Transport Phenomena (Part - 1)

Q. 220. Calculate what fraction of gas molecules
(a) traverses without collisions the distances exceeding the mean free path λ;
(b) has the free path values lying within the interval from λ to 2λ.

Solution. 220. (a) The fraction of gas molecules which traverses distances exceeding the mean free path without collision is just the probability to traverse the distance s = λ without collision.

Thus $P = e^{-1} = \frac{1}{e} = 0{\cdot}37$

(b) This probability is $P = e^{-1} - e^{-2} = 0{\cdot}23$

Q. 221. A, narrow molecular beam makes its way into a vessel filled with gas under low pressure. Find the mean free path of molecules if the beam intensity decreases η-fold over the distance Δl.

Solution. 221. From the formula

$$\frac{1}{\eta} = e^{-\Delta l/\lambda} \quad \text{or} \quad \lambda = \frac{\Delta l}{\ln \eta}$$

Q. 222. Let αdt be the probability of a gas molecule experiencing a collision during the time interval dt; α is a constant. Find:
(a) the probability of a molecule experiencing no collisions during the time interval t;
(b) the mean time interval between successive collisions.

Solution. 222. (a) Let P (f) = probability of no collision in the interval (0, t). Then

$$P(t + dt) = P(t)(1 - \alpha\, dt)$$

Or $\frac{dP}{dt} = -\alpha P(t) \quad \text{or} \quad P(t) = e^{-\alpha t}$

Where we have used P (0) = 1

(b) The mean interval between collisions is also the mean interval of no collision. Then

$$< t > = \frac{\int_0^\infty t e^{-\alpha t} dt}{\int_0^\infty e^{-\alpha t} dt} = \frac{1}{\alpha} \frac{\Gamma(2)}{\Gamma(1)} = \frac{1}{\alpha}$$

Q. 223. Find the mean free path and the mean time interval be- tween successive collisions of gaseous nitrogen molecules
(a) under standard conditions;
(b) at temperature t = 0°C and pressure p = 1.0 nPa (such a pressure can be reached by means of contemporary vacuum pumps).

Solution. 223.

$$\text{(a)} \quad \lambda = \frac{1}{\sqrt{2}\,\pi d^2 n} = \frac{kT}{\sqrt{2}\,\pi d^2 p}$$

$$= \frac{1{\cdot}38 \times 10^{-23} \times 273}{\sqrt{2}\,\pi (0{\cdot}37 \times 10^{-9})^2 \times 10^5} = 6{\cdot}2 \times 10^{-8}\text{ m}$$

$$\tau = \frac{\lambda}{<v>} = \frac{6{\cdot}2 \times 10^{-8}}{454} s = {\cdot}136 \text{ n s}$$

$$\lambda = 6.2 \times 10^6 \text{ m}$$

$$\text{(b)} \quad \eta = 1{\cdot}36 \times 10^4 \text{ s} = 3{\cdot}8 \text{ hours}$$

Q. 224. How many times does the mean free path of nitrogen molecules exceed the mean distance between the molecules under standard conditions?

Solution. 224. The mean distance between molecules is of the order

$$\left(\frac{22{\cdot}4 \times 10^{-3}}{6{\cdot}0 \times 10^{23}}\right)^{1/3} = \left(\frac{224}{6}\right)^{1/3} \times 10^{-9}\text{ meters} \approx 3{\cdot}34 \times 10^{-9}\text{ meters}$$

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This is about 18.5 times smaller than the mean free path calculated in 2.223 (a) above.

Q. 225. Find the mean free path of gas molecules under standard conditions if the Van der Waals constant of this gas is equal to b = 40 ml/mol.

Solution. 225. We know that the Vander Waal’s constant b is four times the molecular volume. Thus

$$b = 4 N_A \frac{\pi}{6} d^3 \quad \text{or} \quad d = \left(\frac{3b}{2\pi N_A}\right)^{1/3}$$

Hence $\lambda = \left(\frac{kT_0}{\sqrt{2}\,\pi p_0}\right)\left(\frac{2\pi N_A}{3b}\right)^{2/3}$

Q. 226. An acoustic wave propagates through nitrogen under standard conditions. At what frequency will the wavelength be equal to the mean free path of the gas molecules?

Solution. 226. The velocity of sound in N_2 is

$$\sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

so, $\frac{1}{\nu} \sqrt{\frac{\gamma RT_0}{M}} = \frac{RT_0}{\sqrt{2}\,\pi d^2 p_0 N_A}$

or, $\nu = \pi d^2 p_0 N_A \sqrt{\frac{2\gamma}{MRT_0}}$

Q. 227. Oxygen is enclosed at the temperature 0°C in a vessel with the characteristic dimension l = 10 mm (this is the linear dimension determining the character of a physical process in question). Find:
(a) the gas pressure below which the mean free path of the molecules λ > 1;
(b) the corresponding molecular concentration and the mean distance between the molecules.

Solution. 227.

(a) $\lambda > l$ if $p < \frac{kT}{\sqrt{2}\,\pi d^2 l}$

Now $\frac{kT}{\sqrt{2}\,\pi d^2 l}$ for O_2 of O is $0{\cdot}7$ Pa.

(b) The corresponding n is obtained by dividing by kT and is $1{\cdot}84 \times 10^{20}$ per m^3 = 1.84 upper c.c. and the corresponding mean distance is $\frac{l}{n^{1/3}}$

$$= \frac{10^{-2}}{(0{\cdot}184)^{1/3} \times 10^{5}} = 1{\cdot}8 \times 10^{-} \, m = 0{\cdot}18\ \mu m.$$

Q. 228. For the case of nitrogen under standard conditions find:
(a) the mean number of collisions experienced by each molecule per second;

(b) the total number of collisions occurring between the molecules within 1 cm^3 of nitrogen per second.

Solution. 228.

$$(a)\quad \nu = \frac{1}{\tau} = \frac{1}{\lambda/<v>} = \frac{<v>}{\lambda}$$

$$= \sqrt{2}\,\pi d^2 n <v> = \cdot 74 \times 10^{10}\,s^{-1} \text{ (see 2.223)}$$

(b) Total number of collisions is

$$\frac{1}{2} n\nu \approx 1{\cdot}0 \times 10^{29}\,s\,cm^{-3}$$

Note, the factor 1/2. When two molecules collide we must not count it twice.

Q. 229. How does the mean free path λ and the number of collisions of each molecule per unit time v depend on the absolute temperature of an ideal gas undergoing
(a) an isochoric process;
(b) an isobaric process?

Solution. 229.

$$(a)\quad \lambda = \frac{1}{\sqrt{2}\,\pi d^2 n}$$

d is a constant and n is a constant for an isochoric process so X is constant for an isochoric process.

$$\nu = \frac{<v>}{\lambda} = \frac{\sqrt{\frac{8RT}{M\pi}}}{\lambda}\,\alpha\,\sqrt{T}$$

(b) $\lambda = \frac{1}{\sqrt{2}\,\pi d^2}\frac{kT}{p}\,\alpha\,T$ for an isobaric process.

$\nu = \frac{<v>}{\lambda}\,\alpha\,\frac{\sqrt{T}}{T} = \frac{1}{\sqrt{T}}$ or an isobaric process.

Q. 230. As a result of some process the pressure of an ideal gas increases n-fold. How many times have the mean free path λ and the number of collisions 'of each

molecule per unit time v changed and how, if the process is
(a) isochoric;
(b) isothermal?

Solution. 230. (a) In an isochoric process λ is constant and

$$v \propto \sqrt{T} \propto \sqrt{pV} \propto \sqrt{p} \propto \sqrt{n}$$

(b) $\lambda = \frac{kT}{\sqrt{2}\,\pi d^2 p}$ must decrease n times in an isothermal process and v must increase n times because <v > is constant in an isothermal process.

Q. 231. An ideal gas consisting of rigid diatomic molecules goes through an adiabatic process. How do the mean free path λ and the number of collisions of each molecule per second v depend in this process on
(a) the volume V;
(b) the pressure p;
(c) the temperature T?

Solution. 231.

(a) $\lambda \propto \frac{1}{n} \Rightarrow \frac{1}{N/V} = \frac{V}{N}$

Thus $\lambda \propto V$ and $v \propto \frac{T^{1/2}}{V}$

But In an adiabatic process $\left(\gamma = \frac{7}{5} \text{ here}\right)$

$TV^{\gamma - 1} =$ constant so $TV^{2/5} =$ constant

or $T^{1/2} \propto V^{-1/5}$ Thus $v \propto V^{-6/5}$

(b) $\lambda \propto \frac{T}{p}$

But $p\left(\frac{T}{p}\right)^{\gamma} =$ constant or $\frac{T}{p} \propto p^{-1/\gamma}$ or $T \propto p^{1-1/\gamma}$

Thus $\lambda \alpha p^{-1/\gamma} = p^{-5/7}$

$$\nu = \frac{\langle v \rangle}{\lambda} \alpha \frac{p}{\sqrt{T}} \alpha p^{1/2 + \frac{1}{2\gamma}} = p^{\frac{\gamma+1}{2\gamma}} = p^{6/7}$$

(c) $\lambda \alpha V$

But $TV^{2/5} =$ constant or $V \alpha T^{-5/2}$

Thus $\lambda \alpha T^{-5/2}$

$$\nu \alpha \frac{T^{1/2}}{V} \alpha T^3$$

Q. 232. An ideal gas goes through a polytrophic process with exponent n. Find the mean free path and the number of collisions of each molecule per second v as a function of
(a) the volume V;
(b) the pressure p;
(c) the temperature T.

Solution. 232. In the polytrophic process of index n

$pV^n =$ constant, $TV^{n-1} =$ constant and $p^{1-n} T^n =$ constant

(a) $\lambda \alpha V$

$$\nu \alpha \frac{T^{1/2}}{V} = V^{\frac{1-n}{2}} V^{-1} = V^{\frac{-n+1}{2}}$$

(b) $\lambda \alpha \frac{T}{p}$, $T^n \alpha p^{n-1}$ or $T \alpha p^{1-\frac{1}{n}}$

so $\lambda \alpha p^{-1/n}$

$$\nu = \frac{\langle v \rangle}{\lambda} \alpha \frac{p}{\sqrt{T}} \alpha p^{1-\frac{1}{2}+\frac{1}{2n}} = p^{\frac{n+1}{2n}}$$

(c) $\lambda \alpha \frac{T}{p}$, $p \alpha T^{\frac{n}{n-1}}$

$$\lambda \alpha T^{1-\frac{n}{n1}} = T^{-\frac{1}{n-1}} = T^{\frac{1}{1-n}}$$

$$v \alpha \frac{p}{\sqrt{T}} \alpha T^{\frac{n}{n-1}-\frac{1}{2}} = T^{\frac{n+1}{2(n-1)}}$$

Q. 233. Determine the molar heat capacity of a polytropic process through which an ideal gas consisting of rigid diatomic molecules goes and in which the number of collisions between the molecules remains constant
(a) in a unit volume;
(b) in the total volume of the gas.

Solution. 233. (a) The number of collisions between the molecules in a unit volume is

$$\frac{1}{2} n v = \frac{1}{\sqrt{2}} \pi d^2 n^2 <v> \alpha \frac{\sqrt{T}}{V^2}$$

This remains constant in the poly process pV^{-3} = constant

Using (Q.122) the molar specific heat for the polytrophic process

$pV^{\alpha} =$ constant,

is $C = R\left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1}\right)$

Thus $C = R\left(\frac{1}{\gamma-1} + \frac{1}{4}\right) = R\left(\frac{5}{2} + \frac{1}{4}\right) = \frac{11}{4} R$

It can also be written as $\frac{1}{4} R\,(1 + 2\,i)$ where $i = 5$

(b) In this case $\frac{\sqrt{T}}{V} =$ constant and so $pV^{-1} =$ constant

so $C = R\left(\frac{1}{\gamma-1} + \frac{1}{2}\right) = R\left(\frac{5}{2} + \frac{1}{2}\right) = 3R$

It can also be written as $\frac{R}{2}(i + 1)$

Q. 234. An ideal gas of molar mass M is enclosed in a vessel of volume V whose thin walls are kept at a constant temperature T. At a moment t = 0 a small hole of

area S is opened, and the gas starts escaping into vacuum. Find the gas concentration n as a function of time t if at the initial moment n (0) = n_0.

Solution. 234. We can assume that all molecules, incident on the hole, leak out. Then,

$$-dN = -d(nV) = \frac{1}{4} n <v> S\, dt$$

or $$dn = -n \frac{dt}{4v/S <v>} = -n \frac{dt}{\tau}$$

Integrating $$n = n_0 e^{-t/\tau}.\ \text{Hence } <v> = \sqrt{\frac{8RT}{\pi M}}$$

Q. 235. A vessel filled with gas is divided into two equal parts 1 and 2 by a thin heat-insulating partition with two holes. One hole has a small diameter, and the other has a very large diameter (in comparison with the mean free path of molecules). In part 2 the gas is kept at a temperature η times higher than that of part 1. How will the concentration of molecules in part 2 change and how many times after the large hole is closed?

Solution. 235. If the temperature of the compartment 2 is η times more than that of compartment 1, it must contain 1/η times less number of molecules since pressure must be the same when the big hole is open. If M - mass of the gas in 1 than the mass of the gas in 2 must be M/η. So immediately after the big hole is closed.

$$n_1^0 = \frac{M}{mV},\ n_2^0 = \frac{M}{mV\eta}$$

Where m = mass of each molecule and n_1^0, n_2^0 are concentrations in 1 and 2. After the big hole is closed the pressures will differ and concentration will become n_1 and n_2 where

$$n_1 + n_2 = \frac{M}{m\,V\eta}(1+\eta)$$

On the other hand

$$n_1 <v_1> = n_2 <v_2> \quad \text{i.e. } n_1 = \sqrt{\eta}\, n_2$$

Thus $$n_2'(1+\sqrt{\eta}) = \frac{m}{mV\eta}(1+\eta) = n_2^0(1+\eta)$$

So $n_2 = n_2^0 \frac{1+\eta}{1+\sqrt{\eta}}$

Q. 236. As a result of a certain process the viscosity coefficient of an ideal gas increases α = 2.0 times and its diffusion coefficient β = 4.0 times. How does the gas pressure change and how many times?

Solution. 236. We know

$$\eta = \frac{1}{3}<v>\lambda\rho = \frac{1}{3}<v>\frac{1}{\sqrt{2}\,\pi d^2} m \,\alpha \sqrt{T}$$

Thus η changing α times implies T changing α^2 times. On the other hand

$$D = \frac{1}{3}<v>\lambda = \frac{1}{3}\sqrt{\frac{8kT}{\pi m}}\frac{kT}{\sqrt{2}\,\pi d^2 p}$$

Thus D changing P times means $\frac{T^{3/2}}{p}$ changing β times

So p must change $\frac{\alpha^3}{\beta}$ times

Q. 237. How will a diffusion coefficient D and the viscosity coefficient η of an ideal gas change if its volume increases n times: (a) isothermally; (b) isobaric ally?

Solution. 237. $D \,\alpha\, \frac{\sqrt{T}}{n} \propto V\sqrt{T}$, $\eta \propto \sqrt{T}$

(a) D will increase n times

η will remain constant if T is constant

(b) $D \,\alpha\, \frac{T^{3/2}}{p} \,\alpha\, \frac{(pV)^{3/2}}{p} = p^{1/2} V^{3/2}$

$\eta \,\alpha\, \sqrt{pV}$

Thus D will increase $n^{3/2}$ times, η will increase $n^{1/2}$ times, if p is constant

Q. 238. An ideal gas consists of rigid diatomic molecules. How will a diffusion

coefficient D and viscosity coefficient η change and how many times if the gas volume is decreased adiabatically n =10 times?

Solution. 238.

$D \alpha V\sqrt{T}$, $\eta \alpha \sqrt{T}$

In an adiabatic process

$TV^{\gamma-1}$ = constant, or $T \propto V^{1-\gamma}$

Now V is decreased $\frac{1}{n}$ times. Thus

$$D \alpha V^{\frac{3-\gamma}{2}} = \left(\frac{1}{n}\right)^{\frac{3-\gamma}{2}} = \left(\frac{1}{n}\right)^{4/5}$$

$$\eta \alpha \text{ of } V^{\frac{1-\gamma}{2}} = \left(\frac{1}{n}\right)^{-1/5} = n^{1/5}$$

So D decreases $n^{4/5}$ times and η increase $n^{1/5}$ times.

Transport Phenomena (Part - 2)

Q. 239. An ideal gas goes through a polytrophic process. Find the polytrophic exponent n if in this process the coefficient
(a) of diffusion;
(b) of viscosity;
(c) of heat conductivity remains constant.

Solution. 239. (a) $D \alpha V\sqrt{T} \alpha \sqrt{pV^3}$

Thus D remains constant in the process pV^3 = constant

So polytrophic index n = 3

(b) $\eta \alpha \sqrt{T} \alpha \sqrt{pV}$

So η remains constant in the isothermal process

pV = constant, n = 1, here

(c) Heat conductivity $k = \eta C_v$ and C_v is a constant for the ideal gas

Thus n = 1 here also,

Q. 240. Knowing the viscosity coefficient of helium under standard conditions, calculate the effective diameter of the helium atom.

Solution. 240.

$$\eta = \frac{1}{3}\sqrt{\frac{8\,kT}{\pi m}}\,\frac{m}{\sqrt{2}\,\pi d^2} = \frac{2}{3}\sqrt{\frac{m\,kT}{\pi^3}}\,\frac{1}{d^2}$$

$$\text{or} \quad d = \left(\frac{2}{3\,\eta}\right)^{1/2}\left(\frac{m\,kT}{\pi^3}\right)^{1/4} = \left(\frac{2}{3\times 18{\cdot}9}\times 10^6\right)^{1/2}\left(\frac{4\times 8{\cdot}31\times 273\times 10^{-3}}{\pi^3\times 36\times 10^{46}}\right)^{1/4}$$

$$= 10^{-10}\left(\frac{2}{3\times 18{\cdot}9}\right)^{1/2}\left(\frac{4\times 83{\cdot}1\times 273}{\pi^3\times {\cdot}36}\right)^{1/4} = 0{\cdot}178 \text{ nm}$$

Q. 241. The heat conductivity of helium is 8.7 times that of argon (under standard conditions). Find the ratio of effective diameters of argon and helium atoms.

Solution. 241.

$$\kappa = \frac{1}{3} < v > \lambda \rho c_V$$

$$= \frac{1}{3}\sqrt{\frac{8kT}{m\pi}}\frac{1}{\sqrt{2}\pi d^2 n} mn \frac{C_V}{M}$$

$\left(C_V \text{ is the specific heat capacity which is } \frac{C_V}{M}\right)$. Now C_v is the same for all monoatomic gases such as He and A. Thus

$$\kappa \alpha \frac{1}{\sqrt{M}\, d^2}$$

or $$\frac{\kappa_{He}}{\kappa_A} = 8{\cdot}7 = \frac{\sqrt{M_A}\, d_A^2}{\sqrt{M_{He}}\, d_{He}^2} = \sqrt{10}\,\frac{d_A^2}{d_{He}^2}$$

$$\frac{d_A}{d_{He}} = \sqrt{\frac{8{\cdot}7}{\sqrt{10}}} = 1{\cdot}658 \approx 1{\cdot}7$$

Q. 242. Under standard conditions helium fills up the space between two long coaxial cylinders. The mean radius of the cylinders is equal to R, the gap between them is equal to ΔR, with AR ≪ R. The outer cylinder rotates with a fairly low angular velocity o about the stationary inner cylinder. Find the moment of friction forces acting on a unit length of the inner cylinder. Down to what magnitude should the helium pressure be lowered (keeping the temperature constant) to decrease the sought moment of friction forces n = 10 times if ΔR = 6 mm?

Solution. 242. In this case

$$N_1 \frac{r_2^2 - r_1^2}{r_1^2 r_2^2} = 4\pi\eta\omega$$

or $$N_1 \frac{2R\,\Delta R}{R^4} = 4\pi\eta\omega \quad \text{or} \quad N_1 = \frac{2\pi\eta\omega R^3}{\Delta R}$$

To decrease N_1,n times η must be decreased n times. Now η does not depend on pressure until the pressure is so low that the mean free path equals, say, $\frac{1}{2}\Delta R.$ Then the mean free path is fixed and η decreases with pressure. The mean free path equals $\frac{1}{2}\Delta R$ when

$$\frac{1}{\sqrt{2}\,\pi d^2 n_0} = \Delta R \ (n_0 = \text{concentration})$$

Corresponding pressure is $p_0 = \frac{\sqrt{2}\,kT}{\pi d^2 \Delta R}$

The sought pressure is n times less

$$p = \frac{\sqrt{2}\,kT}{\pi d^2 n \Delta R} = 70{\cdot}7 \times \frac{10^{-23}}{10^{-18} \times 10^{-3}} \approx 0{\cdot}71 \text{ Pa}$$

The answer is qualitative and depends on die choice $\frac{1}{2}\Delta R$ for the mean free path.

Q. 243. A gas fills up the space between two long coaxial cylinders of radii R_1 and R_2, with $R_1 < R_2$. The outer cylinder rotates with a fairly low angular velocity ω about the stationary inner cylinder. The moment of friction forces acting on a unit length of the inner cylinder is equal to N_1. Find the viscosity coefficient η of the gas taking into account that the friction force acting on a unit area of the cylindrical surface of radius r is determined by the formula σ = $\eta r\,(\partial\omega/\partial r)$**.**

Solution. 243. We neglect the moment of inertia of the gas in a shell. Then the moment of friction forces on a unit length of the cylinder must be a constant as a function of r.

So, $$2\pi r^3 \eta \frac{d\omega}{dr} = N_1 \quad \text{or} \quad \omega(r) = \frac{N_1}{4\pi\eta}\left(\frac{1}{r_1^2} - \frac{1}{r^2}\right)$$

and $$\omega = \frac{N_1}{4\pi\eta}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \quad \text{or} \quad \eta = \frac{N_1}{4\pi\omega}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

Q. 244. Two identical parallel discs have a common axis and are located at a distance h from each other. The radius of each disc is equal to a, with a ≫ h. One disc is rotated with a low angular velocity ω relative to the other, stationary, disc. Find the moment of friction forces acting on the stationary disc if the viscosity coefficient of the gas between the discs is equal to η

Solution. 244. We consider two adjoining layers. The angular velocity gradient is ω/h. So the moment of the frictional force is

$$N = \int_0^a r \cdot 2\pi r\, dr \cdot \eta r \frac{\omega}{h} = \frac{\pi \eta a^4 \omega}{2h}$$

Q. 245. Solve the foregoing problem, assuming that the discs are located in an ultra-rarefied gas of molar mass M, at temperature T and under pressure p.

Solution. 245. In the ultrararefied gas we must determine η by taking $\lambda = \frac{1}{2}h.$ Then

$$\eta = \frac{1}{3}\sqrt{\frac{8kT}{m\pi}} \times \frac{1}{2}h \times \frac{mp}{kT} = \frac{1}{3}\sqrt{\frac{2M}{\pi RT}}\, hp$$

so, $N = \frac{1}{3}\omega a^4 p \sqrt{\frac{\pi M}{2RT}}$

Q. 246. Making use of Poiseuille's equation (1.7d), find the mass μ of gas flowing per unit time through the pipe of length l and radius a if constant pressures p_1 and p_2 are maintained at its ends.

Solution. 246. Take an infinitesimal section of length dx and apply Poiseuilles equation to this. Then

$$\frac{dV}{dt} = \frac{-\pi a^4}{8\eta}\frac{\partial p}{\partial x}$$

From the formula $pV = RT \cdot \frac{m}{M}$

$$pdV = \frac{RT}{M} dm$$

or $\frac{dm}{dt} = \mu = -\frac{\pi a^4 M}{8\eta RT}\frac{pdp}{dx}$

This equation implies that if the flow is isothermal then $p\frac{dp}{dx}$ must be a constant and

so equals $\frac{\left|p_2^2 - p_1^2\right|}{2l}$ in magnitude.

Thus, $$\mu = \frac{\pi a^4 M}{16 \eta R T} \frac{\left| p_2^2 - p_1^2 \right|}{l}$$

Q. 247. One end of a rod, enclosed in a thermally insulating sheath, is kept at a temperature T_1 while the other, at T_2. The rod is composed of two sections whose lengths are l_1 and l_2 and heat conductivity coefficients x_1 and x_2. Find the temperature of the interface.

Solution. 247. Let T = temperature of the interface.

Then heat flowing from left = heat flowing into right in equilibrium.

Thus, $$\kappa_1 \frac{T_1 - T}{l_1} = \kappa_2 \frac{T - T_2}{l_2} \quad \text{or} \quad T = \frac{\left(\frac{\kappa_1 T_1}{l_1} + \frac{\kappa_2 T_2}{l_2} \right)}{\left(\frac{\kappa_1}{l_1} + \frac{\kappa_2}{l_2} \right)}$$

Q. 248. Two rods whose lengths are l_1 and 1_2 and heat conductivity coefficients x_1 and x_2 are placed end to end. Find the heat conductivity coefficient of a uniform rod of length $l_1 + l_2$ whose conductivity is the same as that of the system of these two rods. The lateral surfaces of the rods are assumed to be thermally insulated.

Solution. 248. We have

$$\kappa_1 \frac{T_1 - T}{l_1} = \kappa_2 \frac{T - T_2}{l_2} = \kappa \frac{T_1 - T_2}{l_1 + l_2}$$

or using the previous result

$$\frac{\kappa_1}{l_1} \left(T_1 - \frac{\frac{\kappa_1 T_1}{l_1} + \frac{\kappa_2 T_2}{l_2}}{\frac{\kappa_1}{l_1} + \frac{\kappa_1}{l_2}} \right) = \kappa \frac{T_1 - T_2}{l_1 + l_2}$$

or $$\frac{\kappa_1}{l_1} \frac{\frac{\kappa_2}{l_2} (T_1 - T_2)}{\frac{\kappa_1}{l_1} + \frac{\kappa_2}{l_2}} = \kappa \frac{T_1 - T_2}{l_1 + l_2} \quad \text{or} \quad \kappa = \frac{l_1 + l_2}{\frac{l_1}{\kappa_1} + \frac{l_2}{\kappa_2}}$$

Q. 249. A rod of length l with thermally insulated lateral surface consists of material whose heat conductivity coefficient varies with temperature as x = α/T,

where α is a constant. The ends of the rod are kept at temperatures T_1 and T_2. Find the function T (x), where x is the distance from the end whose temperature is T_1, and the heat flow density

Solution. 249. By definition the heat flux (per unit area) is

$$\dot{Q} = -K\frac{dT}{dx} = -\alpha\frac{d}{dx}\ln T = \text{constant} = +\alpha\frac{\ln T_1/T_2}{l}$$

Integrating $$\ln T = \frac{x}{l}\ln\frac{T_2}{T_1} + \ln T_1$$

Where T_1 = temperature at the end x = 0

So $$T = T_1\left(\frac{T_2}{T_1}\right)^{x/l} \text{ and } \dot{Q} = \frac{\alpha \ln T_1/T_2}{l}$$

Q. 250. Two chunks of metal with heat capacities C_1 and C_2 are interconnected by a rod of length l and cross-sectional area S and fairly low heat conductivity x. The whole system is thermally insulated from the environment. At a moment t = 0 the temperature difference between the two chunks of metal equals $(\Delta T)_0$. Assuming the heat capacity of the rod to be negligible, find the temperature difference between the chunks as a function of time.

Solution. 250. Suppose the chunks have

temperatures T_1, T_2 at time t and $T_1 - dT_1,\ T_2 + dT_2$ at time dt + t.

Then $$C_1\,dT_1 = C_2\,dT_2 = \frac{\kappa S}{l}(T_1 - T_2)\,dt$$

Thus $$d\,\Delta T = -\frac{\kappa S}{l}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\Delta T\,dt \text{ where } \Delta T = T_1 - T_2$$

Hence $$\Delta T = (\Delta T)_0\, e^{-t/\tau} \text{ where } \frac{1}{\tau} = \frac{\kappa s}{l}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

Q. 251. Find the temperature distribution in a substance placed between two parallel plates kept at temperatures T_1 and T_2. The plate separation is equal to 1, the heat conductivity coefficient of the substance $\varkappa \propto \sqrt{T}$.

Solution. 251. $$\dot{Q} = \kappa \frac{\partial T}{\partial x} = -A\sqrt{T}\frac{\partial T}{\partial x}$$

$$= -\frac{2}{3}A\frac{\partial T^{3/2}}{\partial x}, \ (A = \text{constant})$$

$$= \frac{2}{3}A\frac{(T_1^{3/2} - T_2^{3/2})}{l}$$

Thus $$T^{3/2} = \text{constant} - \frac{x}{l}\left(T_1^{3/2} - T_2^{3/2}\right)$$

Or using $T = T_1$ at $x = 0$

$$T^{3/2} = T_1^{3/2} + \frac{x}{l}\left(T_2^{3/2} - T_1^{3/2}\right) \text{ or } \left(\frac{T}{T_1}\right)^{3/2} = 1 + \frac{x}{l}\left(\left(\frac{T_2}{T}\right)^{3/2} - 1\right)$$

$$T = T_1\left[1 + \frac{x}{l}\left\{\left(\frac{T_2}{T_1}\right)^{3/2} - 1\right\}\right]^{2/3}$$

Q. 252. The space between two large horizontal plates is filled with helium. The plate separation equals l = 50 mm. The lower plate is kept at a temperature T_1 = 290 K, the upper, at T_2 = 330 K. Find the heat flow density if the gas pressure is close to standard.

Solution. 252. $$\kappa = \frac{1}{3}\sqrt{\frac{8RT}{\pi M}}\frac{1}{\sqrt{2}\,\pi d^2 n}mn\frac{R\frac{i}{2}}{M} = \frac{R^{3/2}\,i\,T^{3/2}}{3\pi^{3/2}d^2\sqrt{M}\,N_A}$$

Then from the previous problem

$$q = \frac{2i\,R^{3/2}(T_2^{3/2} - T_1^{3/2})}{9\pi^{3/2}d^2\sqrt{M}\,N_A\,l}, \ i = 3 \text{ here.}$$

Q. 253. The space between two large parallel plates separated by a distance l = 5.0 mm is filled with helium under a pressure p = 1.0 Pa. One plate is kept at a temperature t_1 = 17°C and the other, at a temperature t_2 = 37°C. Find the mean free path of helium atoms and the heat flow density.

Solution. 253. At this pressure and average temperature $= 27°C = 300K = T = \frac{(T_1 + T_2)}{2}$

$$\lambda = \frac{1}{\sqrt{2}\,\pi d^2}\frac{\kappa T}{p} = 2330 \times 10^{-5}\text{m} = 23{\cdot}3\text{mm} >> 5{\cdot}0\text{mm} = l$$

The gas is ultrathin and we write $\lambda = \frac{1}{2} l$ here

Then $q = \kappa \frac{dT}{dx} = \kappa \frac{T_2 - T_1}{l}$

Where $\kappa = \frac{1}{3} < v > \times \frac{1}{2} l \times \frac{MP}{RT} \times \frac{R}{\gamma - 1} \times \frac{1}{M} = \frac{p < v >}{6T(\gamma - 1)} l$

And $q = \frac{p < v >}{6T(\gamma - 1)}(T_2 - T_1)$

Where $< v > = \sqrt{\frac{8RT}{M\pi}}$. We have used $T_2 - T_1 << \frac{T_2 + T_1}{2}$ here.

Q. 254. Find the temperature distribution in the space between two coaxial cylinders of radii R_1 and R_2 filled with a uniform heat conducting substance if the temperatures of the cylinders are constant and are equal to T_1 and T_2 respectively.

Solution. 254. In equilibrium $2\pi r \kappa \frac{dT}{dr} = -A =$ constant. So $T = B - \frac{A}{2\pi\kappa} \ln r$

But $T = T_1$ when $r = R_1$ and $T = T_2$. when $r = R_2$.

From this we find $T = T_1 + \frac{T_2 - T_1}{\ln \frac{R_2}{R_1}} \ln \frac{r}{r_1}$

Q. 255. Solve the foregoing problem for the case of two concentric spheres of radii. R_1 and R_2 and temperatures T_1 and T_2.

Solution. 255. In equilibrium $4\pi r^2 \kappa \frac{dT}{dr} = -A =$ constant

$$T = B + \frac{A}{4\pi\kappa}\frac{1}{r}$$

Using T - T_1 when r = R_1 and T = T_2 when r = R_2,

$$T = T_1 + \frac{T_2 - T_1}{\frac{1}{R_2} - \frac{1}{R_1}}\left(\frac{1}{r} - \frac{1}{R_1}\right)$$

Q. 256. A constant electric current flows along a uniform wire with cross-sectional radius R and heat conductivity coefficient x. A unit volume of the wire generates a thermal power w. Find the temperature distribution across the wire provided the steady-state temperature at the wire surface is equal to T_0.

Solution. 256. The heat flux vector is - k grad T and its divergence equals w. Thus

$$\nabla^2 T = -\frac{w}{\kappa}$$

Or $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{w}{\kappa}$ in cylindrical coordinates.

Or $T = B + A \ln r - \frac{\omega}{2x} r^2$

Since T is finite at r = 0, A = 0. Also T = T_0 at r = R

So $B = T_0 + \frac{w}{4\kappa}R^2$

Thus $T = T_0 + \frac{w}{4\kappa}(R^2 - r^2)$

r here is the distance from the axis of wire (axial radius).

Q. 257. The thermal power of density w is generated uniformly inside a uniform sphere of radius R and heat conductivity coefficient x. Find the temperature distribution in the sphere provided the steady-state temperature at its surface is equal to T_0.

Solution. 257. Here again

$$\nabla^2 T = -\frac{w}{\kappa}$$

So in spherical polar coordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{w}{\kappa} \text{ or } r^2\frac{\partial T}{\partial r} = -\frac{w}{3\kappa}r^3 + A$$

Or $T = B - \frac{A}{r} - \frac{w}{6\kappa} r^2$

Again $A = 0 \text{ and } B = T_0 + \frac{w}{6\kappa} R^2$

So finally $T = T_0 + \frac{w}{6\kappa}(R^2 - r^2)$