is not possible.

120. (c)

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

$f(\pi/6) = 0$

Since if we put $\theta = \pi/6$ in above determinant it will evaluate to zero, since I and II row will become same.

$f(\pi/3) = 0$

Since if we put $\theta = \pi/3$ in above determinant it will evaluate to zero, since I and III row will become same.

So $f(\pi/6) = f(\pi/3)$. Also in the interval $[\pi/6, \pi/3]$ the function $f(\theta)$ is continuous and differentiable (***note*** that the given interval doesn't contain any odd multiple of $\pi/2$ where $\tan\theta$ is neither continuous nor differentiable).

Since all the three conditions of **Roll's theorem** are satisfied the conclusion of Rolls theorem is true i.e.

I: $\exists\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) = 0$ is true

Now the statement

II: $\exists\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) \neq 0$

is also true, since the only way it can be false is if $f'(\theta) = 0$ for all values of θ, which is possible only if $f(\theta)$ is a constant which is untrue.

Therefore, both (I) and (II) are correct.

121. Sol.

$$f(x) = x\sin x$$
$$f'(x) = x\cos x + \sin x$$
$$f''(x) = (-x\sin x + \cos x) + \cos x$$

$f''(x) + f(x) + t\cos x = 0$

$\Rightarrow -x\sin x + \cos x + \cos x + x\sin x + t\cos x = 0$

$\Rightarrow (2 + t)\cos x = 0$

$\Rightarrow t + 2 = 0$

$\Rightarrow t = -2$

122. Sol.

$$\frac{d^2y}{dx^2} = 0$$

$$\frac{dy}{dx} = C_1$$

$\Rightarrow$ $C_1 = 2$

$y = C_1x + C_2$

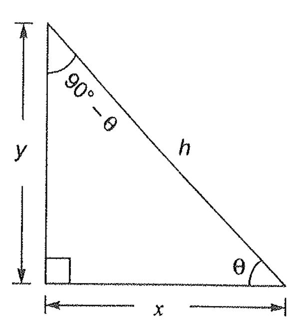
at $x = 0$

$y = 5 = C_2$

$\therefore$ $y = 2x + 5$

at $y(15) = 2 \times 15 + 5 = 35$

123. (c)



$$h = \sqrt{x^2 + y^2}$$

Given that, $x + \sqrt{x^2 + y^2} = k$ (constant)

$$x^2 + y^2 = (k-x)^2$$
$$y^2 = k^2 - 2kx$$

Area, $A = \frac{1}{2} \cdot x \cdot y$

$$A^2 = \frac{x^2}{4}(k^2 - 2kx)$$

Let, $f(x) = A^2 = \frac{x^2}{4}(k^2 - 2kx)$

$$f'(x) = \frac{1}{4}(2k^2x - 6kx^2)$$
$$f'(x) = 0$$
$$2k^2x - 6kx^2 = 0$$
$$x = \frac{k}{3}, 0$$

At $x = \frac{k}{3}$, $f''(x) < 0$

$\therefore$ Area is maximum at $x = \frac{k}{3}$

$\therefore$ $y^2 = k^2 - \frac{2k^2}{3} = \frac{k^2}{3}$

$$y = \frac{k}{\sqrt{3}}$$
$$\tan\theta = \frac{y}{x} = \sqrt{3}$$
$$\theta = 60°$$

124. (c)

$$\frac{\partial z}{\partial x} = y\ln(xy) + \frac{xy}{xy}xy$$
$$\frac{\partial z}{\partial x} = y[\ln(xy) + 1] \quad ...(i)$$
$$\frac{\partial z}{\partial x} = x\ln(xy) + \frac{xy}{xy} \times x$$
$$\frac{\partial z}{\partial x} = x[\ln(xy) + 1] \quad ...(ii)$$

Here $\boxed{x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}}$

125. (a)

$$f(x) = xe^{-x}$$
$$f'(x) = e^{-x} - x\,e^{-x} = 0$$
$$e^{-x}(1-x) = \infty$$

$\Rightarrow$ $x = 1$ (since $e^{-x} = 0$ only when $x = \infty$ which does not belong to the given interval)

Now, we need to check whether at $x = 1$, we have a maximum, minimum or saddle point.

$$f''(x) = -e^{-x} - (e^{-x} - xe^{-x})$$
$$= -2e^{-x} + xe^{-x} = e^{-x}(x-2)$$
$$f''(1) = -e^{-1} \text{ which is } < 0$$

So at $x = 1$, we have a maximum.

The maximum value is

$$f(1) = 1e^{-1} = e^{-1}$$

126. (c)

$$f(x) = (x-1)^{2/3} = (\sqrt[3]{x-1})^2$$

As $f(x)$ is square of $\sqrt[3]{x-1}$ hence its minimum value be 0 where at $x = 1$.

127. (b)

$$f(x) = x^3 - 3x^2 - 24x + 100 \quad x \in [-3, 3]$$
$$f'(x) = 3x^2 - 6x - 24$$
$$f'(x) = 0 \quad \text{at } x = 4, -2$$

Critical points are $\{-3, -2, 3\}$

$$f(-3) = -27 - 27 + 72 + 100 = 118$$
$$f(-2) = -8 - 12 + 48 + 100 = 128$$
$$f(3) = 27 - 27 - 72 + 100 = 28$$

Hence $f(x)$ has minimum value at $x = 3$ which is 28.

128. (a)

$$f(t) = e^{-t} - 2e^{-2t}$$
$$f'(t) = -e^{-t} + 4e^{-2t}$$

For maximum value $f'(t) = 0$

$$f'(t) = 0 = -e^{-t} + 4e^{-2t}$$
$$\Rightarrow \quad 4e^{-2t} = e^{-t}$$
$$4e^{-t} = 1$$
$$\therefore \quad t = \log_e 4$$

129. Sol.

$$f'(x) = \frac{1}{1+x} - 1 = 0$$
$$\frac{1-1-x}{1+x} = 0$$
$$\frac{x}{1+x} = 0$$
$$x = 0$$
$$f''(x) = \frac{-1}{(1+x)^2}$$
$$f''(0) = -1 < 0$$

$f(x)$ have maximum value at $x = 0$

$$f(0) = hg(1+0) - 0 = 0$$
$$f_{max} = 0$$

130. Sol.

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$
$$f'(x) = 6x^2 - 18x + 12$$
$$f'(x) = 0$$
$$6x^2 - 18x + 12 = 0$$
$$x^2 - 3x + 2 = 0$$
$$x = 1, 2$$

Hence critical points are {0, 1, 2, 3,}.

$f(x)$ attains its maximum value at one of these points.

$$f(0) = -3$$
$$f(1) = 2$$
$$f(2) = 1$$
$$f(3) = 6$$

131. (b)

$$I = \int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^2 + \cos(x-1)} dx$$

Taking $x - 1 = z \Rightarrow dx = dz$

for $x = 0, z \to -1$ and $x = 2, z \to 1$

$$\therefore \quad I = \int_{-1}^{1} \frac{z^2 \sin z}{z^2 + \cos z} dz$$

let $\quad f(z) = \dfrac{z^2 \sin z}{z^2 + \cos z}$

$$f(-z) = -\frac{z^2 \sin z}{z^2 + \cos z}$$

$f(z) = -f(z)$ function is ODD.

$$\therefore \quad I = 0$$

132. Sol.

$$\Rightarrow \int_0^{2\pi} |x \sin x| dx = Kp$$

$$\Rightarrow \int_0^{\pi} |x \sin x| dx + \int_{\pi}^{2\pi} |x \sin x| dx = Kp$$

$$\Rightarrow \int_0^{\pi} x \sin x \, dx + \int_{\pi}^{2\pi} -(x \sin x) dx = Kp$$

$$\Rightarrow (-x \sin x + \sin x)\Big|_0^{\pi} - (-x \sin x + \sin x)\Big|_{\pi}^{2\pi} = Kp$$

$$\Rightarrow \quad 4p = Kp$$

$$\Rightarrow \quad K = 4$$

133. (a)

$$\int_0^{\pi} x^2 \cos x \, dx = x^2 (\sin x) - 2x(-\cos x) + 2(-\sin x)\Big|_0^{\pi}$$
$$= \pi^2 \cdot 0 + 2\pi(-1) - 0 = -2\pi$$

134. (b)

$$\bar{F} = yz\,\hat{i}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 0 & 0 \end{vmatrix}$$
$$= \hat{i}(0) - \hat{j}(0 - y) + \hat{k}(0 - z)$$
$$= 0\hat{i} + y\hat{j} - z\hat{k}$$

By Stokes theorem,

$$\int_C \bar{F} \cdot d\bar{r} = \int_S (\nabla \times \bar{F}) \cdot \hat{n} ds$$
$$= \int_S (y\hat{i} - z\hat{k}) \cdot \hat{k} \, ds$$
$$= \int_S -z \, ds \qquad \text{Since } z = 1$$
$$= \int_S -1 \, ds = (-1)S = (-1)\pi = -\pi$$

where S is surface area of $x^2 + y^2 = 1$

$$\therefore \quad S = \pi(1)^2 = \pi$$

135. (b)

$$I = \int_0^2 \left[\int_0^x e^{x+y} dy \right] dx$$

$$\int_0^x e^{x+y} dy = e^{x+y} dy\Big|_0^x = e^{2x} - e^x$$

$$I = \int_0^2 (e^{2x} - e^x) dx$$

$$I = \frac{1}{2} e^{2x}\Big|_0^2 - e^x\Big|_0^2$$

$$= \frac{1}{2}(e^4 - 1) - (e^2 - 1)$$

$$I = \frac{1}{2}e^4 - \frac{1}{2} - e^2 + 1$$

$$= \frac{1}{2}e^4 - e^2 + \frac{1}{2} = \frac{1}{2}(e^4 - 2e^2 + 1)$$

$$I = \frac{1}{2}(e^2 - 1)^2$$

136. (b)

$$\int_0^8 \left(\int_{y/2}^{(y/2)+1} \left(\frac{2x - y}{2} \right) dx \right) dy$$

$$\frac{2x - y}{2} = u$$

$$x - \frac{y}{2} = u$$
$$dx = du$$
$$\text{at} \quad x = \frac{y}{2}$$
$$u = \frac{\frac{2y}{2} - y}{2} = 0$$
$$\text{at} \quad x = \frac{y}{2} + 1$$
$$u = \frac{2\left(\frac{y}{2}+1\right) - y}{2} = \frac{y+2-y}{2} = 1$$

Thus, integral becomes $\int_0^8 \left[\int_0^1 u\,du\right] dy$

$$v = \frac{y}{2}$$
$$dv = \frac{dy}{2} \qquad \Rightarrow \quad dy = 2dv$$

$y = 0$; $v = 0$; $y = 8$; $v = 4$

$$= \int_0^4 \left[\int_0^1 u\,du\right] \times 2dv = \int_0^4 \left[\int_0^1 2u\,du\right] dv$$

137. (b)

For linear dependency, det $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 6 & 4 \end{vmatrix}$ must be zero.

$\therefore \qquad \Delta = 1(12 \quad 6) \quad 1(8 \quad 5) + 1(12 \quad 15)$

$= 6 - 3 - 3 = 0$

$\therefore$ There three vectors are linearly dependent.

138. (a)

$$\vec{F} = x^2z^2\vec{i} - 2xy^2z\vec{j} + 2y^2z^3\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & -2xy^2z^2 & 2y^2z^3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2y^2z^3) + \frac{\partial}{\partial z}(2xy^2z)\right]$$
$$-\vec{j}\left[\frac{\partial}{\partial y}(2y^2z^3) - \frac{\partial}{\partial z}(x^2z^2)\right]$$
$$+\vec{k}\left[\frac{\partial}{\partial x}(-2xy^2z) - \frac{\partial}{\partial y}(x^2z^2)\right]$$

$$\nabla \times \vec{F} = \vec{i}[4yz^3 + 2xy^2] - \vec{j}[2zx^2]$$
$$+ \vec{k}[-2y^2z - 0]$$
$$= (4yz^3 + 2xy^2)\vec{i} - (2x^2z)\vec{j} - (2y^2z)\vec{k}$$

139. (c)

$$\vec{F} = x^2z\hat{i} + xy\hat{j} - yz^2\hat{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(yz^2)$$

$$\nabla \cdot \vec{F} = 2xz + x - 2yz$$

$$\therefore \nabla \cdot \vec{F}\Big|_{(1,-1,1)} = 2 \times 1 \times 1 + 1 - 2 \times -1 \times 1$$
$$= 2 + 1 + 2 = 5$$

140. (a)

$$\vec{v} = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\hat{i}\frac{\partial(fV)}{\partial x} + \hat{j}\frac{\partial(fV)}{\partial y} + \hat{k}\frac{\partial(fV)}{\partial z} = x^2y + y^2z + z^2x$$

$$y\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial z} = x^2y + y^2z + z^2x \qquad \text{...(i)}$$

$$\vec{v}\ \Delta f = (y\hat{i} + z\hat{j} + x\hat{k})\left(\frac{\hat{i}\partial}{\partial x} + \frac{\hat{j}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)f$$

$$\vec{v}\ \Delta f = \frac{y\partial f}{\partial x} + \frac{z\partial f}{\partial y} + \frac{x\partial f}{\partial z} \qquad \text{...(ii)}$$

From equations (i) and (ii)

$$\vec{v} \cdot \nabla f = x^2y + y^2z + z^2x$$

141. (d)

$$y = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^{2x}$$

$$\Rightarrow \qquad \log y = \lim_{x \to \infty} 2x \log\left(1 + \frac{1}{x}\right)$$

Which is in the form of $\infty \times 0$.

To convert this into $\frac{0}{0}$ form, we rewrite as

$$\Rightarrow \qquad \log y = \lim_{x \to \infty} \frac{2\log\left(1 + \frac{1}{x}\right)}{1/x}$$

Now it is in $\frac{0}{0}$ form.

Using L' Hospital's rule

$$\Rightarrow \quad \log y = \lim_{x\to\infty} \frac{\dfrac{2\times -\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{1}{x}} = 2$$

$$\therefore \quad y = e^2$$

142. (c)

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{2x^4}$$

putting the $x \to 0$

we get $\frac{0}{0}$ form

Applying L' Hospital rule

$$\Rightarrow \lim_{x\to 0} \frac{2x\sin(x^2)}{8x^3}$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin(x^2)}{4x^2}$$

$$\Rightarrow \frac{1}{4} \lim_{x\to 0} \frac{\sin(x^2)}{x^2}$$

$$\Rightarrow \frac{1}{4} \lim_{x^2\to 0} \frac{\sin(x^2)}{x^2} = \frac{1}{4}\times 1 = \frac{1}{4}$$

143. Sol.

$$\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + \cos x}\right) = \lim_{x\to 0}\left(\frac{-\sin 0}{2\sin 0 + \cos 0}\right) = \frac{0}{1} = 0$$

(**Note:** Since the function is not evaluating to 0/0 not need to use L' Hospital's rule)

144. (c)

$$y = \lim_{x\to\infty} x^{1/x}$$

$$\log y = \lim_{x\to\infty} \log x^{1/x}$$

$$\log y = \lim_{x\to\infty} \frac{\log x}{x}$$

∞/∞ form, use L' Hospital's rule

$$\log y = \lim_{x\to\infty} \frac{1/x}{1}$$

$$\log y = 0 \quad \Rightarrow y = 1$$

145. (c)

$$\lim_{x\to\infty} (1+x^2)^{e^{-x}}$$

$$\log y = \lim_{x\to\infty} \log(1+x^2)^{e^{-x}} = \lim_{x\to\infty} \frac{\log(1+x^2)}{e^x}$$

∞/∞ form apply L' Hospital's rule

$$\Rightarrow \quad \log y = \lim_{x\to\infty} \frac{\frac{1}{1+x^2}(2x)}{e^x}$$

$$\Rightarrow \quad \log y = \lim_{x\to\infty} \frac{2x}{(1+x^2)e^x}$$

Again we are getting ∞/∞ form apply L' Hospital's rule

$$\log y = \lim_{x\to\infty} \frac{2}{(1+x^2)e^x + e^x\cdot 2x}$$

$$\log y = \frac{2}{\infty} = 0$$

$$\Rightarrow \quad y = 1$$

146. (c)

$$f(x) = \frac{1}{\sqrt[3]{x}}$$

Statement 1: f is continuous in $[-1, 1]$. Let us check this statement.

We need to check continuity at $x = 0$

$$\text{Left limit} = \lim_{x\to 0^-} \frac{1}{\sqrt[3]{x}} = \lim_{h\to 0} \frac{1}{\sqrt[3]{0-h}} = \lim_{h\to 0} \frac{-1}{\sqrt[3]{h}} = -\infty$$

$$\text{Right limit} = \lim_{x\to 0^+} \frac{1}{\sqrt[3]{x}} = \lim_{h\to 0} \frac{1}{\sqrt[3]{0+h}} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h}} = +\infty$$

Left limit $\neq$ Right limit

$\therefore$ Statement 1 is false.

Statement 2: f is not bounded is $[-1, 1]$. Since at $x = 0$ it goes to $-\infty$ and $+\infty$ the function is not bounded.

$\therefore$ Statement 2 is true.

Statement 3: A is non zero and finite.

$$A = \left|\int_{-1}^{0} x^{-1/3}dx\right| + \left|\int_{0}^{1} x^{-1/3}dx\right|$$

$$= \left|\frac{3}{2}\left[x^{2/3}\right]_{-1}^{0}\right| + \left|\frac{3}{2}\left[x^{2/3}\right]_{0}^{1}\right|$$

$$= \left|\frac{3}{2}\right| + \left|\frac{3}{2}\right| = 3$$

So A is non zero and finite.

$\therefore$ Statement 3 is true.

147. (b)

Since $f(1) \neq f(-1)$, Roll's mean value theorem does not apply.

By Lagrange mean value theorem

$$f'(x) = \frac{f(1)-f(-1)}{1-(-1)} = \frac{2}{2} = 1$$

$$-2x + 3x^2 = 1$$

$$x = 1, -\frac{1}{3}$$

x lies in $(-1, 1)$ $\Rightarrow x = -\frac{1}{3}$

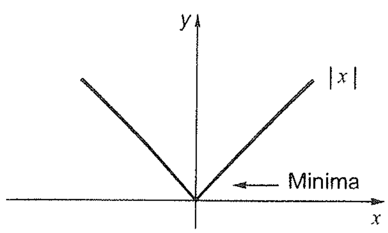
148. (d)

$f(x)$ has a local minimum at $x = x_0$

if $f'(x_0) = 0$

and $f''(x_0) > 0$

149. (a)



150. Sol.

Consider a symmetric matrix $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

Given $a + d = -6$

$$|A| = ad - b^2$$

Now since b^2 is always non-negative, maximum determinant will come when $b^2 = 0$.

So we need to maximize

$$|A| = ad - 0$$

$$= ad = a \times -(6 + a) = -a^2 - 6a$$

$$\frac{d|A|}{da} = -2a - 6 = 0$$

$\Rightarrow$ $a = -3$ is the only stationary point

Since $\left[\frac{d^2|A|}{da}\right]_{a=-3} = -2 < 0$,

we have a maximum at $a = -3$.

Since $a + d = -6$, Corresponding value of $d = -3$.

Now the maximum value of determinant is

$$|A| = ad = -3 \times -3 = 9$$

151. (b)

$$f(x) = e^{-x}(x^2 + x + 1)$$

$$f'(x) = e^{-x}(2x + 1) - e^{-x}(x^2 + x + 1)$$

$$= e^{-x}(x - x^2) = e^{-x}(x)(1 - x)$$

Putting $f'(x) = 0$, we get

$$x = 0 \quad \text{or} \quad x = 1$$

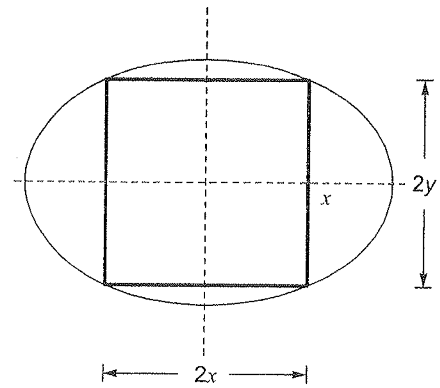
$$f''(x) = e^{-x}(1 - 2x) - e^{-x}(x - x^2) = e^{-x}(1 - 3x + x^2)$$

At $x = 0$, $f''(x) = 1$ (so we have a minimum).

At $x = 1$, $f''(x) = -\frac{1}{e}$ (so we have a maximum).

Only curve (b) shows a single local minimum at $x = 0$ and a single local maximum at $x = 1$.

152. Sol.



$$x^2 + 4y^2 = 1$$

Area of rectangle

$$= 2x \cdot 2y = 4xy$$

Let $f = (\text{Area})^2$

$$= 16x^2 y^2$$

$$= 4x^2(1 - x^2)$$

$(\because\ 1 - x^2 = 4y^2)$

$$f'(x) = 0$$

$$\frac{d}{dx}\left[4(x^2 - x^4)\right] = 0$$

$$4(2x - 4x^3) = 0$$

We get, $x = \pm\frac{1}{\sqrt{2}}$

$$y = \pm\frac{1}{\sqrt{8}}$$

$$\text{Area} = 4xy = 4 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{8}} = 1$$

153. (a)

Partial derivative w.r.t.

$$y\frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

Partial derivative w.r.t.

$$x\frac{\partial}{\partial x}(6y + 4x) = 4$$

From given condition

$$2y = 4$$

$\Rightarrow$ $y = 2$

154. (a)

$$a\,f(x)+b\,f\left(\frac{1}{x}\right) = \frac{1}{x}-25 \qquad ...(1)$$

Put $x = \frac{1}{x}$ in equation (1)

$$a\,f\left(\frac{1}{x}\right)+b\,f(x) = x-25 \qquad ...(2)$$

Equation (1) × a – equation (2) × b

$$(1)\times a:\Rightarrow\; a^2f(x)+ba\,f\left(\frac{1}{x}\right) = \frac{a}{x}-25a$$

$$(2)\times b:\Rightarrow\; ab\,f\left(\frac{1}{x}\right)+b^2f(x) = bx-25b$$

$$a^2f(x)-b^2f(x) = \frac{a}{x}-25a-bx+25b$$

$$\Rightarrow (a^2-b^2)\cdot f(x) = \frac{a}{x}-bx+25(b-a)$$

$$\Rightarrow \qquad f(x) = \frac{1}{a^2-b^2}\left[\frac{a}{x}-bx+25(b-a)\right]$$

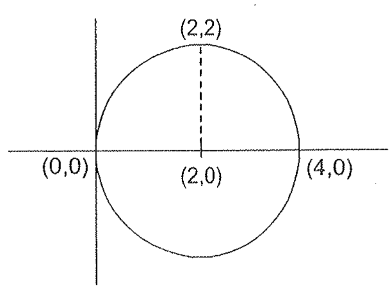
$$\Rightarrow \int_1^2 f(x)\cdot dx = \frac{1}{a^2-b^2}\left[a\cdot\int_1^2\frac{1}{x}\cdot dx-b\int_1^2 x\cdot dx+25(b-a)\int_1^2 1\cdot dx\right]$$

$$= \frac{1}{a^2-b^2}\left[a\ln 2-\frac{3}{2}b+25(b-a)\right]$$

$$= \frac{1}{a^2-b^2}\left[a\ln 2-25a+\frac{47b}{2}\right]$$

$$= \frac{1}{a^2-b^2}\left[a(\ln 2-25)+\frac{47b}{2}\right]$$

155. Sol.



$(x-2)^2+(y^2)=(2)^2$, is a circle of radius 2 m and centre at (2, 0)

Time to reach from (4, 0) to (2, 2) is

$$\text{time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{\left(\frac{2\pi r}{4}\right)}{1.57} = \frac{\left(\frac{2\pi 2}{4}\right)}{1.57} = \frac{\pi}{1.57} = 2 \text{ sec}$$

156. Sol.

$$S = \int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$S = \int_0^{\pi/2}\sqrt{(-\sin t)^2+(\cos t)^2+\left(\frac{2}{\pi}\right)^2}\,dt$$

$$= \int_0^{\pi/2}\sqrt{1+\left(\frac{4}{\pi^2}\right)}\,dt = \sqrt{1+\left(\frac{4}{\pi^2}\right)}\,[t]_0^{\pi/2}$$

$$= \sqrt{1+\left(\frac{4}{\pi^2}\right)}\left(\frac{\pi}{2}\right) = 1.86$$

157. Sol.

$$\text{Volume} = \iint f(x,y)\,dx\,dy = \int_0^1\int_0^y e^x\,dx\,dy$$

$$= \int_0^1 [e^x]\Big|_0^y\,dy = \int_0^1 (e^y-1)\,dy$$

$$= (e^y-y)\Big|_0^1 = (e-1)-(1-0)$$

$$= e-1-1 = e-2 = 0.71828$$

158. (c)

$$I = \int_0^a\int_0^y f(x,y)\,dx\,dy$$

Limit of x:

Lower limt $x = 0$

Upper limt $x = y$

Limit of y:

Lower limt $y = 0$

Upper limt $y = a$

By change of order of integration limt of y

Limit of y:

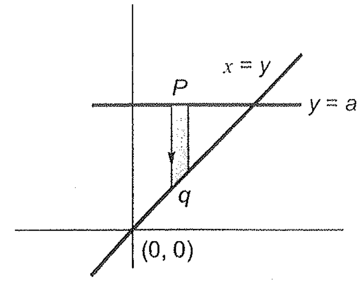
Lower limt $y = x$

Upper limt $y = a$

Limt of x:

Lower limit $x = 0$

Upper limt $x = a$



So, $$I = \int_0^a\int_x^a f(x,y)\,dy\,dx$$

159. Sol.

$$u(x, y, z) = x^2 - 3yz$$

$$\nabla u = 2x\hat{i} - 3z\hat{j} - 3y\hat{k}$$

$$\nabla u|_{at\ (2,-1,4)} = 4\hat{i} + 12\hat{j} - 3\hat{k}$$

Directional derivative,

$$= \left(4\hat{i} + 12\hat{j} - 3\hat{k}\right)\cdot\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$$

$$= \frac{4-12-6}{\sqrt{6}} = -\frac{14}{\sqrt{6}}$$

$$= -\frac{7\sqrt{6}}{3} = -5.715$$

160. (a)

$$\text{Curl of vector} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 & 3z^2 & y^3 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(y^3)\frac{\partial}{\partial z}(3z^2)\right]$$

$$-j\left[\frac{\partial}{\partial x}(y^3)\frac{\partial}{\partial z}(2x^2)\right]$$

$$+k\left[\frac{\partial}{\partial x}(3z^2)\frac{\partial}{\partial y}(2x^2)\right]$$

$$= i[3y^2 - 6z] - j[0] + k[0 + 0]$$

At $x = 1$, $y = 1$ and $z = 1$

Curl $= i(3 \times 1^2 - 6 \times 1) = -3i$

161. (c)

Div Curl $\vec{V} = 0$

$\therefore$ (c) is correct option.

162. (a)

$$f(x, y) = x^2 + 3y^2$$

$$\phi = x^2 + y^2 - 2 \text{ and point } P \Rightarrow (1, 1)$$

Normal to the surface,

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} = 2x\hat{i} + 2y\hat{j}$$

$$\nabla\phi|_{at\ P(1,1)} = 2\hat{i} + 2\hat{j}$$

the normal vector is $\vec{a} = 2\hat{i} + 2\hat{j}$

Magnitude of directional derivative of f along $\vec{a}$ at (1, 1) is $\Rightarrow \nabla\cdot f\cdot\hat{a}$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} = 2x\hat{i} + 6y\hat{j}$$

$$\nabla f|_{(1,1)} = 2\hat{i} + 6\hat{j}$$

$$|\vec{a}| = \sqrt{4+4} = 2\sqrt{2}$$

$$\hat{a} = \frac{2\hat{i}+2\hat{j}}{2\sqrt{2}} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$$

$\therefore$ Magnitude of directional derivative

$$= (2\hat{i} + 6\hat{j})\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$$

$$= \frac{2+6}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

163. Sol.

$\int_c [(3x-8y^2)dx + (4y-6xy)dy]$, C is boundary of region bounded by $x = 0$, $y = 1$, and $z + y = 1$.

Using Green's theorem

$$I = \int_c (Pdx + Qdy)$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Here, $P = 3x - 8y^2$

$Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} = -6y$$

$$\frac{\partial P}{\partial y} = -16y$$

$$I = \iint(-6y - (-16y))dxdy$$

$$= \iint 10y\,dx\,dy$$

$$I = 10\int_0^1 dx \int_0^{1-x} \frac{y^2}{2} = 5\int_0^1 dx(1-x)^2$$

$$I = 5\int_0^1 (1-x)^2 . dx = 1.6666$$

164. Sol.

Accodring to gauge divergence theorem

$$\iint_S \frac{1}{\pi}(9xi - 3yj)\cdot n dS = \frac{1}{\pi}\int \text{divergence } (9xi - 3yj), dv$$

$$= \frac{1}{\pi}[9-3]\times\frac{4}{3}\pi\left[r^3\right]$$

$r = 3$ [given]

$$= \frac{1}{\pi}\times 6\times\frac{4}{3}\pi\times 27 = 216$$

165. Sol.

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$$

Let $x-4=t$ not as $x\to 4$

So the requires limit is $\lim_{t\to 0}\frac{\sin(t)}{t}=1$

166. Sol.

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$$

Let $x-4=t$ not as $x\to 4$

So the requires limit is $\lim_{t\to 0}\frac{\sin(t)}{t}=1$

167. Sol.

$$\int_c \nabla\phi\cdot d\bar{r} = \int_c (yz\bar{i}+xz\bar{\rho}+xy\bar{k})\times$$

$$=\int_c yzdx+xzdy+xydz$$

$$=\int_c d(xyz) = (xyz)$$

Given that $x=t$, $y=t^2$, $z=3t^2$

$$=(t\cdot t^2\cdot 3t^2)\Big|_1^3 = 3(t^5)\Big|_1^3$$

$$=3(3^5-1)=3^6-3$$

$$=729-3=\mathbf{726}$$

168. Sol.

$$\underset{n\to\infty}{\text{Lt}}\sqrt{n^2+n}-\sqrt{n^2+1}$$

$$=\underset{n\to\infty}{\text{Lt}}\frac{\sqrt{n^2+n}-\sqrt{n^2+1}}{\left(\sqrt{n^2+n}+\sqrt{n^2+1}\right)}\left(\sqrt{n^2+n}+\sqrt{n^2+1}\right)$$

$$=\underset{n\to\infty}{\text{Lt}}\frac{n^2+n-n^2-1}{\sqrt{n^2+n}+\sqrt{n^2+1}}$$

$$=\underset{n\to\infty}{\text{Lt}}\frac{n-1}{n\sqrt{1+\frac{1}{n}}+n\sqrt{1+\frac{1}{n^2}}}$$

$$=\underset{n\to\infty}{\text{Lt}}\frac{n\left(1-\frac{1}{n}\right)}{n\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{1}{n^2}}\right)}$$

$$=\frac{1}{1+1}=\frac{1}{2}$$

169. (c)

$$\underset{x\to 0}{\text{Lt}}\frac{\ln(1+4x)}{e^{3x}-1}\quad 0/0 \text{ form}$$

$$\underset{x\to 0}{\text{Lt}}\frac{\frac{1}{1+4x}\cdot 4}{3e^{3x}}=\frac{4}{3}$$

170. (c)

$$\underset{x\to\infty}{\text{Lt}}\sqrt{x^2+x-1}-x$$

$$\underset{x\to\infty}{\text{Lt}}\frac{\left(\sqrt{x^2+x-1}-x\right)\left(\sqrt{x^2+x-1}+x\right)}{\sqrt{x^2+x-1}+x}$$

$$\underset{x\to\infty}{\text{Lt}}\frac{x^2+x-1-x^2}{\sqrt{x^2+x+1}+x}$$

$$\underset{x\to\infty}{\text{Lt}}\frac{x-1}{x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+x}$$

$$\underset{x\to\infty}{\text{Lt}}\frac{1-\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}=\frac{1}{1+1}=\frac{1}{2}$$

171. (d)

(i) $\lim_{x\to\infty}\frac{xy}{x^2+y^2}\lim_{y\to\infty}\left(\frac{0}{0^2+y^2}\right)=0$

(i.e., put $x=0$ and then $y=0$)

(ii) $\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}\lim_{x\to 0}\left(\frac{0}{x^2+0}\right)=0$

(i.e., put $y=0$ and then $x=0$)

(iii) $\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}\lim_{x\to 0}\frac{x(mx)}{x^2+m^2x^2}$

(i.e., put $y=mx$)

$$\lim_{x\to\infty}\left(\frac{m}{1+m^2}\right)=\frac{m}{1+m^2}$$

which depends on m.

172. (b)

P: If $f(x)$ is continuous at $x=x_0$, then it is also differentiable at $x=x_0$

Q: If $f(x)$ is continuous at $x=x_0$, then it may or may not be derivable at $x=x_0$

R: If $f(x)$ is differentiable at $x=x_0$, then it is also continuous at $x=x_0$

P is false

Q is true

R is true Option (b) is correct

173. (c)

$$f(x) = \frac{x^2 - 3x - 4}{x^2 + 3x - 4} \text{ is not continous}$$

when

$$x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$
$$x = -4, 1$$

174. (c)

$F'(x) = f(x)$ which is density function

$F'(x) = f(x) < 0$ when $x < 0$

$\therefore$ $F(x)$ is decreasing for $x < 0$

$F'(x) = f(x) > 0$ when $x > 0$

$\therefore$ $F(x)$ is increasing for $x > 0$

175. Sol.

If $f(x) + f(-x)$ is degree 10

$$f(x) = a_{10}x^{10} + a_9x^9 \ldots\ldots + a_1x + a_0$$
$$f(-x) = a_{10}x^{10} - a_9x^9 \ldots - a_1x + a_0$$
$$f(x) + f(-x) = a_{10}x^{10} + a_8x^8 + \ldots + a_0$$

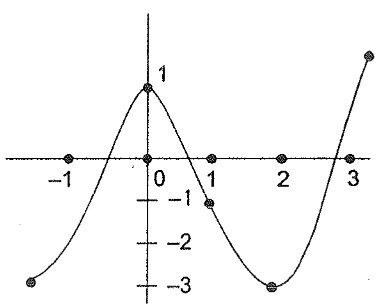
Now $g(x) = f'(x) = 10a_{10}x^9 + 9a_9x^8 + \ldots + a_1$

$g(-x) = f'(-x) = -10a_{10}x^9 + 9a_9x^8 + \ldots + a_1$

$g(x) - g(-x) = 20a_{10}x^9 + \ldots$

Clearly degree of $(g(x) - g(-x))$ is 9.

176. (b)



$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x$$
$$f'(x) = 0$$
$$3x^2 - 6x = 0$$
$$3x(x - 2) = 0$$
$$x = 0, 2$$
$$f''(x) = 6x - 6$$

At $x = 0$ $\quad f''(0) = -6$ maxima

$x = 2$ $\quad f''(2) = 6$ minima

177. Sol.

$$f(x) = 2x^3 - x^4 - 10 \quad \text{in} \quad [-1, 1]$$
$$f'(x) = 6x^2 - 4x^3$$

for minima and maxima

$$f'(x) = 0$$
$$6x^2 - 4x^3 = 0$$
$$2x^2(3 - 2x) = 0$$
$$x = 0, 0, \frac{3}{2}$$
$$f''(x) = 12x - 12x^2 \text{ s}$$

for $x = 0$, $\quad f''(0) = 0$

for $x = \frac{3}{2}$, $\quad f''\left(\frac{3}{2}\right) = 18 - 27 = -9 < 0$ maxima

at $\quad x = -1$, $\quad f(-1) = -2 - 1 - 10 = -13$

at $\quad x = 1$, $\quad f(1) = 2 - 1 - 10 = -9$

At $x = -1$, function attains global minimum value with $f(x)_{min} = -13$.

178. Sol.

$$f(x) = x^3 - 3x^2 + 2x \quad [1, 2]$$
$$f'(x) = 3x^2 - 6x + 2$$

$f'(x) = 0$ for stationary point

stationary points are $1 \pm \frac{1}{\sqrt{3}}$

only $1 + \frac{1}{\sqrt{3}}$ lies in $[1, 2]$

$$f(1) = 0$$
$$f(2) = 0$$
$$f\left(1 + \frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$$

Maximum value is 0.

179. (d)

$$f'(x) = 0$$

$\Rightarrow \quad 2x - 4 = 0$

$\Rightarrow \quad x = 2$ (stationary point)

$$f''(x) = 2 > 0$$

$\Rightarrow$ $f(x)$ is minimum at $x = 2$

i.e., $(2)^2 - 4(2) + 2 = -2$

$\therefore$ The optimum value of $f(x)$ is -2 (minimum)

180. (b)

The quadratic approximation of $f(x)$ at the point $x = 0$ is

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0)$$
$$= (-5) + x\{0\} + \frac{x^2}{2}\{-6\}$$
$$= -3x^2 - 5$$

181. (d)

Given curve

$$x^2 = 4y \quad \ldots(i)$$

and

$$y^2 = 4x \quad \ldots(ii)$$

$$2x = 4\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = 0 = m_1 \text{ (say)}$$

$$2y\frac{dy}{dx} = 4$$

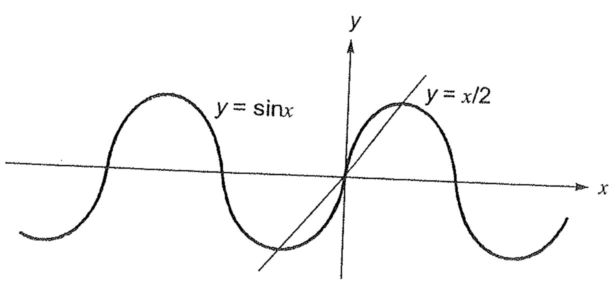
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = \infty = m_2$$

Let $m_2 = \frac{1}{m'}$, where $m' = 0$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{m_1 m' - 1}{m' + m_1}\right| = \left|\frac{0-1}{0+0}\right| = \infty$$

$$\Rightarrow \theta = \frac{\pi}{2} = 90°$$

182. (c)



Hence 3 solutions.

183. Sol.

$$\int_c \bar{F}\cdot\bar{r}'dx = \int_c \bar{F}\cdot d\bar{r} = \int_c F_1 dx + F_2 dy$$

$$= \iint_R \left(\frac{\partial F_2}{dx} - \frac{\partial F_1}{dy}\right) dxdy$$

$$F_1 = y \quad F_2 = 2x$$

$$= \iint_R (2-1)dxdy$$

$$\frac{\partial F_1}{\partial y} = 1 \quad \frac{\partial F_2}{\partial x} = 2$$

$$= \iint dx\,dy$$

$$= \text{Area of the circle with radius } \frac{4}{\sqrt{\pi}}$$

$$= \pi\left(\frac{4}{\sqrt{\pi}}\right)^2 = \pi\frac{16}{\pi} = 16$$

184. Sol.

$$y = mx + c$$

passing through (0, 0)

$$0 = 0 + c \quad \Rightarrow c = 0$$

$$y = mx$$

passing through (2, 6)

$$\therefore \quad 6 = 2m$$

$$\therefore \quad m = 3$$

185. Sol.

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx = -2\int_0^1 \frac{1}{2\sqrt{1-x}}dx$$

$$= -2\left(\sqrt{1-x}\right)\Big|_0^1 = -2(0-1) = 2$$

186. (b)

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}0 = \frac{\pi}{2}$$

and

$$L(\sin x) = \frac{1}{s^2+1}$$

$$\Rightarrow L\left(\frac{\sin x}{x}\right) = \int_s^\infty \frac{1}{s^2+1}\,dx$$

(Using "division by x")

$$= \left[\tan^{-1}s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}(s) = \cot^{-1}(s)$$

$$\Rightarrow \int_0^\infty e^{-sx}\frac{\sin x}{x}dx = \cot^{-1}(s)$$

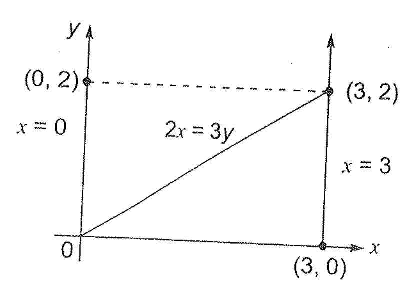
(Using definition of Laplace transform)

Put $s = 0$,

we get $\int_0^\infty \frac{\sin x}{x}dx = \cot^{-1}(0) = \frac{\pi}{2}$

$$\frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx = \pi$$

187. Sol.



$$\text{Volume} = \iiint dz\,dx\,dy$$
$$= \iint z\,dy\,dx$$
$$= \int_0^3 \int_0^{2/3x} (6 - x - y)\,dy\,dx$$
$$= \int_0^3 \left(6y - xy - \frac{y^2}{2}\right)\Bigg|_0^{2/3x} dx$$
$$= \int_0^3 \left(4x - \frac{8}{9}x^2\right) dx$$
$$= \left(4\frac{x^2}{2} - \frac{8}{9}\cdot\frac{x^3}{3}\right)\Bigg|_0^3$$
$$= \left[2x^2 - \frac{8}{9}\left(\frac{x^3}{3}\right)\right]_0^3 = 18 - 8 = 10$$

188. Sol.

Parabola is $x^2 = 8y$

$$y = \frac{x^2}{8} \text{ and straight is } y = 8$$

At the point of intersection, we have

$$\frac{x^2}{8} = 8$$

$\Rightarrow$ $x = -8, 8$ and $y = 8$

$\therefore$ Required area is $\int_{x=-8}^{8}\left(8 - \frac{x^2}{8}\right)dx$

$$= 2\int_0^8 \left(8 - \frac{x^2}{8}\right) dx \quad \left(\because 8 - \frac{x^2}{8} \text{ is even function}\right)$$

$$= 2\left[8x - \frac{x^3}{24}\right]_0^8 = \frac{256}{3} = 85.33 \text{ sq.units}$$

189. Sol.

Put
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = r\,dr\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^2 (r(\cos\theta + \sin\theta) + 10)r\,dr\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^2 (r^2(\cos\theta + \sin\theta) + 10r)\,dr\,d\theta$$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi}(\cos\theta + \sin\theta)\left(\frac{r^3}{3}\right)\Bigg|_0^2 d\theta + 10\int_0^{2\pi}\left(\frac{r^2}{2}\right)\Bigg|_0^2 d\theta\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{8}{3}(\cos\theta + \sin\theta)d\theta + \frac{1}{2\pi}\int_0^{2\pi} 5\cdot(4)d\theta$$

$$= \frac{1}{2\pi}\left[\frac{8}{3}(\sin\theta - \cos\theta)\right]_0^{2\pi} + \frac{1}{2\pi}\cdot 20(2\pi)$$

$$= \frac{1}{2\pi}\left(\frac{8}{3}(0 - 1) - (0 - 1) + 20\right) = 0 + 20 = 20$$

190. Sol.

By Green's theorem

$$\int xy^2dx + x^2ydy = \iint_R \left(\frac{d}{dx}(x^2y) - \frac{d}{dy}(xy^2)\right)dxdy$$

$$= \iint_R (2xy - 2xy) = 0$$

191. (b)

At the point of intersection of the curves,
$y = x^2 + 1$ and $x + y = 3$ i.e., $y = 3 - x$, we have

$$x^2 + 1 = 3 - x$$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow$ $x = -2, 1$ and $3 - x \geq x^2 + 1$

$\therefore$Required area is $\iint_R dydx$

$$= \int_{x=-2}^{1}\left[\int_{y=x^2+1}^{3-x} dy\right]dx$$

$$= \int_{-2}^{1}\{3 - x\} - \left(x^2 + 1\right) dx$$

$$= \left(\frac{-x^3}{3} - \frac{x^2}{2} + 2x\right)_{-2}^{1} = \frac{9}{2}$$

192. Sol.

$$V = \int_{\rho=3}^{5}\int_{\phi=\frac{\pi}{8}}^{\pi/4}\int_{z=3}^{4.5} \rho\,d\rho\,d\phi\,dz = \int_3^{4.5}\int_{\pi/8}^{\pi/4}\left(\frac{\rho^2}{2}\right)\Bigg|_3^5 d\phi\,dz$$

$$= \int_3^{4.5}\int_{\pi/8}^{\pi/4} 8\cdot d\phi\,dz = 8\phi\Big|_{\pi/8}^{\pi/4}\cdot z\Big|_3^{4.5}$$

$$= 8\left(\frac{\pi}{4} - \frac{\pi}{8}\right)(4.5 - 3) = 8\cdot\frac{\pi}{8}\cdot(1.5) = 4.712$$

193. Sol.

$$I = \hat{x}15\cos\omega t + \hat{y}5\sin\omega t$$

$$|I| = \sqrt{(15\cos\omega t)^2 + (5\sin\omega t)^2}$$

$$= \sqrt{225\cos^2\omega t + 25\sin^2\omega t}$$

$$= \sqrt{25 + 200\cos^2\omega t}$$

$|I|$ is minimum when $\cos^2\omega t = 0$

or $\theta = \omega t = 90°$

94. (d)

We know that if $\vec{a}$ and $\vec{b}$ are perpendicular

then $\vec{a}\cdot\vec{b} = 0$

options (a), (b), (c) are perpendicular.

options (d) is not perpendicular.

96. (b)

$$\int_C \bar{F}\cdot\bar{d}r$$

where, $\bar{F} = xy^2\bar{i} + 2x^2y\bar{j} + \bar{k}$

$$\nabla\times\vec{F} = \vec{O}$$

($\vec{F}$ is irrotational $\Rightarrow$ $\vec{F}$ is conservative)

$$\vec{F} = \nabla\phi$$

(ϕ is scalar potational function)

$$\phi_x = 2xy^2$$
$$\phi_y = 2x^2y$$
$$\phi_z = 1$$

$\Rightarrow$ $\phi = x^2y^2 + z + C$

where, $\vec{F}$ is conservative

$$\int_C \bar{F}\cdot\bar{d}r = \int_{(0,0,0)}^{(1,1,1)} d\phi = \left[x^2y^2+z\right]_{(0,0,0)}^{(1,1,1)} = 2$$

7. Sol.

$$F = 5xz\bar{i} + (3x^2+2y)\bar{j} + x^2z\bar{k}$$

$$= \int_C \bar{F}\cdot\bar{d}r$$

$$= \int_C 5xz\,dx + (3x^2+2y)\,dy + x^2z\,dz$$

$x = t,\quad y = t^2,\quad z = t,\ t = 0$ to 1

$dx = dt$

$dy = 2t\,dt,\quad dz = dt$

$$= \int_0^1 5t^2dt + (3t^2+2t^2)2t\,dt + t^3dt$$

$$= \int_0^1 (5t^2+11t^3)\,dt$$

$$= \left[\frac{5t^3}{3}+\frac{11t^4}{4}\right]_0^1 = \frac{5}{3}+\frac{11}{4}$$

$$= \frac{53}{12} = 4.41$$

8. (b)

$$f(x) = e^{-x-e^{-x}} = e^{-x}\cdot e^{-e^{-x}}$$

$$y(x) = \int f(x)dx = \int e^{-x}\cdot e^{-e^{-x}}dx$$

Let $e^{-x} = t$

$$-e^{-x}\,dx = dt$$

$$\int f(x)dx = \int e^{-t}\cdot(-dt)$$

$$= \frac{e^{-t}}{-1}\cdot(-d)$$

$$= e^{-t}$$

$$= e^{-\left(e^{-x}\right)} = e^{-e^{-x}}$$

199. Sol.

$$\bar{F} = -y\bar{i} + x\bar{j}$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$

$$= 0 + 0 = 0$$

200. Sol.

$$\vec{F} = (x+y)i + (x+z)j + (y+z)k$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(y+z)$$

$$= 1 + 0 + 1 = 2$$

By Gauss divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\overrightarrow{dS} = \iint_V \nabla\cdot\vec{F}\,dV = \iint_V 2\,dV$$

$= 2V$ where V is volume of $x^2 + y^2 + z^2 = 9$

$$= 2\left(\frac{4}{3}\pi(3)^3\right) = 226.08$$

201. (d)

$$\lim_{x\to 0}\frac{x^3-\sin x}{x} \quad \left(\frac{0}{0}\text{form}\right)$$

$$\lim_{x\to 0}\frac{3x^2-\cos x}{1} = 0 - \cos 0$$

$$= 0 - 1 = -1$$

202. (c)

$$x = \cos\left(\frac{\pi u}{2}\right);\quad y = \sin\left(\frac{\pi u}{2}\right)$$

$$x^2 + y^2 = 1$$

It represents a circle in x-y plane.

$0 \le u \le 1$ (given range)

$\therefore$ $0 \le x \le 1,\ 0 \le y \le 1$

so, $0 \le \theta \le \frac{\pi}{2}$

Thus, we will get a quarter circle in x-y plane and when rotate by 360°, we get a hemisphere

$\therefore$ Area of hemisphere $= 2\pi(r)^2 = 2\pi\times(1)^2 = 2\pi$

203. Sol.

By vector identities

div (curl $\vec{F}$) = 0

204. Sol.

Since V is non-zero vector of dimension 3 × 1

Therefore,

$$\rho(A) \le \min\{\rho(V), \rho(V^T)\}$$
$$\le \min\{1, 1\}$$
$$\le 1$$

Since V is non-zero. Hence $\rho(A) = 1$

205. Sol.

$$\bar{x}_1 = 2\bar{i} + 6\bar{j} + 14\bar{k}$$

$$\bar{x}_2 = -12\bar{i} + 8\bar{j} + 16\bar{k}$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$$

$$= \frac{(2\bar{i}+6\bar{j}+14\bar{k})(-12\bar{i}+8\bar{j}+16\bar{k})}{\sqrt{4+36+196}\sqrt{144+64+256}}$$

$$= \frac{-24+48+224}{(15.36)(21.54)} = \frac{248}{330.8544} = 0.7495$$

$$\theta = \cos^{-1}(0.7495) = 41.45°$$

$$= 0.723 \text{ radians}$$

206. Sol.

$$x + 2y = 11,$$
$$x = 11 - 2y$$
$$2x^2 + y^2 = 34$$
$$2(11 - 2y)^2 + y^2 = 34$$
$$24^2 + 8y^2 - 88y + y^2 - 34 = 0$$
$$9y^2 - 88y + 208 = 0$$
$$y = 5.77, 4$$
$$x = -0.54,$$
$$x = 11 - 2(4) = 3$$
$$x = 3, y = 4$$
$$x + y = 3 + 4 = 7$$

207. Sol.

$$y^2 - 2y + 1 - x$$
$$(y-1)^2 = x$$
$$y - 1 = \sqrt{x}$$
$$y = 1 + \sqrt{x}$$
$$y = 1+\sqrt{4} = 3$$
$$\therefore \quad = 3$$

$$\sqrt{x} + y = 5$$
$$\sqrt{x} + 1 + 5x = 5$$
$$2\sqrt{x} = 4$$
$$\sqrt{x} = 2$$
$$x = 4$$

$$x + \sqrt{y} = 4 + \sqrt{3} = 5.732$$

208. (a)

$$f(x) = 1 - x; \quad x < 0$$
$$g(x) = -x; \quad x < 0$$ (Both are continuous for $x < 0$)

$\therefore$ fog(x) is continous for $x < 0$

The composite function of two continous functions is always continous.

Therefore the number of discontinuities are zero

209. Sol.

$$I = C\iint xy^2 dx\,dy$$

$$= C\int_{x=1}^{5}\int_{y=0}^{2x} xy^2 dy\,dx$$

$$= C\int_1^5 x.\left(\frac{y^3}{3}\right)\Bigg|_0^{2x} dx$$

$$= C\int_1^5 x.\frac{8x^3}{3}dx = C\int_1^5 8\frac{x^4}{3}dx$$

$$= C.\frac{8}{3}\left(\frac{x^5}{5}\right)\Bigg|_1^5$$

$$= \frac{8C}{3}\left(5^4 - \frac{1}{5}\right) = \frac{8C}{3}(625 - 0.2)$$

$$= \frac{8}{3}(6\times10^{-4})(625 - 0.2)$$

$$= 0.99968$$

210. (a)

$$f(x) = \begin{cases} e^x & x < 1 \\ \ln x + ax^2 + bx & x \ge 1 \end{cases}$$

$$\text{L.H.D} = \underset{x\to1^-}{Lt}\frac{f(x) - f(1)}{x - 1}$$

$$= \underset{x\to1^-}{Lt}\frac{e^x - (a+b)}{x-1} = \underset{x\to1}{Lt}\frac{e^x}{1} = e$$

$$\text{R.H.D} \quad \underset{x\to1^+}{Lt}\frac{f(x) - f(1)}{x-1}$$

$$= \underset{x\to1^+}{Lt}\frac{\ln x + ax^2 + bx - a + b}{x - 1}$$

$$= \underset{x\to1}{Lt}\frac{\frac{1}{x} + 2ax + b}{1} = 1 + 2a + b$$

L.H.D ≠ R.H.D.

$\therefore$ $f(x)$ is not derivable at $x = 1$

11. **Sol.**

If θ is the angle between

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$a_1 = 1 \quad b_1 = 1 \quad c_1 = 1 \quad d_1 = -1$

$a_2 = 2 \quad b_2 = -1 \quad c_2 = 2 \quad d_2 = 0$

$$\cos\theta = \frac{(1)(2)+(1)(-1)+(1)(2)}{\sqrt{(1)^2+(1)^2+(1)^2}\sqrt{(2)^2+(-1)^2+(2)^2}}$$

$$= \frac{2-1+2}{\sqrt{3}\sqrt{9}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.73^\circ$$

12. **Sol.**

$$f(x) = \frac{1}{3}x\left(x^2-3\right) = \frac{1}{3}\left(x^3-3x\right)$$

$$f'(x) = \frac{1}{3}\left(3x^2-3\right) = x^2-1$$

$$f'(x) = x^2 - 1 = 0$$

$\Rightarrow \quad x = \pm 1$

$$f''(x) = 2x$$

At $x = 1, f''(1) = 2 = 0 \quad \Rightarrow$ minima

At $x = -1, f''(-1) = -2 < 0 \quad \Rightarrow$ maxima

Minimum value of $f(x)$ in $[-100, 100]$ is given by

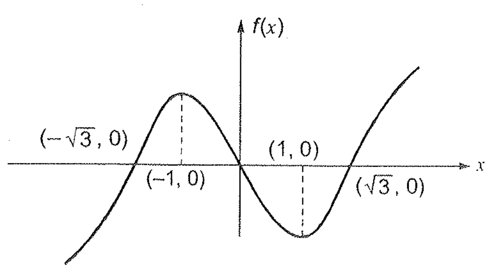
Minimum $\{f(-100), f(100), f(1)\}$

Minimum $\{-333433.3, 333233.3, -0.666\}$

$= -3335433.3$

Hence the minimum value occurs at $x = -100$

Also graph of the function will be like



3. (c)

$$\text{Integral } I_1 = \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dydx$$

$$= \int_0^1\left[\int_0^1\left(\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right)dy\right]dx$$

$$= \int_0^1\left[2x\left(\frac{-1}{2(x+y)^2}\right)\Bigg|_0^1 + \left(\frac{1}{x+y}\right)\Bigg|_0^1\right]dx$$

$$= \int_0^1 \frac{1}{(x+1)^2}dx = -\left[\frac{1}{x+1}\right]_0^1 = 0.5$$

and Integral

$$I_2 = \int_0^1\left(\int_0^1 \frac{x-y}{(x+y)^3}dx\right)dy$$

$$= \int_0^1\left(\int_0^1 \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}dx\right)dy$$

$$= \int_0^1\left[\frac{-1}{(x+y)} + \frac{2y}{2(x+y)}\right]_0^1 dy$$

$$= \int_0^1 \frac{-1}{(1+y)^2}dy = -\left[\frac{-1}{y+1}\right]\Bigg|_0^1 = -0.5$$

Option (c) is correct.

214. (b)

$$\vec{F} = \hat{a}_x(3y - k_1z) + \hat{a}_y(k_2x - 2z) - \hat{a}_z(k_3y + z)$$

$\nabla\times\vec{F} = 0$ (irrotational)

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y - k_1z & k_2x - 2z & -(k_3y + z) \end{vmatrix}$$

$$= \hat{a}_x\left[\frac{\partial}{\partial y}[-(k_3y+z)] - \frac{\partial}{\partial z}(k_2x - 2z)\right]$$

$$-\hat{a}_y\left[\frac{\partial}{\partial x}[-(k_3y+z)] - \frac{\partial}{\partial z}(3y - k_1z)\right]$$

$$+\hat{a}_z\left[\frac{\partial}{\partial x}(k_2x - 2z) - \frac{\partial}{\partial y}(3y - k_1z)\right]$$

$$\hat{a}_x[-k_3 + 2] - \hat{a}_y[k_1] + \hat{a}_z[k_2 - 3] = 0$$

$\Rightarrow \quad k_3 = 2, \; k_1 = 0, \; k_2 = 3$

or $\quad k_1 = 0, \; k_2 = 3, \; k_3 = 2$

215. **Sol.**

$A(0, 2, 1)$ and $B(4, 1, -1)$

The equation of the line AB is

$$\frac{x-0}{4-0} = \frac{y-2}{1-2} = \frac{3-1}{-1-1} = t \text{ say}$$

$x = 4t \quad ; \quad y = -t + 2 \quad ; \quad z = -2t + 1$

$dx = 4dt \quad ; \quad dy = -dt \quad ; \quad dz = -2dt$

t varies from 0 to 1

$$I = \int_0^1 2(-2t+1)\,4dt + 2(-t+2)\,(-dt) + 2(4t)\,(-2dt)$$

$$= \int_0^1 (-16t + 8 + 2t - 4 - 16t)dt$$

$$= \int_0^1 (-30t + 4)dt$$

$$= \left(-30\frac{t^2}{2} + 4t\right)\Bigg|_0^1 = -15 + 4 = -11$$

216. (c)

About $x = 0$

$$f(x) = e^x\, e^{x^2}$$

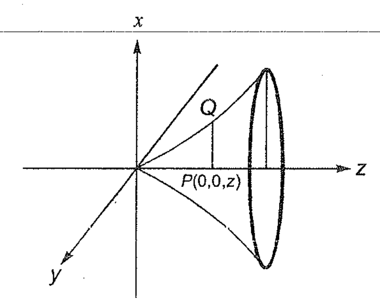
$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\right)$$

$$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots + x + x^3 + \frac{x^5}{2} + \frac{x^7}{6} + \ldots$$

$$+ \frac{x^2}{2} + \frac{x^4}{2} + \frac{x^6}{4} + \frac{x^8}{12} \ldots + \frac{x^3}{6} + \frac{x^5}{6} + \frac{x^7}{12} + \frac{x^9}{36} + \ldots$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$$

217. Sol.



Let $x^2 + y^2 = t^2$

$t^2 = z^3$

Here revolution is about z axis

volume of region $R = \int_0^1 \pi (PQ)^2 dz$

Here PQ is radius of circle at some z, which is given by

$$PQ = \sqrt{x^2 + y^2}$$

$$(PQ)^2 = x^2 + y^2 = z^3$$

so,volume of region R

$$= \int_0^1 \pi t^2 dz = \int_0^1 \pi z^3 dz = \frac{\pi z^4}{4}\Bigg|_0^1 = \frac{\pi}{4} = 0.7853$$

218. (c)

Given, $\quad f(x) = R\sin\left(\frac{\pi x}{2}\right) + S \qquad \ldots(1$

$$f'\left(\frac{1}{2}\right) = \sqrt{2} \qquad \ldots(2)$$

$$\int_0^1 f(x)dx = \frac{2R}{\pi} \qquad \ldots(3)$$

Now we need to find R and S.

$$f'(x) = R\cos\left(\frac{\pi x}{2}\right)\frac{\pi}{2}$$

$$f'\left(\frac{1}{2}\right) = R\cos\left(\frac{\pi}{4}\right) \times \frac{\pi}{2} = \sqrt{2}$$

$$\Rightarrow \quad \frac{R}{\sqrt{2}} \times \frac{\pi}{2} = \sqrt{2}$$

$$\Rightarrow \quad R = \frac{4}{\pi}$$

Now, $\int f(x)dx = \int\left(R\sin\frac{\pi x}{2} + S\right)dx$

Putting $\quad R = \frac{4}{\pi}$ we get

$$\int f(x)dx = \int \frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right)dx + \int S\,dx$$

$$= \frac{4}{\pi} \times -\frac{\cos\left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}} + Sx = \frac{-8}{\pi^2}\cos\left(\frac{\pi x}{2}\right) + Sx$$

Putting limit 0 and 1

$$\int_0^1 f(x)dx = \frac{-8}{\pi^2}\left(\cos\frac{\pi}{2} - \cos(0)\right) + S(1-0) = \frac{2R}{\pi}$$

$$\Rightarrow \frac{-8}{\pi^2}(0-1) + S = \frac{2R}{\pi}$$

Put $R = \frac{4}{\pi}$ and solve for S

$\Rightarrow \qquad S = 0$

So, $R = \frac{4}{\pi}$ and $S = 0$ is answer.

219. (c)

$$\lim_{x\to 1}\frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \frac{1-2+1}{1-3+2}$$

$= 0/0$ form

So, use L′ Hospitals rule $= \lim_{x\to 1}\frac{7x^6 - 10x^4}{3x^2 - 6x}$

$$= \frac{7-10}{3-6} = \frac{-3}{-3} = 1$$

220. (c)

$$w = f(x, y)$$

By chain rule,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\times\frac{dx}{dt}+\frac{\partial w}{\partial y}\times\frac{dy}{dt}$$

221. Sol.

$$\vec{V} = x^2\hat{i}+2y^3\hat{j}+z^4\hat{k}$$

$$\vec{\nabla}.\vec{V} = \left[\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right].\left[x^2\hat{i}+2y^3\hat{j}+z^4\hat{k}\right]$$

$$= 2x + 6y^2 + 4z^3$$

At (1, 2, 3),

$$\vec{\nabla}.\vec{V} = 2 + 6(2)^2 + 4(3)^3$$

$$= 134$$

222. (a)

$$\tan\theta = \frac{dy}{dx} = \ln x+1$$

$$\tan 45° = \ln x + 1$$

$$1 = \ln x + 1$$

$$\Rightarrow \quad \ln x = 0$$

$$\therefore \quad x = 1$$

Putting $x = 1$ in the eq. of curve,

we get $y = 0$.

223. (a)

Let, $\sin^{-1}x = t$

$$\frac{dx}{\sqrt{1-x^2}} = dt$$

$$I = \int_0^{\pi/2} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{\pi/2} = \frac{\pi^3}{24}$$

224. (a)

Space headway,

$$S = 60t - 60t^2$$

$$\frac{dS}{dt} = 60 - 120t = 0$$

$$t = 0.5 \text{ hr} = 30 \text{ minutes}$$

$$\frac{d^2S}{dt^2} = -120 \times 0 \text{ (Maxima)}$$

$\therefore$Maximum space head

$$S_{max} = 60 \times 0.5 - 60 \times (0.5)^2 = 15 \text{ km}$$

225. Sol.

$$\underset{x\to 0}{Lt}\frac{\tan x}{x^2-x} \qquad \text{(Applying L'Hospital rule)}$$

$$\underset{x\to 0}{Lt}\frac{\sec^2 x}{2x^2-1} = \frac{\sec^2 0}{0-1} = \frac{1}{-1} = -1$$

226. (a)

$$f(x) = \frac{x^3}{3}-x$$

We will find the first 2nd derivative

$$f'(x) = \frac{3x^2}{3}-1 = x^2-1$$

and $f''(x) = 2x$

to determine minimum value of x,

put $f'(x) = x^2 - 1 = 0$ gives $x = 1$ or -1

For $x = 1$ only, $f''(x) > 0$ which means minimum value of the function exists for $x = 1$.

Alt: this question can be directly solved by putting given values of x.

227. (a)

$$\int_0^{2\pi}\frac{3}{9+\sin^2\theta}d\theta = \int_0^{2\pi}\frac{3\sec^2\theta}{9\sec^2\theta+\tan^2\theta}d\theta$$

$$= 4\int_0^{\pi/2}\frac{3\sec^2\theta}{9\sec^2\theta+\tan^2\theta}d\theta = 4\int_0^{\pi/2}\frac{3\sec^2\theta}{9+10\tan^2\theta}d\theta$$

$$= \frac{12}{10}\int_0^{\pi/2}\frac{\sec^2\theta}{\frac{9}{10}+\tan^2\theta}d\theta$$

Let, $\tan\theta = t$, $\sec^2\theta\,d\theta = dt$

Limits	
$\theta = 0$	$t = 0$
$\theta = \frac{\pi}{2}$	$t \to \infty$

$$= \frac{12}{10}\times\int_0^{\pi/2}\frac{dt}{\left(\sqrt{\frac{9}{10}}\right)^2+t^2}$$

$$= \frac{12}{10}.\left[\frac{1}{\frac{3}{\sqrt{10}}}\tan^{-1}\frac{t}{\frac{3}{\sqrt{10}}}\right]_0^{\infty}$$

$$= \frac{4}{\sqrt{10}}\left[\frac{\pi}{2}-0\right] = \frac{2\pi}{\sqrt{10}}$$

228. (b)

Dot product of two vectors A and B is defined as

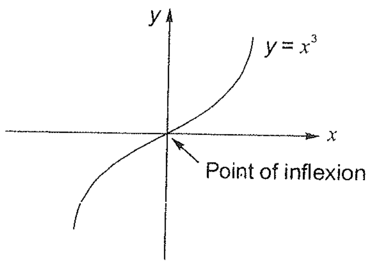
$$A \cdot B = |A|\,|B| \cos\theta_n,$$

1. Dot product will be less than or equal to the product of magnitude of two vectors.
2. For perpendicular vector $\theta_n = 90°$, dot product $= 0$.
3. If $\theta_n < 90°$ the dot product is positive and for $\theta_n > 90°$ the dot product is negative.
4. $A \cos\theta_n$ is projection of vector A on vector B thus dot product is product of one vector and projection of the vector on first one.

Thus 1, 3 and 4 statements are correct.

229. (d)

$$f(x) = x^3 \text{ at } x = 0$$



At $x = 0$, the function $y = x^3$ has neither minima nor maxima.

230. (b)

The value of $\int_0^{\pi} x\cos^2 x\,dx$

$$= \int_0^{\pi}\left(\frac{x}{2} + \frac{x\cos 2x}{2}\right)dx$$

$$= \frac{x^2}{4}\Big|_0^{\pi} + \frac{1}{2}\left(\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right)$$

$$= \frac{\pi^2}{4} + \frac{1}{2}\left\{\left(0+\frac{1}{4}\right) - \left(0+\frac{1}{4}\right)\right\}$$

$$= \frac{\pi^2}{4} + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4}\right)$$

$$= \frac{\pi^2}{4}$$

231. (c)

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\int \frac{1}{y}dy = \int \frac{-1}{x}dx$$

$$\ln y = -\ln x + c$$

when $y = 1, x = 1$

$$c = 0$$

$$\Rightarrow \quad y = \frac{1}{x} = x^{-1}$$

232. Sol.

$$\bar{F} = x^2\bar{i} + y^2\bar{j}$$

$$\int \bar{F}\cdot d\bar{r} = \int (x^2\bar{i} + y^2\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j})$$

$$= \int x^2dx + y^2dy$$

(0, 0) to (1, 1) line is $y = x$

$$= \int x^2dx + x^2dx = \int_0^1 2x^2dx$$

$$= 2\left(\frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{3} \simeq 0.67$$

233. (a)

$$\int_a^b f(x)dx = f(x)(b-a)$$

234. (c)

By Gauss Divergence Theorem

$$\iint_s \vec{r}\cdot\hat{n}ds = \iiint_v \bar{V}\cdot\vec{r}dV$$

$$= \iiint_v 3\,dV = 3V$$

235. (c)

$$\vec{u} = e^x\cos y\,\hat{i} + e^x\cdot\sin y\,\hat{j}$$

$$\nabla\cdot\vec{u} = \frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial y}(u_2)$$

$$= \frac{\partial}{\partial x}(e^x\cdot\cos y) + \frac{\partial}{\partial y}(e^x\cdot\sin y)$$

$$= e^x\cos y + e^x\cos y$$

$$\nabla\cdot\vec{u} = 2e^x\cdot\cos y$$

236. (c)

$$\phi = \ln r$$

$$\nabla\phi = \nabla(\ln r)$$

$$f(r) = \ln(r)\ ;\ f'(r) = \frac{1}{r}$$

$$\nabla f(r) = \frac{f'(r)}{r}\cdot\vec{r} = \left(\frac{1}{r}\right)\times\left(\frac{1}{r}\right)\cdot\vec{r} = \frac{\vec{r}}{r^2}$$

237. Sol.

$$f(x) = \int_0^x e^{-t^2/2}dt$$

$$f'(x) = e^{-x^2/2} \text{ and } f''(x) = e^{-x^2/2}(-x)$$
$$f''(0) = 0$$

$$a_2 = \frac{f''(0)}{2!} = 0$$

238. (d)

$$f(x, y) = \frac{ax^2 + by^2}{xy} = a\left(\frac{x}{y}\right) + b\left(\frac{y}{x}\right)$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \left.\left[\frac{a}{y} - \frac{by}{x^2}\right]\right|_{(1,2)} = \frac{a}{2} - 2b$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = \left.\left[-\frac{ax}{y^2} + \frac{b}{x}\right]\right|_{(1,2)} = -\frac{a}{4} + b$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

So, $\frac{a}{2} - 2b = -\frac{a}{4} + b$

$$\frac{3a}{4} = 3b$$

$$a = 4b$$

239. (a)

$$\phi = xy^2 + yz^2 + zx^2$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\left(y^2 + 2xz\right) + \bar{j}\left(2xy + z^2\right) + \bar{k}\left(2yz + x^2\right)$$

$$\nabla\phi_{(2,-1,1)} = \bar{i}(1+4) + \bar{j}(-4+1) + \bar{k}(-2+4)$$

$$= 5\bar{i} - 3\bar{j} + 2\bar{k}$$

$$\bar{P} = \bar{i} + 2\bar{j} + 2\bar{k}$$

$$|\bar{P}| = \sqrt{1+4+4} = 3$$

The directional derivative of $\phi(x, y, z)$ at $(2, -1, 1)$

in the direction of $\bar{P}$ is $\nabla\phi_{\text{at } P} \cdot \frac{\bar{P}}{|\bar{P}|}$

$$= \left(5\bar{i} - 3\bar{j} + 2\bar{k}\right) \cdot \left(\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}\right)$$

$$= \frac{5 - 6 + 4}{3} = 1$$

240. Sol.

$$\int_{0.25}^{1.25} f(x)\,dx = \int_{0.25}^{1} (x - [x])dx + \int_{1}^{1.25} (x - [x])\,dx$$

$$= \int_{0.25}^{1.25} (x\,dx) - \left[\int_{0.25}^{1} [x]dx + \int_{1}^{1.25} [x]dx\right]$$

$$= \left.\left(\frac{x^2}{2}\right)\right|_{0.25}^{1.25} - \left(\int_{0.25}^{1} 0\,dx + \int_{1}^{1.25} 1\,dx\right)$$

$$= \frac{(1.25)^2}{2} - \left(\frac{(0.25)^2}{2}\right) - (0 + 0.25)$$

$$= \frac{1}{2}\left[(1.5625 - 0.0625)\right] - 0.25$$

$$= 0.5$$

241. (d)

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2 & x \geq 0 \\ -2 & x < 0 \end{cases}$$

The first derivation of f (i.e) $f'(x)$ is not derivable at $x = 0$.

242. Sol.

$$x^2 + y^2 = 9$$
$$x = 3\cos\theta$$
$$y = 3\sin\theta$$
$$dx = -3\sin\theta\, d\theta$$
$$dy = 3\cos\theta\, d\theta$$

θ varies from 0 to $\pi/2$

$$\int\left(y^2 + 2xy\right)dx + \left(2xy + x^2\right)dy$$

$$= \int_0^{\pi/2} \begin{array}{l}(9\sin^2\theta + 18\sin\theta\cos\theta)(-3\sin\theta\, d\theta) \\ +(18\sin\theta\cos\theta + 9\cos^2\theta)(3\cos\theta)\,d\theta\end{array}$$

$$= \int_0^{\pi/2} \begin{pmatrix} -27\sin^3\theta - 54\sin^2\theta\cos\theta \\ +54\sin\theta\cos^2\theta + 27\cos^3\theta \end{pmatrix} d\theta$$

$$= 0$$

243. Sol.

$$f(x) = 3x^3 - 7x^2 + 5x + 6 \qquad \text{in}[0, 2]$$
$$f'(x) = 9x^2 - 14x + 5$$
$$f''(x) = 18x - 14$$

$$f'(x) = 0$$
$$x^2 - 14x + 5 = 0$$
$$x = 1, 0.55$$
$$x = 1$$
$$f''(1) = 18 - 14 = 4 > 0 \text{ minima}$$
$$x = 0.55$$
$$f''(0.55) = -4.1 < 0 \text{ maxima}$$

Maximum $\{f(0), f(0.55), f(2)\}$

Maximum $\{6, 7.13, 12\} = 12$

244. (d)

From trial and error method, we can found that, option (d) is correct.

245. (d)

$$x = y = z$$
$$\Rightarrow \quad dx = dy = dz$$
$$\int_C \vec{F}\cdot d\vec{l} = \int_C \left(x^2 - 2y\right)dx - 4yzdy + 4xz^2dz$$
$$= \int_0^1 \left(x^2 - 2x - 4x^2 + 4x^3\right)dx$$
$$= -3\left[\frac{x^3}{3}\right]_0^1 - 2\left[\frac{x^2}{2}\right]_0^1 + 4\left[\frac{x^4}{4}\right]_0^1$$
$$= -1 - 1 + 1 = -1$$

246. Sol.

$$\int_0^{\pi/4} x\cos(x^2)\,dx$$

Let, $t = x^2$

$$dt = 2x\,dx$$
$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

when $x = 0$, $t = 0$ and when $x = \frac{\pi}{4}, t = \left(\frac{\pi}{4}\right)^2$

So required integral reduce to

$$\int_0^{(\pi/4)^2} \cos t\,dt = [\sin t]_0^{(\pi/4)^2}$$
$$= \sin\left(\frac{\pi}{4}\right)^2 - \sin(0)$$
$$= \sin\left(\frac{\pi}{4}\right)^2 = 0.28898$$
$$\simeq 0.289$$

(rounded to 3 decimal places)

247. (b)

$u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ is non-homogeneous

$F(u) = e^u = \frac{x^2 + y^2}{x + y}$ is homogeneous function of degree

$$n = x\,u_x + y\,u_y$$
$$= n\frac{F(u)}{F^1(u)} = 1\frac{e^u}{e^u} = 1$$