4.2 Quantum definitions

Quantum uncertainty relations

De Broglie relation	$p=\frac{h}{\lambda}$	(4.1)	$p,\boldsymbol{p}$	particle momentum
	$\boldsymbol{p}=\hbar\boldsymbol{k}$	(4.2)	h	Planck constant
			$\hbar$	$h/(2\pi)$
			λ	de Broglie wavelength
			$\boldsymbol{k}$	de Broglie wavevector
Planck–Einstein relation	$E=h\nu=\hbar\omega$	(4.3)	E	energy
			ν	frequency
			ω	angular frequency ($=2\pi\nu$)
Dispersion[a]	$(\Delta a)^2=\langle(a-\langle a\rangle)^2\rangle$	(4.4)	a,b	observables[b]
	$=\langle a^2\rangle-\langle a\rangle^2$	(4.5)	$\langle\cdot\rangle$	expectation value
			$(\Delta a)^2$	dispersion of a
General uncertainty relation	$(\Delta a)^2(\Delta b)^2\geq\frac{1}{4}\langle\mathrm{i}[\hat{a},\hat{b}]\rangle^2$	(4.6)	$\hat{a}$	operator for observable a
			$[\cdot,\cdot]$	commutator (see page 26)
Momentum–position uncertainty relation[c]	$\Delta p\,\Delta x\geq\frac{\hbar}{2}$	(4.7)	x	particle position
Energy–time uncertainty relation	$\Delta E\,\Delta t\geq\frac{\hbar}{2}$	(4.8)	t	time
Number–phase uncertainty relation	$\Delta n\,\Delta\phi\geq\frac{1}{2}$	(4.9)	n	number of photons
			ϕ	wave phase

[a]Dispersion in quantum physics corresponds to variance in statistics.
[b]An observable is a directly measurable parameter of a system.
[c]Also known as the "Heisenberg uncertainty relation."

Wavefunctions

Probability density	$\mathrm{pr}(x,t)\,\mathrm{d}x=\lvert\psi(x,t)\rvert^2\,\mathrm{d}x$	(4.10)	pr	probability density
			ψ	wavefunction
Probability density current[a]	$j(x)=\frac{\hbar}{2\mathrm{i}m}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)$	(4.11)	$\boldsymbol{j},j$	probability density current
	$\boldsymbol{j}=\frac{\hbar}{2\mathrm{i}m}\left[\psi^*(\boldsymbol{r})\nabla\psi(\boldsymbol{r})-\psi(\boldsymbol{r})\nabla\psi^*(\boldsymbol{r})\right]$	(4.12)	$\hbar$	(Planck constant)/(2π)
	$=\frac{1}{m}\Re(\psi^*\hat{\boldsymbol{p}}\psi)$	(4.13)	x	position coordinate
			$\hat{\boldsymbol{p}}$	momentum operator
			m	particle mass
			$\Re$	real part of
			t	time
Continuity equation	$\nabla\cdot\boldsymbol{j}=-\frac{\partial}{\partial t}(\psi\psi^*)$	(4.14)		
Schrödinger equation	$\hat{H}\psi=\mathrm{i}\hbar\frac{\partial\psi}{\partial t}$	(4.15)	H	Hamiltonian
Particle stationary states[b]	$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+V(x)\psi(x)=E\psi(x)$	(4.16)	V	potential energy
			E	total energy

[a]For particles. In three dimensions, suitable units would be particles $\mathrm{m^{-2}\,s^{-1}}$.
[b]Time-independent Schrödinger equation for a particle, in one dimension.

Operators

Hermitian conjugate operator	$\int (\hat{a}\phi)^* \psi \, dx = \int \phi^* \hat{a}\psi \, dx$	(4.17)	$\hat{a}$ Hermitian conjugate operator ψ, ϕ normalisable functions * complex conjugate
Position operator	$\hat{x^n} = x^n$	(4.18)	x, y position coordinates
Momentum operator	$\hat{p_x^n} = \frac{\hbar^n}{\mathbf{i}^n} \frac{\partial^n}{\partial x^n}$	(4.19)	n arbitrary integer ≥ 1 p_x momentum coordinate
Kinetic energy operator	$\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$	(4.20)	T kinetic energy $\hbar$ (Planck constant)/(2π) m particle mass
Hamiltonian operator	$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$	(4.21)	H Hamiltonian V potential energy
Angular momentum operators	$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$	(4.22) (4.23)	L_z angular momentum along z axis (sim. x and y) L total angular momentum
Parity operator	$\hat{P}\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r})$	(4.24)	$\hat{P}$ parity operator $\boldsymbol{r}$ position vector

Expectation value

Expectation value[a]	$\langle a \rangle = \langle \hat{a} \rangle = \int \Psi^* \hat{a} \Psi \, dx$ $= \langle \Psi \vert \hat{a} \vert \Psi \rangle$	(4.25) (4.26)	$\langle a \rangle$ expectation value of a $\hat{a}$ operator for a Ψ (spatial) wavefunction x (spatial) coordinate
Time dependence	$\frac{d}{dt}\langle \hat{a} \rangle = \frac{\mathbf{i}}{\hbar}\langle [\hat{H}, \hat{a}] \rangle + \left\langle \frac{\partial \hat{a}}{\partial t} \right\rangle$	(4.27)	t time $\hbar$ (Planck constant)/(2π)
Relation to eigenfunctions	if $\hat{a}\psi_n = a_n \psi_n$ and $\Psi = \sum c_n \psi_n$ then $\langle a \rangle = \sum \vert c_n \vert^2 a_n$	(4.28)	ψ_n eigenfunctions of $\hat{a}$ a_n eigenvalues n dummy index c_n probability amplitudes
Ehrenfest's theorem	$m \frac{d}{dt}\langle \boldsymbol{r} \rangle = \langle \boldsymbol{p} \rangle$ $\frac{d}{dt}\langle \boldsymbol{p} \rangle = -\langle \nabla V \rangle$	(4.29) (4.30)	m particle mass $\boldsymbol{r}$ position vector $\boldsymbol{p}$ momentum V potential energy

[a]Equation (4.26) uses the Dirac "bra-ket" notation for integrals involving operators. The presence of vertical bars distinguishes this use of angled brackets from that on the left-hand side of the equations. Note that $\langle a \rangle$ and $\langle \hat{a} \rangle$ are taken as equivalent.

Dirac notation

Matrix element[a]	$a_{nm} = \int \psi_n^* \hat{a} \psi_m \, dx$ (4.31) $= \langle n\|\hat{a}\|m\rangle$ (4.32)	n,m a_{nm} ψ_n $\hat{a}$ x	eigenvector indices matrix element basis states operator spatial coordinate
Bra vector	bra state vector $= \langle n\|$ (4.33)	$\langle\cdot\|$	bra
Ket vector	ket state vector $= \|m\rangle$ (4.34)	$\|\cdot\rangle$	ket
Scalar product	$\langle n\|m\rangle = \int \psi_n^* \psi_m \, dx$ (4.35)		
Expectation	if $\Psi = \sum_n c_n \psi_n$ (4.36) then $\langle a\rangle = \sum_m \sum_n c_n^* c_m a_{nm}$ (4.37)	Ψ c_n	wavefunction probability amplitudes

[a]The Dirac bracket, $\langle n|\hat{a}|m\rangle$, can also be written $\langle \psi_n|\hat{a}|\psi_m\rangle$.