Long Answer Questions (PYQ)

[4 Mark / 6 Mark]

Q.1. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3.$$

Ans.

LHS $\Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$

Applying $R_2 \leftrightarrow R_3$, then $R_1 \leftrightarrow R_2$, we have

$$\Delta = \begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = \begin{vmatrix} x+y+z & y+z+x & z+x+y \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Taking out $(x+y+z)$ from first row, we have

$$\Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have

$$\Delta = (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ 0 & (y+z+x) & 2y \\ (x+y+z) & z+x+y & z-x-y \end{vmatrix}$$

Expanding along first row, we have

$$\Delta = (x+y+z)\ (x+y+z)^2 = (x+y+z)^3 = \text{RHS}$$

Q.2. Prove that: $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

Ans.

$$\text{LHS } \Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix} \quad [\text{Applying } R_3 \to R_1 + R_3]$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{Taking out } (\alpha+\beta+\gamma) \text{ from } R_3]$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{Applying } C_1 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix}$$

Taking out $(\beta - \alpha)$ and $(\gamma - \alpha)$ from C_2 and C_3 respectively

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)1.\begin{vmatrix} 1 & 1 \\ \beta+\alpha & \gamma+\alpha \end{vmatrix} \quad [\text{Expanding along } R_3]$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta) = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma) = \text{RHS}$$

Q.3. Show that: $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0,$ **where *a*, *b*, *c* are in AP.**

Ans.

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$\because$ a, b, c are in AP

$\therefore$ $2b = a + c$

Applying $R_1 \to R_1 + R_3 - 2R_2$, we have

$$= \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$\therefore$ $\Delta = 0$ $[\because R_1 = 0]$

Q.4.

Prove that: $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$

OR

Prove that: $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3.$

Ans.

$$\text{LHS}\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$=\begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking $2(a+b+c)$ common from C_1, we get

$$=2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$=2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

Taking $(a+b+c)$ common from R_2 and R_3, we get

$$=2(a+b+c)^3\begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$=2(a+b+c)^3[1-0]=2(a+b+c)^3=\text{RHS}$$

OR

For solution replace $a \to x$, $b \to y$ and $c \to z$ in above solution.

Q.5. Using Properties of determinants, prove the following:

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}=a^3+b^3+c^3-3\,\text{abc}$$

Ans.

$$\text{Let } = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix} \quad [\text{Taking out } (a+b+c) \text{ from } C_1]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - 2R_1]$$

Expanding along C_1, we get

$$= (a+b+c) \,.\, 1.\{(b-c)(a+b-2c) - (c-a)(c+a-2b)\}$$

$$= (a+b+c)\ (ab + b^2 - 2bc - ac - bc + 2c^2 - c^2 - ac + 2bc + ac + a^2 - 2ab)$$

$$= (a+b+c)\ (a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc$$

Q.6. Using properties of determinant, solve for *x*:

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Ans.

Let $\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$

$$\Delta = \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Delta = (3a-x) \begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix}$$

$$= (3a-x) \begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

Expanding along C_1, we get

$$= (3a-x)(4x^2 - 0) = 4x^2(3a-x)$$

Now, given that $\Delta = 0$

Therefore, $4x^2(3a-x) = 0 \quad \Rightarrow \quad x = 0, \quad \text{or} \quad x = 3a.$

Hence, required values of x are $x = 0, 3a$.

Q.7. Using property of determinant, prove the following:

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$$

Ans.

$$\text{LHS} = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

$$= \begin{vmatrix} 3(a+b) & 3(a+b) & 3(a+b) \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} \quad [\text{Applying } R_1 = R_1 + R_2 + R_3]$$

$$= 3(a+b)\begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} \quad [\text{Taking } 3(a+b) \text{ common from } R_1]$$

$$= 3(a+b)\begin{vmatrix} 0 & 0 & 1 \\ b & -b & a+b \\ b & 2b & a \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_3, C_2 \to C_2 - C_3]$$

Expanding along R_1 we get

$$= 3(a+b)\{1\ (2b^2 + b^2)\} = 9b^2(a+b) = \text{RHS}$$

Q.8. By using properties of determinant, prove the following:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda - x)^2$$

Ans.

$$\text{LHS} = \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5x+\lambda & 2x & 2x \\ 5x+\lambda & x+\lambda & 2x \\ 5x+\lambda & 2x & x+\lambda \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+\lambda & 2x \\ 1 & 2x & x+\lambda \end{vmatrix} \quad [\text{Taking out } (5x+\lambda) \text{ common from } C_1]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 0 & \lambda-x & 0 \\ 0 & 0 & \lambda-x \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

Expanding along C_1, we get

$$= (5x+\lambda)\ (\lambda - x)2 = \text{RHS.}$$

Q.9. Using properties of determinant, prove that:

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z)$$

Ans.

$$\text{LHS} = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (a+x+y+z) \begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix} \quad [\text{Taking out } (a+x+y+z) \text{ common from } C_1]$$

$$= (a+x+y+z) \begin{vmatrix} 0 & -a & 0 \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix} \quad [\text{Apply } R_1 \to R_1 - R_2]$$

Expanding along R_1, we get

$$= (a+x+y+z)\,\{0 + a\,(a+z-z)\}$$

$$= a^2(a+x+y+z) = \text{RHS}$$

Q.10. Using properties of determinant, prove the following:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

Ans.

$$\text{LHS} = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$$

$$= x^2 \begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 10x+8y & 8 & 3 \end{vmatrix} \quad [\text{Taking out } x \text{ from } C_2 \text{ and } C_3]$$

$$= x^2 \begin{vmatrix} x+y & 1 & 1 \\ 3x+2y & 2 & 0 \\ 7x+5y & 5 & 0 \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1]$$

Expanding along C_3, we get

$$x^2 \, [1 \, \{(3x + 2y) \, 5 - 2 \, (7x + 5y)\} - 0 + 0]$$

$$= x^2 \, (15x + 10y - 14x - 10y)$$

$$= x^2 \, (x) = x^3 = \text{RHS}$$

Q.11. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$$

Ans.

Let $|A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

Using the transformation $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$|A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 3 & -2+3p \end{vmatrix}$$

Using $R_3 \to R_3 - 3R_2$

$$\Rightarrow \quad |A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along column C_1, we get

$$|A| = 1$$

Q.13. Prove the following using properties of determinant:

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3\,\text{abc} - a^3 - b^3 - c^3)$$

Ans.

LHS

$$= \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$= 2(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad [\text{Taking } 2(a+b+c) \text{ common from } R_1]$$

$$= 2(a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1;\ C_3 \to C_3 - C_1]$$

$$= 2(a+b+c)\left[1\left(bc - b^2 - c^2 + bc - bc + ac + ab - a^2\right)\right] \quad [\text{Expanding along } R_1]$$

$$= 2(a+b+c)\left(bc + ac + ab - a^2 - b^2 - c^2\right)$$

$$= -2(a+b+c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right) = -2\left(a^3 + b^3 + c^3 - 3abc\right)$$

$$= 2\left(3abc - a^3 - b^3 - c^3\right) = \text{RHS}$$

Q.15. Using properties of determinants, prove the following:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

OR

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2$$

Ans.

$$\text{LHS} = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \qquad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \qquad [\text{Taking } (5x+4) \text{ common from } R_1]$$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix} \qquad [\text{Applying } C_2 \to C_2 - C_1;\ C_3 \to C_3 - C_1]$$

$$= (5x+4)\ [1\ \{(4-x)2 - 0\} + 0 + 0] \qquad [\text{Expanding along } R_1]$$

$$= (5x+4)\ (4-x)^2 = \text{RHS}$$

OR

Solve as above by putting λ instead of 4.

Q.20. Using properties of determinant, solve the following for *x*:

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

Ans.

$$\text{Given: }\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} = 0 \qquad [\text{Applying } C_2 \to C_2 - 2C_1 \text{ and } C_3 \to C_3 - 3C_1]$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} = 0 \qquad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad (x-2)(84-72) - 1(84-36) + 2(24-12) = 0 \qquad [\text{Expanding along } R_1]$$

$$\Rightarrow \quad 12x - 24 - 48 + 24 = 0 \quad \Rightarrow \quad 12x = 48 \quad \Rightarrow \quad x = 4$$

Q.21. Prove, using properties of determinant:

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

Ans.

$$\text{LHS} = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix} \qquad [\text{Taking } (3y+k) \text{ common from } C_1]$$

$$= (3y+k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

Expanding along C_1 we get

$$= (3y+k)\{1(k^2-0) - 0 + 0\} = (3y+k) \cdot k^2 = k^2(3y+k)$$

Q.22. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

OR

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Q.23. Using properties of determinants, show that Δ*ABC* is an isosceles if:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

Ans.

We have

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$\Rightarrow \quad \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ \cos^2 A+\cos A - \cos^2 C - \cos C & \cos^2 B+\cos B - \cos^2 C - \cos C & \cos^2 C+\cos C \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ (\cos A - \cos C)(\cos A+\cos C+1) & (\cos B - \cos C)(\cos B+\cos C+1) & \cos^2 C+\cos C \end{vmatrix}$$

Taking common $(\cos A - \cos C)$ from C_1 and $(\cos B - \cos C)$ from C_2, we get

$$\Rightarrow \quad (\cos A - \cos C)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A+\cos C+1 & \cos B+\cos C+1 & \cos^2 C+\cos C \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 - C_2$, we get

$$\Rightarrow \quad (\cos A - \cos C)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1+\cos C \\ \cos A - \cos B & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix}$$

Expanding along R_1, we get

$\Rightarrow \quad (\cos A - \cos C)\ (\cos B - \cos C)\ (\cos B - \cos A) = 0$

$\Rightarrow \quad \cos A - \cos C = 0 \quad$ *i.e.*, $\cos A = \cos C$

or, $\quad \cos \mathrm{B} - \cos \mathrm{C} = 0 \quad$ *i.e.*, $\cos B = \cos C$

or, $\quad \cos \mathrm{B} - \cos \mathrm{A} = 0 \quad$ *i.e.*, $\cos B = \cos A$

$A = C$ or $B = C$ or $B = A$

Hence, ΔABC is an isosceles triangle.

Q.30. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (1 - a^3)^2$$

Ans.

$$\text{LHS } \Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 + a^2 + a & a + 1 + a^2 & a^2 + a + 1 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$\Delta = (1 + a + a^2) \begin{vmatrix} 1 & 1 & 1 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} \quad [\text{Taking out } (1 + x + x^2) \text{ from first row}]$$

$$\Delta = (1 + a + a^2) \begin{vmatrix} 0 & 1 & 1 \\ a^2 - 1 & 1 & a \\ a - a^2 & a^2 & 1 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2]$$

$$\Delta = (1 + a + a^2) \begin{vmatrix} 0 & 0 & 1 \\ a^2 - 1 & 1 - a & a \\ a - a^2 & a^2 - 1 & 1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_3]$$

Expanding along R_1 we have

$$= (1 + a + a^2) \left[(a^2 - 1)^2 - a(1 - a)^2\right]$$

$$= (1 + a + a^2) \left[(a + 1)^2 (a - 1)^2 - a(a - 1)^2\right]$$

$$= (1 + a + a^2)(a - 1)^2 \left[a^2 + 1 + a\right] = (1 + a + a^2)(a - 1)^2 \left[a^2 + 1 + a\right]$$

$$= (a - 1)^2 (1 + a + a^2)^2 = (1 - a)^2 (1 + a + a^2)^2$$

$$= \left[(1 - a)(1 + a + a^2)\right]^2 = (1 - a^3)^2 = \text{RHS}$$

Q.32. If $a \neq b \neq c$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0,$ then using properties of determinants, prove that $a + b + c = 0$.

Ans.

We have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} (a+b+c) & b & c \\ (a+b+c) & c & a \\ (a+b+c) & a & b \end{vmatrix} = 0$$

Taking $(a + b + c)$ common from C_1, we get

$$(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

Expanding along C_1, we get

$$(a+b+c)\,[1\,\{(c-b)(b-c)-(a-c)(a-b)\}-0+0]=0$$

$$\Rightarrow \quad (a+b+c)\,[(c-b)(b-c)-(a-c)(a-b)]=0$$

$$\Rightarrow \quad (a+b+c)\,[(b-c)(b-c)+(a-c)(a-b)]=0$$

$$\Rightarrow \quad (a+b+c)\,[(b-c)^2+(a-c)(a-b)]=0$$

$$\Rightarrow \quad (a+b+c)\,[(b^2+c^2-2bc+a^2-ab-ac+bc]=0$$

$$\Rightarrow \quad (a+b+c)\,[a^2+b^2+c^2-bc-ab-ac]=0$$

$$\Rightarrow \quad (a+b+c)\tfrac{1}{2}[2a^2+2b^2+2c^2-2bc-2ab-2ac]=0$$

$$\Rightarrow \quad (a+b+c)\,[(a-b)^2+(b-c)^2+(c-a)^2]=0$$

$$\Rightarrow \quad (a+b+c)=0\ [\because a\neq b\neq c \Rightarrow (a-b)^2+(b-c)^2+(c-a)^2\neq 0]$$

Long Answer Questions (OIQ)

[4 Mark / 6 Mark]

Q.1. Without expanding, show that:

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$$

Ans.

$$\text{Given, } \Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 - C_2 - C_3]$$

$$= \begin{vmatrix} 1-1 & \cot^2\theta & 1 \\ -1+1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} \qquad [\because \operatorname{cosec}^2\theta - \cot^2\theta = 1]$$

$$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0 \qquad [\because \text{All elements of } C_1 \text{ are } 0]$$

Q.2. If *a*, *b*, *c* are real numbers, then prove that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

where ω is a complex number and cube root of unity.

Ans.

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \quad [\text{Taking out } (a+b+c) \text{ from } C_1]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix} \quad [\text{Expanding along } C_1]$$

$$= (a+b+c)\{-(b-c)^2 - (a-c)(a-b)\}$$

LHS $= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Also, RHS $= -(a+b+c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$$= -(a+b+c)(a^2 + ab\omega^2 + ac\omega + ab\omega + b^2\omega^3 + bc\omega^2 + ac\omega^2 + bc\omega^4 + c^2\omega^3)$$

$$= -(a+b+c)[(a^2 + b^2 + c^2 + ab(\omega^2 + \omega) + bc(\omega^2 + \omega^4) + ca(\omega + \omega^2)] \quad [\because \omega^3 = 1]$$

$$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = \text{LHS} \quad [\because \omega^2 + \omega + 1 = 0 \text{ and } \omega^4 = \omega^3 . \omega = \omega]$$

Q.3. Find the equation of the line joining *A* (1, 3) and *B* (0, 0) using determinants and find *k* if *D* (*k*, 0) is a point such that the area of Δ*ABD* is 3 sq units.

Ans.

Let $P(x, y)$ be any point on the line AB. Then,

$ar(\Delta ABP) = 0$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ x & y & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad \frac{1}{2}\{1(0-y) - 3(0-x) + 1(0-0)\} = 0$$

$\Rightarrow \quad 3x - y = 0$, which is the required equation of line AB.

Now, area $(\Delta ABD) = 3$ sq units

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3 \quad \Rightarrow \quad \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 6$$

$$\Rightarrow \quad 1(0-0) - 3(0-k) + 1(0-0) = \pm 6 \Rightarrow 3k = \pm 6 \quad \Rightarrow \quad k = \pm 2$$

Q.4. In a triangle *ABC*, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0,$$

then prove that Δ*ABC* is an isosceles triangle.

Ans.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin A \\ \sin A + \sin^2 A & \sin^2 B - \sin^2 A + \sin B - \sin A & \sin^2 C - \sin^2 A + \sin C - \sin A \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin A \\ \sin A + \sin^2 A & (\sin B - \sin A)(\sin B + \sin A + 1) & (\sin C - \sin A)(\sin C + \sin A + 1) \end{vmatrix}$$

$$= (\sin B - \sin A)(\sin C - \sin A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & 1 & 1 \\ \sin A + \sin^2 A & \sin B + \sin A + 1 & \sin C + \sin A + 1 \end{vmatrix}$$

Expanding along R_1, we get

$(\sin B - \sin A)\ (\sin C - \sin A)\ [\sin C + \sin A + 1 - \sin B - \sin A - 1]$

$= (\sin B - \sin A)\ (\sin C - \sin A)\ (\sin C - \sin B)$

$\because \quad \Delta = 0$

$\Rightarrow \quad (\sin B - \sin A)\ (\sin C - \sin B)\ (\sin C - \sin A) = 0$

$\Rightarrow \quad \sin B - \sin A = 0$ or $\sin C - \sin B = 0$ or $\sin C - \sin A = 0$

$\Rightarrow \quad B = A$ or $C = B$ or $C = A$

$\Rightarrow \quad \Delta ABC$ is an isosceles triangle.

Q.5. Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then find $\lim_{t \to 0} \frac{f(t)}{t^2}$.

Ans.

Given, $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix} = \begin{vmatrix} \cos t & t & 1 \\ 0 & -t & 0 \\ \sin t & t & t \end{vmatrix}$ [Applying $R_2 \to R_2 - 2R_3$]

$$= t \begin{vmatrix} \cos t & 1 & 1 \\ 0 & -1 & 0 \\ \sin t & 1 & t \end{vmatrix}$$

Expanding along R_2, we get

$$t\left[(-1)\ (t\cos t - \sin t)\right] = -t^2 \cos t + t \sin t$$

$$\therefore \quad \lim_{t \to 0} \frac{f(t)}{t^2} = \lim_{t \to 0} \frac{-t^2 \cos t + t \sin t}{t^2}$$

$$= \lim_{t \to 0} \left(\frac{-t^2 \cos t}{t^2} + \frac{t \sin t}{t^2} \right)$$

$$= \lim_{t \to 0} \left(-\cos t + \frac{\sin t}{t} \right) = -1 + \lim_{t \to 0} \frac{\sin t}{t} = -1 + 1 = 0$$