Chapter 1 Relations and Functions

EXERCISE 1.1

Question 1:
Determine whether each of the following relations are reflexive, symmetric and transitive.

(i) Relation R in the set $A=\{1,2,3,\ldots,13,14\}$ defined as

$$R=\{(x,y):3x-y=0\}$$

(ii) Relation R in the set of N natural numbers defined as

$$R=\{(x,y):y=x+5 \text{ and } x<4\}$$

(iii) Relation R in the set $A=\{1,2,3,4,5,6\}$ as

$$R=\{(x,y):y \text{ is divisible by } x\}$$

(iv) Relation R in the set Z of all integers defined as

$$R=\{(x,y):x-y \text{ is an integer}\}$$

(v) Relation R in the set A of human beings in a town at a particular time given by

(a) $R=\{(x,y):x \text{ and } y \text{ work at the same place}\}$

(b) $R=\{(x,y):x \text{ and } y \text{ live in the same locality}\}$

(c) $R=\{(x,y):x \text{ is exactly 7cm taller than } y\}$

(d) $R=\{(x,y):x \text{ is wife of } y\}$

(e) $R=\{(x,y):x \text{ is father of } y\}$

Solution:

(i) $R=\{(1,3),(2,6),(3,9),(4,12)\}$

R is not reflexive because $(1,1),(2,2)\ldots$ and $(14,14)\notin R$.

R is not symmetric because $(1,3)\in R$, but $(3,1)\notin R.[\text{since } 3(3)\neq 0]$.

R is not transitive because $(1,3),(3,9)\in R$, but $(1,9)\notin R.[3(1)-9\neq 0]$.

Hence, R is neither reflexive nor symmetric nor transitive.

(ii) $R=\{(1,6),(2,7),(3,8)\}$

R is not reflexive because $(1,1)\notin R$.

R is not symmetric because $(1,6)\in R$ but $(6,1)\notin R$.

R is not transitive because there isn't any ordered pair in R such that $(x,y),(y,z)\in R$, so $(x,z)\notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

(iii) $R=\{(x,y):y \text{ is divisible by } x\}$

We know that any number other than 0 is divisible by itself.

Thus, $(x,x)\in R$

So, R is reflexive.

$(2,4) \in R$ [because 4 is divisible by 2]
But $(4,2) \notin R$ [since 2 is not divisible by 4]
So, R is not symmetric.
Let (x,y) and $(y,z) \in R$. So, y is divisible by x and z is divisible by y.
So, z is divisible by x $\Rightarrow (x,z) \in R$
So, R is transitive.
So, R is reflexive and transitive but not symmetric.

(iv) $R = \{(x,y) : x - y \text{ is an integer}\}$
For $x \in Z, (x,x) \notin R$ because $x - x = 0$ is an integer.
So, R is reflexive.
For, $x,y \in Z$, if $x,y \in R$, then $x - y$ is an integer $\Rightarrow (y-x)$ is an integer.
So, $(y,x) \in R$
So, R is symmetric.
Let (x,y) and $(y,z) \in R$, where $x,y,z \in Z$.
$\Rightarrow (x-y)$ and $(y-z)$ are integers.
$\Rightarrow x - z = (x-y)+(y-z)$ is an integer.
So, R is transitive.
So, R is reflexive, symmetric and transitive.

(v)

a) $R = \{(x,y) : x \text{ and } y \text{ work at the same place}\}$
R is reflexive because $(x,x) \in R$
R is symmetric because ,
If $(x,y) \in R$, then x and y work at the same place and y and x also work at the same place. $(y,x) \in R$.
R is transitive because,
Let $(x,y),(y,z) \in R$
x and y work at the same place and y and z work at the same place.
Then, x and z also works at the same place. $(x,z) \in R$.
Hence, R is reflexive, symmetric and transitive.

b) $R = \{(x,y) : x \text{ and } y \text{ live in the same locality}\}$
R is reflexive because $(x,x) \in R$
R is symmetric because,
If $(x,y) \in R$, then x and y live in the same locality and y and x also live in the same locality $(y,x) \in R$.
R is transitive because,

Let $(x, y),(y, z) \in R$

x and y live in the same locality and y and z live in the same locality.

Then x and z also live in the same locality. $(x, z) \in R$.

Hence, R is reflexive, symmetric and transitive.

c) $R=\{(x, y): x \text{ is exactly 7cm taller than } y\}$

R is not reflexive because $(x, x) \notin R$.

R is not symmetric because,

If $(x, y) \in R$, then x is exactly $7cm$ taller than y and y is clearly not taller than x. $(y, x) \notin R$.

R is not transitive because,

Let $(x, y),(y, z) \in R$

x is exactly $7cm$ taller than y and y is exactly $7cm$ taller than z.

Then x is exactly $14cm$ taller than z. $(x, z) \notin R$

Hence, R is neither reflexive nor symmetric nor transitive.

d) $R=\{(x, y): x \text{ is wife of } y\}$

R is not reflexive because $(x, x) \notin R$.

R is not symmetric because,

Let $(x, y) \in R$, x is the wife of y and y is not the wife of x. $(y, x) \notin R$.

R is not transitive because,

Let $(x, y),(y, z) \in R$

x is wife of y and y is wife of z, which is not possible.

$(x, z) \notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

e) $R=\{(x, y): x \text{ is father of } y\}$

R is not reflexive because $(x, x) \notin R$.

R is not symmetric because,

Let $(x, y) \in R$, x is the father of y and y is not the father of x. $(y, x) \notin R$.

R is not transitive because,

Let $(x, y),(y, z) \in R$

x is father of y and y is father of z, x is not father of z. $(x, z) \notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

Question 2:

Show that the relation R in the set R of real numbers, defined as $R=\{(a,b):a\le b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution:

$R=\{(a,b):a\le b^2\}$

$\left(\frac{1}{2},\frac{1}{2}\right)\notin R$ because $\frac{1}{2}>\left(\frac{1}{2}\right)^2$

R is not reflexive.

$(1,4)\in R$ as $1<4$. But 4 is not less than 1^2.

$(4,1)\notin R$

R is not symmetric.

$(3,2)(2,1.5)\in R$ [Because $3<2^2=4$ and $2<(1.5)^2=2.25$]

$3>(1.5)^2=2.25$

$\therefore (3,1.5)\notin R$

R is not transitive.

R is neither reflective nor symmetric nor transitive.

Question 3:

Check whether the relation R defined in the set $\{1,2,3,4,5,6\}$ as $R=\{(a,b):b=a+1\}$ is reflexive, symmetric or transitive.

Solution:

$A=\{1,2,3,4,5,6\}$

$R=\{(a,b):b=a+1\}$

$R=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

$(a,a)\notin R, a\in A$

$(1,1),(2,2),(3,3),(4,4),(5,5)\notin R$

R is not reflexive.

$(1,2)\in R$, but $(2,1)\notin R$

R is not symmetric.

$(1,2),(2,3) \in R$
$(1,3) \notin R$
R is not transitive.

R is neither reflective nor symmetric nor transitive.

Question 4:

Show that the relation R in R defined as $R=\{(a,b): a \leq b\}$ is reflexive and transitive, but not symmetric.

Solution:

$R=\{(a,b): a \leq b\}$
$(a,a) \in R$
R is reflexive.

$(2,4) \in R$ (as $2<4$)
$(4,2) \notin R$ (as $4>2$)
R is not symmetric.

$(a,b),(b,c) \in R$
$a \leq b$ and $b \leq c$
$\Rightarrow a \leq c$
$\Rightarrow (a,c) \in R$
R is transitive.
R is reflexive and transitive but not symmetric.

Question 5:

Check whether the relation R in R defined as $R=\{(a,b): a \leq b^3\}$ is reflexive, symmetric or transitive.

Solution:

$R=\{(a,b): a \leq b^3\}$

$\left(\frac{1}{2},\frac{1}{2}\right) \notin R$, since $\frac{1}{2} > \left(\frac{1}{2}\right)^3$
R is not reflexive.

$(1,2) \in R\left(\text{as } 1 < 2^3 = 8\right)$

$(2,1) \notin R\left(\text{as } 2^3 > 1 = 8\right)$

R is not symmetric.

$\left(3,\frac{3}{2}\right),\left(\frac{3}{2},\frac{6}{5}\right) \in R,$ since $3 < \left(\frac{3}{2}\right)^3$ and $\frac{2}{3} < \left(\frac{6}{2}\right)^3$

$\left(3,\frac{6}{5}\right) \notin R \, 3 > \left(\frac{6}{5}\right)^3$

R is not transitive.

R is neither reflexive nor symmetric nor transitive.

Question 6:

Show that the relation R in the set $\{1,2,3\}$ given by $R = \{(1,2),(2,1)\}$ is symmetric but neither reflexive nor transitive.

Solution:

$A = \{1,2,3\}$

$R = \{(1,2),(2,1)\}$

$(1,1),(2,2),(3,3) \notin R$

R is not reflexive.

$(1,2) \in R$ and $(2,1) \in R$

R is symmetric.

$(1,2) \in R$ and $(2,1) \in R$

$(1,1) \in R$

R is not transitive.

R is symmetric, but not reflexive or transitive.

Question 7:

Show that the relation R in the set A of all books in a library of a college, given by $R = \{(x,y): x \text{ and } y \text{ have same number of pages}\}$ is an equivalence relation.

Solution:

$R = \{(x,y): x \text{ and } y \text{ have same number of pages}\}$

R is reflexive since $(x,x) \in R$ as x and x have same number of pages.

R is reflexive.

$(x, y) \in R$

x and y have same number of pages and y and x have same number of pages $(y, x) \in R$
R is symmetric.

$(x, y) \in R, (y, z) \in R$
x and y have same number of pages, y and z have same number of pages.
Then x and z have same number of pages.
$(x, z) \in R$
R is transitive.

R is an equivalence relation.

Question 8:

Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{is even}\}$ is an equivalence relation. Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

Solution:

$a \in A$
$|a - a| = 0 (\text{which is even})$

R is reflective.

$(a, b) \in R$
$\Rightarrow |a - b| \; [\text{is even}]$
$\Rightarrow |-(a - b)| = |b - a| \; [\text{is even}]$
$(b, a) \in R$
R is symmetric.

$(a, b) \in R$ and $(b, c) \in \text{R}$
$\Rightarrow |a - b|$ is even and $|b - c|$ is even
$\Rightarrow (a - b)$ is even and $(b - c)$ is even
$\Rightarrow (a - c) = (a + b) + (b - c)$ is even

$\Rightarrow |a-b|$ is even

$\Rightarrow (a,c) \in R$

R is transitive.

R is an equivalence relation.

All elements of $\{1,3,5\}$ are related to each other because they are all odd. So, the modulus of the difference between any two elements is even.

Similarly, all elements $\{2,4\}$ are related to each other because they are all even.

No element of $\{1,3,5\}$ is related to any elements of $\{2,4\}$ as all elements of $\{1,3,5\}$ are odd and all elements of $\{2,4\}$ are even. So, the modulus of the difference between the two elements will not be even.

Question 9:

Show that each of the relation R in the set $A = \{x \in Z : 0 \le x \le 12\}$, given by

i. $R = \{(a,b) : |a-b| \text{ is a mutiple of } 4\}$

ii. $R = \{(a,b) : a = b\}$

Is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution:

$A = \{x \in Z : 0 \le x \le 12\} = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$

i. $R = \{(a,b) : |a-b| \text{ is a mutiple of } 4\}$

$a \in A, (a,a) \in R \qquad [|a-a| = 0 \text{ is a multiple of } 4]$

R is reflexive.

$(a,b) \in R \Rightarrow |a-b|$ [is a multiple of 4]

$\Rightarrow |-(a-b)| = |b-a|$ [is a multiple of 4]

$(b,a) \in R$

R is symmetric.

$(a,b) \in R$ and $(b,c) \in$ R

$\Rightarrow |a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4

$\Rightarrow (a-b)$ is a multiple of 4 and $(b-c)$ is a multiple of 4

$\Rightarrow (a-c) = (a-b)+(b-c)$ is a multiple of 4

$\Rightarrow |a-c|$ is a multiple of 4

$\Rightarrow (a,c) \in R$

R is transitive.

R is an equivalence relation.

The set of elements related to 1 is $\{1,5,9\}$ as

$|1-1| = 0$ is a multiple of 4.

$|5-1| = 4$ is a multiple of 4.

$|9-1| = 8$ is a multiple of 4.

ii. $R = \{(a,b) : a = b\}$

$a \in A, (a,a) \in R$ [since a=a]

R is reflective.

$(a,b) \in R$

$\Rightarrow a = b$

$\Rightarrow b = a$

$\Rightarrow (b,a) \in R$

R is symmetric.

$(a,b) \in R$ and $(b,c) \in \text{R}$

$\Rightarrow a = b$ and $b = c$

$\Rightarrow a = c$

$\Rightarrow (a,c) \in R$

R is transitive.

R is an equivalence relation.

The set of elements related to 1 is $\{1\}$.

Question 10:

Give an example of a relation, which is

i. Symmetric but neither reflexive nor transitive.
ii. Transitive but neither reflexive nor symmetric.
iii. Reflexive and symmetric but not transitive.
iv. Reflexive and transitive but not symmetric.
v. Symmetric and transitive but not reflexive.

Solution:

i.

$A=\{5,6,7\}$

$R=\{(5,6),(6,5)\}$

$(5,5),(6,6),(7,7)\notin R$

R is not reflexive as $(5,5),(6,6),(7,7) \notin R$

$(5,6),(6,5)\in R$ and $(6,5)\in R$, R is symmetric.

$\Rightarrow (5,6),(6,5)\in R,$ but $(5,5)\notin R$

R is not transitive.

Relation R is symmetric but not reflexive or transitive.

ii. $R=\{(a,b): a<b\}$

$a\in R,(a,a)\notin R$ [since a cannot be less than itself]

R is not reflexive.

$(1,2)\in R(\text{as } 1<2)$

But 2 is not less than 1

$\therefore (2,1)\notin R$

R is not symmetric.

$(a,b),(b,c)\in R$

$\Rightarrow a<b$ and $b<c$

$\Rightarrow a<c$

$\Rightarrow (a,c)\in R$

R is transitive.

Relation R is transitive but not reflexive and symmetric.

iii. $A=\{4,6,8\}$

$A=\{(4,4),(6,6),(8,8),(4,6),(6,8),(8,6)\}$

R is reflexive since $a\in A,(a,a)\in R$

R is symmetric since $(a,b)\in R$

. $\Rightarrow (b,a)\in R \quad for\, a,b\in R$

R is not transitive since $(4,6),(6,8)\in R, but\,(4,8)\notin R$

R is reflexive and symmetric but not transitive.

iv. $R=\{(a,b): a^3>b^3\}$

$(a,a)\in R$

R is reflexive.

$(2,1)\in R$

$But\,(1,2)\notin R$

$\therefore$ R is not symmetric.

$(a,b),(b,c) \in R$
$\Rightarrow a^3 \geq b^3$ and $b^3 < c^3$
$\Rightarrow a^3 < c^3$
$\Rightarrow (a,c) \in R$
$\therefore$ R is transitive.
R is reflexive and transitive but not symmetric

v.

$A = *\{1*, 3*, 5\}$ Define a Relation R

On A.

$R : A* \rightarrow A$

$R = *\{(1,3)*(3,1)*(1,1)*(3,3)\}$

Relation R is not Reflexive as $(5*, 5) \not\subset R$

Relation R is Symmetric as

$(1,3) \in R \Rightarrow *(3,1) *\in R$

Relation R is Transitive

$(a, b) \in R*, (b, c) \in R* \Rightarrow *(a, c) \in R$

$(3,1) \in R*, (1,1) \in R* \Rightarrow *(3,1) \in R$

Alternative Answer

$R = *(a, b) : a$ is brother of b {suppose a and b are male}

$Ref \rightarrow a$ is not brother of a

So, $(a, a) \not\subset R$

Relation R is not Reflexive

Symmetric$\rightarrow$ a is brother of b so

b is brother of a

$a, b \in \mathrm{R} \Rightarrow (b, a) \in \mathrm{R}$

Transitive$\rightarrow$ a is brother of b and

b is brother of c so

a is brother of c

$(a, b) \in R*, (b, c) \in R \Rightarrow (a, c) \in R$

Question 11:

Show that the relation R in the set A of points in a plane given by

$R=\{(P,Q):\text{Distance of the point P from the origin is same as the distance of the point Q from the origin}\}$

, is an equivalence relation. Further, show that the set of all points related to a point $P\neq(0,0)$ is the circle passing through P with origin as centre.

Solution:

$R=\{(P,Q):\text{Distance of the point P from the origin is same as the distance of the point Q from the origin}\}$

Clearly, $(P,P)\in R$

R is reflexive.

$(P,Q)\in R$

Clearly R is symmetric.

$(P,Q),(Q,S)\in R$

$\Rightarrow$ The distance of P and Q from the origin is the same and also, the distance of Q and S from the origin is the same.

$\Rightarrow$ The distance of P and S from the origin is the same.

$(P,S)\in R$

R is transitive.

R is an equivalence relation.

The set of points related to $P\neq(0,0)$ will be those points whose distance from origin is same as distance of P from the origin.

Set of points forms a circle with the centre as origin and this circle passes through P.

Question 12:

Show that the relation R defined in the set A of all triangles as $R=\{(T_1,T_2):T_1\text{ is similar to }T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides $3,4,5$, T_2 with sides $5,12,13$ and T_3 with sides $6,8,10$. Which triangle among T_1,T_2,T_3 are related?

Solution:

R is reflexive since every triangle is similar to itself.

If $(T_1,T_2)\in R$, then is similar to .

is similar to .

$\Rightarrow(T_2,T_1)\in R$

R is symmetric.

$(T_1,T_2),(T_2,T_3)\in R$

is similar to T_2 and T_2 is similar to T_3.

$\therefore$ T_1 is similar to T_3.

$\Rightarrow (T_1, T_3) \in R$

$\therefore$ R is transitive.

$$\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \left(\frac{1}{2}\right)$$

$\therefore$ Corresponding sides of triangles T_1 and T_3 are in the same ratio.

Triangle T_1 is similar to triangle T_3.

Hence, T_1 is related to T_3.

Question 13:

Show that the relation R defined in the set A of all polygons as

$R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides $3, 4 and 5$?

Solution:

$R = \{(P_1, P_2) : P_1 \text{and } P_2 \text{ have same number of sides}\}$

$(P_1, P_2) \in R$ as same polygon has same number of sides.

$\therefore$ R is reflexive.

$(P_1, P_2) \in R$

$\Rightarrow P_1$ and P_2 have same number of sides.

$\Rightarrow P_2$ and P_1 have same number of sides.

$\Rightarrow (P_2, P_1) \in R$

$\therefore$ R is symmetric.

$(P_1, P_2), (P_2, P_3) \in R$

$\Rightarrow P_1$ and P_2 have same number of sides.

P_2 and P_3 have same number of sides.

$\Rightarrow P_1$ and P_3 have same number of sides.

$\Rightarrow (P_1, P_3) \in R$

$\therefore$ R is transitive.

R is an equivalence relation.

The elements in A related to right-angled triangle (T) with sides $3, 4, 5$ are those polygons which have three sides.

Set of all elements in a related to triangle T is the set of all triangles.

Question 14:

Let L be the set of all lines in XY plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1 \text{is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Solution:

$R = \{(L_1, L_2) : L_1 \text{is parallel to } L_2\}$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_2) \in R$

If $(L_1, L_2) \in R$, then

$\Rightarrow L_1$ is parallel to L_2.

$\Rightarrow L_2$ is parallel to L_1.

$\Rightarrow (L_2, L_1) \in R$
$\therefore$ R is symmetric.

$(L_1, L_2), (L_2, L_3) \in R$
$\Rightarrow L_1$ is parallel to L_2
$\Rightarrow L_2$ is parallel to L_3
$\therefore L_1$ is parallel to L_3.
$\Rightarrow (L_1, L_3) \in R$
$\therefore$ R is transitive.

R is an equivalence relation.

Set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.
Slope of the line $y = 2x + 4$ is $m = 2$.
Line parallel to the given line is in the form $y = 2x + c$, where $c \in R$.
Set of all lines related to the given line is given by $y = 2x + c$, where $c \in R$.

Question 15:

Let R be the relation in the set $\{1, 2, 3.4\}$ given by
$R = \{(1,2)(2,2),(1,1),(4,4),(1,3),(3,3),(3,2)\}$.
Choose the correct answer.

A. R is reflexive and symmetric but not transitive.
B. R is reflexive and transitive but not symmetric.
C. R is symmetric and transitive but not reflexive.
D. R is an equivalence relation.

Solution:

$R = \{(1,2)(2,2),(1,1),(4,4),(1,3),(3,3),(3,2)\}$
$(a, a) \in R$ for every $a \in \{1, 2, 3.4\}$
$\therefore$ R is reflexive.

$(1,2) \in R$ but $(2,1) \notin R$
$\therefore$ R is not symmetric.

$(a,b),(b,c) \in R$ for all $a, b, c \in \{1,2,3,4\}$
$\therefore$ R is not transitive.

R is reflexive and transitive but not symmetric.

The correct answer is B.

Question 16:

Let R be the relation in the set N given by $R=\{(a,b):a=b-2,b>6\}$. Choose the correct answer.

A. $(2,4)\in R$

B. $(3,8)\in R$

C. $(6,8)\in R$

D. $(8,7)\in R$

Solution:

$R=\{(a,b):a=b-2,b>6\}$

Now,

$b>6,(2,4)\notin R$

$3\neq 8-2$

$\therefore (3,8)\notin R$ and as $8\neq$ 7-2

$\therefore (8,7)\notin R$

Consider $(6,8)$

$8>6$ and $6=8-2$

$\therefore (6,8)\in R$

The correct answer is C.

EXERCISE 1.2

Question 1:

Show that the function $f : R_{\bullet} \rightarrow R_{\bullet}$ defined by $f(x) = \frac{1}{x}$ is one –one and onto, where $R_{\bullet}$ is the set of all non –zero real numbers. Is the result true, if the domain $R_{\bullet}$ is replaced by N with co-domain being same as $R_{\bullet}$?

Solution:

$f : R_{\bullet} \rightarrow R_{\bullet}$ is by $f(x) = \frac{1}{x}$

For one-one:

$x, y \in R_{\bullet}$ such that $f(x) = f(y)$

$\Rightarrow \frac{1}{x} = \frac{1}{y}$

$\Rightarrow x = y$

$\therefore f$ is one-one.

For onto:

For $y \in R$, there exists $x = \frac{1}{y} \in R_{\bullet} \left[\text{as } y \notin 0\right]$ such that

$f(x) = \frac{1}{\left(\frac{1}{y}\right)} = y$

$\therefore f$ is onto.

Given function f is one-one and onto.

Consider function $g: N \to R$, defined by $g(x) = \frac{1}{x}$

We have, $g(x_1) = g(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

$\therefore g$ is one-one.

g is not onto as for $1.2 \in R$, there exist any x in N such that $g(x) = \frac{1}{1.2}$

Function g is one-one but not onto.

Question 2:

Check the injectivity and surjectivity of the following functions:

i. $f: N \to N$ given by $f(x) = x^2$
ii. $f: Z \to Z$ given by $f(x) = x^2$
iii. $f: R \to R$ given by $f(x) = x^2$
iv. $f: N \to N$ given by $f(x) = x^3$
v. $f: Z \to Z$ given by $f(x) = x^3$

Solution:

i. For $f: N \to N$ given by $f(x) = x^2$

$x, y \in N$

$f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$

$\therefore f$ is injective.

$2 \in N$. But, there does not exist any x in N such that $f(x) = x^2 = 2$
$\therefore f$ is not surjective
Function f is injective but not surjective.

ii. $f : Z \to Z$ given by $f(x) = x^2$
$f(-1) = f(1) = 1$ but $-1 \neq 1$
$\therefore f$ is not injective.

$-2 \in Z$ But, there does not exist any $x \in Z$ such that $f(x) = -2 \Rightarrow x^2 = -2$
$\therefore f$ is not surjective.
Function f is neither injective nor surjective.

iii. $f : R \to R$ given by $f(x) = x^2$
$f(-1) = f(1) = 1$ but $-1 \neq 1$
$\therefore f$ is not injective.

$-2 \in Z$ But, there does not exist any $x \in Z$ such that $f(x) = -2 \Rightarrow x^2 = -2$
$\therefore f$ is not surjective.
Function f is neither injective nor surjective.

iv. $f : N \to N$ given by $f(x) = x^3$
$x, y \in N$
$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$
$\therefore f$ is injective.

$2 \in N$. But, there does not exist any x in N such that $f(x) = x^3 = 2$
$\therefore f$ is not surjective
Function f is injective but not surjective.

v. $f : Z \to Z$ given by $f(x) = x^3$
$x, y \in Z$
$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$
$\therefore f$ is injective.

$2 \in Z$. But, there does not exist any x in Z such that $f(x) = x^3 = 2$
$\therefore f$ is not surjective.
Function f is injective but not surjective.

Question 3:

Prove that the greatest integer function $f: R \to R$ given by $f(x)=[x]$ is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Solution:

$f: R \to R$ given by $f(x)=[x]$

$f(1.2)=[1.2]=1, f(1.9)=[1.9]=1$

$\therefore f(1.2)=f(1.9)$, but $1.2 \neq 1.9$

$\therefore f$ is not one-one.

Consider $0.7 \in R$

$f(x)=[x]$ is an integer. There does not exist any element $x \in R$ such that $f(x)=0.7$

$\therefore f$ is not onto.

The greatest integer function is neither one-one nor onto.

Question 4:

Show that the modulus function $f: R \to R$ given by $f(x)=|x|$ is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Solution:

$f: R \to R$ is $f(x)=|x|=\begin{cases} \text{x, if x} \geq 0 \\ -x, \text{if x} < 0 \end{cases}$

$f(-1)=|-1|=1$ and $f(1)=|1|=1$

$\therefore f(-1)=f(1)$ but $-1 \neq 1$

$\therefore f$ is not one-one.

Consider $-1 \in R$

$f(x)=|x|$ is non-negative. There exist any element x in domain R such that $f(x)=|x|=-1$

$\therefore f$ is not onto.

The modulus function is neither one-one nor onto.

Question 5:

Show that the signum function $f: R \rightarrow R$ given by $f(x)=\begin{cases}1, \text{ if } x>0 \\ 0, \text{ if } x=0 \\ -1, \text{ if } x<0\end{cases}$ is neither one-one nor onto.

Solution:

$f: R \rightarrow R$ is $f(x)=\begin{cases}1, \text{if } x>0 \\ 0, \text{ if } x=0 \\ -1, \text{ if } x<0\end{cases}$

$f(1)=f(2)=1$, but $1 \neq 2$

$\therefore f$ is not one-one.

$f(x)$ takes only 3 values $(1,0,-1)$ for the element -2 in co-domain

R, there does not exist any x in domain R such that $f(x)=-2$.

$\therefore f$ is not onto.

The signum function is neither one-one nor onto.

Question 6:

Let $A=\{1,2,3\}$, $B=\{4,5,6,7\}$ and let $f=\{(1,4),(2,5),(3,6)\}$ be a function from A to B. Show that f is one-one.

Solution:

$A=\{1,2,3\}$, $B=\{4,5,6,7\}$

$f: A \rightarrow B$ is defined as $f=\{(1,4),(2,5),(3,6)\}$

$\therefore f(1)=4, f(2)=5, f(3)=6$

It is seen that the images of distinct elements of A under f are distinct.

$\therefore f$ is one-one.

Question 7:

In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

i. $f: R \rightarrow R$ defined by $f(x)=3-4x$

ii. $f: R \rightarrow R$ defined by $f(x)=1+x^2$

Solution:

i. $f: R \to R$ defined by $f(x) = 3 - 4x$

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 3 - 4x_1 = 3 - 4x_2$

$\Rightarrow -4x = -4x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

For any real number (y) in R, there exists $\frac{3-y}{4}$ in R such that $f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y$

$\therefore f$ is onto.

Hence, f is bijective.

ii. $f: R \to R$ defined by $f(x) = 1 + x^2$

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 1 + x_1^2 = 1 + x_2^2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

Consider $f(1) = f(-1) = 2$

$\therefore f$ is not one-one.

Consider an element -2 in co domain R.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in R$.

$\therefore f$ is not onto.

Hence, f is neither one-one nor onto.

Question 8:

Let A and B be sets. Show that $f: A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is a bijective function.

Solution:

$f: A \times B \to B \times A$ is defined as $(a, b) = (b, a)$.

$(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$
$\Rightarrow b_1 = b_2$ and $a_1 = a_2$
$\Rightarrow (a_1, b_1) = (a_2, b_2)$
$\therefore f$ is one-one.
$(b,a) \in B \times A$ there exist $(a,b) \in A \times B$ such that $f(a,b) = (b,a)$
$\therefore f$ is onto.
f is bijective.

Question 9:

Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

Solution:

$f : N \to N$ be defined as $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{if } n \text{ is even} \end{cases}$ for all $n \in N$.

$f(1) = \frac{1+1}{2} = 1$ and $f(2) = \frac{2}{2} = 1$

$f(1) = f(2)$, where $1 \neq 2$

$\therefore f$ is not one-one.

Consider a natural number n in co domain .

Case I: n is odd

$\therefore n = 2r + 1$ for some $r \in N$ there exists $4r + 1 \in N$ such that

$f(4r+1) = \frac{4r+1+1}{2} = 2r + 1$

Case II: n is even

$\therefore n = 2r$ for some $r \in N$ there exists $4r \in N$ such that

$f(4r) = \frac{4r}{2} = 2r$

$\therefore f$ is onto.

f is not a bijective function.

Question 10:

Let A = **R** – {3} and B = **R** – {1}. Consider the function f: A → B defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

Solution:

$A = R - \{3\}, B = R - \{1\}$ and $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$

$x, y \in A$ such that $f(x) = f(y)$

$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$

$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$

$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$

$\Rightarrow -3x - 2y = -3y - 2x$

$\Rightarrow 3x - 2x = 3y - 2y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

Let $y \in B = R - \{1\}$, then $y \neq 1$

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now,

$f(x) = y$

$\Rightarrow \frac{x-2}{x-3} = y$

$\Rightarrow x - 2 = xy - 3y$

$\Rightarrow x(1-y) = -3y + 2$

$\Rightarrow x = \frac{2-3y}{1-y} \in A \qquad [y \neq 1]$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right) - 2}{\left(\frac{2-3y}{1--y}\right) - 3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

$\therefore f$ is onto.

Hence, the function is one-one and onto.

Question 11:

Let $f: R \to R$ defined as $f(x) = x^4$. Choose the correct answer.

A. f is one-one onto
B. f is many-one onto
C. f is one-one but not onto
D. f is neither one-one nor onto

Solution:

$f: R \to R$ defined as $f(x) = x^4$

$x, y \in R$ such that $f(x) = f(y)$

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

$\therefore f(x) = f(y)$ does not imply that $x = y$.

For example $f(1) = f(-1) = 1$

$\therefore f$ is not one-one.

Consider an element 2 in co domain R there does not exist any x in domain R such that $f(x) = 2$.

$\therefore f$ is not onto.

Function f is neither one-one nor onto.
The correct answer is D.

Question 12:

Let $f: R \to R$ defined as $f(x) = 3x$. Choose the correct answer.

A. f is one-one onto
B. f is many-one onto
C. f is one-one but not onto
D. f is neither one-one nor onto

Solution:

$f: R \to R$ defined as $f(x) = 3x$

$x, y \in R$ such that $f(x) = f(y)$

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

For any real number y in co domain R, there exist $\frac{y}{3}$ in R such that $f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$

$\therefore f$ is onto.

Hence, function f is one-one and onto.

The correct answer is A.

MISCELLANEOUS EXERCISE

Question 1:

Show that function $f:R\rightarrow\{x\in R:-1<x<1\}$ defined by $f(x)=\frac{x}{1+|x|}$, $x\in R$ is one-one and onto function.

Solution:

$f:R\rightarrow\{x\in R:-1<x<1\}$ is defined by $f(x)=\frac{x}{1+|x|}$, $x\in R$.

For one-one:

$f(x)=f(y)$ where $x,y\in R$

$$\Rightarrow \frac{x}{1+|x|}=\frac{y}{1+|y|}$$

If x is positive and y is negative,

$$\frac{x}{1+|x|}=\frac{y}{1+|y|}$$
$$\Rightarrow 2xy=x-y$$

Since, x is positive and y is negative,

$x>y\Rightarrow x-y>0$

$2xy$ is negative.

$2xy\neq x-y$

Case of x being positive and y being negative, can be ruled out.

x and y have to be either positive or negative.

If x and y are positive,

$$f(x)=f(y)$$
$$\Rightarrow \frac{x}{1+x}=\frac{y}{1+y}$$
$$\Rightarrow x-xy=y-xy$$
$$\Rightarrow x=y$$

$\therefore f$ is one-one.

For onto:

Let $y \in R$ such that $-1 < y < 1$.

If x is negative, then there exists $x = \dfrac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If x is positive, then there exists $x = \dfrac{y}{1-y} \in R$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1+\left(\frac{y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

$\therefore f$ is onto.

Hence, f is one-one and onto.

Question 2:

Show that function $f : R \to R$ be defined by $f(x) = x^3$ is injective.

Solution:

$f : R \to R$ is defined by $f(x) = x^3$

For one-one:

$f(x) = f(y)$ where $x, y \in R$

$x^3 = y^3$.......................(1)

We need to show that $x = y$

Suppose $x \neq y$, their cubes will also not be equal.

$\Rightarrow x^3 \neq y^3$

This will be a contradiction to (1).

$\therefore x = y$. Hence, f is injective.

Question 3:

Given a non-empty set X, consider $P(X)$ which is the set of all subsets of X.

Define the relation R in $P(X)$ as follows:

For subsets A, B in $P(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $P(X)$? Justify you answer.

Solution:

Since every set is a subset of itself, ARA for all .
$\therefore$ R is reflexive.

Let $ARB \Rightarrow A \subset B$
This cannot be implied to $B \subset A$.

If $A = \{1,2\}$ and $B = \{1, 2,3\}$, then it cannot be implied that B is related to A.
$\therefore$ R is not symmetric.

If ARB and BRC, then $A \subset B$ and $B \subset C$.
$\Rightarrow A \subset C$
$\Rightarrow ARC$
$\therefore$ R is transitive.

R is not an equivalence relation as it is not symmetric.

Question 4:

Find the number of all onto functions from the set $\{1,2,3,\ldots,n\}$ to itself.

Solution:

Onto functions from the set $\{1,2,3,\ldots,n\}$ to itself is simply a permutation on n symbols $1,2,3,\ldots,n$.

Thus, the total number of onto maps from $\{1,2,3,\ldots,n\}$ to itself is the same as the total number of permutations on n symbols $1,2,3,\ldots,n$, which is $n!$.

Question 5:

Let A = {− 1, 0, 1, 2}, B = {− 4, − 2, 0, 2} and *f*, *g* : A → B be functions defined by $f(x) = x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$. Are *f* and *g* equal? Justify your answer. (Hint: One may note that two functions $f : A \to B$ and $g : A \to B$ such that $f(a) = g(a)$ $\forall a \in A$, are called equal functions).

Solution:

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$

Also, $f, g : A \to B$ is defined by $x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$.

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$

$$g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(-1) = g(-1)$$

$$f(0) = (0)^2 - 0 = 0$$

$$g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(0) = g(0)$$

$$f(1) = (1)^2 - 1 = 0$$

$$g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(1) = g(1)$$

$$f(2) = (2)^2 - 2 = 2$$

$$g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(2) = g(2)$$

$$\therefore f(a) = g(a) \quad \forall a \in A$$

Hence, the functions f and g are equal.

Question 6:

Let $A=\{1,2,3\}$. Then number of relations containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is,

A. 1
B. 2
C. 3
D. 4

Solution:

The given set is $A=\{1,2,3\}$.

The smallest relation containing $(1,2)$ and $(1,3)$ which are reflexive and symmetric but not transitive is given by,

$R=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(3,1)\}$

This is because relation R is reflexive as $\{(1,1),(2,2),(3,3)\}\in R$.

Relation R is symmetric as $\{(1,2),(2,1)\}\in R$ and $\{(1,3)(3,1)\}\in R$.

Relation R is transitive as $\{(3,1),(1,2)\}\in R$ but $(3,2)\notin R$.

Now, if we add any two pairs $(3,2)$ and $(2,3)$ (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 7:

Let $A=\{1,2,3\}$. Then number of equivalence relations containing $(1,2)$ is,

A. 1
B. 2
C. 3
D. 4

Solution:

The given set is $A=\{1,2,3\}$.

The smallest equivalence relation containing $(1,2)$ is given by;

$R_1=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

Now, we are left with only four pairs i.e., $(2,3),(3,2),(1,3)$ and $(3,1)$.

If we odd any one pair [say $(2,3)$] to R_1, then for symmetry we must add $(3,2)$. Also, for transitivity we are required to add $(1,3)$ and $(3,1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing $(1,2)$ is two.

The correct answer is B.