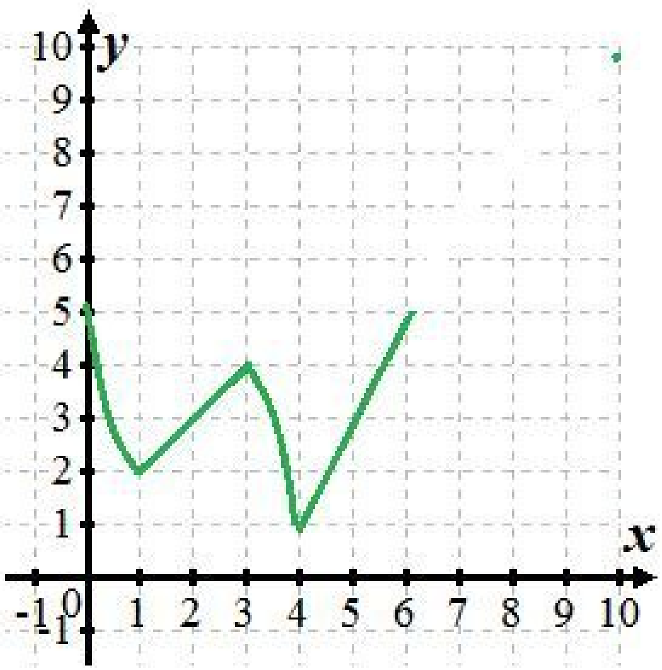
Exercise 3.3

Chapter 3 Applications of Differentiation Exercise 3.3 1E

(a)

Sketch the following diagram:



From the diagram, whenever $x_1 < x_2$ then $f(x_1) < f(x_2)$ in the intervals $(1,3);(4,6)$

Therefore, the function is increasing in the intervals $\boxed{(1,3);(4,6)}$.

(b)

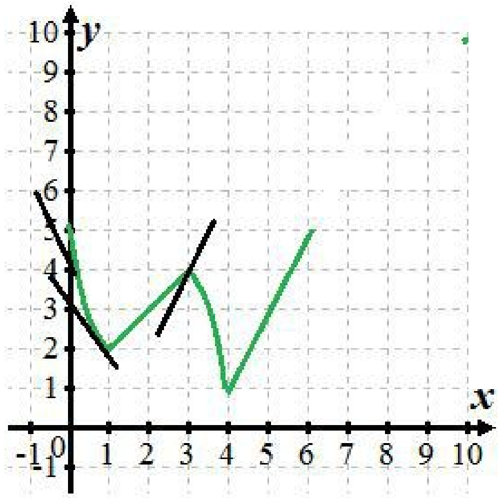
From the diagram, whenever $x_1 < x_2$ then $f(x_1) > f(x_2)$ in the intervals $(0,1);(3,4)$

Therefore, the function is increasing in the intervals $\boxed{(0,1);(3,4)}$.

(c)

The objective is to find interval on which f is concave upward.

Sketch the following diagram:



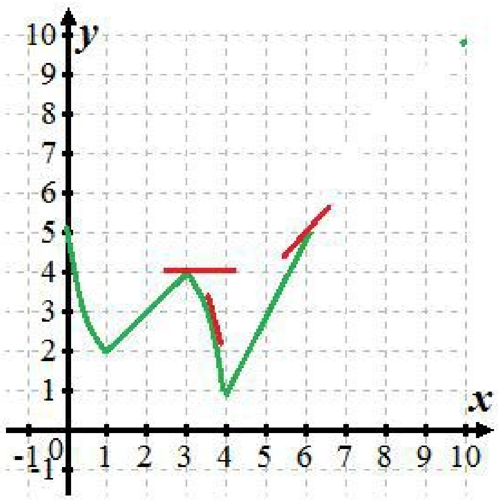
From this diagram, the function f lies above all tangents in the interval $(0,2)$

Therefore, the function f is concave upward on an interval $\boxed{(0,2)}$.

(d)

The objective is to find interval on which f is concave downward.

Sketch the following diagram:



From this diagram, the function f lies below all tangents in the intervals $(2,4);(4,6)$

Therefore, the function f is concave downward on an intervals $\boxed{(2,4);(4,6)}$.

(e)

The function f is changed from concave upward to concave downward only on the interval $(2,3)$.

Therefore, the inflection point is $\boxed{(2,3)}$.

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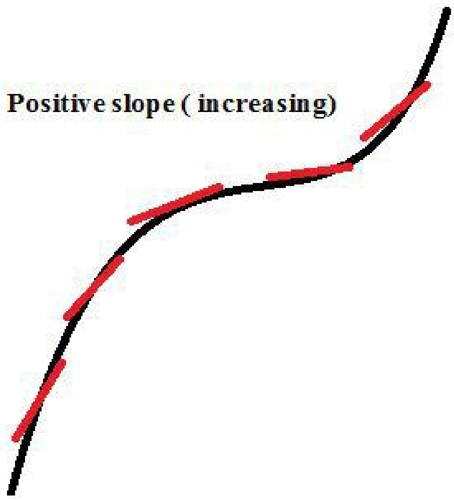
Refer the graph given in the text book.

a)

The tangent lines have positive slop in the intervals $(0,1)$ and $(3,7)$.

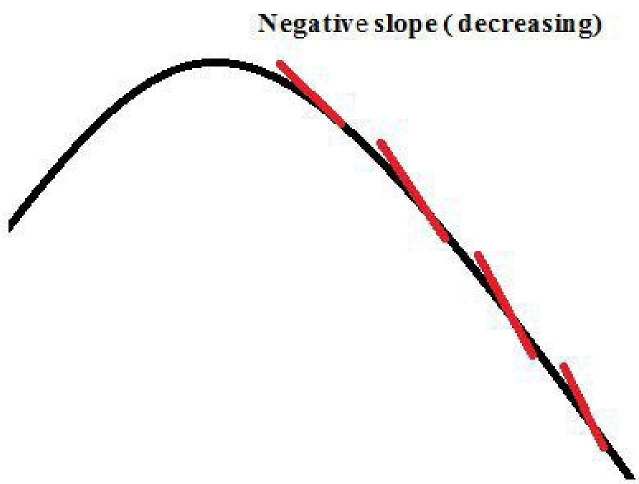
Therefore f is increasing on the open intervals $(0,1)$ and $(3,7)$.



b)

The tangent lines have negative slope in the interval $(1,3)$.

Therefore the graph f is decreasing on the open interval $\boxed{(1,3)}$



c)

The graph of f is concave upward in the intervals $\boxed{(2,4)}$ and $\boxed{(5,7)}$.

d)

The graph of f is concave downward in the intervals $\boxed{(0,2)}$ and $\boxed{(4,5)}$.

e)

The graph is changing from concave downward to upward at $x=2$, upward to downward at $x=4$ and downward to upward at $x=5$

Therefore the coordinates of points of inflection are $\boxed{(2,2)}$, $\boxed{(4,2.9)}$ and $\boxed{(5,4)}$.

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(A)

Suppose a formula for a function f is given so we can check, where f is increasing or decreasing by the first derivative test such as.

(1) If $f'(x) > 0$ on an interval then f is increasing

(2) If $f'(x) < 0$ on an interval then f is decreasing

(B) The concavity of f can be checked by second derivative test such as

(1) If $f''(x) > 0$ for all x in an interval I then the graph of f is concave upward on that interval

(2) If $f''(x) < 0$ for all x in an interval I then the graph of f is concave downward on that interval.

(C) At which point the concavity of f is changing called inflection points of f. It means at an inflection point the concavity will change from upward to downward or from downward to upward.

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(A)

First derivative test: - suppose that f is a continuous function, and a is any critical number of f then

(1) If f' changes from positive to negative at a then f has a local maximum at a.

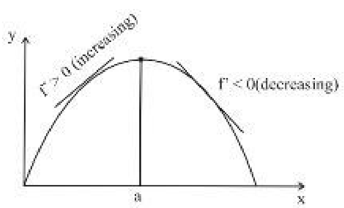
(2) If f' changes from negative to positive at a then f has a local minimum at a.

(3) If f' doesn't change sign at a then f has no local maximum or local minimum at a.

The first derivative test can be used as I / d test such as.

(1) $f' > 0$ On an interval, the f is increasing.

(2) $f' < 0$ On an interval, the f is decreasing.



Local Maximum

Fig. 1

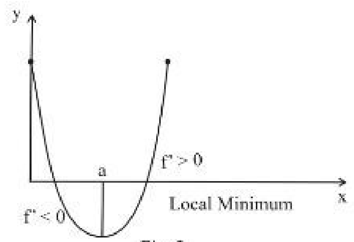


Fig. 2

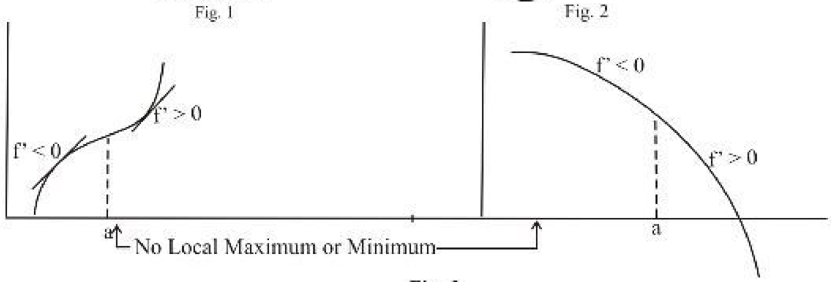


Fig. 3

(B) Second derivative test: - if $f' = 0$ then we apply second derivative test as.

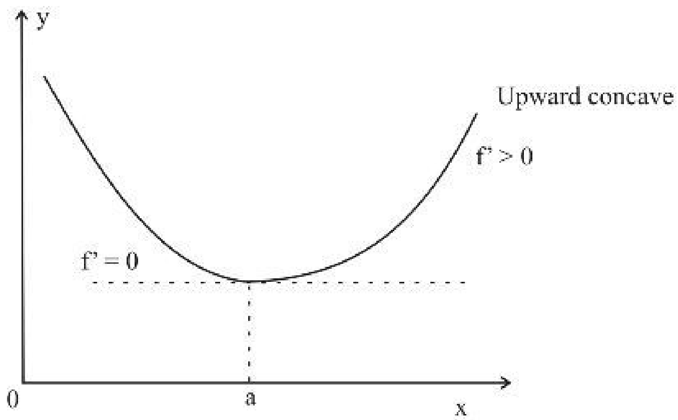
(1) If $f' = 0$ and $f'' > 0$ then f has a local minimum at a

(2) If $f' = 0$ *and* $f'' < 0$ then f has a local maximum at a

We can check the concavity of f with help of f'' such as

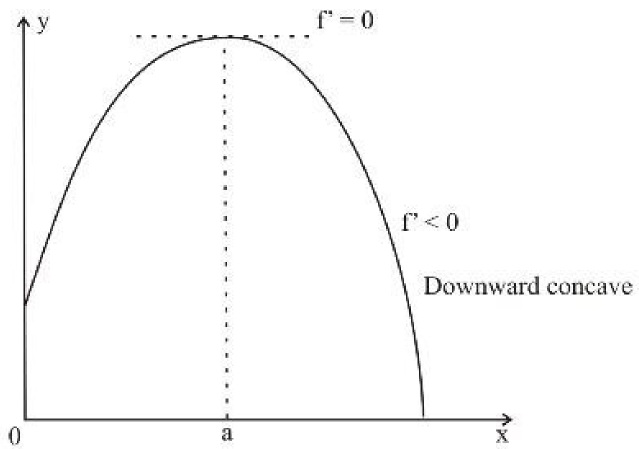
(1) If $f'' > 0$ then f has upward concavity on the interval I

(2) If $f'' < 0$ then f has downward concavity on the interval I



local minimum

Fig. 4



local maximum

Fig. 5

If $f'' = 0$ then second derivative test is in conclusive
So in that case we use first derivative test.

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(A) From the graph we see that the derivative is positive on (1, 5) and negative on (0, 1) and (5, 6)
Then $f(x)$ is increasing on (1, 5) and decreasing on (0, 1) and (5, 6)

(B) Since the sign of the derivative changes from negative to positive at x = 1 and positive to negative at x = 5
So $f(x)$ has a local max. at x = 5 and local minimum at x = 1

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(b)

At $x = 2$, the value of $f'(x)$ changes the sign from positive to negative then $f(x)$ has local minimum at:

$\boxed{x = 2}$.

Also, at $x = 4$, the value $f'(x)$ changes the sign from negative to positive, then $f(x)$ has local maximum at:

$\boxed{x = 4}$.

But at the points $x = 1$, $x = 3$, and $x = 5$, the sign of $f'(x)$ doesn't change.

So there is no maximum or minimum there.

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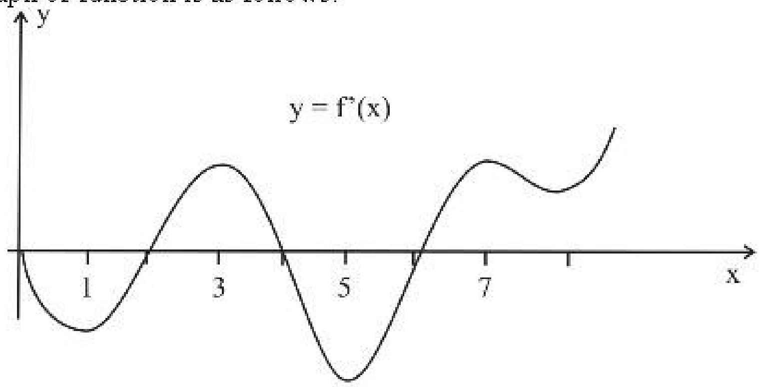
The graph of function is as follows:



(A)

f is increasing on the interval (2, 4) and (6, 9) because on these intervals the value of $f'(x)$ is positive which means $f'(x) > 0$.

(B) At $\boxed{x=2 \text{ and } x=6}$, f has local minimum because at these points the value of $f'(x)$ is changing from negative to positive.

And at $\boxed{x=4}$, f has local maximum because at this point the value of $f'(x)$ is changing from positive to negative.

(C) On the intervals $\boxed{(0,1),(3,5), \text{ and}(7,8)}$ the slope of tangents of the graph of $f'(x)$ is negative, so the value of $f''(x)$ on these intervals is negative. It means $f'' < 0$ so here the function $f(x)$ is $\boxed{\text{concave downward}}$.

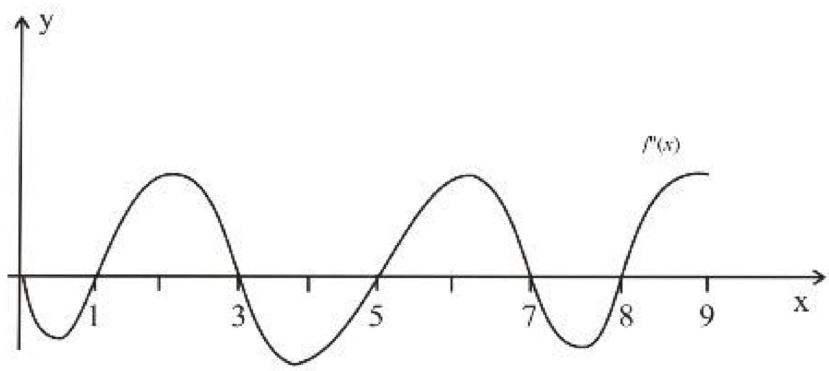
On the internals $\boxed{(1,3),(5,7), \text{and}(8,9)}$ the slope of tangents of the graph of $f'(x)$ is positive, so the value of $f''(x)$ on these intervals is positive. It means here the graph of the function $f(x)$ is $\boxed{\text{concave upward}}$.

(D)

If we draw the rough sketch of $f''(x)$ with the help of given the graph of $f'(x)$ then we can see that the x-coordinates of the inflection points are $\boxed{1,3,5,7, \text{and } 8}$.

On these points, the value of $f''(x)$ is changing from negative to positive or from positive to negative.

Rough graph of $f''(x)$



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(a)

Consider the following function:

$f(x)=2x^3+3x^2-36x$.

The function is continuous and differentiable for real values, since the function is a polynomial.

$$\begin{aligned} f'(x) &= 3(2x^{3-1})+2(3x)-36 && \text{Since } \frac{d}{dx}(x^n)=nx^{n-1} \\ &= 6x^2+6x-36 \\ &= 6(x^2+x-6) && \text{Take common } 6 \\ &= 6(x^2+3x-2x-6) && \text{Factorize} \\ &= 6(x(x+3)-2(x+3)) \\ &= 6(x+3)(x-2) \end{aligned}$$

Increasing/Decreasing test:

1. If $f'(x)>0$ on an interval *I*, then f is increasing on that interval *I*.

2. If $f'(x)<0$ on an interval *I*, then f is decreasing on that interval *I*.

To use the Increasing / Decreasing test, it is needed to know whether $f'(x)>0$, and whether $f'(x)<0$.

This depends upon the signs of the two factors of $f'(x)<0$, namely, $6(x+3)$ and $(x-2)$.

Divide the real line into intervals, whose end-points are critical numbers, -3, 2, and arrange the work on a chart.

Interval	$x+3$	$x-2$	$f'(x)$	f
$x<-3$	$-$	$-$	$+$	Increasing on $(-\infty,-3)$
$-3<x<2$	$+$	$-$	$-$	Decreasing on $(-3,2)$
$x>2$	$+$	$+$	$+$	Increasing on $(2,\infty)$

From the above chart, the function f is increasing on $\boxed{(-\infty,-3)\cup(2,\infty)}$ and f is decreasing on the interval, $\boxed{(-3,2)}$.

(b)

First Derivative Test:

Suppose, *c* is a critical number of a continuous function *f*.

1. If f' changes from positive to negative at *c*, then *f* has local maximum at *c*.

2. If f' changes from negative to positive at *c*, then *f* has local minimum at *c*.

3. If f' does not change sign at *c*, then *f* has neither local maximum nor local minimum at *c*.

From the above chart, if the derivative of the function f' changes from positive to negative at -3, then f has a local maximum at -3.

$$\begin{aligned} f(-3) &= 2(-3)^3+3(-3)^2-36(-3) \\ &= -54+27+108 \\ &= 81 \end{aligned}$$

The function f' changes from negative to positive at $\frac{1}{2}$, then f has a local minimum at $\frac{1}{2}$.

$$\begin{aligned} f(2) &= 2(2)^3+3(2)^2-36(2) \\ &= 16+12-72 \\ &= -44 \end{aligned}$$

Therefore, the function has a local maximum $\boxed{f(-3)=81}$ and local minimum $\boxed{f(2)=-44}$.

(c)

Second Derivative Test:

Suppose f'' is continuous near *c*.

1. If $f'(c)=0$ and $f''(c)>0$ then *f* has local minimum at *c*.

2. If $f'(c)=0$ and $f''(c)<0$ then *f* has local maximum at *c*.

Concavity Test:

1. If $f''(x)>0$ for all x in *I*, Then the graph of f is concave upward on *I*.

2. If $f''(x)<0$ for all x in *I*, Then the graph of f is concave downward on *I*.

A point P on a curve, $y=f(x)$ is called an inflection point, if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at *P*.

To perform the concavity test, find $f''(x)$.

$$f(x)=2x^3+3x^2-36x$$
$$f'(x)=6x^2+6x-36 \qquad \text{From part}(a)$$

and $f''(x)=12x+6$.

Suppose, $f''(x)=0$.

$$\begin{aligned} 12x+6&=0 \\ 12x&=-6 \\ x&=\frac{-6}{12} \\ &=-\frac{1}{2} \end{aligned}$$

Thus, $f''(x)=0$ at $x=-\frac{1}{2}$.

$$\begin{aligned} f\left(-\frac{1}{2}\right)&=2\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2-36\left(-\frac{1}{2}\right) \\ &=-\frac{1}{4}+\frac{3}{4}+18 \\ &=\frac{1}{2}+18 \\ &=\frac{37}{2} \end{aligned}$$

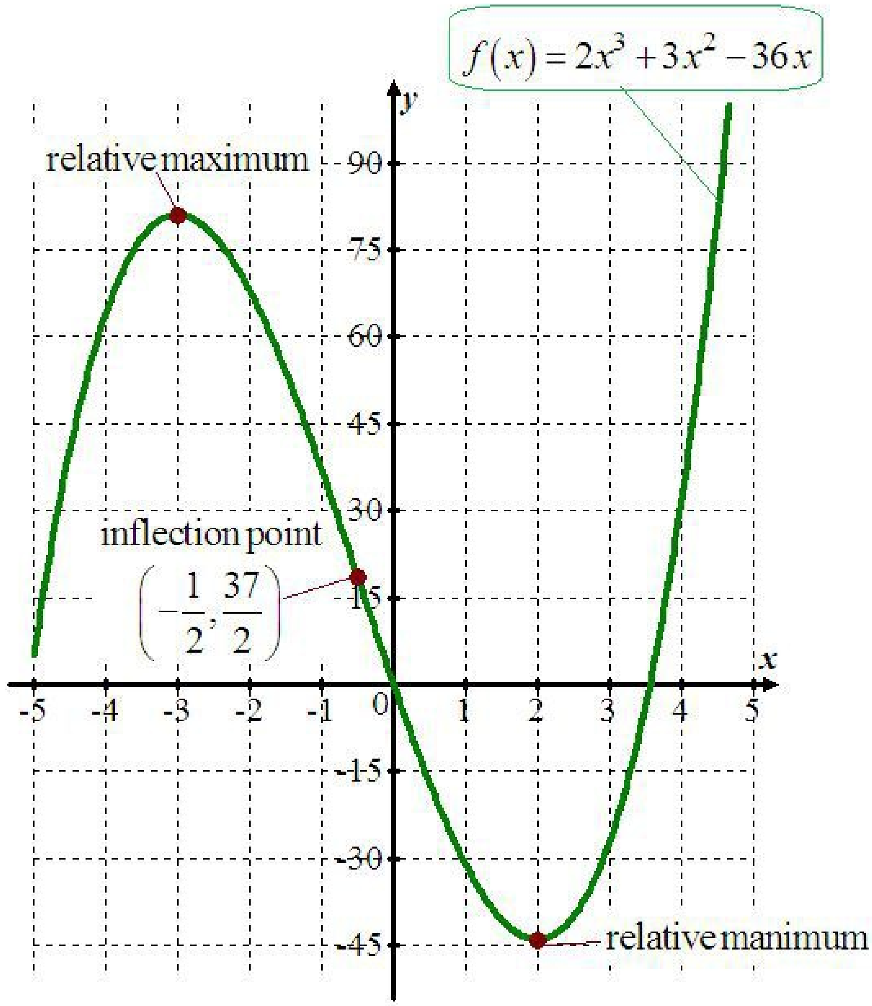
Clearly, $f''(x)<0$ for $x<-\frac{1}{2}$ and $f''(x)>0$ for $x>-\frac{1}{2}$.

Hence, $f(x)$ is concave down in $\boxed{\left(-\infty,-\frac{1}{2}\right)}$ and $f(x)$ is concave up in $\boxed{\left(-\frac{1}{2},\infty\right)}$.

Thus, the inflection point occurs at $x=-\frac{1}{2}$, and the inflection point is,

$$\boxed{\left(-\frac{1}{2},f\left(-\frac{1}{2}\right)\right)=\left(-\frac{1}{2},\frac{37}{2}\right)}.$$

The following figure explains the relative maximum and inflection point:



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First we discuss increasing, decreasing, maximum, minimum, concavity and inflection points of a function.

1. **Increasing/Decreasing test:**

(a) If $f'(x)>0$ on an interval *I*, then f is increasing on that interval *I*.

(b) If $f'(x)<0$ on an interval *I*, then f is decreasing on that interval *I*.

2. **First Derivative Test**: Suppose that *c* is a critical number of a continuous function *f*.

(a) If f' changes from positive to negative at *c*, then *f* has local maximum at *c*.

(b) If f' changes from negative to positive at *c*, then *f* has local minimum at *c*.

(c) If f' does not change sign at *c*, then *f* has neither local maximum nor local minimum at *c*.

3. **Second Derivative Test**: Suppose f'' is continuous near *c*.

(a) If $f'(c)=0$ and $f''(c)>0$ then *f* has local minimum at *c*.

(b) If $f'(c)=0$ and $f''(c)<0$ then *f* has local maximum at *c*.

4. **Concavity Test:**

(a) If $f''(x)>0$ for all x in *I*, Then the graph of f is concave upward on *I*.

(b) If $f''(x)<0$ for all x in *I*, Then the graph of f is concave downward on *I*.

5. A point P on a curve $y=f(x)$ is called an **inflection point** if f is continuous there and and the curve changes from concave upward to concave downward or from concave downward to concave upward at *P*.

(a)

Consider the function

$f(x)=4x^3+3x^2-6x+1$

Then $f'(x)=12x^2+6x-6$

$=6(x+1)(2x-1)$

To use the Increasing / Decreasing test we have to know where $f'(x)>0$ and where $f'(x)<0$.

This depends on the signs of the two factors of $f'(x)<0$, namely, $6(x+1)$ and $(2x-1)$. We divide the real line into intervals whose endpoints are the critical numbers -1, $\frac{1}{2}$ and arrange our work on a chart.

Interval	$x+1$	$2x-1$	$f'(x)$	f
$x<-1$	–	–	+	Increasing on $(-\infty,-1)$
$-1<x<\frac{1}{2}$	+	–	–	Decreasing on $\left(-1,\frac{1}{2}\right)$
$x>\frac{1}{2}$	+	+	+	Increasing on $\left(\frac{1}{2},\infty\right)$

From the above chart, the function f is increasing on $\boxed{(-\infty,-1)\cup\left(\frac{1}{2},\infty\right)}$ and f is decreasing on the interval $\boxed{\left(-1,\frac{1}{2}\right)}$.

(b)

From the above chart the derivative of the given function i.e., f' changes from positive to negative at -1, then f has a local maximum at -1.

Then

$f(-1)=4(-1)^3+3(-1)^2-6(-1)+1$
$=-4+3+6+1$
$=6$

The f' changes from negative to positive at $\frac{1}{2}$, then f has a local minimum at $\frac{1}{2}$.

Then

$f\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)^3+3\left(\frac{1}{2}\right)^2-6\left(\frac{1}{2}\right)+1$
$=0.5+0.75-3+1$
$=-0.75$

Therefore, the function has a local maximum: $\boxed{f(-1)=6}$

and local minimum: $\boxed{f\left(\frac{1}{2}\right)=-0.75}$

(c)

To perform the concavity test we must find $f''(x)$.

$f''(x) = 24x + 6$

Then $f''(x) = 0$ at $x = -\frac{1}{4}$

And

$$f\left(-\frac{1}{4}\right) = 4\left(-\frac{1}{4}\right)^3 + 3\left(-\frac{1}{4}\right)^2 - 6\left(-\frac{1}{4}\right) + 1$$
$$= -\frac{1}{16} + \frac{3}{16} + \frac{6}{4} + 1$$
$$= \frac{42}{16}$$
$$= 2.625$$

So, inflection point occurs at $x = -\frac{1}{4}$ and inflection point is

$$\boxed{\left(-\frac{1}{4}, f\left(-\frac{1}{4}\right)\right) = (-0.25, 2.625)}$$

Clearly $f''(x) < 0$ for $x < -\frac{1}{4}$ and $f''(x) > 0$ for $x > -\frac{1}{4}$

Hence $f(x)$ is concave down in $\boxed{\left(-\infty, -\frac{1}{4}\right)}$

And $f(x)$ is concave up in $\boxed{\left(-\frac{1}{4}, \infty\right)}$.

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(A) We have $f(x) = x^4 - 2x^2 + 3$

Then $f'(x) = 4x^3 - 4x$

$= 4x(x^2 - 1)$

$f'(x) = 0$ When $x = 0$ or $x = \pm 1$

To use I / D test we have to get the intervals on which $f'(x) > 0$ and $f'(x) < 0$.
This depends on the sign of the two factors of $f'(x)$ namely 4x and $(x^2 - 1)$. We make a chart for our work

Interval	4 x	$(x^2 - 1)$	$f'(x)$	f
x < -1	- ve	+ ve	-ve	Decreasing on $(-\infty, -1)$
-1 < x < 0	- ve	- ve	+ve	Increasing on (-1, 0)
0<x < 1	+ ve	- ve	-ve	Decreasing on $(0,1)$
x>1	+ ve	+ ve	+ ve	Increasing on $(1,\infty)$

Conclusion: - we conclude that f is increasing on the interval $(-1, 0)$ and $(1,\infty)$
And f is decreasing on the interval $(-\infty, -1)$ and $(0, 1)$

(B) From the part (a) we see that at x = 0 the sign of $f'(x)$ is changing from positive to negative so at x = 0, f has local maximum that is

$\boxed{f(0) = 3} \Rightarrow$ Local maximum

And now at x = -1 and 1, the sign of $f'(x)$ changing from negative to positive so at x = -1 and 1, $f(x)$ has local minima that is

$f(-1) = f(1) = 1 - 2 + 3 = 2$

$\boxed{f(\pm 1) = 2} \Rightarrow$ Local minimum values

(C) $f''(x) = 12x^2 - 4$

Now $f''(x) > 0$ when $12x^2 - 4 > 0$ or $x^2 > 1/3$ or $\boxed{|x| > 1/\sqrt{3}}$

So $f(x)$ has concavity upward on the $\boxed{\text{intervals } (-\infty, -1/\sqrt{3}) \text{ and } (1/\sqrt{3}, \infty)}$

And $f''(x) < 0$ when $12x^2 - 4 < 0$ or $x^2 < 1/3$ or $\boxed{|x| < 1/\sqrt{3}}$

So $f(x)$ has concavity downward on the $\boxed{\text{interval } (-1/\sqrt{3}, 1/\sqrt{3})}$

For getting inflection points put $f''(x) = 0$

$$\Rightarrow 12x^2 - 4 = 0$$

$$\Rightarrow \boxed{x = \pm 1/\sqrt{3}} \text{ And so } f(\pm 1/\sqrt{3}) = 1/9 - 2/3 + 3 = 22/9$$

So the coordinates of the inflection points are $(\pm 1/\sqrt{3}, 22/9)$

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Consider the function $f(x) = \frac{x}{x^2+1}$

The derivative of the function $f(x)$ is

$$f'(x) = \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

(a)

The function f is increasing on the intervals where $f'(x) > 0$.

$$\frac{1-x^2}{(x^2+1)^2} > 0$$

$$1 - x^2 > 0$$

$$1 > x^2$$

$$x^2 < 1$$

$$-1 < x < 1$$

or $x \in (-1, 1)$

Therefore, f is increasing on the interval $\boxed{(-1,1)}$.

And f is decreasing on the intervals where $f'(x) < 0$

$$\frac{1-x^2}{(x^2+1)^2} < 0$$

$$1 - x^2 < 0$$

$$1 < x^2$$

$$x \in (-\infty, -1) \cup (1, \infty)$$

Therefore, f is decreasing on $\boxed{(-\infty, -1) \text{ and } (1, \infty)}$.

(b)

The function f has a local maximum where it changes from increasing to decreasing, where f' changes from positive to negative at $x=1$.

Similarly where f' changes from negative to positive, f has local minimum at $x=-1$.

Therefore, f has **local maximum** at $\boxed{x=1}$ and f has **local minimum** at $\boxed{x=-1}$.

(c)

The second derivative of the function $f(x)$ is

$$f''(x)=\frac{(x^2+1)^2(-2x)-(1-x^2)2(x^2+1)2x}{(x^2+1)^4}$$

$$=\frac{(x^2+1)\left[(x^2+1)(-2x)-(1-x^2)(4x)\right]}{(x^2+1)^4}$$

$$=\frac{(x^2+1)\left[-2x^3-2x-4x+4x^3\right]}{(x^2+1)^4}$$

$$=\frac{(x^2+1)\left[2x^3-6x\right]}{(x^2+1)^4}$$

$$=\frac{2x(x^2-3)}{(x^2+1)^3}$$

Set the function

$$f''(x)=0$$

$$\frac{2x(x^2-3)}{(x^2+1)^3}=0$$

$$x=0,\pm\sqrt{3}$$

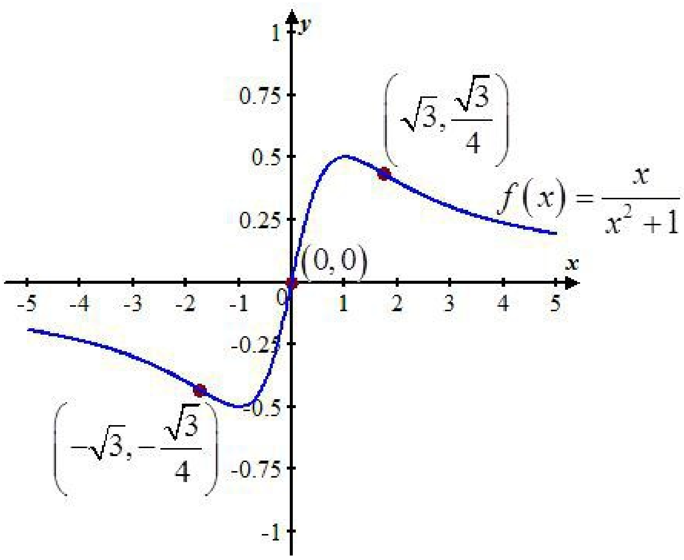
Therefore, where f' is increasing, its derivative f'' is positive.

f is concave upward on $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$.

Similarly f is concave downward when f' is decreasing on $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$.

The points $\left(-\sqrt{3},-\frac{\sqrt{3}}{4}\right)$, $(0,0)$ and $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$ are inflection points because the curve changes from concave upward to concave downward and again concave downward to concave upward.

The graph is shown below:



(a)

Consider the function,

$f(x) = \sin x + \cos x, 0 \le x \le 2\pi$.

The object is to find the intervals on which f is increasing or decreasing.

Differentiate the function $f(x) = \sin x + \cos x$ on both sides with respect to x, and then the derivative of the function is,

$f'(x) = \cos x - \sin x$

For maximum or minimum values set $f'(x) = 0$, then this implies that

$$\begin{aligned} f'(x) &= 0 \\ \cos x - \sin x &= 0 \\ \cos x &= \sin x \\ x &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \end{aligned}$$

So the critical points of the function are $\frac{\pi}{4}$ or $\frac{5\pi}{4}$.

Now check the values of $f'(x)$ in the interval $[0, 2\pi]$ as shown in the below table:

Interval	sing of $f'(x)$	f decreasing or increasing
$\left(0, \frac{\pi}{4}\right)$	+	f is increasing on $\left(0, \frac{\pi}{4}\right)$
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	−	f is decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$
$\left(\frac{5\pi}{4}, 2\pi\right)$	+	f is increasing on $\left(\frac{5\pi}{4}, 2\pi\right)$

From the table, it is observed that the function is increasing on the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$. The function is decreasing on the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

(b)

Consider the function,

$f(x) = \sin x + \cos x$.

From the part (a), observe the table it confirms that the derivative of the function is changes from positive to negative at $x = \frac{\pi}{4}$. So the function has a maximum value at $x = \frac{\pi}{4}$. Here, the derivative changes from negative to positive at $x = \frac{5\pi}{4}$. So the function has a minimum value at $x = \frac{5\pi}{4}$. Since the critical points of the function are $x = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

Substitute $x = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$ into the function $f(x) = \sin x + \cos x$, and then the function values are

$$
\begin{aligned}
f\left(\frac{\pi}{4}\right) &= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= \frac{2}{\sqrt{2}} \\
&= \sqrt{2}
\end{aligned}
$$

$$
\begin{aligned}
f\left(\frac{5\pi}{4}\right) &= \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} \\
&= \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}} \\
&= \frac{-2}{\sqrt{2}} \\
&= -\sqrt{2}
\end{aligned}
$$

Hence, the local maximum and local minimum values of the function are respectively

$\boxed{\left\{\sqrt{2}, -\sqrt{2}\right\}}$.

(c)

Consider the function,

$f(x) = \sin x + \cos x$.

The first and second derivatives of the function are respectively

$$
\begin{aligned}
f'(x) &= \cos x - \sin x \\
f''(x) &= -\sin x - \cos x \\
&= -(\sin x + \cos x)
\end{aligned}
$$

For inflection points set $f''(x) = 0$, this implies that

$$
\begin{aligned}
-(\sin x + \cos x) &= 0 \\
\sin x &= -\cos x
\end{aligned}
$$

The values satisfy the equation $\sin x = -\cos x$ in the interval $[0, 2\pi]$ is

$x = \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

So, the inflection points situated at $x = \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

The sign of the second derivative $\left(f''(x)\right)$ are shown in the below table:

Interval	sing of $f''(x) = -(\sin x + \cos x)$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$-$	f is concave down on $\left(0, \frac{\pi}{4}\right)$
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$+$	f is concave up on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$
$\left(\frac{7\pi}{4}, 2\pi\right)$	$-$	f is concave down on $\left(\frac{5\pi}{4}, 2\pi\right)$

From the table, it confirms that the function is concave up on the interval $\boxed{\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)}$ and the function is concave down on the interval $\boxed{\left(0, \frac{3\pi}{4}\right) \text{ and } \left(\frac{7\pi}{4}, 2\pi\right)}$.

Substitute $x = \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ into $f(x) = \sin x + \cos x$, and then the functional values at those points are

$$
\begin{aligned}
f\left(\frac{3\pi}{4}\right) &= \sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} \\
&= \sin\left(\pi - \frac{\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right) \\
&= \sin\frac{\pi}{4} - \cos\frac{\pi}{4} \\
&= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
f\left(\frac{7\pi}{4}\right) &= \sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} \\
&= \sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right) \\
&= -\sin\frac{\pi}{4} + \cos\frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= 0
\end{aligned}
$$

Hence, the inflection points are $\boxed{\left(\frac{3\pi}{4}, 0\right) \text{ and } \left(\frac{7\pi}{4}, 0\right)}$.

Chapter 3 Applications of Differentiation Exercise 3.3 14E

A) Find Intervals where increasing or decreasing;

$f(x) = \cos^2 x - 2\sin x,\ 0 \le x \le 2\pi$.

$f'(x) = -2\cos x * \sin x - 2\cos x = -2\cos x(1 + \sin x)$.

$f'(x) = 0$ when $\cos x = 0$ or $\sin x = -1$

or when $x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = \frac{3\pi}{2}$

now since on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, f'(x) < 0

on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, f'(x)>0

Thus, f is increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and f is decreasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

B) Find local max and min values of f;

f changes from decreasing to increasing at $x = \frac{\pi}{2}$ and from increasing to decreasing at $x = \frac{3\pi}{2}$.

Thus, $f\left(\frac{\pi}{2}\right) = -2$ is a local minimum value;

and $f\left(\frac{3\pi}{2}\right) = 2$ is a local maximum value.

C) Find the intervals of concavity and the inflection points.

$f''(x) = 2\sin x(1+\sin x) - 2\cos^2 x$

$= 2\sin x + 2\sin^2 x - 2(1-\sin^2 x)$,

$= 4\sin^2 x + 2\sin x - 2$

$= 2(2\sin x - 1)(\sin x + 1)$

so $f''(x) > 0 \Leftrightarrow \sin x > \frac{1}{2} \Leftrightarrow \frac{\pi}{6} < x < \frac{5\pi}{6}$,

and $f''(x) < 0 \Leftrightarrow \sin x < \frac{1}{2}$ and $\sin x \neq -1 \Leftrightarrow 0 < x < \frac{\pi}{6}$

or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$.

With all that said, f is concave up on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

and concave down on $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$.

There are inflection points at $\left(\frac{\pi}{6}, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}, -\frac{1}{4}\right)$.

Chapter 3 Applications of Differentiation Exercise 3.3 15E

Consider the function $f(x) = 1 + 3x^2 - 2x^3$

Now

$f'(x) = 6x - 6x^2 = 6x(1-x)$
$f''(x) = 6 - 12x$

First derivative test:

Suppose that *c* is a critical number of continuous function *f*.

(a) If f' changes from positive to negative at *c*, then *f* has a local maximum at *c*.

(b) If f' changes from negative to positive at *c*, then *f* has a local minimum at *c*.

(c) If f' does not change sign at *c*, then *f* has no local maximum or minimum at *c*.

Now $f'(x) = 6x(1-x)$, so critical points are $x = 0, 1$.

Interval	$f'(x)$
$x < 0$	−
$0 < x < 1$	+
$x > 1$	−

Clearly $f'(x)=6x(1-x)$ changes sign from negative to positive at $x=0$

Hence $f(x)$ has local minimum at $\boxed{x=0}$.

And $f'(x)$ Changes sign from positive to negative at $x=1$

Hence $f(x)$ has local maximum at $\boxed{x=1}$.

Thus, local maximum is

$$\begin{aligned} f(1) &= 1+3-2 \\ &= \boxed{2} \end{aligned}$$

And local minimum is

$$\begin{aligned} f(0) &= 1+0-0 \\ &= \boxed{1} \end{aligned}$$

Second derivative test:

Suppose f'' is continuous at *c*.

(a) If $f'(c)=0$ and $f''(c)>0$, then *f* has a local minimum at *c*.

(b) If $f'(c)=0$ and $f''(c)<0$, then *f* has a local maximum at *c*.

If $f'(x)=0$ then $x=0, x=1$

Since $f''(x)=6-12x$

So,

$$f''(0)=6>0$$
$$f''(1)=-6<0$$

Hence $f(x)$ has local maximum at $x=1$ and local minimum at $x=0$

Thus, local maximum value is $f(1)=\boxed{2}$

And local minimum value is $f(0)=\boxed{1}$

Preference: For this function, the two tests are equally easy.

Chapter 3 Applications of Differentiation Exercise 3.3 16E

The first derivative test requires the test of the change in the sign of the function at the critical point.

The second derivative requires the test of the sign of the value obtained in the second derivative at the critical point.

Consider the function:

$$f(x)=\frac{x^2}{x-1}$$

Determine the derivative of the above function:

$$\begin{aligned} f'(x) &= \frac{(x-1)(2x)-x^2(1)}{(x-1)^2} \\ &= \frac{2x^2-2x-x^2}{(x-1)^2} \\ &= \frac{x^2-2x}{(x-1)^2} \\ &= \frac{x(x-2)}{(x-1)^2} \end{aligned}$$

Perform the first derivative test:

$$f'(x) < 0$$
$$x \in (\infty, 0) \text{ or } (2, \infty)$$

The function f' changes from positive to negative at $x = 0$.

So, the value $f(0) = 0$ is local maximum.

The function f' changes from negative to positive at $x = 2$.

So, the value $f(2)$ is local minimum.

$$f(2) = \frac{4}{2-1}$$
$$= 4$$

Perform the second derivative test:

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x)2(x-1)}{(x-1)^4}$$

Put $f'(x) = 0$ to find $x = 0$ and $x = 2$.

Put $x = 0$ in the function $f''(x)$:

$$f''(0) = \frac{(-2)}{1}$$
$$= -2$$
$$f''(0) < 0$$

So, the point $x = 0$ is the point of local maximum.

Put $x = 2$ in the function $f''(x)$:

$$f''(2) = \frac{(2) - 0}{1}$$
$$= 2$$
$$f''(2) > 0$$

So, the point $x = 2$ is the point of local minimum.

Consider the function,

$f(x) = \sqrt{x} - x^{1/4}$.

The objective is to determine the local maximum and minimum of f using both the first and second derivative tests.

Using first derivative test:

To determine the critical points, find the derivative of the function and set it equal to zero.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\sqrt{x} - x^{1/4}\right) \\ &= \frac{1}{2\sqrt{x}} - \frac{1}{4x^{3/4}} \end{aligned}$$

Equate $f'(x) = 0$ and solve for x:

$$\begin{aligned} \frac{1}{2\sqrt{x}} - \frac{1}{4x^{3/4}} &= 0 \\ \frac{2x^{1/4} - 1}{4x^{3/4}} &= 0 \\ 2x^{1/4} - 1 &= 0 \\ x^{1/4} &= \frac{1}{2} \end{aligned}$$

$$x = \frac{1}{16}$$

So, the only critical value is $x = \frac{1}{16}$.

Thus, the intervals are .

According to the first derivative test, we have to see where $f'(x) > 0$ and where $f'(x) < 0$.

Interval	$f'(x)$	$f(x)$
$\left(0, \frac{1}{16}\right)$	negative	decreasing over $\left(0, \frac{1}{16}\right)$
$\left(\frac{1}{16}, \infty\right)$	positive	increasing over $\left(\frac{1}{16}, \infty\right)$

So, at the critical point $x = \frac{1}{16}$, the function *f* is changing from decreasing to increasing, so the function *f* has a local minimum at $\boxed{x = \frac{1}{16}}$.

And, the value at $x = \frac{1}{16}$ is,

$$\begin{aligned} f\left(\frac{1}{16}\right) &= \sqrt{\frac{1}{16}} - \left(\frac{1}{16}\right)^{1/4} \\ &\approx -\frac{1}{4} \end{aligned}$$

>

Therefore, the local minimum of $f(x)$ is $-\frac{1}{4}$ at $x = \frac{1}{16}$.

Now, determine the local maximum or minimum, by using the second derivative test.

Compute $f''(x)$ by using $f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{4x^{3/4}}$ as follows:

$$f''(x)=\frac{d}{dx}\left(\frac{1}{2\sqrt{x}}-\frac{1}{4x^{3/4}}\right)$$
$$=-\frac{1}{4x^{3/2}}+\frac{3}{16x^{7/4}}$$

Now, substitute the critical point $x=\frac{1}{16}$ in $f''(x)=-\frac{1}{4x^{3/2}}+\frac{3}{16x^{7/4}}$, to get:

$$f''(x)=-\frac{1}{4\left(\frac{1}{16}\right)^{3/2}}+\frac{3}{16\left(\frac{1}{16}\right)^{7/4}}$$
$$=8>0$$

Thus, the function *f* has a local minimum at $x=\frac{1}{16}$, and the local minimum is

$$f\left(\frac{1}{16}\right)=\sqrt{\frac{1}{16}}-\left(\frac{1}{16}\right)^{1/4}$$
$$\approx-\frac{1}{4}$$

Therefore, the local minimum of $f(x)$ is $-\frac{1}{4}$ at $x=\frac{1}{16}$.

Hence, the local minimum of $f(x)$ is $\left(\frac{1}{16},-\frac{1}{4}\right)$ from both methods, in this case, the second derivative test is preferable.

Chapter 3 Applications of Differentiation Exercise 3.3 18

(A)

The given function is

$$f(x)=x^4(x-1)^3$$

Then using product rule, we have

$$f'(x)=4x^3(x-1)^3+3x^4(x-1)^2$$
$$=(x-1)^2x^3\left[4(x-1)+3x\right]$$
$$=x^3(x-1)^2\left[4x-4+3x\right]$$
$$=x^3(x-1)^2(7x-4)$$

At the critical point, $f'(x)=0$

i.e., $x^3(x-1)^2(7x-4)=0$

i.e., $x=0,\ 1,\ \frac{4}{7}$ i.e. the critical points are $\boxed{0, 1, \text{and } \frac{4}{7}}$.

(B)

Now $f''(x)=3x^2(x-1)^2(7x-4)+2x^3(x-1)(7x-4)+7x^3(x-1)^2$

$$=x^2(x-1)\left[3(x-1)(7x-4)+2x(7x-4)+7x(x-1)\right]$$
$$=x^2(x-1)6\left(7x^2-8x+2\right)$$

Now we check for the critical points, i.e.,

When $x=0$, $f''(x)=0$

$\Rightarrow$Second derivative test is inconclusive at $x=0$.

When $x = 1$, $f''(x) = 0$

$\Rightarrow$Second derivative test is inconclusive at $x = 1$.

When $x = \frac{4}{7}$,

$$f''(x) = \left(\frac{4}{7}\right)^2 \left(\frac{-3}{7}\right)\left(\frac{-2}{4}\right) > 0$$

$\Rightarrow f(x)$ has a local minimum at $x = \frac{4}{7}$.

(C) To apply first derivative test, consider the intervals $(-\infty, 0)$, $\left(0, \frac{4}{7}\right)$, $\left(\frac{4}{7}, 1\right)$, $(1, \infty)$.

So we have

Interval	Behavior of $f'(x)$
$-\infty < x < 0$	Positive
$0 < x < \frac{4}{7}$	Negative
$\frac{4}{7} < x < 1$	Positive
$1 < x < \infty$	Positive

Then first derivative test conduces that,

Since $f'(x)$ changes sign from positive to negative at $x = 0$, so $x = 0$ is a point of local maximum.

Since $f'(x)$ changes sign from negative to positive at $x = \frac{4}{7}$, then $x = \frac{4}{7}$ is a point of local minimum.

Since $f'(x)$ does not change sign at $x = 1$, then $x = 1$ is neither a point of maximum nor minimum.

Chapter 3 Applications of Differentiation Exercise 3.3 19E

Consider f'' is continuous on $(-\infty, \infty)$

(a)

Given $f'(2) = 0$ and $f''(2) = -5$

By second derivative test,

Suppose f'' is continuous near *c*,

(a) If $f'(x) = 0$ and $f''(x) > 0$ then the function *f* has local maximum at *c*.

(b) If $f'(x) = 0$ and $f''(x) < 0$ then the function *f* has local minimum at *c*.

Here, for $x = 2$, $f'(2) = 0$, $f''(2) < 0$

Hence, the function *f* has local maximum at 2.

(b)

Given $f'(6) = 0$ and $f''(6) = 0$

In this case $x = 6$ is an inflection point in which there is a transition between concave upward to concave downward or concave downward to concave upward. In this, particular case, point of inflection has a slope of 0 when $x = 6$, since the first derivative is zero.

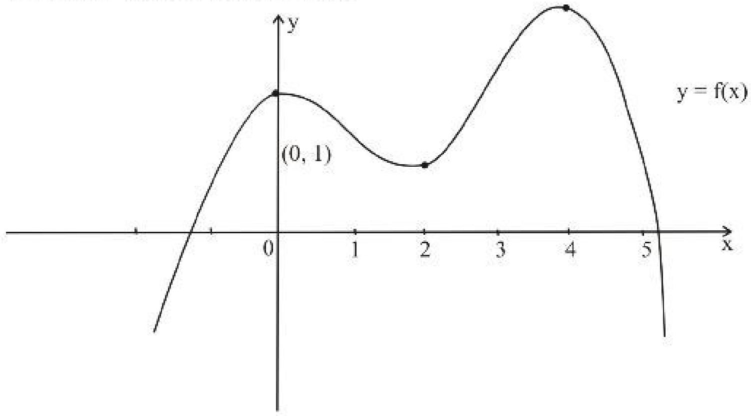
The line with slope zero is horizontal line.

From these result, it can only say that *f* has horizontal tangent at 6.

Chapter 3 Applications of Differentiation Exercise 3.3 20E

The first condition is $f'(0) = f'(2) = f'(4) = 0$
This tells that f has horizontal tangent at 0, 2 and 4
Second condition is $f'(x) > 0$ if $x < 0$ or $2 < x < 4$
Means f has positive slope or increasing on the interval $(-\infty, 0)$ and (2, 4)
Third condition is $f'(x) < 0$ if $0 < x < 2$ or $x > 4$
This means f is decreasing on the interval (0. 2) and $(4, \infty)$
It means f has local maximum at x = 0 and 4 and f has local minimum at $x = 2$
Forth condition is $f''(x) > 0$ if $1 < x < 3$ means f has concavity upward on interval (1, 3) and $f''(x) < 0$ if $x < 1$ or $x > 3$ means f has concavity downward on interval $(-\infty, 1)$ and $(3, \infty)$

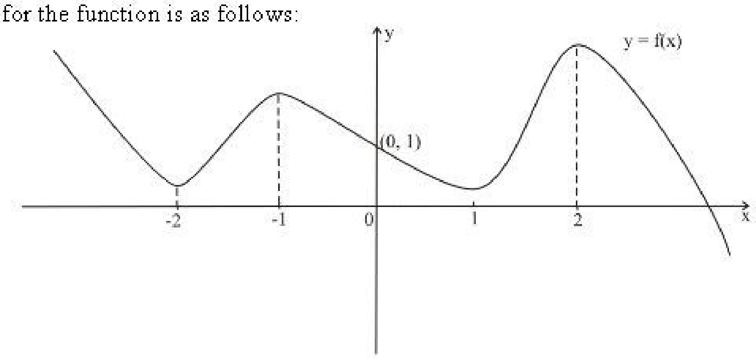
So graph of $f(x)$ will be an follows



Chapter 3 Applications of Differentiation Exercise 3.3 22E

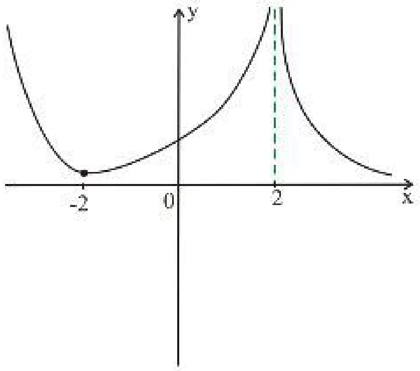
First condition is $f'(1) = f'(-1) = 0$ means at 1 and -1 f has horizontal tangent.
$f'(x) < 0$ *if* $|x| < 1$ means f is decreasing when $-1 < x < 1$.
$f'(x) > 0$ *if* $1 < |x| < 2$ means f is increasing on the intervals (-2, -1) and (1, 2).
$f'(x) = -1$ *if* $|x| > 2$ means f has a tangent with slope -1 on $(-\infty, -2)$ and $(2, \infty)$.
$f''(x) < 0$ *if* $-2 < x < 0$ means f is concave downward on (-2, 0).
f has local minimum at x = -2, and x =1.
f has local maximum at x = -1 and x =2.
Inflection point is (0, 1) means at this point the concavity of graph changes.

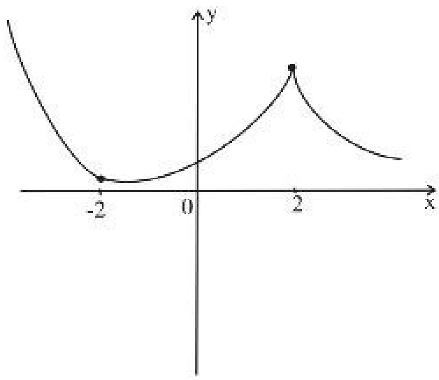
The graph for the function is as follows:



Chapter 3 Applications of Differentiation Exercise 3.3 23E

$f'(x) > 0$ *if* $|x| < 2$, f is increasing on $(-2, 2)$
$f'(x) < 0$ *if* $|x| > 2$ f is decreasing on $(-\infty, -2)$ and $(2, \infty)$
So f has local minimum at x = -2 and local maximum at x = 2
$f'(-2) = 0$ Means at x = -2, f has horizontal tangent
$\lim_{x \to 2} |f'(x)| = \infty$ Means at x = 2, f is not defined or not differentiable.
$f'' > 0$ if $x \neq 2$ f has concavity upward for all x when $x \neq 2$ so we can get two graphs of f, one is not defined at 2 and another is having corner at 2





Chapter 3 Applications of Differentiation Exercise 3.3 24

$f(0) = 0$ and $f'(0) = f'(2) = f'(4) = f'(6) = 0$

So f has horizontal tangents at x = 0, 2, 4 and 6
Now we make a chart of information's about $f'(x)$

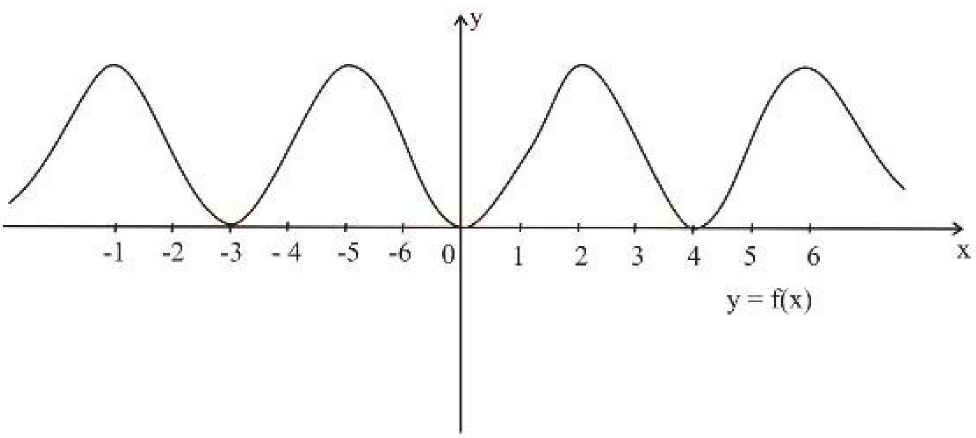
Interval	$f'(x)$	f
0 < x < 2	+	f is increasing on (0, 2)
2 < x < 4	-	f is decreasing on (2, 4)
4 < x < 6	+	f is increasing on (4, 6)
x > 6	-	f is decreasing on $(6,\infty)$

So f has local maxima at x = 2 and 6 and f has local minima at x = 4
Again we make a chart for $f''(x)$

Interval	$f''(x)$	f
0 < x < 1	+	Concave upward on (0,1)
1 < x < 3	-	Concave downward on (2, 4)
3 < x < 5	+	Concave upward on (3, 5)
x > 5	-	Concave downward on $(5,\infty)$

So inflection points have the x-coordinates 1, 3 and 5

Now $f(-x) = f(x)$ means f is even function so graph of will be symmetric with respect to the y-axis.
With the help of these informations we can draw the graph

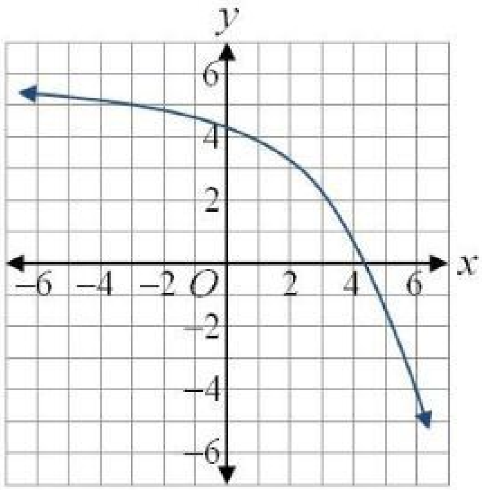


Chapter 3 Applications of Differentiation Exercise 3.3 25E

Consider the problem statement provided in the textbook.

The function *f* is always decreasing if $f'(x) < 0$ for all *x* and the function *f* is always concave down if $f''(x) < 0$ for all *x*.

The function *f* is decreasing and concave down. Therefore, the graph of the problem is:



Chapter 3 Applications of Differentiation Exercise 3.3 26

we consider the interval [2,4] such that f(3) = 2 , f ' (3) = 1/2while f '(x) >0 , f ''(x) <0

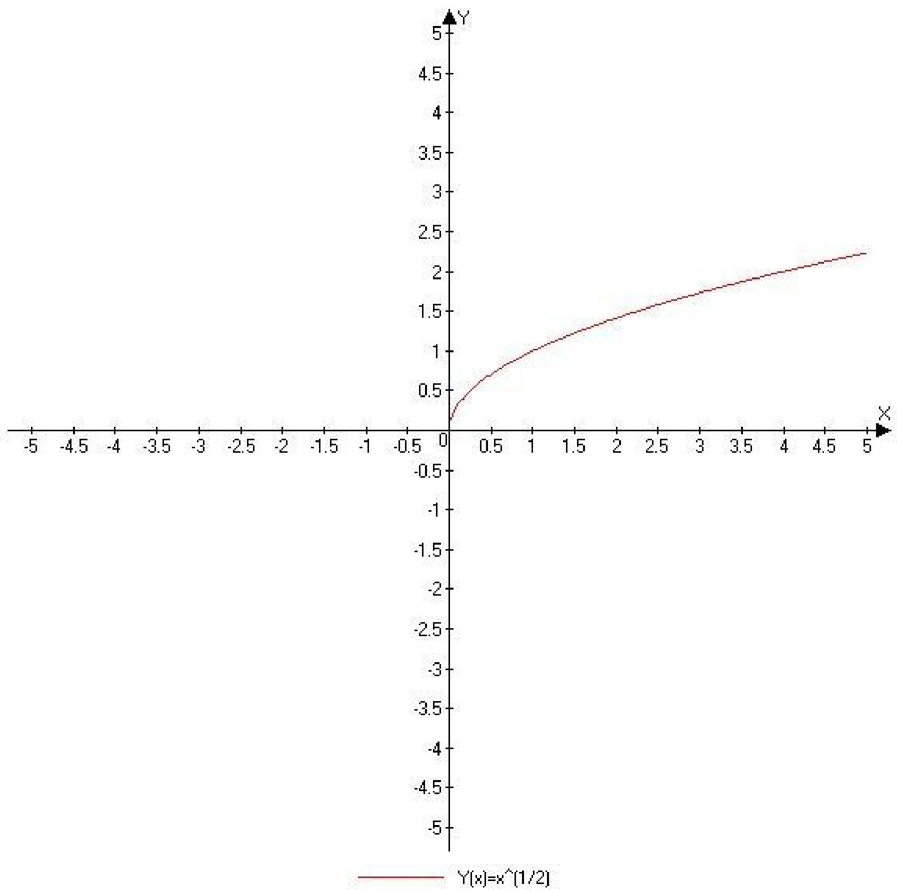
f ' (x) means the ratio between the images and the elements in the neighbourhood of 3 must be 1 : 2.

from this , we follow that the rate of change in the y coordinate is half of the rate of change in the x coordinate . consequently, the concavity of the graph will be down wards .

i.e. f ''(x) <0.

keeping all these conditions in view, we graph as follows :

(a)



(b) this type of function will have two solutions in the interval having 3 at the middle .

because the graph is increasing from a certain point and having the concavity downwards .

As consequence the curve reaches the x axis again starting from x axis.

(c) f ' (2) = 1/3 < 1/2 = f ' (3) is possible while the graph has the concavity downwards,

f ' (x) < f ' (3) for all x < 3 in the neighbourhood of 3 must hold.

Chapter 3 Applications of Differentiation Exercise 3.3 27E

(A)

f is increasing on (0, 2), (4, 6) and $(8, \infty)$
f is decreasing on (2, 4), (6, 8)

(B)

Local maxima at $x = 2$ and 6
Local minima at $x = 4$ and 8

(C)

f is concave upward where $f'(x)$ has positive slope that is on the intervals (3, 6) and $(6, \infty)$

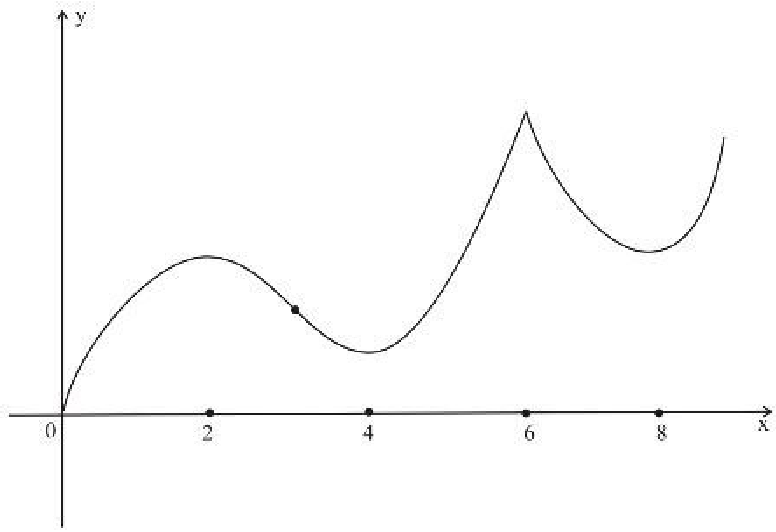
f is concave downward where $f'(x)$ has negative slope that is on (0, 3)

(D)

Inflection point is at $\boxed{x = 3}$ because here $f''(x)$ is changing its sign

(E)

Assuming $f(0)=0$ we draw the graph of $f(x)$



Chapter 3 Applications of Differentiation Exercise 3.3 28E

(A)

f is decreasing on the intervals (0, 1) and (6, 8)
f is increasing on the intervals (1, 6) and (8, ∞)

(B)

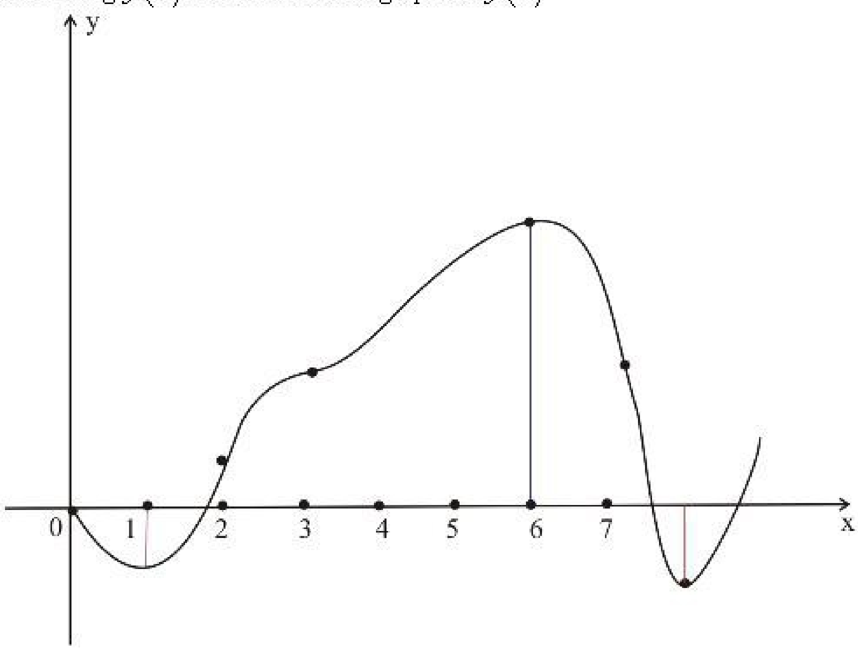
f has local maximum at x = 6
f has local minimum at x = 1 and 8

(C) f is concave upward where $f'(x)$ has positive slope that is on (0, 2), (3, 5) and $(7,\infty)$.

f is concave downward where $f'(x)$ has negative slope that is on (2, 3) and (5, 7)

(D) Inflection points are at x = 2, 3, 5 and 7

(E) Assuming $f(0)=0$ we draw the graph of $f(x)$



Given that the function is $f(x) = x^3 - 12x + 2$

Then $f'(x) = 3x^2 - 12$

(a) $f'(x) > 0$

$\Rightarrow \quad 3x^2 - 12 > 0$

$\Rightarrow \quad x^2 - 4 > 0$

$\Rightarrow \quad (x+2)(x-2) > 0$

$f(x)$ is increasing on $(-\infty, -2), (2, \infty)$

$f'(x) < 0 \;\Rightarrow\; (x+2)(x-2) < 0$

$f(x)$ is decreasing on $(-2, 2)$.

(b) If $f'(x) = 0$ then $x = \pm 2$

Critical points are $x = \pm 2$

Since $f''(x) = 6x$, $f''(2) = 12 > 0$

$x = 2$ is local minimum

$f''(-2) = -12 < 0$

$x = -2$ is local maximum.

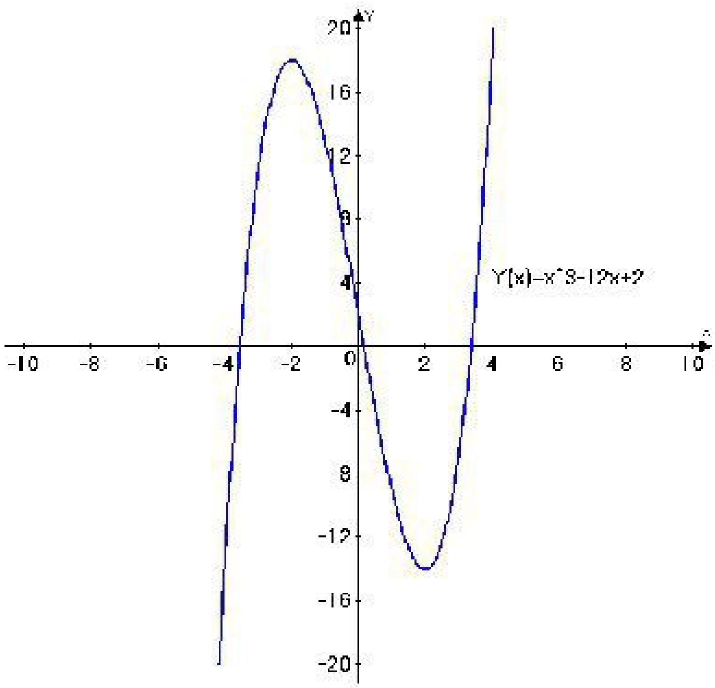
(c) $f''(x) < 0 \Rightarrow 6x < 0 \Rightarrow x < 0$

Concave up on $(0, \infty)$

$f''(x) < 0 \Rightarrow 6x < 0 \Rightarrow x < 0$

Concave down on $(-\infty, 0)$.

(d) The graph is



Given that the function is $f(x) = 36x + 3x^2 - 2x^3$

Then $f'(x) = 36 + 6x - 6x^2$

(a) $f'(x) > 0$

$\Rightarrow \quad 36 + 6x - 6x^2 > 0$

$\Rightarrow \quad x^2 - x - 6 < 0$

$\Rightarrow \quad (x-3)(x+2) < 0$

$\Rightarrow \quad (-2, 3)$

$f(x)$ is increasing on $(-2, 3)$

$f'(x) < 0$

$\Rightarrow \quad (x-3)(x+2) > 0$

$\Rightarrow \quad x \in (-\infty, -2), (3, \infty)$

$f(x)$ is decreasing on $(-\infty, -2), (3, \infty)$

(b) If $f'(x) = 0$ then $x = -2, 3$

Critical points are $-2, 3$

Since $f''(x) = 6 - 12x$,

$f''(-2) = 6 + 24$

$= 30 > 0$

$x = -2$ is a local minimum of $f(x)$

Since $f''(x) = 6 - 12x$,

$f''(3) = 6 - 36$

$= -30 < 0$

$x = 3$ is a local maximum of $f(x)$ and

$f(3) = 108 + 27 - 54$

$= 108 - 27$

$= 81$

(c) $f''(x) > 0$

$\Rightarrow \quad 6 - 12x > 0$

$\Rightarrow \quad 6 > 12x$

$\Rightarrow \quad x < \frac{1}{2}$

Concave upon $\left(-\infty, \frac{1}{2}\right)$

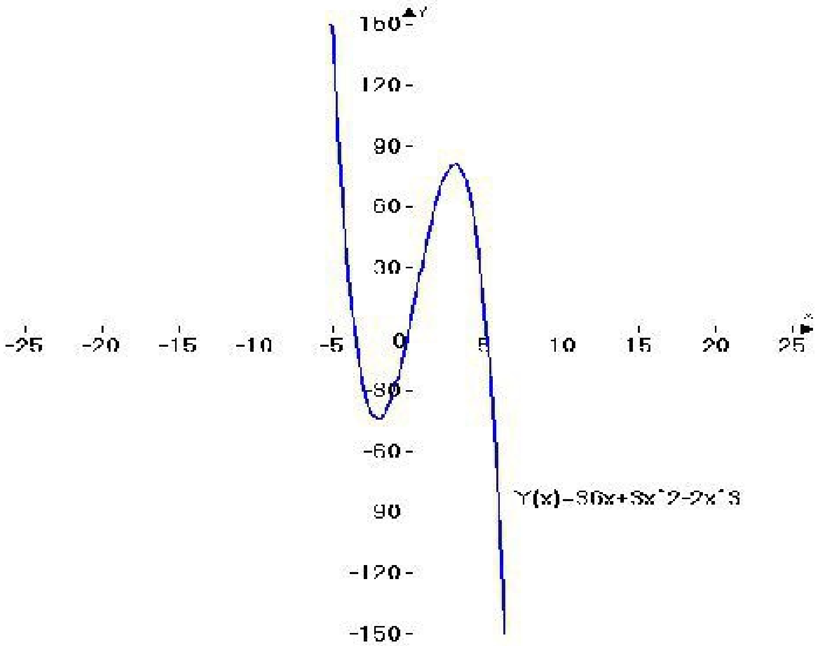
$f''(x) < 0$

$\Rightarrow \quad 6 - 12x < 0$

$\Rightarrow \quad x > \frac{1}{2}$

Concave down on $\left(\frac{1}{2}, \infty\right)$.

(d) The graph is



Chapter 3 Applications of Differentiation Exercise 3.3 31E

f(x) = 2 + 2x2 -x4

(a) to know the intervals in which the function is increasing or decreasing ,

f ' (x) = 4x - 4x3 = 4x (1- x2) = 4x (1-x)(1 +x)

f ' (x) = 0 ==> x = -1, 0 , 1.

so, f has three critical points . so, considering these points as the end points of the intervals we can findthe function where is it increasing and decreasing.

we can see by substitution method that f ' (x) > 0 when ever x is in (- ∞ , -1) or (0,1) .

and f ' (x) < 0 when x is in (-1,0) or (1, ∞).

but we know that if f '(a) > 0 , then f is increasing at a and if f ' (a) < 0 , then f is decreasing at a.

from this , we confirm the given function increases in (-∞ , -1) and (0,1) while it is decreasing in (-1,0) and (1, ∞).

(b) substituting the critical points in the function, we can say the local maximum and local minimum values of f.

f(-1) = 3 , f (1) = 3 , f (0) = 2 .

from these, we say that f has local maximum at -1 , 1 and is equal to 3 while the function has local minimum at 0 and is equal to 2.

(c) now, f '' (x) = 4 - 12 x2

f '' (x) =0 ==> 4(1- 3x2) = 0 ==> -4(3x2 - 1) = 0 ==> x = ± 1/ √3

∴ f has inflection points -1/√3 , 1/ √3
from these , we follow that the concavity of f changes at these points.

f '' (x) is a quadratic expression whose value will be > 0 when x lies away from the roots. and < 0 when x lies between the roots.

i.e. f ''(x) > 0 when x is in (-∞ , -1/√3) or (1/√3 , ∞)

and f '' (x) < 0 when x is in (-1/√3 , 1/√3).

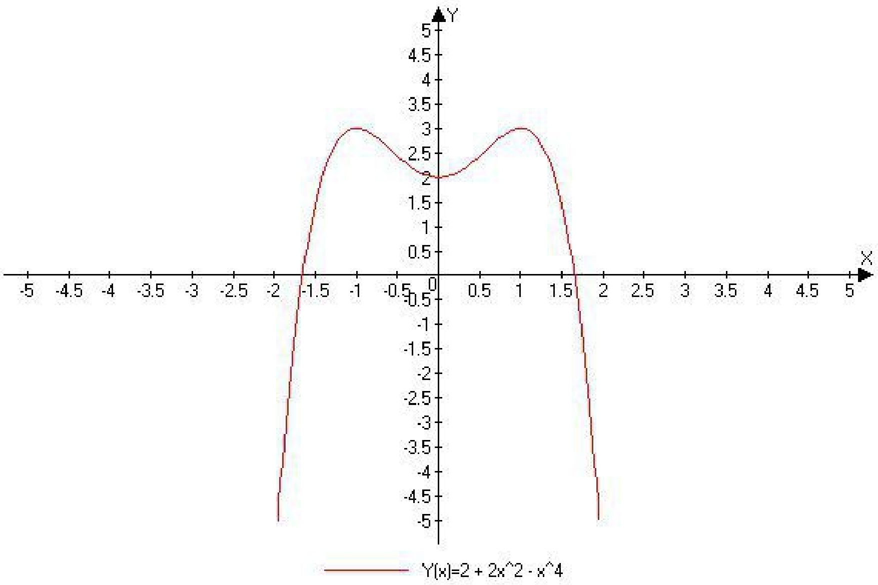
but f '' has - sign. ∴ f '' > 0 when x is in (-1/√3 , 1/√3). and < 0 when x is in (-∞ , -1/√3) or (1/ √3 , ∞)

on the other hand, we know that f '' (x) > 0 ==> f has concavity upwards and otherwise down wards.

∴ the given function has concavity downwards in (-∞ , -1/√3) or (1/√3 , ∞)

and concavity upwards in (-1/√3 , 1/ √3)

further, the points of infliction has the coordinates (-1/√3 , 23/9) and (1/√3 , 23/9) which will be observed from the graph following :



Chapter 3 Applications of Differentiation Exercise 3.3 32E

(A)

We have $g(x) = 200 + 8x^3 + x^4$

Then

$$g'(x) = 24x^2 + 4x^3$$

For getting critical numbers put $g'(x) = 0$.

$$\Rightarrow 24x^2 + 4x^3 = 0$$
$$\Rightarrow 4x^2(6 + x) = 0$$

So critical numbers are x = 0 and -6

Now break the interval $(-\infty,\infty)$ in to the intervals $(-\infty,-6),(-6,0)$, and $(0,\infty)$. So we have

Intervals	$g'(x)$	$g(x)$
$(-\infty,-6)$	Negative	Decreasing on $(-\infty,-6)$
$(-6,0)$	Positive	Increasing on $(-6,0)$
$(0,\infty)$	Positive	Increasing on $(0,\infty)$

(B)

$g'(x)$ is changing for negative to positive at x = -6.

So $g(x)$ has local minimum at x = -6, that is,

$$g(-6)=200+8(-6)^3+(-6)^4=-232$$

Local minimum is $\boxed{g(-6)=-232}$.

$g'(x)$ has no change in sign after x = -6, so $g(x)$ has no local maximum.

(C)

We have $g'(x)=24x^2+4x^3$

So $g''(x)=48x+12x^2$

For getting inflection points put $g''(x)=0$.

So $\Rightarrow 48x+12x^2=0$

Or $12x(4+x)=0$

The above condition is satisfied only when x = 0 or x = -4.

At x = 0, we have $g(0)=200$.

And at x = -4 we have $g(-4)=200+8(-4)^3+(-4)^4=-56$

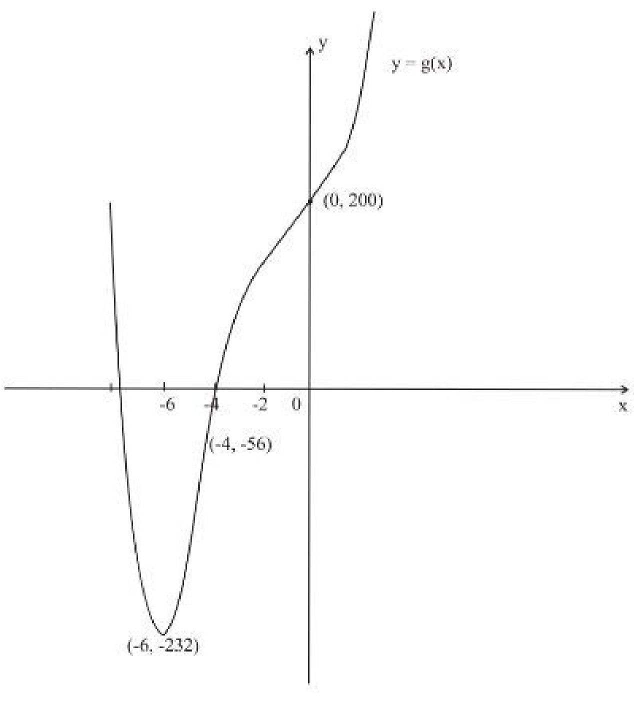
So the inflection points are (0, 200) and (-4, -56)

Now we consider the intervals $(-\infty,-4),(-4,0)$ and $(0,\infty)$.

So we have

Intervals	$g''(x)$	$g(x)$
$(-\infty,-4)$	Positive	Concave upward on $(-\infty,-4)$
$(-4,0)$	Negative	Concave downward on $(-4,0)$
$(0,\infty)$	Positive	Concave upward on $(0,\infty)$

(D)The graph of the function is as follows:



Chapter 3 Applications of Differentiation Exercise 3.3 33E

(a)

Consider the function,

$$h(x)=(x+1)^5-5x-2 \quad \text{......} (1)$$

Differentiate the given function with respect to *x* as follows:

$$\begin{aligned}h'(x)&=\frac{d(x+1)^5}{dx}-5\frac{dx}{dx}-\frac{d2}{dx}\\&=5(x+1)^4-5\times1-0\\&=5\left[(x+1)^4-1\right]\\&=5\left[\left\{(x+1)^2-1\right\}\left\{(x+1)^2+1\right\}\right]\end{aligned}$$

Simplify further,

$$\begin{aligned}h'(x)&=5\left[\left\{(x+1)-1\right\}\left\{(x+1)+1\right\}\left\{(x+1)^2+1\right\}\right]\\&=5(x+1-1)(x+1+1)\left\{(x+1)^2+1\right\}\\&=5x(x+2)\left\{(x+1)^2+1\right\}\end{aligned}$$

Compute the intervals of increasing and decreasing by equating $h'(x)=0$ and solve for *x*,

$$\begin{aligned}h'(x)&=0\\5x(x+2)\left\{(x+1)^2+1\right\}&=0\\x=0,\ (x+2)&=0\\\text{and } \left\{(x+1)^2+1\right\}&=0\end{aligned}$$

Simplify further,

$$x=0$$
$$x+2=0 \Rightarrow x=-2$$
$$(x+1)^2+1=0 \Rightarrow (x+1)^2=-1 \quad (\text{not possible})$$

Points $x=0$ and $x=-2$ divides the domain into three intervals $(-\infty,-2),(-2,0)$ and $(0,\infty)$.

The following chart represents the behaviour of $h'(x)$ in three intervals:

Interval	$5x$	$x+2$	$(x+1)^2+1$	$h'(x)$	$h(x)$
$(-\infty,-2)$	−	−	+	+	Increasing
$(-2,0)$	−	+	+	−	Decreasing
$(0,\infty)$	+	+	+	+	Increasing

Therefore, the given function "**will be increasing in the intervals** $(-\infty,-2)$ **and** $(0,\infty)$**, and will be decreasing in the interval** $(-2,0)$."

(b)

Use the First Derivative Test,

"If f' changes from positive to negative at a point *c*, then *f* has a local maximum at *c* and if f' changes from negative to positive at a point *c*, then *f* has a local minimum at *c*."

From the above chart, $h'(x)$ changes from positive to negative at point $x=-2$, so the given function has a local maximum at $x=-2$ and the maximum value will be,

$$\begin{aligned} h(-2) &= (-2+1)^5 - 5(-2) - 2 \\ &= (-1)^5 + 10 - 2 \\ &= -1 + 10 - 2 \\ &= 7 \end{aligned}$$

$h'(x)$ changes from negative to positive at point $x=0$. So the given function has a local minimum at $x=0$ and the minimum value will be,

$$\begin{aligned} h(0) &= (0+1)^5 - 5(0) - 2 \\ &= (1)^5 - 0 - 2 \\ &= 1 - 0 - 2 \\ &= -1 \end{aligned}$$

Therefore, local maximum value of given function is $\boxed{h(-2)=7}$ and local minimum value is $\boxed{h(0)=-1}$

(c)

Use the Concavity Test,

"If $f''(x)>0$ for all *x* in *I*, then the graph of *f* is concave upward on *I* and if $f''(x)<0$ for all *x* in *I*, then the graph of *f* is concave downward on *I*."

Differentiate $h'(x)=5\left[(x+1)^4-1\right]$ with respect to *x* as follows:

$$\begin{aligned} h''(x) &= 5\left[\frac{d(x+1)^4}{dx} - \frac{d1}{dx}\right] \\ &= 5\left[4\times(x+1)^3 - 0\right] \\ &= 5\left[4(x+1)^3\right] \\ &= 20(x+1)^3 \end{aligned}$$

Compute the intervals of concavity by equating $h''(x)=0$ and solve for *x*,

$$h''(x)=0$$
$$20(x+1)^3=0$$
$$(x+1)^3=0$$
$$x+1=0 \Rightarrow x=-1$$

Points $x=-1$ divides the domain into two intervals $(-\infty,-1)$ and $(-1,\infty)$.

The following chart represents the behaviour of $h''(x)$ in two intervals:

Interval	$h''(x)$	$h(x)$
$(-\infty,-1)$	$-$	Concave Downwards
$(-1,\infty)$	$+$	Concave Upwards

Therefore, the given function "**will concave downwards in the interval $(-\infty,-1)$ and will concave upwards in the interval $(-1,\infty)$**."

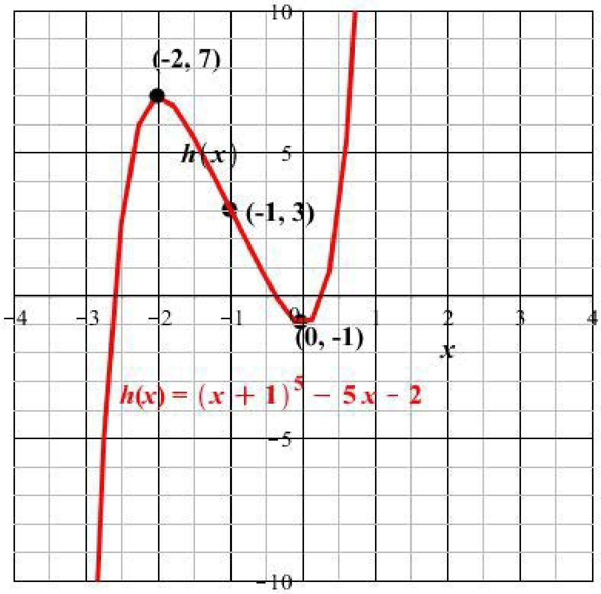
As the curve of given function changes from concave downward to concave upward at point $x=-1$, it will be the *x*-ordinate of inflection point.

$$\begin{aligned} h(-1) &= (-1+1)^5-5(-1)-2 \\ &= (0)^5+5-2 \\ &= 0+5-2 \\ &= 3 \end{aligned}$$

Therefore, the inflection point will be $\boxed{(-1,3)}$.

(d)

From the above information, the graph of the given function $h(x)=(x+1)^5-5x-2$ is found to be as follows:



Consider the function,

$h(x) = 5x^3 - 3x^5$

The formulas for differentiation,

$\frac{d}{dx}(x^n) = nx^{n-1}$

Increasing and Decreasing Test:

If $h'(x) > 0$ on an interval, then h is increasing on that interval, and if $h'(x) < 0$ on an interval, then h is decreasing on that interval.

Concavity Test:

If $h''(x) > 0$ for all on x in I, then the graph of h is concave upward on I, and if $h''(x) < 0$ for all on x in I, then the graph of h is concave downward on I.

The Second Derivative Test:

Suppose h'' is continuous near c.

If $h'(c) = 0$ and $h''(c) > 0$, then h has a local minimum at c.

And, if $h'(c) = 0$ and $h''(c) < 0$, then h has a local maximum at c.

(a) To find the intervals of increase or decrease, find the first derivative of h,

$h(x) = 5x^3 - 3x^5$

$h'(x) = 15x^2 - 15x^4$

Now,

$$h'(x) > 0$$
$$15x^2 - 15x^4 > 0$$
$$x^2(1 - x^2) > 0$$

It is clear that $x^2 > 0$, so $x^2(1 - x^2) > 0$ is possible only when $1 - x^2 > 0$ or $x^2 - 1 < 0$

So, $x \in (-1,1)$

And, every polynomial function is continuous and differentiable on the set of real numbers R, so we cannot say increasing or decreasing of a function at a single point.

Therefore by the Increasing and Decreasing Test, $h(x)$ is increasing on $\boxed{(-1,1)}$.

And if $h'(x) < 0$ then, from the above simplification, $h(x)$ is decreasing on $\boxed{(-\infty,-1),(1,\infty)}$

(b) To find the local maximum and minimum values of h, solve the equation $h'(x) = 0$, and find the second order derivative of h

$$h'(x) = 0$$
$$15x^2 - 15x^4 = 0$$
$$x^2(1 - x^2) = 0$$
$$x = 0, -1, 1$$

And the second order derivative of h,

$h''(x) = 30x - 60x^3 \quad (\text{Since } h'(x) = 15x^2 - 15x^4)$

Then,

$h''(0)=0$

$h''(1)=30-60$

$=-30<0$

$h''(-1)=-30+60$

$=30>0$

Therefore, by the Second Derivative Test, h has a local maximum at 1, and h has a local minimum at -1

So, the local maximum and minimum values of h are,

$$\begin{aligned} h(1)&=5(1)^3-3(1)^5\\ &=\boxed{2}\\ h(-1)&=5(-1)^3-3(-1)^5\\ &=\boxed{-2} \end{aligned}$$

(c) To find the intervals of concavity, and the inflection points of h use the Concavity Test.

From the part (b),

$h''(x)=30x-60x^3$

$=30(x-2x^3)$

So, $h''(x)>0$ when

$x-2x^3>0$

$x(1-2x^2)>0$

So, the roots $-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}$ will divide the real line into four intervals as,

$\left(-\infty,-\frac{1}{\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},0\right),\left(0,\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},\infty\right)$

Observe that, for large negative values of x, $x(1-x^2)>0$

So, $h''(x)>0$ in $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$

And alternately, $h''(x)<0$ in $\left(-\frac{1}{\sqrt{2}},0\right)$, $h''(x)>0$ in $\left(0,\frac{1}{\sqrt{2}}\right)$, and $h''(x)<0$ in $\left(\frac{1}{\sqrt{2}},\infty\right)$

Therefore $h(x)$ is concave up in $\boxed{\left(-\infty,-\frac{1}{\sqrt{2}}\right),\left(0,\frac{1}{\sqrt{2}}\right)}$

And, $h(x)$ is concave down in $\boxed{\left(-\frac{1}{\sqrt{2}},0\right),\left(\frac{1}{\sqrt{2}},\infty\right)}$

And observe that, $h''(x)$ changes its sign at $-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}$

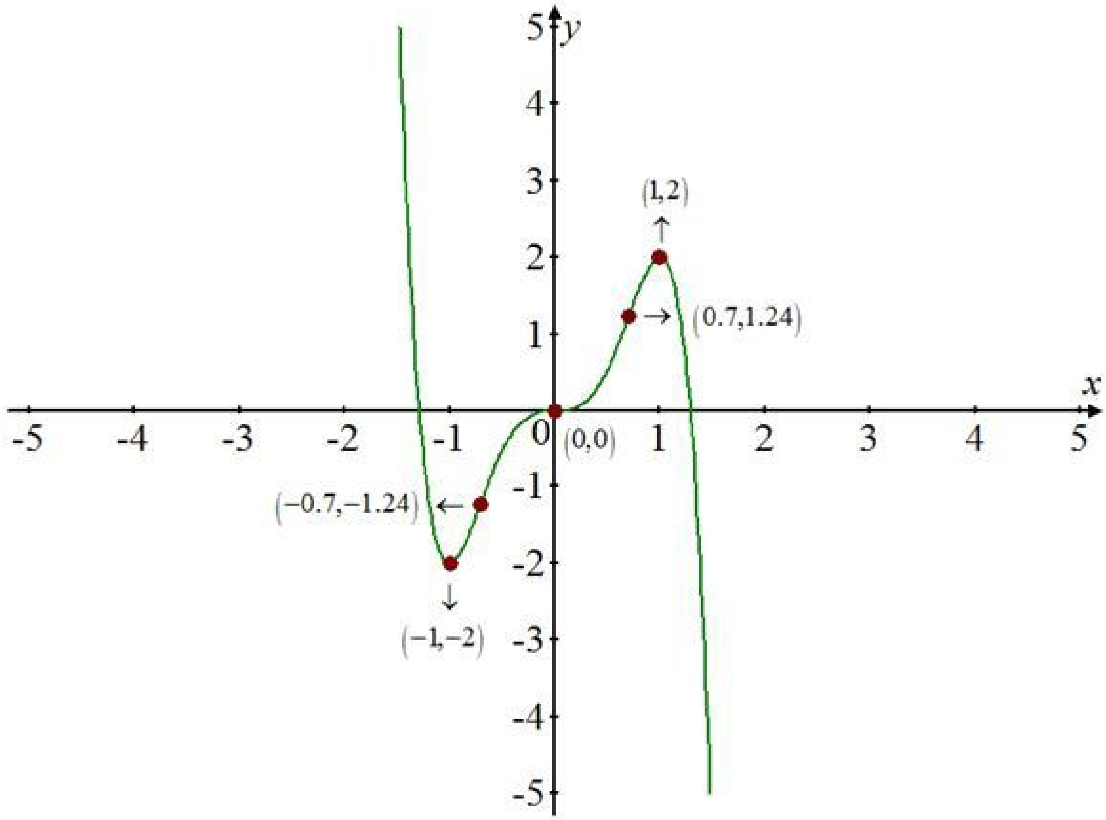
So, the values of h at $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$,

$$h\left(-\frac{1}{\sqrt{2}}\right) = 5\left(-\frac{1}{\sqrt{2}}\right)^3 - 3\left(-\frac{1}{\sqrt{2}}\right)^5$$
$$= -\frac{7}{4\sqrt{2}}$$
$$\approx -1.24$$
$$h(0) = 0$$
$$h\left(\frac{1}{\sqrt{2}}\right) = 5\left(\frac{1}{\sqrt{2}}\right)^3 - 3\left(\frac{1}{\sqrt{2}}\right)^5$$
$$= \frac{7}{4\sqrt{2}}$$
$$\approx 1.24$$

Therefore the inflection points of h are,

$$\boxed{\left(-\frac{1}{\sqrt{2}}, -1.24\right), (0,0), \left(\frac{1}{\sqrt{2}}, 1.24\right)}$$

(d) Sketch of the graph of h is shown below:



Chapter 3 Applications of Differentiation Exercise 3.3 35E

(a)

Consider the following function:

$$f(x)=x\sqrt{6-x} \quad \cdots\cdots(1)$$

Differentiate (1) on both sides.

$$f'(x)=\sqrt{6-x}+x\cdot\frac{1}{2\sqrt{6-x}}(-1)$$
$$=\sqrt{6-x}-\frac{x}{2\sqrt{6-x}}$$

Find the intervals of increase or decrease.

$$f'(x)>0$$

$$\sqrt{6-x}-\frac{x}{2\sqrt{6-x}}>0$$
$$\frac{2(6-x)-x}{2\sqrt{6-x}}>0$$
$$12-2x-x>0$$
$$12-3x>0$$

The function $f(x)$ is increasing on $(-\infty,4)$.

$$f'(x)<0$$

$$12-3x<0 \text{ and } 6-x>0$$

For instance, $f'(x)<0$ for $4<x<6$, so f is decreasing on $(4,6)$.

(b)

The First Derivative Test is a consequence of the I/D Test.

If the function $f'(x)$ changes from positive to negative at $x=4$.

The First Derivative Test tells us that there is a local maximum at $x=4$ and the local maximum value is $x=4$.

$$f(4)=4\sqrt{6-4}$$
$$=4\sqrt{2}$$

Therefore, local maximum is $f(4)=4\sqrt{2}$.

(c)

If $f''(x)<0$ for all x in I, then the graph of f is concave downward on I.

The function f lies below all of its tangents on I, it is called concave downward on I.

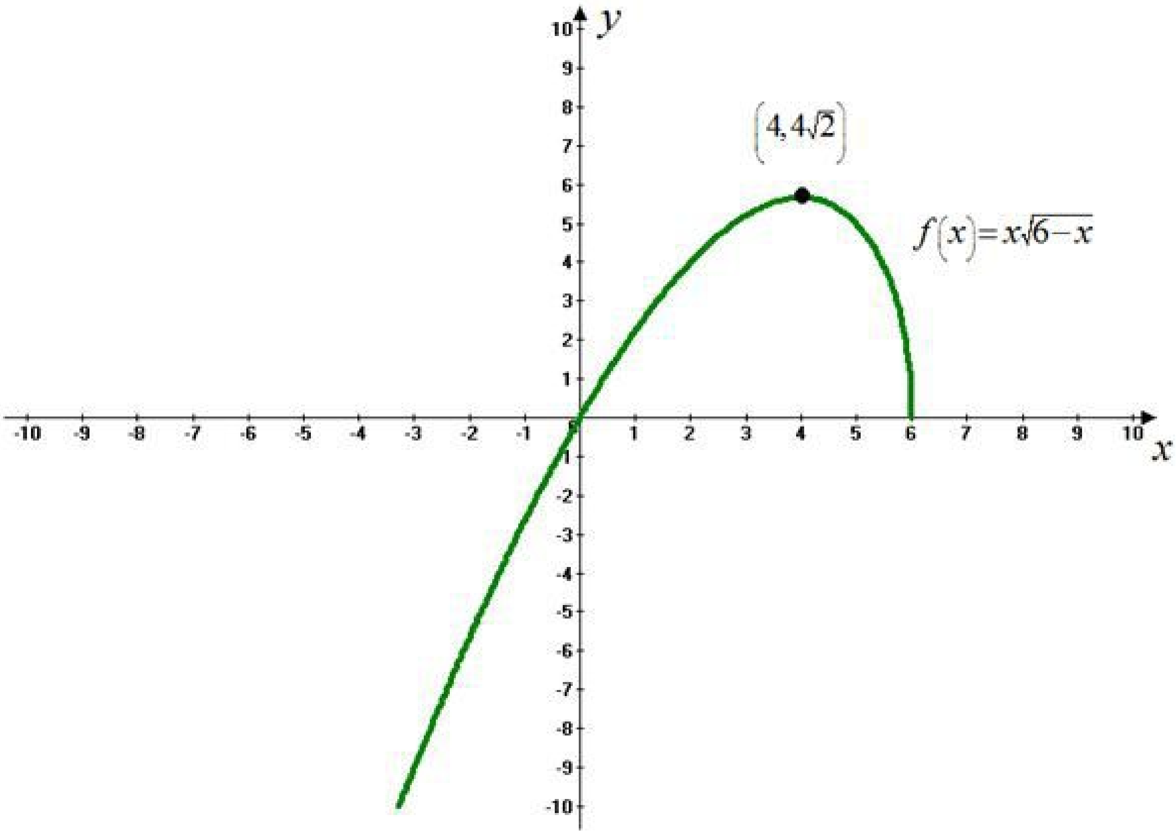
The function $f''(x)<0$ is on the interval $(-\infty,6)$.

Therefore, Concave down on $(-\infty,6)$.

No inflection point because curve not changes from concave upward to concave downward or from concave downward to concave upward at the point.

(d)

Sketch the graph of $f(x) = x\sqrt{6-x}$:



Curve sketching is the representation of the function on the plane and shows the characteristics of the function, such as the maximum and minimum values (absolute and relative), interval of increase or decrease, range and domain of the function.

Consider the function:

$$f(x) = 5x^{\frac{2}{3}} - 2x^{\frac{5}{3}}$$

Determine the first derivative of the function:

$$f'(x) = \frac{10}{3}x^{\frac{-1}{3}} - \frac{10}{3}x^{\frac{2}{3}}$$

Determine the second derivative of the function:

$$f''(x) = \frac{-20}{9}x^{\frac{-4}{3}} - \frac{20}{9}x^{\frac{-1}{3}}$$

a.

Substitute the value of the derivative as 0:

$$f'(x)=0$$
$$\frac{1-x}{\sqrt[3]{x}}=0$$
$$x=1$$

So, the critical points are $x=0,1$.

Determine the sign of the function at $x<0$:

$$x<0$$
$$f'(x)<0$$

Determine the sign of the function at $0<x<1$:

$$x<0$$
$$f'(x)>0$$

Determine the sign of the function at $x>1$:

$$x<0$$
$$f'(x)<0$$

So, the function $f(x)$ is **decreasing** on $\boxed{(-\infty,0) \text{ and } (1,\infty)}$ and the function is increasing on $\boxed{(1,\infty)}$.

b.

Determine the sign of the derivative for $x<0$:

$$x<0$$
$$f'(x)<0$$

Determine the sign of the derivative for $0<x<1$:

$$0<x<1$$
$$f'(x)>0$$

Determine the sign of the derivative for $x>1$:

$$x>1$$
$$f'(x)>0$$

Determine the value of the function at $x=0$:

$$f(0)=5\times(0)^{\frac{2}{3}}-2\times(0)^{\frac{5}{3}}$$
$$=0$$

Since, the sign changes from negative to positive at 0, so $f(0)=0$ is the local minima.

Since, the sign does not change for $x=1$, so there are no local minima or maxima at $x=1$.

Hence, the **local minimum value is** $\boxed{0}$.

c.

The critical points are $x=0,1$.

Determine the sign of the second derivative at $x<0$:

$$x<0$$
$$f''(x)<0$$

Determine the sign of the second derivative at $0<x<1$:

$$0<x<1$$
$$f''(x)<0$$

Determine the sign of the second derivative at $x>1$:

$$x>1$$
$$f''(x)<0$$

Hence, the function is **concave is downward over the interval** $\boxed{x\in\mathbb{R}}$.

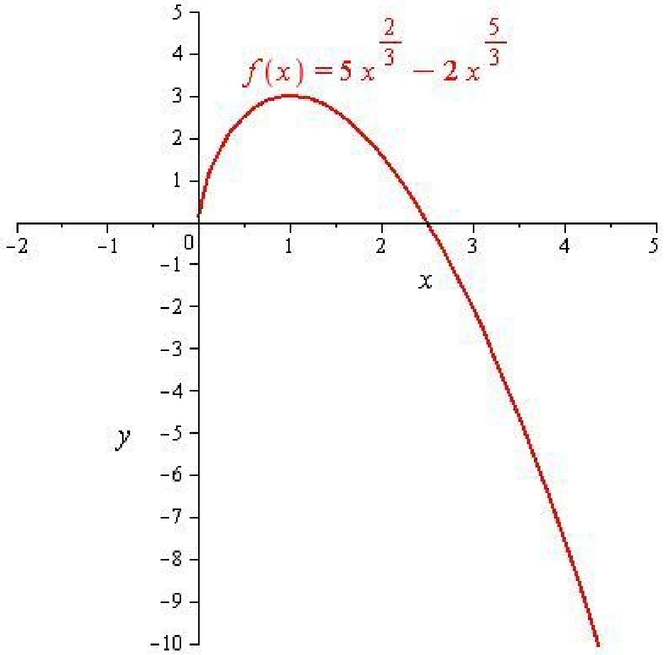
Determine the interval of inflection point of the function.

The function does not change from concave up to concave down or from concave down to concave up anywhere in the graph.

Hence, the function **has no inflection point.**

d.

Consider the graph of the function as shown below:



Chapter 3 Applications of Differentiation Exercise 3.3 36E

(A)

We have $C(x) = x^{1/3}(x+4)$

By product rule, we have

$$C'(x) = \frac{1}{3x^{2/3}}(x+4) + x^{1/3}$$

$$= \frac{x+4+3x}{\left(3x^{2/3}\right)}$$

$$\Rightarrow C'(x) = \frac{4}{3}\left(\frac{(x+1)}{x^{2/3}}\right)$$

So $C'(x) = 0$ when $x+1=0$ *or* $x=-1$

$C'(x)$ does not exist when $x=0$.

So the critical numbers are 0 and -1.

So we see:

Interval	$C'(x)$	$C(x)$
$-\infty < x < -1$	Negative	Decreasing on $(-\infty,-1)$
$-1 < x < 0$	Positive	Increasing on $(-1,0)$
$0 < x < \infty$	Positive	Increasing on $(0,\infty)$

(B) The sign of $C'(x)$ is changing from negative to positive, so at $x = -1$ $C(x)$ has a local minimum there. So $C(-1) = -3$ is a local minimum.

Now since there is no change in sign after $x = -1$ so it has $\boxed{\text{no } local \text{ maximum}}$.

(C) $$C'(x)=\frac{4(x+1)}{3x^{2/3}}$$

So by Quotient rule, we have

$$C''(x)=\frac{4}{3}\left[\frac{x^{2/3}.1-(x+1).\frac{2}{3}.x-\frac{1}{3}}{\left(x^{2/3}\right)^2}\right]$$

$$=\frac{4}{3}\left[\frac{\frac{3x-(x+1)2}{3x^{1/3}}}{x^{4/3}}\right]$$

$$=\frac{4}{3}\left[\frac{x-2}{3x^{5/3}}\right]=\frac{4}{9}\left(\frac{(x-2)}{x^{5/3}}\right)$$

Now $C''(x)=0$ when $x-2=0$

Or $x=2$

And $C''(x)=$ does not exist when $x=0$.

So we have

$C(0)=0$ and $C(2)=\sqrt[3]{2}(2+4)=6\sqrt[3]{2}$

Thus we see,

Interval	$C''(x)$	$C(x)$
$-\infty < x < 0$	Positive	Concave upward on $(-\infty, 0)$
$0 < x < 2$	Negative	Concave downward on (0, 2)
$2 < x < \infty$	Positive	Concave upward on $(2, \infty)$

So the inflection points are (0, 0) and $\left(2, 6\sqrt[3]{2}\right)$.

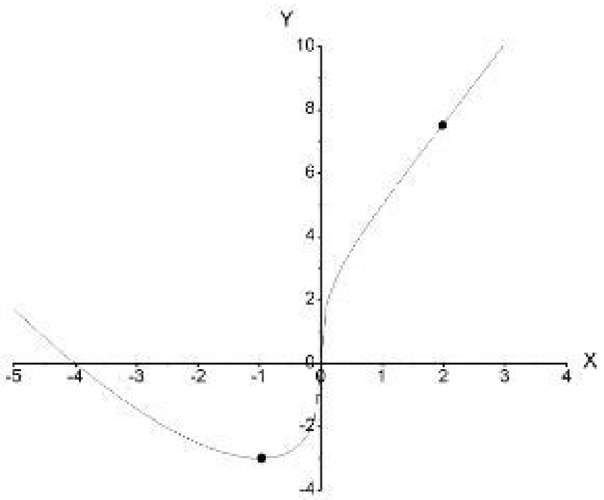


Fig.1

Chapter 3 Applications of Differentiation Exercise 3.3 38E

a) Find the intervals of increase or decrease

G(x) is increasing when G'(x) is positive, and

G(x) is decreasing when G'(x) is negative.

G(x)= x - 4√x

Domain of this function is (0, ∞)

$$G'\left(x\right) = \frac{d}{dx}\left(x - 4x^{\frac{1}{2}}\right) = 1 - 2x^{-\frac{1}{2}} = 1 - \frac{2}{\sqrt{x}}$$

G'(x) = 0 , when x = 4

G'(x) < 0 on the interval 0 < x < 4.

G'(x) > 0 on the interval x > 4.

So,

G(x) is decreasing on the interval (0, 4)

G(x) is decreasing on the interval (4, ∞)

b) Find the local maximum and minimum values.

G(x) has local maximum when G'(x) changes its sign from positive to negative.

G(x) has local minimum when G'(x) changes its sign from negative to positive.

G'(x) < 0 on the interval 0 < x < 4.

G'(x) > 0 on the interval x > 4.

So,

G(x) has local minimum at x = 4.

Since G'(x) doesn't change its sign from positive to negative,

there is no local maximum.

c) Find the intervals of concavity and the inflection points.

G(x) is concave up when G"(x) is positive.

G(x) is concave down when G"(x) is negative.

G(x) has inflection points when G"(x) changes its sign.

G(x)= x - 4√x

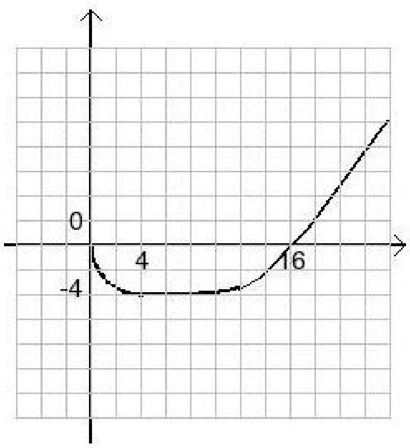
$$G'(x) = \frac{d}{dx}\left(x - 4x^{\frac{1}{2}}\right) = 1 - 2x^{-\frac{1}{2}} = 1 - \frac{2}{\sqrt{x}}$$

$$G''(x) = \frac{d}{dx}\left(1 - \frac{2}{\sqrt{x}}\right) = \frac{d}{dx}\left(1 - 2x^{-\frac{1}{2}}\right) = x^{-\frac{3}{2}} = \frac{1}{\sqrt[3]{x^2}}$$

Since the domain is x>0, G"(x) is always positive.

So, G(x) is concave up everywhere and there is no inflection points.

d) Use the information from parts a-c to sketch the graph.



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Let $f(\theta) = 2\cos\theta + \cos^2\theta$ and $0 \le \theta \le 2\pi$.

(a)

Recall that, **Increasing/Decreasing Test**:

If $f'(x) > 0$ on an interval, then the function f is increasing on that same interval.

If $f'(x) < 0$ on an interval, then the function f is decreasing on that same interval.

Find the first derivative of the function $f(\theta)$ with respect to θ:

$$\begin{aligned} f'(\theta) &= \frac{d}{d\theta}(2\cos\theta) + \frac{d}{d\theta}(\cos^2\theta) \\ &= -2\sin\theta + 2\cos^{2-1}\theta \cdot \frac{d}{d\theta}(\cos\theta) \\ &= -2\sin\theta - 2\cos\theta \cdot \sin\theta \\ &= -2\sin\theta(1 + \cos\theta) \end{aligned}$$

To use the Increasing/Decreasing Test, have to know that where $f'(x)>0$ and $f'(x)<0$.

First find the critical numbers of the function f by equating $f'(\theta)$ to zero.

$$\begin{aligned} f'(\theta) &= 0 \\ -2\sin\theta - 2\cos\theta\cdot\sin\theta &= 0 \\ -2\sin\theta(1+\cos\theta) &= 0 \\ \sin\theta(1+\cos\theta) &= 0 \end{aligned}$$

Solve the trigonometric equation.

$\sin\theta = 0$ or $1+\cos\theta = 0$

$\sin\theta = 0$ or $\cos\theta = -1$

If $\sin\theta = 0$, then $\theta = 0$ or π or 2π. Since $0 \le \theta \le 2\pi$

If $\cos\theta = -1$, then $\theta = \pi$. Since $0 \le \theta \le 2\pi$

Thus gives, $0, \pi,$ and 2π are the critical numbers of the function f.

Now, divide the interval $0 \le \theta \le 2\pi$ into intervals whose end points are the critical numbers of the function f, namely, $0, \pi,$ and 2π.

Use the following intervals to check the signs of the factors of $f'(\theta)$ as well as $f'(\theta)$:

$0 < \theta < \pi, \pi < \theta < 2\pi$.

Construct a chart by arranging our work, a plus sign indicates in that chart represents that the given expression is positive, and a minus sign indicates in the chart represents that the given expression is negative in a particular interval.

Interval	$-2\sin\theta$	$1+\cos\theta$	$f'(\theta)$	f
$0<\theta<\pi$	−	+	−	Decreasing on $(0,\pi)$
$\pi<\theta<2\pi$	+	+	+	Increasing on $(\pi,2\pi)$

Therefore, the function f is decreasing on the interval $\boxed{(0,\pi)}$ and increasing on the interval $\boxed{(\pi,2\pi)}$.

(b)

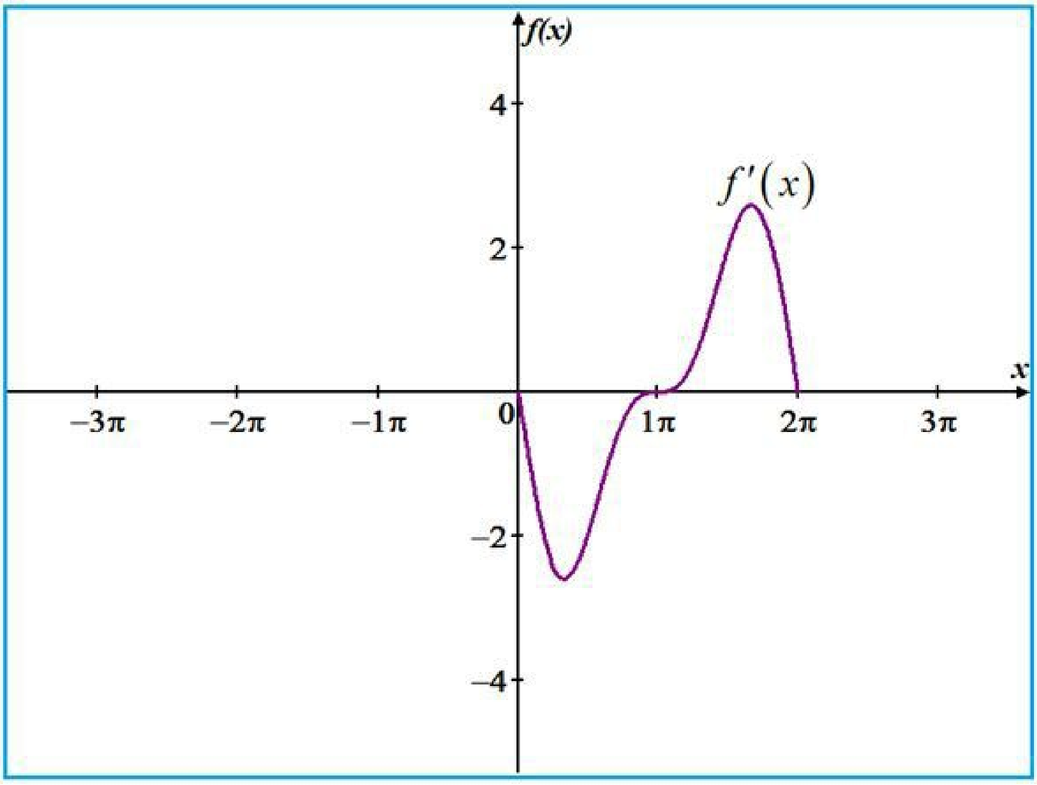
Recall that the First Derivative Test: Let f be a function, and c be the critical number of the function f.

• If the first derivative f' changes from positive to negative at c, then the function f has a local maximum at c.

• If the first derivative f' changes from negative to positive at c, then the function f has a local minimum at c.

• If the first derivative f' does not change sign at the value c, then the function f has no local maximum or minimum at c.

Sketch the graph to identify the sign of $f'(\theta)$ on different intervals.



Construct a table to observe the sign changes of f' in corresponding intervals $(0,\pi)$ and

(c)

Recall that Concavity Test:

• If the second derivative $f''(x)>0$ for all x in an interval I, then the graph of f is concave upward on I.

• If the second derivative $f''(x)<0$ for all x in an interval I, then the graph of f is concave downward on I.

Find the second derivative of the function $f(\theta)$:

The second derivative of the function $f(\theta)$ is the first derivative of the $f'(\theta)$:

$$\begin{aligned} f''(\theta) &= \frac{d}{d\theta}\left(f'(\theta)\right) \\ &= \frac{d}{d\theta}\left(-2\sin\theta\left(1+\cos\theta\right)\right) \end{aligned}$$

Use the formula:

$$\frac{d}{dx}\left(f(x)g(x)\right)=f(x)g'(x)+g(x)f'(x)$$

$$f''(\theta)=(-2\sin\theta)\frac{d}{d\theta}(1+\cos\theta)+(1+\cos\theta)\frac{d}{d\theta}(-2\sin\theta)$$
$$=(-2\sin\theta)(-\sin\theta)+(1+\cos\theta)(-2\cos\theta)$$
$$=2\sin^2\theta-2\cos\theta-2\cos^2\theta$$
$$=-2\left(\cos^2\theta-\sin^2\theta\right)-2\cos\theta$$
$$=-2\cos 2\theta-2\cos\theta$$
$$=-2\left(\cos 2\theta+\cos\theta\right)$$

Suppose that $f''(\theta)=0$, then obtained:

$$-2\left(\cos 2\theta+\cos\theta\right)=0$$
$$\cos 2\theta+\cos\theta=0$$
$$2\cos^2\theta-1+\cos\theta=0$$
$$-1+\cos\theta+2\cos^2\theta=0$$

Write the left hand side expression into product two terms:

$$-1+\cos\theta+2\cos^2\theta=0$$
$$-1+2\cos\theta-\cos\theta+2\cos^2\theta=0$$
$$2\cos\theta-1+\cos\theta\left(2\cos\theta-1\right)=0$$
$$\left(1+\cos\theta\right)\left(2\cos\theta-1\right)=0$$

So, $\cos\theta+1=0$ or $2\cos\theta-1=0$.

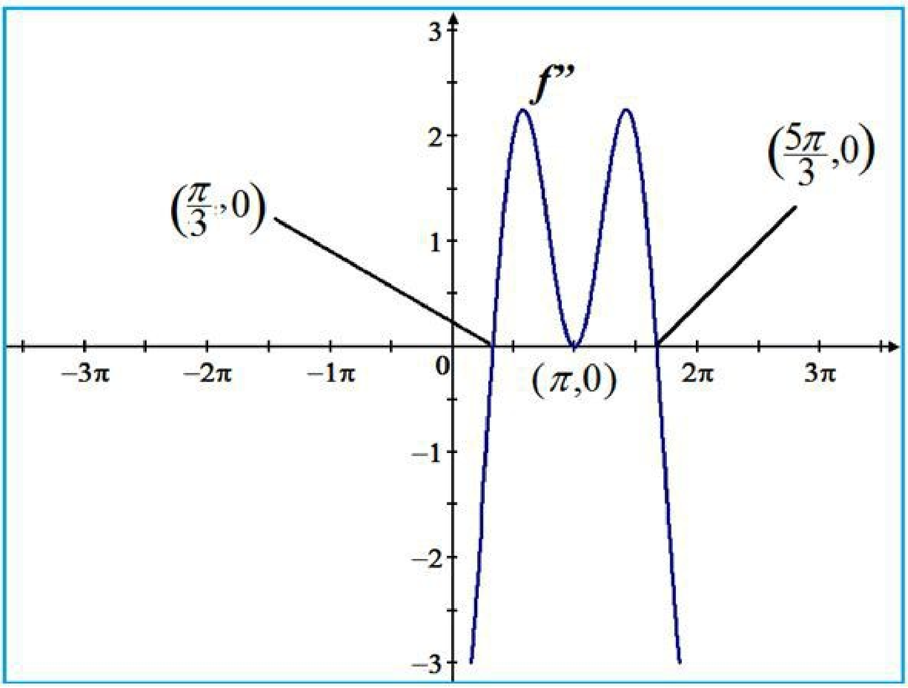
Simplify the equations:

$$\cos\theta=-1 \text{ or } \cos\theta=\frac{1}{2}$$

Therefore, $\theta=\frac{\pi}{3},\frac{5\pi}{3}$ or $\theta=\pi$. Since $0\le\theta\le 2\pi$

Since $f''(\theta)=0$ for $\theta=\frac{\pi}{3},\frac{5\pi}{3},$ and π, divide the real line into intervals with these numbers as endpoints and construct the table to determine whether the graph is concave upwards or concave downwards.

Sketch the graph to identify the sign of $f''(\theta)$ on different intervals:



Construct the table to determine the sign of the f'' on different intervals:

Interval	$f''(\theta) = -2(\cos 2\theta + \cos\theta)$	f
$0 < \theta < \frac{\pi}{3}$	$-$	Downwards
$\frac{\pi}{3} < \theta < \pi$	$+$	Upwards
$\pi < \theta < \frac{5\pi}{3}$	$+$	Upwards
$\frac{5\pi}{3} < \theta < 2\pi$	$-$	Downwards

Hence, the graph of the function f is concave upwards on the interval $\boxed{\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)}$ and concave downwards on the intervals $\boxed{\left(0, \frac{\pi}{3}\right) \text{ and } \left(\frac{5\pi}{3}, 2\pi\right)}$.

(d)

To sketch the graph of the function $f(\theta) = 2\cos\theta + \cos^2\theta$ and $0 \le \theta \le 2\pi$ by using the following details obtained in **(a)**, **(b)**, and **(c)** subparts about the graph of the function:

Since the function f is concave downwards in the intervals $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$ and moving upwards in the intervals $\left(\frac{\pi}{3}, \pi\right)$ and $\left(\pi, \frac{5\pi}{3}\right)$.

Determine the values of $f(\theta)$ at different θ values in the interval $0 \le \theta \le 2\pi$:

If $\theta = \frac{\pi}{3}$, then obtained:

$$\begin{aligned} f\left(\frac{\pi}{3}\right) &= 2\cos\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) \\ &= 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 \\ &= 1 + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

If $\theta = \pi$, then obtained:

$$\begin{aligned} f(\pi) &= 2\cos(\pi) + \cos^2(\pi) \\ &= 2(-1) + (-1)^2 \\ &= -2 + 1 \\ &= -1 \end{aligned}$$

If $\theta = \frac{5\pi}{3}$, then obtained:

$$\begin{aligned} f\left(\frac{5\pi}{3}\right) &= 2\cos\left(\frac{5\pi}{3}\right) + \cos^2\left(\frac{5\pi}{3}\right) \\ &= 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 \\ &= 1 + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

Thus the points $\left(\frac{\pi}{3}, \frac{5}{4}\right), (\pi, -1)$, and $\left(\frac{5\pi}{3}, \frac{5}{4}\right)$ are line on the required sketch.

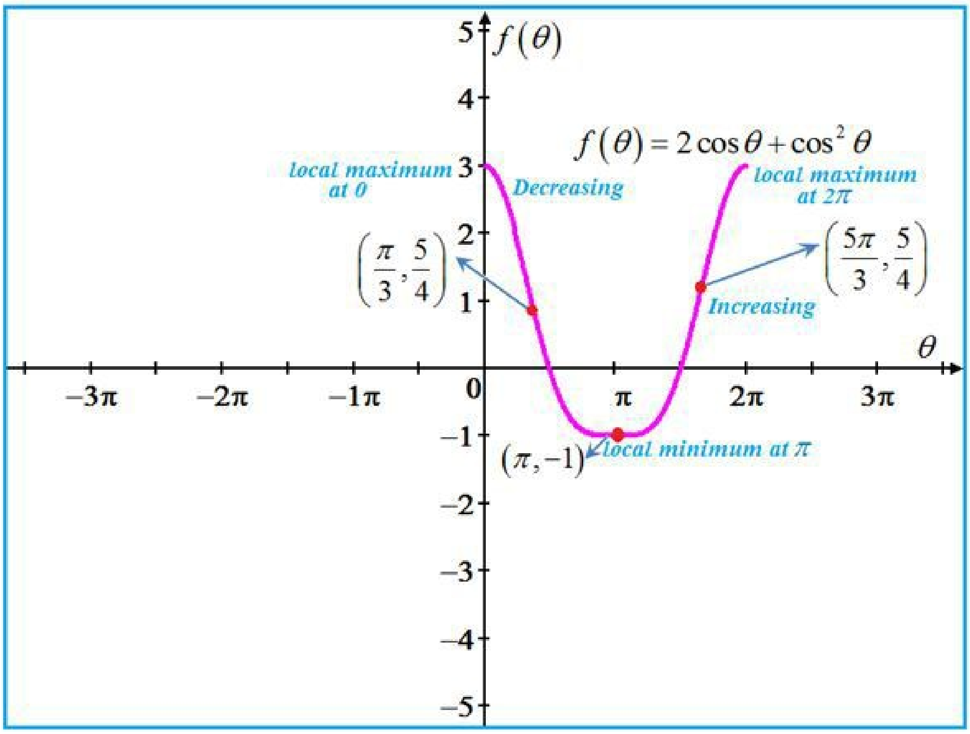
The function f has a critical number at 0 and $f''(0) < 0$, so use the Second derivative test, then the function f has a local maximum at 0.

The function f has a critical number at 2π and $f''(2\pi) < 0$, so use the Second derivative test then the function f has a local maximum at 2π.

The function f has a critical number at π and $f''(\pi)$, so use the Second derivative test then the function f has a local minimum at π.

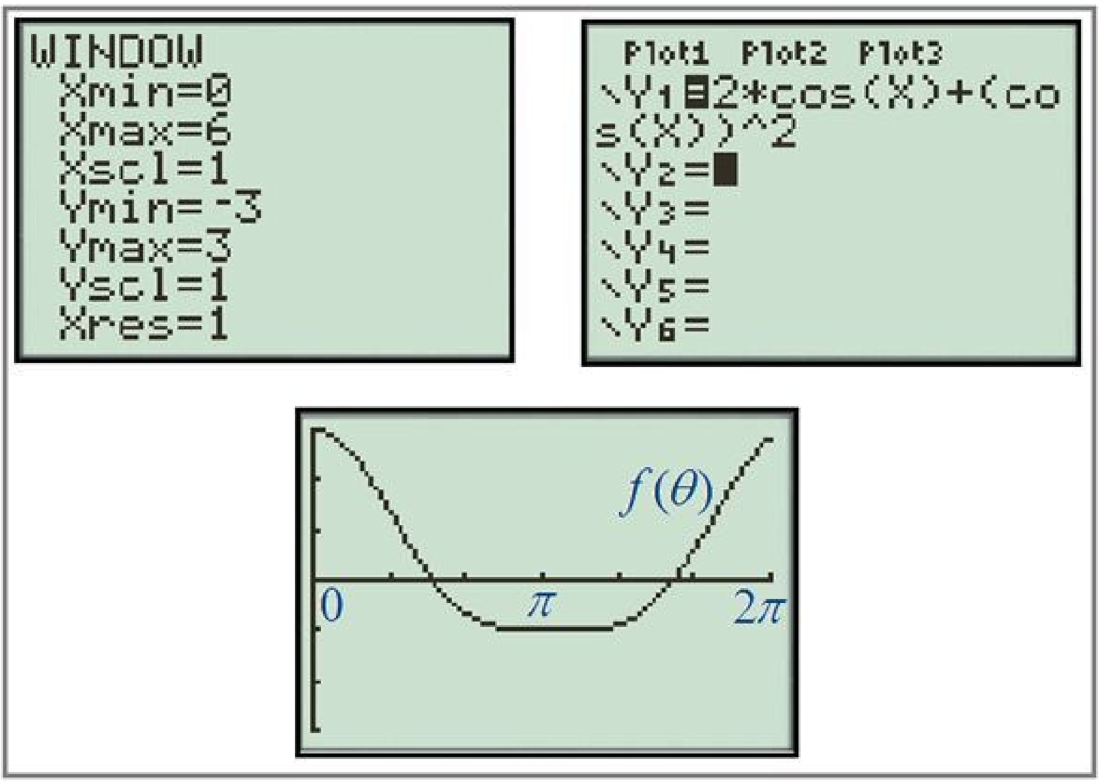
Since the function f is decreasing on the interval $(0, \pi)$ and increasing on the interval $(\pi, 2\pi)$.

To sketch the graph of the function f, considering all facts about the function and join the obtained points on the graph:



To check the graph of the function $f(\theta)$ by using TI calculator:

Insert the function by click on the $\boxed{Y=}$, set the window settings as shown in below after click on $\boxed{\text{WINDOW}}$, and then obtained the graph of the function by click on $\boxed{\text{GRAPH}}$. Observe the screen shots of the calculator displays.



So, its verified that both graphs are same in the subpart **(d)**.

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Consider the function $S(x) = x - \sin x$, $0 \le x \le 4\pi$

(a)

The objective is to find interval of increase and decrease.

First compute $S'(x)$.

$$S(x) = x - \sin x$$
$$S'(x) = 1 - \cos x$$

To find the critical numbers set $S'(x)$ equal to zero:

$$1 - \cos x = 0$$
$$\cos x = 1$$

This equation is satisfied when $x = n\pi$, where n is an integer. We're only concerned with x values in the interval $0 \le x \le 4\pi$; thus, $x = 0, \pi, 2\pi, 3\pi$ and 4π are our critical values.

Next, we'll construct a table, to use the Increasing/Decreasing Test, where we look at the sign of $S'(x)$.

Interval	$S'(x)=1-\cos x$	$S(x)=x-\sin x$
$0<x<\pi$	+	Increasing on $(0,\pi)$
$\pi<x<2\pi$	+	Increasing on $(\pi,2\pi)$
$2\pi<x<3\pi$	+	Increasing on $(2\pi,3\pi)$
$3\pi<x<4\pi$	+	Increasing on $(3\pi,4\pi)$

From the table, we concluded that $S(x)$ is increasing on the intervals $(0,\pi)$, $(\pi,2\pi)$, $(2\pi,3\pi)$ and $(3\pi,4\pi)$. Thus, it is increasing on the full interval of $(0,4\pi)$.

(b)

The objective is to find the local minimum and local maximum values.

Observe the table in part (a), our function is increasing on the whole interval. Therefore, we didn't see $S'(x)$ change from negative to positive or vice versa.

Therefore, $S(x)$ does not have a local maximum or local minimum.

(c)

The objective is to find the interval of concavity and the inflection points.

Compute $S''(x)$:

$$S'(x) = 1 - \cos x$$
$$\begin{aligned} S''(x) &= -(-\sin x) \\ &= \sin x \end{aligned}$$

Find where $S''(x) = 0$, that is, where $\sin x = 0$.

Since $\sin x$ is periodic, we know there will be roots that are spaced out evenly, specifically $x = n\pi$ where n is an integer. Since the problem states we should only focus on $0 \le x \le 4\pi$, we know that $x = 0, \pi, 2\pi, 3\pi$ and 4π.

We can then divide the real line into intervals with $0, \pi, 2\pi, 3\pi$ and 4π as endpoints and construct the chart, below, in a way similar to the increasing/decreasing chart in part (a). The concavity is determined using the Concavity Test; if $S''(x) > 0$ for all x in the given interval, then S is concave upward on that interval; if $S''(x) < 0$ for all x in the given interval, then S is concave downward on that interval.

Interval	$S''(x) = \sin x$	Concavity
$(0, \pi)$	+	upward
$(\pi, 2\pi)$	−	downward
$(2\pi, 3\pi)$	+	upward
$(3\pi, 4\pi)$	−	downward

From the above table, we concluded that $S(x) = x - \sin x$ is concave upward on $(0, \pi)$ and $(2\pi, 3\pi)$, and concave downward on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$.

The inflection point is where the concavity changes; we can examine this in the chart above. Observe, at $x = \pi$ we see concavity change from upward to downward. Evaluating $S(x) = x - \sin x$ at this value will give us the y-coordinate of our inflection point.

$$\begin{aligned} S(\pi) &= (\pi) - \sin(\pi) \\ &= \pi - 0 \\ &= \pi \end{aligned}$$

Also, at $x = 2\pi$ the concavity changes from downward to upward. Evaluating $S(x) = x - \sin x$ at this value will give us the y-coordinate of our other inflection point.

$$\begin{aligned} S(2\pi) &= (2\pi) - \sin(2\pi) \\ &= 2\pi - 0 \\ &= 2\pi \end{aligned}$$

And, observe, at $x = 3\pi$ we see concavity change from upward to downward. Evaluating $S(x) = x - \sin x$ at this value will give us the y-coordinate of our inflection point.

$$\begin{aligned} S(3\pi) &= (3\pi) - \sin(3\pi) \\ &= 3\pi - 0 \\ &= 3\pi \end{aligned}$$

So, (π, π), $(2\pi, 2\pi)$ and $(3\pi, 3\pi)$ are inflection points of our function.

(d)

The objective is to use the information from (a)-(c) sketch the graph:

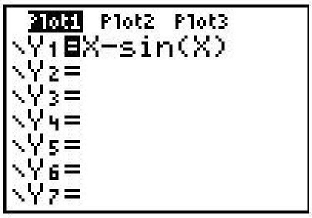
Let's summarize what we found out in parts (a) to (c):

- $S(x)$ is increasing on the full interval of $(0,4\pi)$.
- $S(x)$ does not have a local maximum or local minimum.
- $S(x)=x-\sin x$ is concave upward on $(0,\pi)$ and $(2\pi,3\pi)$, and concave downward on $(\pi,2\pi)$ and $(3\pi,4\pi)$.
- (π,π), $(2\pi,2\pi)$ and $(3\pi,3\pi)$ are inflection points.

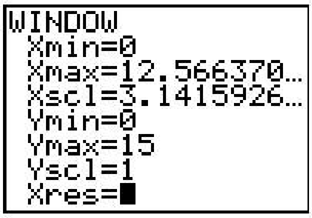
Use TI-83 calculator:

First, press Y= button, to enter the function $y=x-\sin x$ in Y_1.

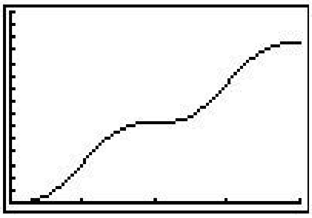
The output will be displayed as shown below:



Press WINDOW button and set the scale as shown below:



Finally, press GRAPH to get the desired graph of the equation as shown below:



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The Increasing/Decreasing Test is stated as follows:

- If $f'(x)>0$ on an interval, then the function f is increasing on the same interval.
- If $f'(x)<0$ on an interval, then the function f is decreasing on the same interval.

Let $f'(x)=(1+x)^2(x-3)^5(x-6)^4$.

To use the Increasing/Decreasing Test, it must be known where $f'(x)>0$ and $f'(x)<0$. This depends on the signs of the three factors $(1+x)^2,(x-3)^5$, and $(x-6)^4$ of the function $f'(x)$.

First find the critical numbers of the function f by equating $f'(x)$ to zero.

$$(1+x)^2(x-3)^5(x-6)^4=0$$

$$x=-1 \text{ or } x=3 \text{ or } x=6$$

Thus $-1,3,$ and 6 are the critical numbers of the function f.

Divide the real line into intervals whose end points are the critical numbers of the function f, namely, $-1, 3, \text{ and } 6$.

Use the following intervals to check the signs of the factors of $f'(x)$ as well as $f'(x)$:

$x < -1, -1 < x < 3, 3 < x < 6, \text{ and } x > 6$.

Construct a chart by arranging the intervals, a plus sign in the chart represents that the given expression is positive. A minus sign indicates in the chart represents that the given expression is negative in a particular interval.

Interval	$(1+x)^2$	$(x-3)^5$	$(x-6)^4$	$f'(x)$	f
$x < -1$	+	−	+	−	Decreasing on $(-\infty, -1)$
$-1 < x < 3$	+	−	+	−	Decreasing on $(-1, 3)$
$3 < x < 6$	+	+	+	+	Increasing on $(3, 6)$
$x > 6$	+	+	+	+	Increasing on $(3, \infty)$

Therefore, the function f is increasing on the intervals $\boxed{(3,6)}$ and $\boxed{(3,\infty)}$.

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Consider:

$y = x^3 - 3a^2x + 2a^3$.

Use the methods that can be used to determine where (what interval(s)) the function is increasing or decreasing, computing the local maximum and minimum value(s), determining where (what interval(s)) the function is concave upward or concave downward, and computing the inflection point(s) to sketch the curve.

Let,

$f(x) = x^3 - 3a^2x + 2a^3$

To find the intervals where $f(x) = x^3 - 3a^2x + 2a^3$ are increasing or decreasing, first compute $f'(x)$ using the Power Rule:

$f'(x) = 3x^2 - 3a^2$

Next, use the Increasing/Decreasing Test to determine where $f(x)$ is increasing and decreasing.

Factor $f'(x) = 3x^2 - 3a^2$ so that it is easier to find where $f'(x) > 0$ and where $f'(x) < 0$.

$$\begin{aligned} f'(x) &= 3x^2 - 3a^2 \\ &= 3(x^2 - a^2) \end{aligned}$$

To find the critical numbers set $f'(x)$ equal to zero:

$3(x^2 - a^2) = 0$

3 is never equal to zero for any value of x.

Thus, $(x^2 - a^2) = 0$.

Solving for this equation will yield critical numbers.

Observe, $x^2 - a^2$ is a difference of squares and thus can be factored as follows:

$x^2 - a^2 = (x - a)(x + a)$

Therefore, we can solve for x in $(x - a)(x + a) = 0$ to obtain our critical numbers. Again, with a product equal to zero, at least one of the factors must be equal to zero.

Thus, solve for x in each factor.

$$\begin{aligned} x - a &= 0 \\ x &= a \\ x + a &= 0 \\ x &= -a \end{aligned}$$

Next, construct a table where we look at the signs of the factors of $f'(x)$ to determine the sign of $f'(x)$.

A plus sign in the chart indicates that the expression is positive; a minus sign indicates that it is negative.

The last column of the chart gives the conclusion based on the Increasing/Decreasing Test.

Interval	$x - a$	$x + a$	$f'(x)$	$f(x)$
$x < -a$	$-$	$-$	$+$	Increasing on $(-\infty, -a)$
$-a < x < a$	$-$	$+$	$-$	Decreasing on $(-a, a)$
$x > a$	$+$	$+$	$+$	Increasing on (a, ∞)

It is concluded that $f(x)$ is increasing on $(-\infty, -a)$ and (a, ∞), and decreasing on $(-a, a)$.

From above and use the First Derivative Test to find the local minimum and the local maximum.

Here, $f'(x)$ changes from positive to negative at $-a$.

Evaluate the function $f(x) = x^3 - 3a^2x + 2a^3$ at $-a$ to determine the local maximum.

The First Derivative Test states we have a local maximum if $f'(x)$ changes from positive to negative.

$$\begin{aligned} f(-a) &= (-a)^3 - 3a^2(-a) + 2a^3 \\ &= -a^3 + 3a^3 + 2a^3 \\ &= 4a^3 \end{aligned}$$

So, $f(-a) = 4a^3$ is a local maximum value by the First Derivative Test.

We see that $f'(x)$ changes from negative to positive at a.

Evaluate the function $f(x)=x^3-3a^2x+2a^3$ at a to determine the local minimum

The First Derivative Test states we have a local minimum if $f'(x)$ changes from negative to positive.

$$\begin{aligned} f(a) &= (a)^3-3a^2(a)+2a^3 \\ &= a^3-3a^3+2a^3 \\ &= 0 \end{aligned}$$

So, $f(a)=0$ is a local minimum value by the First Derivative Test.

It is concluded that $f(-a)=4a^3$ is a local maximum and $f(a)=0$ is a local minimum for the function $f(x)=x^3-3a^2x+2a^3$.

To use the Concavity Test, first calculate $f''(x)$ using the Power Rule

on $f'(x)=3x^2-3a^2$:

$$\begin{aligned} f''(x) &= 3(2)x \\ &= 6x \end{aligned}$$

Find where $f''(x)=0$, i.e. where $6x=0$. Obviously, $x=0$ is the solution here.

Then divide the real line into intervals with 0 as endpoints and construct the chart, below, in a way similar to the increasing/decreasing char above.

The concavity is determined using the Concavity Test; if $f''(x)>0$ for all x in the given interval, then f is concave upward on that interval; if $f''(x)<0$ for all x in the given interval, then f is concave downward on that interval.

Interval	$f''(x)=6x$	Concavity
$(-\infty,0)$	$-$	downward
$(0,\infty)$	$+$	upward

It is concluded that $f(x)=x^3-3a^2x+2a^3$ is concave downward on the interval $(-\infty,0)$, and is concave upward on the interval $(0,\infty)$.

Lastly, identify the inflection point(s) of the function $f(x)=x^3-3a^2x+2a^3$.

The inflection point is where the concavity changes.

Observe, where $x=0$, the concavity change from downward to upward.

Evaluating $f(x)=x^3-3a^2x+2a^3$ at this value, gives the *y*-coordinate of the inflection point.

$$\begin{aligned} f(0) &= (0)^3-3a^2(0)+2a^3 \\ &= 0-0+2a^3 \\ &= 2a^3 \end{aligned}$$

Thus, $(0,2a^3)$ is the inflection point.

Now summarize what it is found about $f(x)=x^3-3a^2x+2a^3$:

(1) Increasing on $(-\infty,-a)$ and (a,∞), and decreasing on $(-a,a)$.

(2) $f(-a)=4a^3$ is a local maximum and $f(a)=0$ is a local minimum for our function.

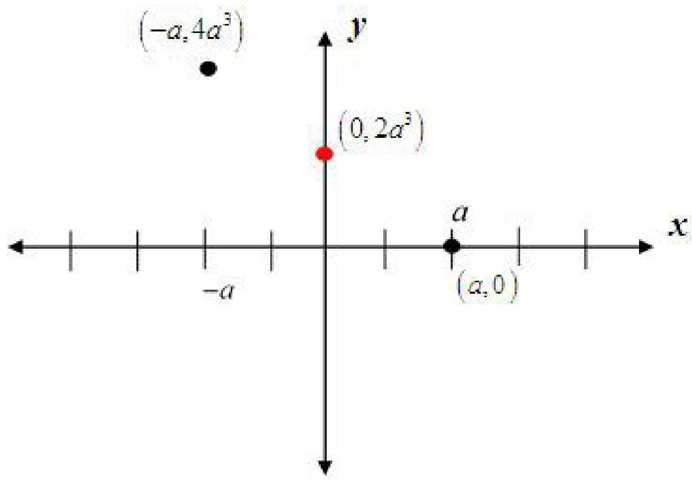
(3) $f(x)=x^3-3a^2x+2a^3$ is concave downward on the interval $(-\infty,0)$, and is concave upward on the interval $(0,\infty)$.

(4) $(0,2a^3)$ is our inflection point.

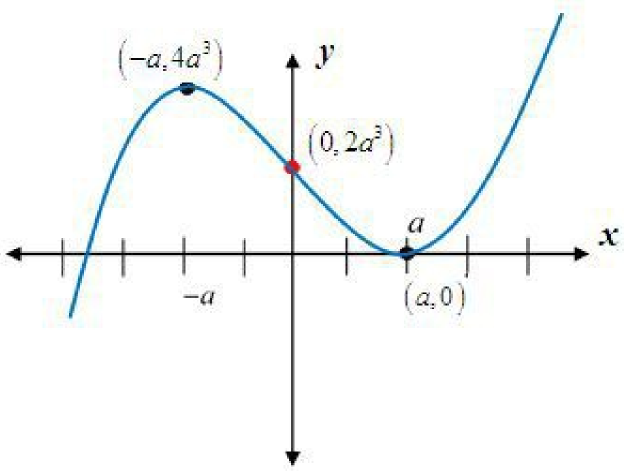
Now begin to put these observations on the Cartesian plane, so that to sketch the curve. First plot the local maximum $(-a,4a^3)$ and minimum $(a,0)$ (shown in black) and the inflection point $(0,2a^3)$ (shown in red below).

These are points that the function will definitely cross through.

Plotting these on the Cartesian plane will look as follows:

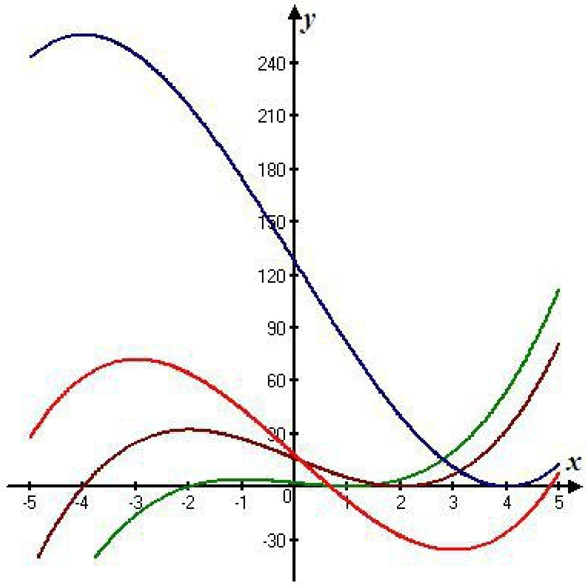


Next construct the curve $f(x)=x^3-3a^2x+2a^3$ with the remaining conditions: increasing on $(-\infty,-a)$ and (a,∞), and decreasing on $(-a,a)$, and is concave downward on the interval $(-\infty,0)$, and is concave upward on the interval $(0,\infty)$.



The following are the graph of $f(x) = x^3 - 3a^2x + 2a^3$ curves for $a = 1, 2, 3, 4$.

For $a = 1$, the curve showing in green color, for $a = 2$, the curve showing in maroon color, for $a = 3$, the curve showing in red color and for $a = 4$, the curve showing in blue color.



From the above graph, the other curves in this family, of the form $f(x) = x^3 - 3a^2x + 2a^3$, have a similar shape with concavity in similar locations and the same pattern with increasing, then decreasing, then increasing again.

The other curves in this family are concave downward and then concave upward.

The curves differ in their maximums, minimums, and inflection points.

So the higher the value of a, the higher the *y*-coordinate of the maximum and the higher the *x*-coordinate of the minimum.

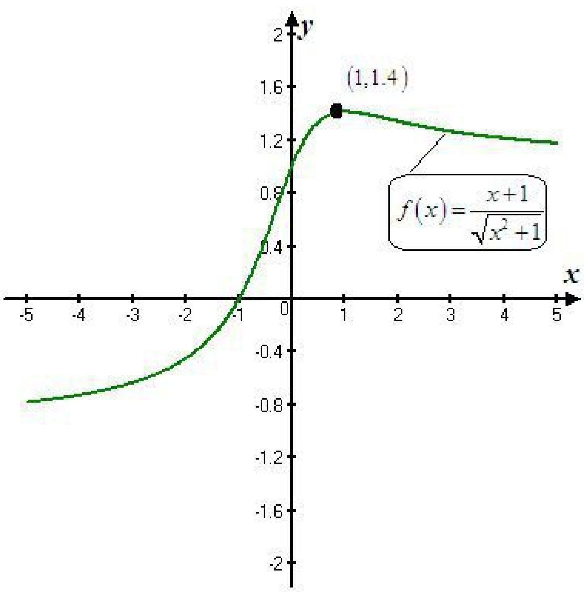
Chapter 3 Applications of Differentiation Exercise 3.3 4E

Consider the function:

$$f(x) = \frac{x+1}{\sqrt{x^2+1}}$$

(a)

The following is the graph of $f(x) = \frac{x+1}{\sqrt{x^2+1}}$:



To estimate the maximum, find the highest point on the graph, if there is one.

In the above graph of the function, it appears the maximum is around $(1,1.4)$.

To estimate the minimum, find the lowest point on the graph, if there is one.

In this case, it does not appear that there is a definite minimum.

Use the Second Derivative Test to find the exact maximum and minimum of $f(x)$.

Begin by finding the first derivative of $f(x)=\frac{x+1}{\sqrt{x^2+1}}$.

Rewrite $f(x)$ by changing the root into an exponent.

$$\begin{aligned} f(x) &= \frac{x+1}{\sqrt{x^2+1}} \\ &= \frac{x+1}{(x^2+1)^{\frac{1}{2}}} \\ &= (x+1)(x^2+1)^{-\frac{1}{2}} \end{aligned}$$

Now use the Product Rule, Power Rule, and Chain Rule to compute the derivative.

$$\begin{aligned} f'(x) &= 1(x^2+1)^{-\frac{1}{2}}+(x+1)\left(-\frac{1}{2}\right)(x^2+1)^{-\frac{3}{2}}(2x) \quad d(uv)=udv+vdu \\ &= (x^2+1)^{-\frac{1}{2}}-x(x+1)(x^2+1)^{-\frac{3}{2}} \\ &= (x^2+1)^{-\frac{1}{2}}-(x^2+x)(x^2+1)^{-\frac{3}{2}} \\ &= \frac{1}{(x^2+1)^{\frac{1}{2}}}-\frac{x^2+x}{(x^2+1)^{\frac{3}{2}}} \end{aligned}$$

Simplify further by finding a common denominator.

$$\begin{aligned} f'(x) &= \frac{1}{(x^2+1)^{\frac{1}{2}}}-\frac{x^2+x}{(x^2+1)^{\frac{3}{2}}} \\ &= \left(\frac{x^2+1}{x^2+1}\right)\left(\frac{1}{(x^2+1)^{\frac{1}{2}}}\right)-\frac{x^2+x}{(x^2+1)^{\frac{3}{2}}} \\ &= \frac{x^2+1}{(x^2+1)^{\frac{3}{2}}}-\frac{x^2+x}{(x^2+1)^{\frac{3}{2}}} \\ &= \frac{x^2+1-(x^2+x)}{(x^2+1)^{\frac{3}{2}}} \end{aligned}$$

$$=\frac{1-x}{(x^2+1)^{\frac{3}{2}}}$$

Now find the critical numbers by setting $f'(x)=0$.

That is, $\frac{1-x}{\left(x^2+1\right)^{\frac{3}{2}}}=0$.

Set the numerator equal to zero to solve for the critical numbers:

$$1-x=0$$
$$1=x$$

Thus, $x=1$ is the critical number.

Now calculate the second derivative, which will be used in the Second Derivative Test. Take the derivative of $f'(x)=\frac{1-x}{\left(x^2+1\right)^{\frac{3}{2}}}$ to find $f''(x)$ using the Quotient Rule:

$$f''(x)=\frac{(-1)\left(x^2+1\right)^{\frac{3}{2}}-(1-x)\left(\frac{3}{2}\right)\left(x^2+1\right)^{\frac{1}{2}}(2x)}{\left(\left(x^2+1\right)^{\frac{3}{2}}\right)^2} \qquad d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$$

$$=\frac{-\left(x^2+1\right)^{\frac{3}{2}}-3x(1-x)\left(x^2+1\right)^{\frac{1}{2}}}{\left(x^2+1\right)^3}$$

$$=\frac{-\left(x^2+1\right)^{\frac{3}{2}}-\left(3x-3x^2\right)\left(x^2+1\right)^{\frac{1}{2}}}{\left(x^2+1\right)^3}$$

$$=\frac{\left(x^2+1\right)^3\left(-\left(x^2+1\right)^{\frac{3}{2}-3}-\left(3x-3x^2\right)\left(x^2+1\right)^{\frac{1}{2}-3}\right)}{\left(x^2+1\right)^3}$$

$$=\frac{\left(x^2+1\right)^3\left(-\left(x^2+1\right)^{-\frac{3}{2}}-\left(3x-3x^2\right)\left(x^2+1\right)^{-\frac{5}{2}}\right)}{\left(x^2+1\right)^3}$$

$$=-\left(x^2+1\right)^{-\frac{3}{2}}-\left(3x-3x^2\right)\left(x^2+1\right)^{-\frac{5}{2}}$$

Next, evaluate $f''(x)$ at the critical number, $x=1$:

$$f''(1) = -\left((1)^2+1\right)^{-\frac{1}{2}} - \left(3(1)-3(1)^2\right)\left((1)^2+1\right)^{-\frac{5}{2}}$$
$$= -(2)^{-\frac{3}{2}} - (3-3)(2)^{-\frac{5}{2}}$$
$$= -\frac{1}{\sqrt{8}} - 0(2)^{-\frac{15}{6}}$$
$$= -\frac{1}{\sqrt{8}}$$

Notice, $f''(1)<0$ since $-\frac{1}{\sqrt{8}}<0$.

By the Second Derivative Test, the local maximum at $x=1$.

Substitute $x=1$ into $f(x)$ to find the value of this local maximum.

$$f(1) = \frac{(1)+1}{\sqrt{(1)^2+1}}$$
$$= \frac{2}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$
$$= \frac{2\sqrt{2}}{2}$$
$$= \sqrt{2}$$

Therefore, at $x=1$, $f(x)=\frac{x+1}{\sqrt{x^2+1}}$ has local maximum of $\boxed{\sqrt{2}}$.

(b)

To estimate the values of x at which $f(x)=\frac{x+1}{\sqrt{x^2+1}}$ increases most rapidly, look for the steepest slope on the graph.

It seems that the $f(x)$ continues most rapidly right before the maximum.

Now estimate that the x value at which this occurs, say $x=0.4$.

Now find the exact value where $f(x)=\frac{x+1}{\sqrt{x^2+1}}$ increases most rapidly.

Recall, $f'(x)$ tells the function's rate of increase with respect to x.

So, to find the x value where $f(x)$ increases most rapidly, first look for zeros of $f''(x)$ and then plug them into $f'(x)$ and see which has the greatest positive value,

since " $f(x)$ increases most rapidly" means e looking for the largest possible positive value of $f'(x)$.

Recall, $f''(x) == -\left(x^2+1\right)^{-\frac{3}{2}} - \left(3x-3x^2\right)\left(x^2+1\right)^{-\frac{5}{2}}$.

The zeros of $f''(x)$ can be found by setting $f''(x)$ equal to zero and solving for x:

$$-\left(x^2+1\right)^{-\frac{3}{2}} - \left(3x-3x^2\right)\left(x^2+1\right)^{-\frac{5}{2}} = 0$$

$$-\left(x^2+1\right)^{-\frac{3}{2}} = \left(3x-3x^2\right)\left(x^2+1\right)^{-\frac{5}{2}}$$

$$\frac{-\left(x^2+1\right)^{-\frac{3}{2}}}{\left(x^2+1\right)^{-\frac{5}{2}}} = \frac{\left(3x-3x^2\right)\left(x^2+1\right)^{-\frac{5}{2}}}{\left(x^2+1\right)^{-\frac{5}{2}}}$$

$$-\left(x^2+1\right) = 3x-3x^2$$

$$-x^2-1-3x+3x^2 = 0$$

$$2x^2-3x-1 = 0$$

The equation is a quadratic equation

Substitute $a=2, b=-3, c=-1$ in $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+8}}{4}$$

$$= \frac{3 \pm \sqrt{17}}{4}$$

The zeros of $f''(x)$ are $x = \frac{3 \pm \sqrt{17}}{4}$.

Next, plug in both of these roots into $f'(x) = \frac{1-x}{\left(x^2+1\right)^{\frac{3}{2}}}$ and evaluate.

$$f'\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right) = \frac{1-\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right)}{\left(\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right)^2+1\right)^{\frac{3}{2}}}$$

$$= \frac{1-\frac{3}{4}+\frac{1}{4}\sqrt{17}}{\left(\left(\frac{9}{16}-\frac{6\sqrt{17}}{16}+\frac{17}{16}\right)+1\right)^{\frac{3}{2}}}$$

All the steps to get $\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right)^2 = \left(\frac{9}{16}-\frac{6\sqrt{17}}{16}+\frac{17}{16}\right)$ are not shown. Show as many steps as you would like to solidify understanding.

$$f'\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right) = \frac{1-\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right)}{\left(\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right)^2+1\right)^{\frac{3}{2}}}$$

$$= \frac{1-\frac{3}{4}+\frac{1}{4}\sqrt{17}}{\left(\left(\frac{9}{16}-\frac{6\sqrt{17}}{16}+\frac{17}{16}\right)+1\right)^{\frac{3}{2}}}$$

$$= \frac{\frac{1}{4}+\frac{1}{4}\sqrt{17}}{\left(\frac{26-6\sqrt{17}}{16}+1\right)^{\frac{3}{2}}}$$

while

$$f'\left(\frac{1}{4}\left(3+\sqrt{17}\right)\right)=\frac{1-\left(\frac{1}{4}\left(3+\sqrt{17}\right)\right)}{\left(\left(\frac{1}{4}\left(3+\sqrt{17}\right)\right)^2+1\right)^{\frac{3}{2}}}$$

$$=\frac{1-\frac{3}{4}+\frac{1}{4}\sqrt{17}}{\left(\left(\frac{9}{16}+\frac{6\sqrt{17}}{16}+\frac{17}{16}\right)+1\right)^{\frac{3}{2}}}$$

$$=\frac{\frac{1}{4}+\frac{1}{4}\sqrt{17}}{\left(\frac{26+6\sqrt{17}}{16}+1\right)^{\frac{3}{2}}}$$

Observe $f'\left(\frac{1}{4}\left(3-\sqrt{17}\right)\right)>f'\left(\frac{1}{4}\left(3+\sqrt{17}\right)\right)$.

Therefore,

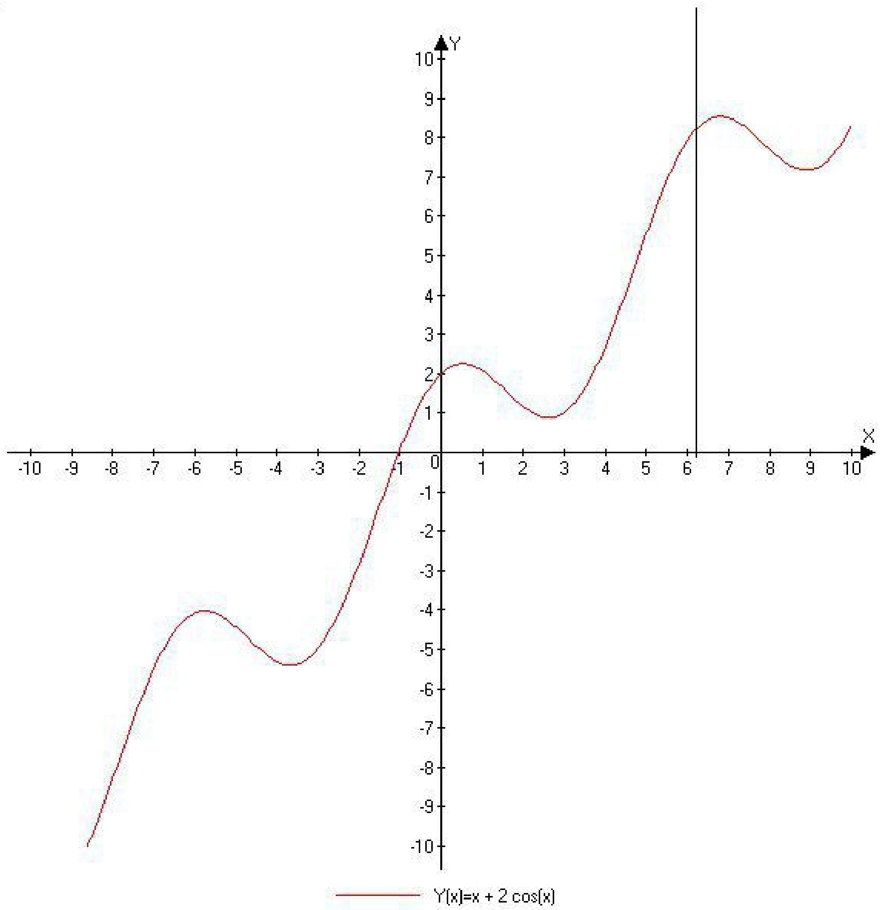
$x=\boxed{\frac{1}{4}\left(3-\sqrt{17}\right)}$ is the x value at which $f(x)=\frac{x+1}{\sqrt{x^2+1}}$ increases most rapidly.

Chapter 3 Applications of Differentiation Exercise 3.3 44E

f(x) = x + 2 cos x while $0\leq x\leq 2\pi$

we replace 2π by the radians 6.28 and graph to decide the local maximmum and minimum.

(a)



(b) the function has local maximum at 2π and f(2π) = 8 while the local minimum at 160o

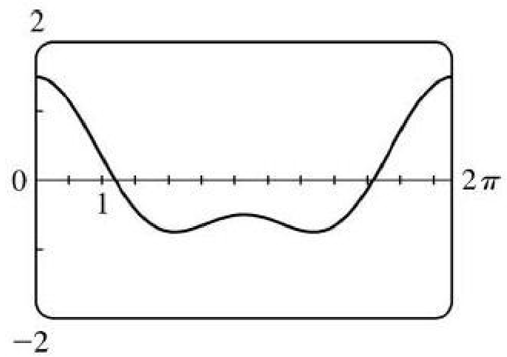
and the minimum value is f(160o) = 1.

$f(x) = \cos x + \frac{1}{2}\cos 2x,\ 0 \le x \le 2\pi$

$f'(x) = -\sin x - \sin 2x$

$f''(x) = -\cos x - 2\cos 2x$

Graph of f looks like:



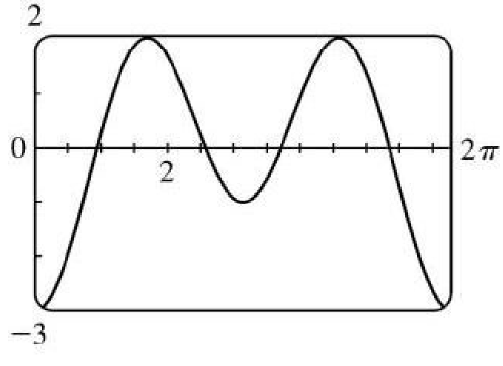
A) So from the graph of f, it seems that f is

concave down on (0, 1), (2.5, 3.7), (5.3, 2π)

concave up on (1, 2.5), (3.7, 5.3)

points of inflection appear to be at (1, 0.4), (2.5, -0.6), (3.7, -0.6), and (5.3, 0.4)

B) Graph f "



By zooming in on the graph of f '',

f is concave down on (0, 0.94) (2.57, 3.71) (5.35, 2π)

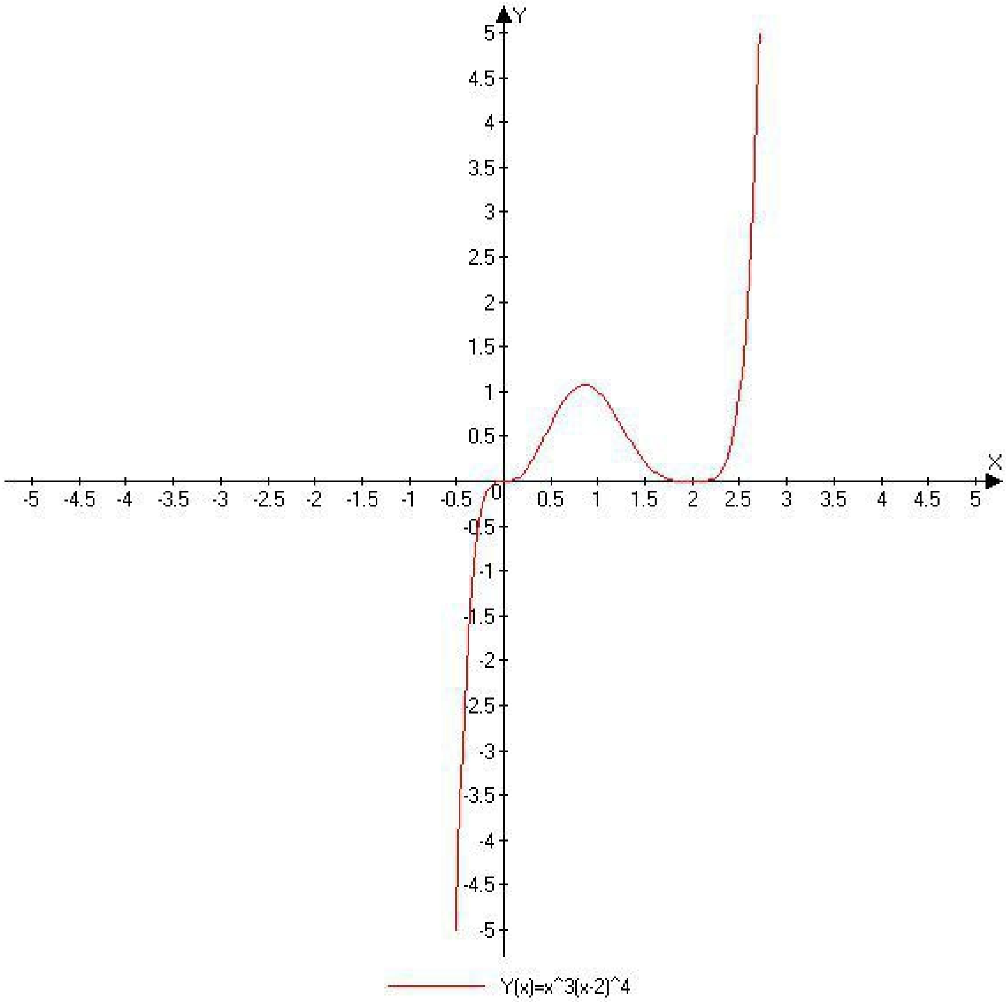
f is concave up on (0.94, 2.57) (3.71, 5.35)

Inflection points (0.94, 0.44) (2.57, -0.63) (3.71, -0.63) and (5.35, 0.44)

Chapter 3 Applications of Differentiation Exercise 3.3 46E

f(x) = x3(x-2) 4
(b) we first graph the function and then confirm the intervals of concavity and points of inflection.

y ' = (x-2)3 (7x3 - 6x2) , y '' = 3(x-2)2 (14x3 - 24x2 + 8x)



observe that this function has 6 critical points and 5 points of infliction.

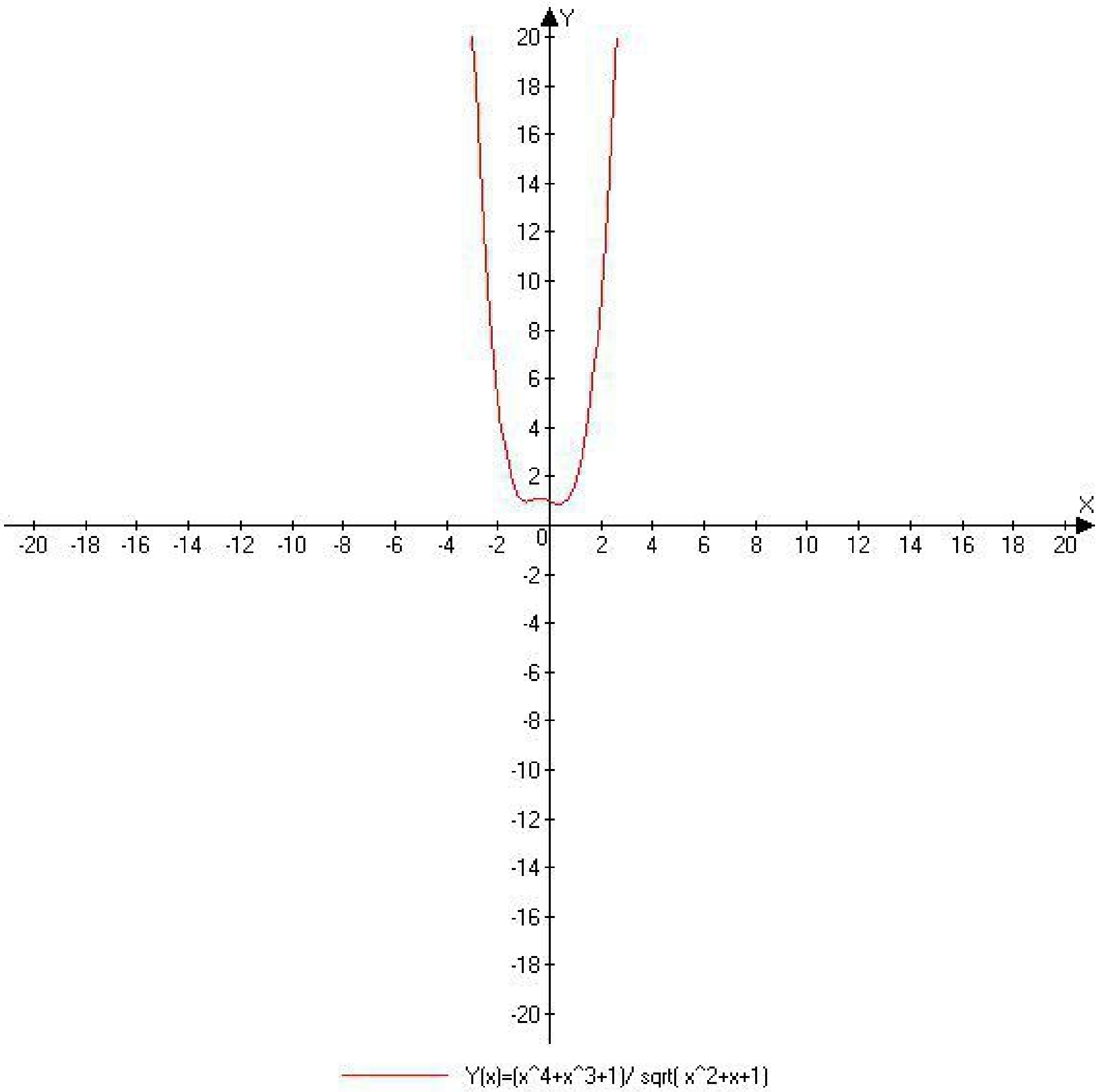
(a) the function has concavity down ward from (-∞ , 0), (0.5,1.2),(1.4,1.5) while concavity upwards in (0,0.5) and ((1.5 , ∞).

the points of infliction are 0, 0.5, 1.2, 1.4, 1.5

Chapter 3 Applications of Differentiation Exercise 3.3 47E

$$f(x) = \frac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$$

we first draw the graph and then find the inflection points so that the intervals of concavity.



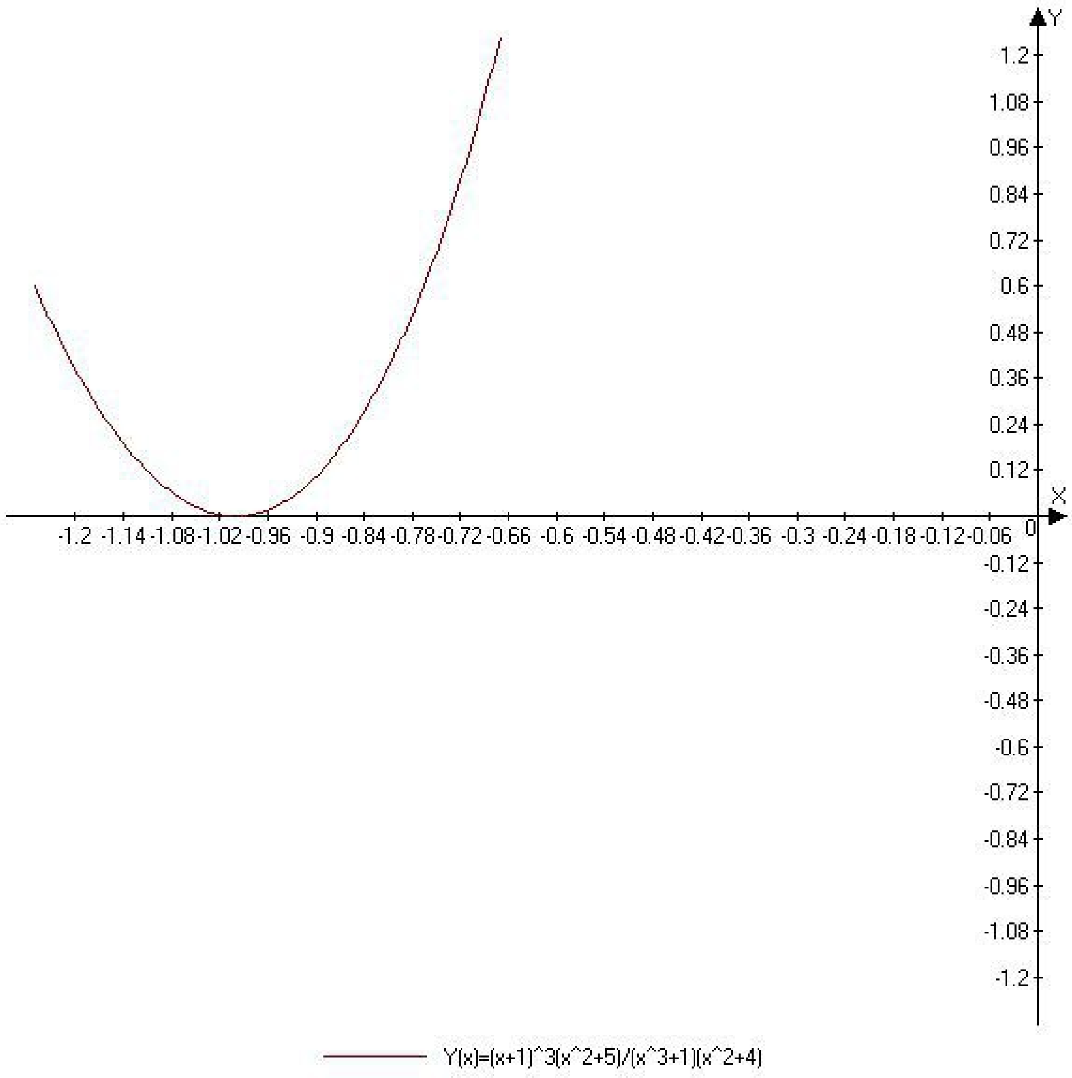
clearly , we can observethat the function has inflection at 2 places namely -0.6 and 0.

the function has concavity upwards on (-∞ , -0.6) and (0 , ∞) while the concavity downwards on (-0.6 , 0).

Chapter 3 Applications of Differentiation Exercise 3.3 48E

$$f\left(x\right) = \frac{(x+1)^3\left(x^2+5\right)}{\left(x^3+1\right)\left(x^2+4\right)}$$

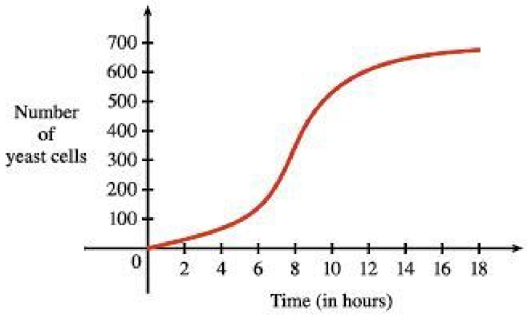
first we graph this function then confirm the points of infliction and thus the intervals of concavity.



observe that the function has only concavity upwards .

so, there are no points of inflection to this function.

Chapter 3 Applications of Differentiation Exercise 3.3 49E



(a)

The rate of increase is initially very small, increases to a maximum at t ≈ 8 h, then decreases toward 0

(b)

t=8 ~Because the graph after 8 is concave downward.~

(c)

concave up: (0,8) the graph of f lies above all of its tangents on an interval (0,8)

concave down (8,18) the graph of f lies below all of its tangents on an interval (8,18)

(d)

(8, 350) is the inflection point, at this point the graph becomes concave down from concave up.

Chapter 3 Applications of Differentiation Exercise 3.3 50E

Consider that $f(t)$ is the temperature at time t where we live and at time $t=3$ we feel uncomfortably hot.

(a)

Since $f'(3)=2$ is positive, it means that the temperature raises and since $f''(3)=4$ is positive it means that the rate of change in rising of temperature is positive.

Therefore, the temperature increases very fast.

Hence we feel $\boxed{\text{Very uncomfortable}}$.

(b)

Since $f'(3)=2$ is positive, it means that the temperature raises and since $f''(3)=-4$ is negative it means that the rate of change in rising of temperature is negative

Therefore, the temperature increases slowly.

Hence we feel $\boxed{\text{uncomfortable}}$.

(c)

Since $f'(3) = -2$ is negative, it means that the temperature is decreasing and since $f''(3) = 4$ is positive it means that the rate of change in decreasing of temperature is positive.

Therefore, the temperature decreases very fast.

Hence we feel $\boxed{\text{Verycomfortable}}$.

(d)

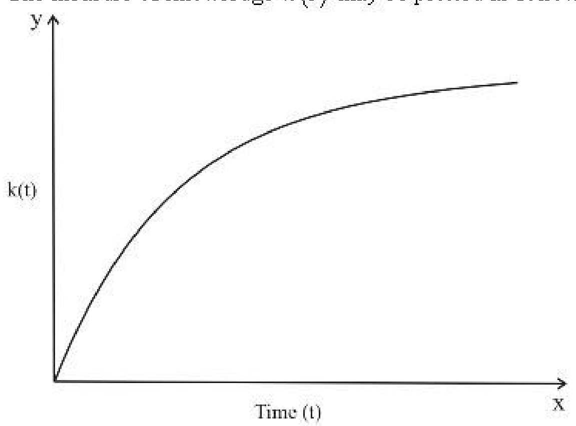
Since $f'(3) = -2$ is negative, it means that the temperature is decreasing and since $f''(3) = -4$ is negative it means that the rate of change in decreasing of temperature is positive.

Therefore, the temperature decreases slowly.

Hence we feel $\boxed{\text{Comfortable}}$.

Chapter 3 Applications of Differentiation Exercise 3.3 51E

The measure of knowledge $k(t)$ may be plotted as follows



In the beginning of the graph the slope of graph will be higher and with respect to time the slope of graph will decrease

So $\quad k(3) - k(2) > k(8) - k(7)$

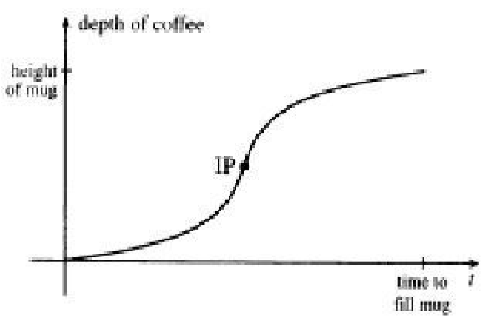
Or $\quad k(3) - k(2)$ will be larger

The graph is concave downward

Chapter 3 Applications of Differentiation Exercise 3.3 52

Since the base of the mug is wide, so depth of the coffee increases slowly. But as the mug becomes narrow, the depth of the coffee increases more quickly so graph of depth of the coffee is concave upward.

Where the mug is narrowest, the graph has an inflection point. After the point, the mug again starts to become wide, so the rate of increase starts to decrease therefore the graph is concave downward.



We have a cubic function $f(x) = ax^3 + bx^2 + cx + d$

According to the question $f(x)$ has local maximum value of 3 at -2

So $a(-2)^3 + b(-2)^2 + c(-2) + d = 3$

$\Rightarrow -8a + 4b - 2c + d = 3$ --- (1)

Now $f(x)$ has local minimum value of 0 at 1

So $a(1)^3 + b(1)^2 + c(1) + d = 0$

$\Rightarrow a + b + c + d = 0$ --- (2)

Now we have

$f(x) = ax^3 + bx^2 + cx + d$

Then $f'(x) = 3ax^2 + 2bx + c$

Now $f(x)$ has local maximum value at -2 and local minimum value at 1

So -2 and 1 will be the critical numbers

And at x = -2 and 1, $f'(x) = 0$

So we have $f'(x) = 3ax^2 + 2bx + c$

So $f'(-2) = 3a(-2)^2 + 2b(-2) + c = 0$

$\Rightarrow 12a - 4b + c = 0$ --- (3)

And $f'(1) = 3a + 2b + c = 0$ --- (4)

Now we have four equations

$-8a + 4b - 2 + d = 3$ --- (1)

$a + b + c + d = 0$ --- (2)

$12a - 4b + c = 0$ --- (3)

$3a + 2b + c = 0$ --- (4)

Now by subtracting equation (2) from equation (1)

$-9a + 3b - 3c = 3$

Or $-3a + b - c = 1$ --- (5)

Now by subtracting equation (4) from equation (3)

We have

$9a - 6b = 0$ --- (6)

And by adding the equations (4) and (5)

We have

$3b = 1$ or $\boxed{b = \frac{1}{3}}$

Now put this value of b in equation (6)

$9a - 6.\frac{1}{3} = 0$

Or $9a = 2$ or $\boxed{a = \frac{2}{9}}$

Now put these value of a and b in equation (4)

$3.\frac{2}{9} + 2.\frac{1}{3} + c = 0$ Or $\boxed{c = -\frac{4}{3}}$

Again put the values of a, b and c in equation (2)

We have

$\frac{2}{9} + \frac{1}{3} - \frac{4}{3} + d = 0$

Or $\frac{2+3-12}{9} + d = 0$

Or $\boxed{d = \frac{7}{9}}$

Now we put the values of all coefficient a, b, c and d in the cubic equation ax^3+bx^2+cx+d

So we have $f(x)=\frac{2}{9}x^3+\frac{1}{3}x^2-\frac{4}{3}+\frac{7}{9}$

Or $\boxed{f(x)=\frac{1}{9}\left[2x^3+3x^2-12+7\right]}$

This is the required cubic polynomial which has the local maximum 3 at -2 and local minimum 0 at 1.

Chapter 3 Applications of Differentiation Exercise 3.3 54E

Consider the function,

$$y=\frac{1+x}{1+x^2}$$

The objective is to show that the curve has three points of inflection and they all lie on one straight line.

The inflection points of the curve occur when the second derivative of the function equals zero.

$$y=\frac{1+x}{1+x^2}$$

$$y'=\frac{1(1+x^2)-(1+x)2x}{(1+x^2)^2}$$

$$=\frac{1+x^2-2x-2x^2}{(1+x^2)^2}$$

$$=\frac{1-2x-x^2}{(1+x^2)^2}$$

$$y''=\frac{(-2-2x)(1+x^2)^2-(1-2x-x^2)2(1+x^2)2x}{(1+x^2)^4}$$

Next, find where $y''=0$, Since quotient equal to zero, need to set the numerator equal to zero to solve that equality

$$2(1+x^2)\left[(-1-x)(1+x^2)-(1-2x-x^2)2x\right]=0$$

$$x^3+3x^2-3x-1=0$$

Solving for x yields $\boxed{x=1, x=-0.268, x=-3.732}$

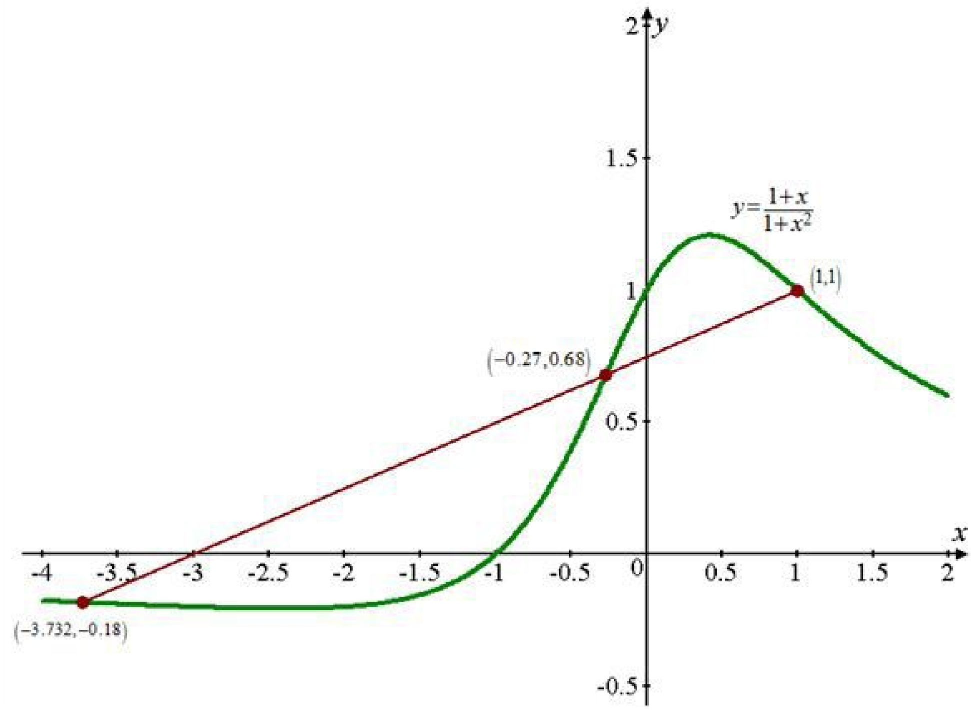
Interval	$y''=\frac{x^3+3x^2-3x-1=0}{(x^2+1)^4}$	Concavity
$(-\infty,-3.732)$	$-$	downward
$(-3.732,-0.27)$	$+$	upward
$(-0.27,1)$	$-$	downward
$(1,\infty)$	$+$	upward

Observe that the concavity change at $x\approx-3.732,-0.27,1$.

Therefore, the curve has exactly 3 points of inflection.

The inflection points are $(-3.732,-0.18)(-0.27,0.68),(1,1)$

Sketch the graph of $y=\dfrac{1+x}{1+x^2}$ is shown below:



Therefore, the three inflection points lie on one straight line.

Chapter 3 Applications of Differentiation Exercise 3.3 55E

Given that the function is $f(x)=x^3+ax^2+bx$

Then $f'(x)=3x^2+2ax+b$

Since $f(x)$ has the local minimum value $\dfrac{-2}{9}\sqrt{3}$

$x=\dfrac{1}{\sqrt{3}}$, so $f'\left(\dfrac{1}{\sqrt{3}}\right)=0$

$$\Rightarrow \quad 3\frac{1}{3}+2a\frac{1}{\sqrt{3}}+b=0$$

$$\Rightarrow \quad \frac{2a}{\sqrt{3}}+b=-1$$

Since $f\left(\dfrac{1}{\sqrt{3}}\right)=\dfrac{1}{3\sqrt{3}}+\dfrac{a}{3}+\dfrac{b}{\sqrt{3}}=\dfrac{-2}{9}\sqrt{3}$

$$\Rightarrow \quad \frac{a}{3}+\frac{b}{\sqrt{3}}=\frac{-2}{9}\sqrt{3}-\frac{1}{3\sqrt{3}}$$

$$=\frac{-2\sqrt{3}}{9}-\frac{\sqrt{3}}{9}$$

$$=\frac{-3\sqrt{3}}{9}$$

$$=\frac{-1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{a}{3}+\frac{b}{\sqrt{3}}=\frac{-1}{\sqrt{3}}$$

$$\Rightarrow \quad a+b\sqrt{3}=-\sqrt{3} \qquad \ldots\ldots (2)$$

From equations (1) and (2), we get

$a=0$ and $b=-1$

$\boxed{a=0 \text{ and } b=-1}$

(b) If $x=0$ then

$$y=f(x)$$
$$=(0)^3-(0)$$
$$=0$$

$(0,0)$ is a point on the curve x^3-x.

The slope is $f'(x)=3x^2-1$

If $x=0$ then $f'(0)=-1$

The tangent line is

$$(y-0)=(-1)(x-0)$$
$$\Rightarrow \quad y=-x.$$

The tangent line $y=-x$ at $(0,0)$ has the smallest slope.

Chapter 3 Applications of Differentiation Exercise 3.3 56E

Given curve is $x^2y+ax+by=0$

$$\Rightarrow \quad y=\frac{-ax}{x^2+b}$$

$$\Rightarrow \quad y'=\frac{(x^2+b)(-a)-(-ax)(2x)}{(x^2+b)^2}$$

Since $(2,2.5)$ is an inflection point, so

$$y'(2)=0$$
$$\Rightarrow \quad (4+b)(-a)+8a=0$$
$$\Rightarrow \quad -4a-ab+8a=0$$
$$\Rightarrow \quad 4a-ab=0 \quad \ldots\ldots (1)$$

If $x=2$ then

$$y(2)=\frac{-2a}{4+b}$$
$$=\frac{5}{2}$$
$$\Rightarrow \quad -4a=20+5b$$
$$\Rightarrow \quad 4a+5b=-20 \quad \ldots\ldots (2)$$

From equations (1) and (2), we have

$a=-10$ and $b=4$

$$y=\frac{10x}{x^2+4}$$

$$\Rightarrow \quad y'=\frac{(x^2+4)(10)-(10x)(2x)}{(x^2+4)^2}$$
$$=0$$
$$\Rightarrow \quad x^2+4-2x^2=0$$
$$\Rightarrow \quad 4-x^2=0$$
$$\Rightarrow \quad x=\pm 2$$

-2 is also inflection point of the given curve.

Chapter 3 Applications of Differentiation Exercise 3.3 57E

Given that the curve is $y = x\sin x$

Then $y' = x\cos x + \sin x$

If $y' = 0$ c then $x\cos x + \sin x = 0$

$$\Rightarrow \quad x = -\frac{\sin x}{\cos x}$$
$$= -\tan x$$
$$\Rightarrow \quad x^2 + 4 = \tan^2 x + 4$$

And $\quad 4x^2 = 4\tan^2 x$

$$\frac{4x^2}{x^2+4} = \frac{4\tan^2 x}{\tan^2 x + 4}$$
$$= \tan^2 x \sin^2 x$$
$$= x^2 \sin^2 x$$
$$= y^2$$
$$\frac{4x^2}{x^2+4} = y^2$$
$$\Rightarrow \quad y^2\left(x^2+4\right) = 4x^2$$

Hence the inflection points of the curve $y = x\sin x$ lie on the curve $y^2\left(x^2+4\right) = 4x^2$

Chapter 3 Applications of Differentiation Exercise 3.3 58E

We have f and g are twice differentiable and concave upward on interval I so

$f'' > 0$ *and* $g'' > 0$ *on* I --- (1)

(A) Now let $h = f + g$

So $h' = (f+g)'$

$h' = f' + g'$

$\Rightarrow$[By the law of the differentiation of sum of the two functions]

Then $h'' = (f' + g')'$

Or $h'' = f'' + g''$

We have $f'' > 0$ *and* $g'' > 0$ *on* I

Then $h'' = f'' + g'' > 0$ *on* I

Or $(f+g)'' > 0$ *on* I

So $f + g$ is concave upward when f and g are concave upward

Proved

(B) Let f is positive and concave upward on I

Means

$f'' > 0$ *on* I *and* $f > 0$

Now let $g(x) = \left[f(x)\right]^2$

Then $g'(x) = 2.f(x).f'(x)$

And $g''(x) = 2\left[f(x).f''(x) + f'(x).f'(x)\right]$

$$= 2\left[f(x).f''(x) + f'(x).\left[f'(x)\right]^2\right]$$
$$= 2f(x).f''(x) + 2\left(f'(x)\right)^2$$

Now $\left(f'(x)\right)^2 > 0$ always for $f'(x) < 0$ or $f'(x) > 0$ and we have $f(x) > 0$ and $f''(x) > 0$

So $f(x).f''(x) > 0$

Thus we have $g''(x) > 0$ when $f''(x) > 0$ and f > 0 on I

So $g(x)$ is also concave upward on I

(A)

We have $f > 0$ and $g > 0$
And $f' > 0$ and $g' > 0$ [both are increasing]

And we have $f'' > 0$ and $g'' > 0$ [both are concave upward]

Now let $h = fg$
Then $h' = f'g + g'f$
$h'' = f''g + g'f' + g'f' + f.g''$
$\boxed{h'' = f''g + 2g'f' + f.g''}$ --- (1)
Now we have $f' > 0$ and $g' > 0$ then $f'.g' > 0$
And $g > 0$ and $f'' > 0$ then $f''.g > 0$
And also $f > 0$ *and* $g'' > 0$ *then* $f.g'' > 0$
Then all the factors of h'' are greater than 0
So $h'' > 0$ Or $(g.f)'' > 0$
So $g.f$ is also concave upward.

(B) Now we have if $f' < 0$ and $g' < 0$
Means both are decreasing but the product of the two negative values will always be positive so
$f'.g' > 0$
We already have
$f'' > 0$ And $g'' > 0$ and $f > 0$ $g > 0$
So $f''.g > 0$ and $g''.f > 0$

Thus $h'' = f''.g + 2g'.f' + g''.f > 0$
Or $h'' > 0$
So $(g.f)'' > 0$
So $(g.f)$ is concave upward when f and g both are decreasing.

(C) Example 1

Let $f(x) = x^3$ and $g(x) = \dfrac{1}{x}$

So $f'(x) = 3x^2 > 0$ and $g'(x) = -\dfrac{1}{x^2} < 0$

So here $f(x)$ is increasing and $g(x)$ is decreasing where $x \neq 0$ and $x \in (0,\infty)$
Now the second derivatives are

$f''(x) = 6x > 0$ and $g''(x) = \dfrac{2}{x^3} < 0$

So $f(x)$ and $g(x)$ both are concave upward on the interval $(0,\infty)$

Now $h(x) = f(x).g(x)$

Or $h(x) = x^3.\dfrac{1}{x} = x^2$

Then $h'(x) = 2x$ and $h''(x) = 2 > 0$

Then $(f(x).g(x))'' = 2 > 0$
So $f.g$ will be concave upward when f is increasing and g is decreasing function.

Example 2
Now let $f(x) = x + x^2$ and $g(x) = x - x^2$
Then $f'(x) = 1 + 2x$ and $g'(x) = 1 - 2x$
Where $1 < x < \infty$
So here $f'(x) > 0$ and $g'(x) < 0$ in the interval $1 < x < \infty$
So $f(x)$ is increasing and $g(x)$ is decreasing

$$h(x) = f(x).g(x) = (x + x^2)(x - x^2)$$

$$h(x) = x^2 - x^4$$

Now $h'(x) = 2x - 4x^3$

Or $h''(x) = 2 - 12x^2$

So $h''(x) < 0$ for $1 < x < \infty$

So $(f.g)$ is concave downward

Example 3:

Now Let $f(x) = x^2$ and $g(x) = \frac{1}{x}$

Then $f'(x) 2x > 0$ and $g'(x) = -\frac{1}{x^2} < 0$

When $0 < x < \infty$

So $f(x)$ is increasing and $g(x)$ is decreasing

Now let $h(x) = f(x).g(x)$

$$= x^2 . \frac{1}{x} = x$$

$$h(x) = x$$

Then $h'(x) = 1$ and $h''(x) = 0$

So $(f.g)$ is linear (there is no concavity)

So we have $h'' = f''.g + 2f'.g' + f.g''$

And if $f' > 0$ and $g' < 0$ then and $f'' > 0, g'' > 0$

$$h'' = f''.g - 2f'.g' + f.g''$$

So h will be concave upward when

$$f''.g + f.(g'') > 2f'.g'$$

And concave downward when $f''.g + f(g'') > 2f'.g'$

And linear when $f''.g + fg'' = 2f'.g'$

Chapter 3 Applications of Differentiation Exercise 3.3 60E

Function $f(x)$ *and* $g(x)$ are both concave upward on $(-\infty, \infty)$ which means that

$$f''(x) > 0 \text{ and } g''(x) > 0 \quad on \ (-\infty, \infty)$$

It is required to find he condition under, which the function $h(x) = f(g(x))$ will be concave upward.

Differentiating $h(x)$ we get

$$h'(x) = f'(g(x)).g'(x)$$

And $h''(x) = f'(g(x)).g''(x) + f''(g(x)).(g'(x))^2$

For $h(x)$ to be concave upward, we should have $h''(x) > 0$

i.e. $f'(g(x)).g''(x) + f''(g(x)).(g'(x))^2 > 0$ --- (1)

In inequality (1), we have been given

$$g''(x) > 0 \ and \ f''(x) > 0 \ ie. \ f''(g(x)) > 0$$

So the inequality (1) will be true, when

$$f'(g(x)) > 0$$

Or $h'(x) = f'(g(x)).g'(x) > 0$

Thus the condition for $h''(x)$ to be +ve is that

$$h'(x) > 0 \text{ And } g'(x) > 0$$

Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.

To show $\tan x > x$ for $0 < x < \frac{\pi}{2}$, it is enough to show that $f(x) = \tan x - x$ is increasing on $\left(0, \frac{\pi}{2}\right)$.

Now compute $f'(x)$ for $f(x) = \tan x - x$.

$$f'(x) = \sec^2 x - 1$$

Next, find the critical numbers by setting $f'(x)$ equal to zero:

That is, $\sec^2 x - 1 = 0$

The x values which satisfy this have the form $x = n\pi$ where n is an integer.

However, this does allow for any critical numbers in the interval $\left(0, \frac{\pi}{2}\right)$.

Therefore, $f(x) = \tan x - x$ has no critical numbers.

You'll notice that $f'(x)$ is never undefined on the interval $\left(0, \frac{\pi}{2}\right)$, since $f(x) = \tan x - x$ is only undefined for $x = \frac{\pi}{2} + n\pi$ where n is an integer.

Therefore, $f'(x)$ must be either always negative or always positive on $\left(0, \frac{\pi}{2}\right)$.

Rewrite $f'(x) = \sec^2 x - 1$, by using the trig identity $\sec^2 x = \tan^2 x + 1$, will give some insight on which behavior $f'(x)$ has:

$$\begin{aligned} f'(x) &= \sec^2 x - 1 \\ &= \left(\tan^2 x + 1\right) - 1 \\ &= \tan^2 x + 1 - 1 \\ &= \tan^2 x \end{aligned}$$

So, $f'(x)$ is a perfect square and is thus always positive.

Therefore, $f'(x)$ is always positive on the interval $\left(0, \frac{\pi}{2}\right)$.

That is, $f(x) = \tan x - x$ is increasing on the interval $\left(0, \frac{\pi}{2}\right)$.

Thus, $f(x) > f(0)$ since $f(x) = \tan x - x$ is increasing and $x > 0$ because 0 is not contained in the open interval $\left(0, \frac{\pi}{2}\right)$.

Therefore,

$$f(x) = \tan x - x > 0$$

$\tan x > x$ for all x in $\left(0, \frac{\pi}{2}\right)$.

Thus, it is shown that $\tan x > x$ for all x in $\left(0, \frac{\pi}{2}\right)$.

Chapter 3 Applications of Differentiation Exercise 3.3 62

Consider the function,

$$f(x)=2\sqrt{x}-3+\frac{1}{x} \qquad (x\neq 0)$$

To prove the required result, differentiate $f(x)$ with respect to x.

$$f(x)=2\sqrt{x}-3+\frac{1}{x}$$
$$f'(x)=2\left(\frac{1}{2\sqrt{x}}\right)-0-\frac{1}{x^2}$$
$$=\frac{1}{\sqrt{x}}-\frac{1}{x^2}$$
$$>0 \text{ for } x>1$$

Since $f'(x)>0$ for $x>1$

Therefore $f(x)$ is increasing on $x>1$.

By definition of increasing function $f(x)>f(1)$

Then,

$$f(x)>\left(2\sqrt{1}-3+\frac{1}{1}\right)$$
$$=2-3+1$$
$$=0$$

Thus $f(x)>0$

Since $f(x)=2\sqrt{x}-3+\frac{1}{x}$, then $f(x)>0$ becomes

$$2\sqrt{x}-3+\frac{1}{x}>0$$
$$2\sqrt{x}>3-\frac{1}{x}$$

Hence $2\sqrt{x}>3-\frac{1}{x}$ proved

Chapter 3 Applications of Differentiation Exercise 3.3 63E

Let $\quad f(x)=ax^3+bx^2+cx+d$

Then $\quad f'(x)=3ax^2+2bx+c$

And $\quad f''(x)=6ax+2b$

At points of inflection, $\quad f''(x)=0$

$$\Rightarrow \quad 6ax+2b=0$$
$$\Rightarrow \quad x=\frac{-b}{3a} \qquad ---(1)$$

Since $f''(x)>0$ for $x>-b/(3a)$ and $f''(x)<0$ for $x<-b/(3a)$

So there is only one point of inflection

Since graph of the cubic polynomial has x-intercepts $x_1, x_2 \& x_3$.

Then $x_1, x_2 \& x_3$ are roots of equation $f(x) = 0$

So $f(x) = ax^3 + bx^2 + cx + d$ must have the factors as

$$f(x) = a(x - x_1)(x - x_2)(x - x_3)$$

Or $f(x) = a(x^2 - xx_1 - xx_2 + x_1x_2)(x - x_3)$

Or $f(x) = a(x^3 - x^2x_1 - x^2x_2 + xx_1x_2 - x^2x_3 + xx_1x_3 + xx_2x_3 - x_1x_2x_3)$

Or $f(x) = ax^3 - ax^2(x_1 + x_2 + x_3) + ax(x_1x_2 + x_1x_3 + x_2x_3) - ax_1x_2x_3$

Comparing with $f(x) = ax^3 + bx^2 + cx + d$, we have

$$-a(x_1 + x_2 + x_3) = b$$

Or $(x_1 + x_2 + x_3) = -b/a$

So from (1), x- coordinate of the point of inflection is

$$x = \frac{x_1 + x_2 + x_3}{3}$$

Chapter 3 Applications of Differentiation Exercise 3.3 64E

Consider:

$$P(x) = x^4 + cx^3 + x^2.$$

Find for what values of c the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points, one inflection point and none.

Inflection point:

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and at the point the concavity of a function changes.

Determine the intervals of concavity for $P(x) = x^4 + cx^3 + x^2$.

In order to do this, find the second derivative $P''(x)$.

Using the Power Rule, compute the first derivative of $P(x) = x^4 + cx^3 + x^2$:

$$P'(x) = 4x^3 + 3cx^2 + 2x \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

Now compute the second derivative using the Power Rule again:

$$\begin{aligned} P''(x) &= 4(3)x^2 + 3c(2)x + 2 \\ &= 12x^2 + 6cx + 2 \end{aligned}$$

let $P''(x) = 0$.

That is,

$$12x^2 + 6cx + 2 = 0.$$

Solve for x by using the Quadratic Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where

$a = 12, b = 6c$ and c=2:

$$\begin{aligned} x &= \frac{-6c \pm \sqrt{(6c)^2 - 4(12)(2)}}{2(12)} \\ &= \frac{-6c \pm \sqrt{36c^2 - 96}}{24} \end{aligned}$$

The number of inflection points depending on the discriminant $36c^2-96$.

If the discriminant is zero, there is one real inflection point.

So, set $36c^2-96=0$ and solve for c:

$$36c^2-96=0$$
$$36c^2=96$$
$$c^2=\frac{96}{36} \quad \text{Simplify}$$

Continuous to the above step.

$$c^2=\frac{8}{3}$$
$$c=\pm\sqrt{\frac{8}{3}}$$
$$c=\pm\frac{\sqrt{8}}{\sqrt{3}}$$
$$=\pm\frac{\sqrt{4\cdot 2}}{\sqrt{3}}$$
$$=\pm\frac{2\sqrt{2}}{\sqrt{3}}$$

Therefore, if $c=\pm\frac{2\sqrt{2}}{\sqrt{3}}$ it will have one inflection point.

If the discriminant is positive, it will have two real inflection points.

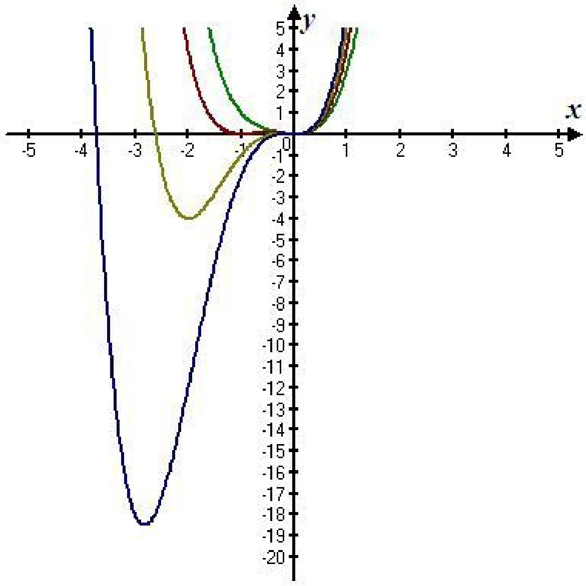
$36c^2-96>0$.

Therefore, if $c>\frac{2\sqrt{2}}{\sqrt{3}}, c<-\frac{2\sqrt{2}}{\sqrt{3}}$ it will have two real inflection points.If the discriminant is negative, no real inflection points.

Therefore, if $c<\frac{2\sqrt{2}}{\sqrt{3}}, c>-\frac{2\sqrt{2}}{\sqrt{3}}$ it will not real inflection points.

Now graph P for various values of c, say $c=1,2,3,4$.

The green line has the lowest c value $c=1$; maroon is $c=2$; yellow is $c=3$; blue is $c=4$



Observe, as c decreases the function becomes strictly positive and strictly concave up. Whereas with higher values of c, from negative infinity to zero, the function is negative and concave downward, and from zero to positive infinity the function is positive and concave upward.

Chapter 3 Applications of Differentiation Exercise 3.3 65E

Let $(c, f(c))$ be the point of inflection on the graph of function *f*.
So by the definition of inflection points we see that at c the value of f'' is changing from positive to negative or vice versa.
Now let $g = f'$ then $g' = f''$.
By using first derivative test, we see that, "if g' is changing from positive to negative at c then g has local maximum at c and if g' is changing from negative to positive at c then g has local minimum at c."

Then by Fermat's theorem we see that, "if g has local minimum or local maximum at c and $g'(c)$ exist then $g'(c) = 0$."

So we have $g'(c) = 0$

Or $(f'(c))' = 0$

Or $\boxed{f''(c) = 0}$

Proved

Chapter 3 Applications of Differentiation Exercise 3.3 66E

We have $f(x) = x^4$

Then $f'(x) = 4x^3$

And $f''(x) = 12x^2$

we have $f''(0) = 0$ at $x = 0$.

Now we check the concavity of $f(x)$ in the interval $(-\infty, 0)$ and $(0, \infty)$

Intervals	$f''(x)$	Concavity of $f(x)$
$(-\infty, 0)$	+ve	Upward
$(0, \infty)$	+ve	Upward

(Because x^2 will be positive for all negative or positive values)
So there is no change in concavity of f at $x = 0$ so (0, 0) can not be an inflection point.

Chapter 3 Applications of Differentiation Exercise 3.3 67E

Consider the function

$$g(x) = x|x|$$

By the definition of modulus function,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

So,

$$g(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Find the derivatives of the function $g(x)$.
Differentiate the function with respect to x.
Then

$$g'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

Again differentiate the function with respect to x
Then the second derivative is

$$g''(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ -2 & \text{if } x < 0 \end{cases}$$

For all the values $x<0$ the second derivative is negative and for all the values $x\ge 0$ the second derivative of the function is positive,
In the interval notation,
On the interval $(-\infty,0), g''(x)<0$ and on the interval $(0,\infty), g''(x)>0$

The concavity of the graph of $g(x)$ is changing from negative to positive.

So, **the point $(0,\ 0)$ will be the inflection point of $g(x)$.**

It is proved that the function will have inflection point at $(0,\ 0)$. So prove that the function does not have the second derivative at $x=0$.

Now let $k(x)=x$ and $f(x)=|x|$

And let $g(x)=k(x).f(x)$

So by the rule of differentiation of the product of the two functions
"The function $g(x)$ is differentiable at $x=0$ if $k(x)$ and $f(x)$ both are differentiable at 0"

Find the differentiability of $k(x)$ and $f(x)$.
By the definition of differentiation,
The derivative of a function *f* at a number *a*, denoted by $f'(a)$ is

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

If this limit exists
Here $k(x)=x$

$$k'(a)=\lim_{h\to 0}\frac{k(x+h)-k(x)}{h}$$ By the definition of derivative

$$=\lim_{h\to 0}\frac{x+h-x}{h}$$ Use $k(x)=x$

$$=\lim_{h\to 0}\frac{h}{h}$$

$$=\lim_{c\to 0}1$$

$$=1$$

The limit exists. Hence, $k(x)=x$ is differentiable at 0.

Chapter 3 Applications of Differentiation Exercise 3.3 68E

We have f''' is continuous, $f'(c)=f''(c)=0$, and $f'''(c)>0$.
So at c, f has no local minimum and local maximum.
Now let $g(x)=f'(x)$, so that $g'(x)=f''(x)$ and $g''(x)=f'''(x)$.

We have $g'(x)=f''(x)=0$ at c.

So $g'(c)=f''(c)=0$

And $g''(c)=f''(c)>0$

So $g''(c)>0$

By the second derivative test $g(x)$ has the local minimum at c. It means $f'(x)$ has the local minimum at c.
$f''(x)$ is changing from negative to positive at c [by first derivative test for $f'(x)$].
Then by the definition of the inflection point, f has a point of inflection at c. Also at the inflection point, f cannot have local minimum or local maximum.

Chapter 3 Applications of Differentiation Exercise 3.3 69E

Begin by breaking the large interval I into two subintervals,

I_1 and I_2; This way we can examine what happens on and around c more easily.

Let those two subintervals be the following:

$$I_1 = \{x \mid x \in I \text{ and } x \le c\}$$
$$I_2 = \{x \mid x \in I \text{ and } x \ge c\}$$

Since f is differentiable, and thus continuous, on the interval I, for any arbitrary a and b In I_1, where $a < b$, the Mean Value Theorem tells us that there exists a number $r \in (a,b)$ Such that $f(b) - f(a) = f'(r)(b-a)$. Thus, f is monotonically increasing On the first subinterval I_1.

In a similar way, for any arbitrary a and b in I_2, where $a < b$, the Mean Value Theorem tells us that there exists a number $r \in (a,b)$ such that $f(b) - f(a) = f'(r)(b-a)$. Thus, f is monotonically increasing on the first subinterval I_2.

Therefore, f is monotonically increasing on the entire interval $I = I_1 \cup I_2$.

Chapter 3 Applications of Differentiation Exercise 3.3 70E

Given that the curve is $f(x) = cx + \dfrac{1}{x^2+3}$

Then $f'(x) = c - \dfrac{2x}{(x^2+3)^2}$

Assume $g(x) = \dfrac{2x}{(x^2+3)^2}$

Then $g'(x) = \dfrac{(x^2+3)^2 \cdot 2 - (2x)2(x^2+3)(2x)}{(x^2+3)^2}$

Similarly $g(x)$ has a local minimum at $x = 1$ and

$$g(1) = \frac{1}{8}$$

$$\boxed{\frac{-1}{8} \le \frac{2x}{(x^2+3)^2} \le \frac{1}{8}}$$

Since $f(x)$ is increasing on $(-\infty, \infty)$, so $f'(x) > 0$

$$x \in (-\infty, \infty)$$

$$c - \frac{1}{8} \ge 0$$

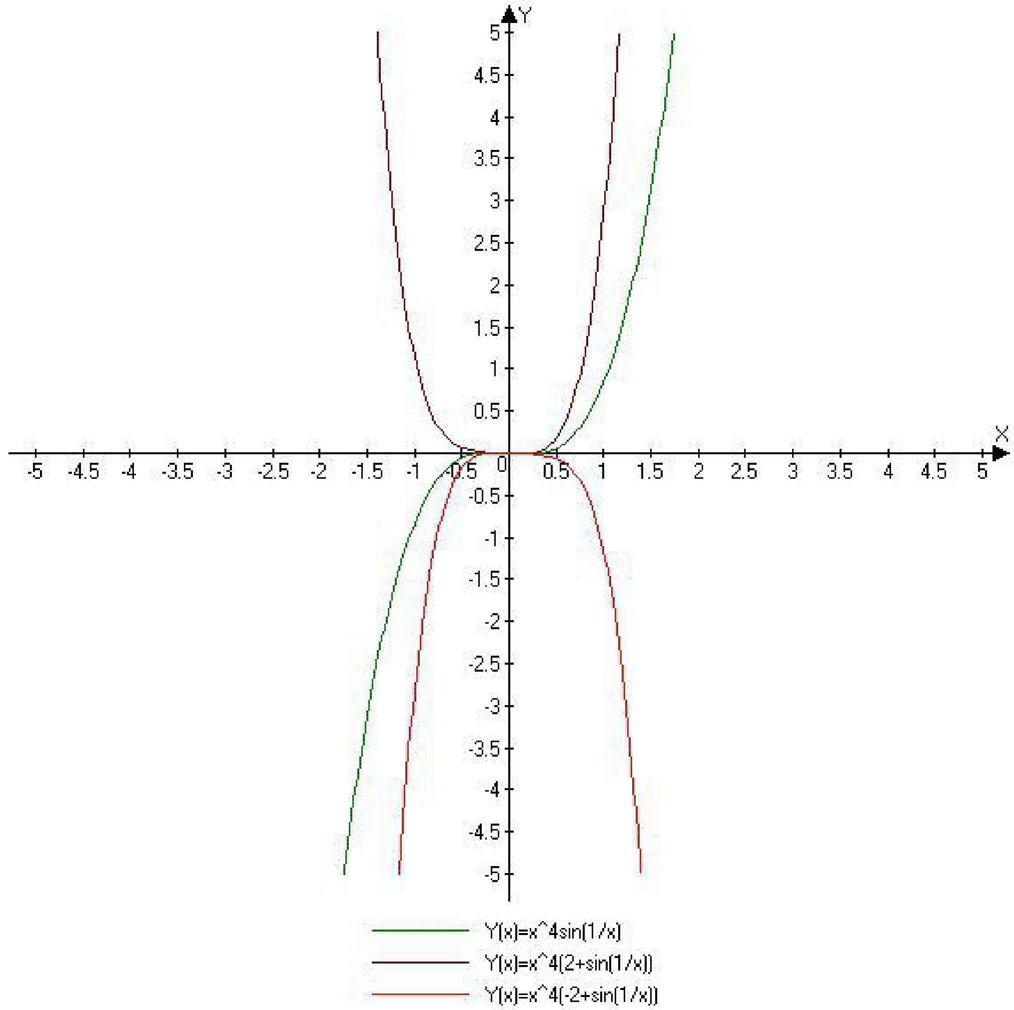
$$\Rightarrow \quad c > \frac{1}{8}$$

$$\boxed{c > \frac{1}{8} \text{ is the function } f(x) = cx + \frac{1}{x^2+3} \text{ increasing on } (-\infty, \infty).}$$

Chapter 3 Applications of Differentiation Exercise 3.3 71E

$f(x) = x^4 \sin\frac{1}{x}$, $g(x) = x^4\left(2 + \sin\frac{1}{x}\right)$, $h(x) = x^4\left(-2 + \sin\frac{1}{x}\right)$

(a) first we draw these functions in the same rectangle to show 0 is the critical value of all these functions.



observe that all the graphs are becoming parallel to x axis as they cross the origin.

they are changing their sign at 0 or the concavity of the curves from upwards to down wards or the other at 0.

∴ 0 is either a critical point or a point of infliction to these functions.

(b) while the curves are becoming parallel to x axis while they cross 0, we cannot say that 0 is neither minimum nor maximum value for each of these functions.