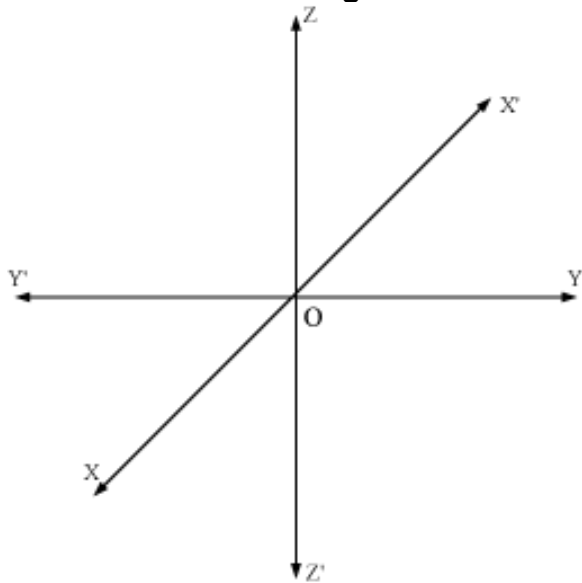# Introduction to Three Dimensional Geometry
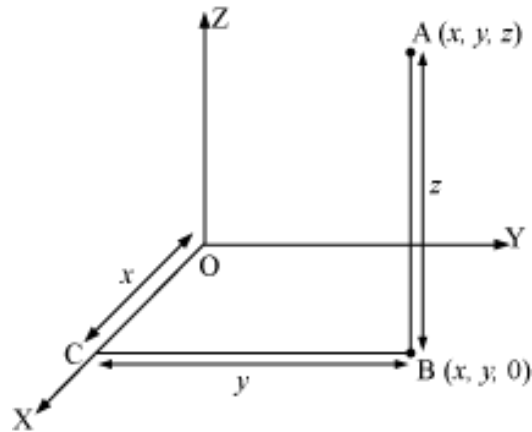
## Rectangular Coordinate System

- If we draw three planes intersecting at O such that they are mutually perpendicular to each other, then these will intersect along the lines X′OX, Y′OY and Z′OZ. These lines constitute the **rectangular coordinate system** and are respectively known as the *x*, *y*, and *z*-axes.

- Point O is called the **origin** of the coordinate system.



- The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downward in the direction of OZ′ are taken as negative.

- The distances measured to the right of ZX-plane along OY are taken as positive and those measured to the left of ZX-plane along OY′ are taken as negative.

- The distances measured in front of YZ-plane along OX are taken as positive and those measured at the back of YZ-plane along OX′ are taken as negative.

- The planes XOY, YOZ, and ZOX are known as the three **coordinate planes** and are respectively called the XY-plane, the YZ-plane, and the ZX-plane.

- The three coordinate planes divide the space into eight parts known as **octants**. These octants are named as XOYZ, X′OYZ, X′ OY′Z, XOY′Z, XOYZ′, X′OYZ′, X′OY′Z′, and XOY′Z′ and are denoted by I, II, III, IV, V, VI, VII, and VIII respectively.

- If a point A lies in the first octant of a coordinate space, then the lengths of the perpendiculars drawn from point A to the planes XY, YZ and ZX are represented by *x*, *y*,

and $z$ respectively and are called the **coordinates** of point A.

This means that the coordinates of point A are $(x, y, z)$. However, if point A would have been in any other quadrant, then the signs of $x$, $y$, and $z$ would change accordingly.



- The coordinates of the origin are $(0, 0, 0)$.

- The sign of the coordinates of a point determines the octant in which the point lies. The following table shows the signs of the coordinates in the eight octants.

| Octants → <br><br> Coordinates↓ | I | II | III | IV | V | VI | VII | VIII |
|---|---|---|---|---|---|---|---|---|
| $x$ | + | − | − | + | + | − | − | + |
| $y$ | + | + | − | − | + | + | − | − |
| $z$ | + | + | + | + | − | − | − | − |

- The coordinates of a point lying on different axes are as follows:

- The coordinates of a point lying on the $x$-axis will be of the form $(x, 0, 0)$.

- The coordinates of a point lying on the *y*-axis will be of the form (0, *y*, 0).

- The coordinates of a point lying on the *z*-axis will be of the form (0, 0, *z*).

- The coordinates of a point lying on different planes are as follows:

- The coordinates of a point lying in the XY-plane will be of the form (*x*, *y*, 0).

- The coordinates of a point lying in the YZ-plane will be of the form (0, *y*, *z*).

- The coordinates of a point lying in the ZX-plane will be of the form (*x*, 0, *z*).

- Let's now try and solve the following puzzle to check whether we have understood the basic concepts that we just studied.

## Solved Examples

### Example 1

State whether the following statements are true or false.

1. The point (−5, 0, 1) lies on the *y*-axis.

2. The point (1, 7, −1) lies in octant V, whereas the point (−10, −8, 6) lies in octant VIII.

3. The *x*, *y*, and *z* coordinates of the point (5, 7, 18) are 5, 7 and 18 respectively.

4. The point (0, 0, 19) lies in the ZX-plane.

5. The point (−2, 11, 8) lies in octant II.

### Solution:

1. False. The point (−5, 0, 1) does not lie on the *y*-axis since the point that lies on the *y*-axis is of the form (0, *y*, 0).

2. False. Point (−10, −8, 6) lies in octant III.

3. True.

4. False. A point lying in the ZX-plane is of the form (*x*, 0, *z*).
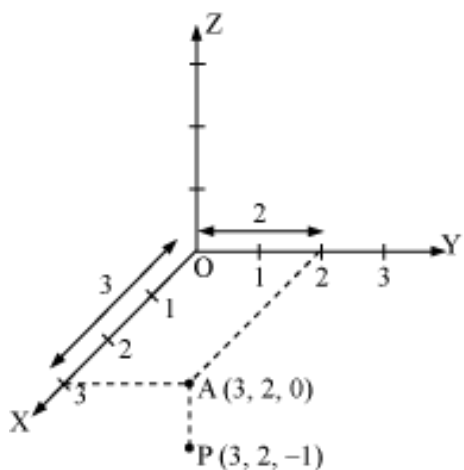
5. True.

### Example 2

Locate point (3, 2, −1) in a three-dimensional space.

**Solution:**

We have to locate point (3, 2, −1) in a three-dimensional space. In order to do this, we will first draw the three axes.

Then, starting from the origin, when we move 2 units in the positive *y*-direction and then 3 units in the positive *x*-direction, we will reach at the point A(3, 2, 0).

From point A(3, 2, 0), we move 1 unit in the negative *z*-direction to reach the required point i.e., P(3, 2, −1).



## Distance between Two Points in Three-Dimensional Space

- The distance formula that is used for finding the distance between two points A($x_1$, $y_1$, $z_1$) and B($x_2$, $y_2$, $z_2$) lying in three-dimensional space is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Example: The distance between points A(3, 7, 8) and B(5, 0, 1) is given by

$$AB = \sqrt{(5-3)^2 + (0-7)^2 + (1-8)^2} = \sqrt{102}$$

**Solved Examples**

**Example 1:**

Prove that points (−4, 8, 1), (2, 4, −1) and (−2, 2, 5) are the vertices of an equilateral triangle.

**Solution:**

Let the given points be A(−4, 8, 1), B(2, 4, −1) and C(−2, 2, 5) respectively.

We know that in an equilateral triangle, the lengths of all the sides are the same.

On using the distance formula, we obtain

AB = $\sqrt{\left[2-(-4)\right]^2+(4-8)^2+(-1-1)^2}=\sqrt{(6)^2+(-4)^2+(-2)^2}=\sqrt{36+16+4}=\sqrt{56}$

BC = $\sqrt{(-2-2)^2+(2-4)^2+\left[5-(-1)\right]^2}=\sqrt{(-4)^2+(-2)^2+(6)^2}=\sqrt{16+4+36}=\sqrt{56}$

CA = $\sqrt{\left[-4-(-2)\right]^2+(8-2)^2+(1-5)^2}=\sqrt{(-2)^2+(6)^2+(-4)^2}=\sqrt{4+36+16}=\sqrt{56}$

Thus,

AB = BC = CA = $\sqrt{56}$ units

Thus, the given vertices are the vertices of an equilateral triangle.

**Example 2:**

Find a point lying in the XY-plane such that the sum of the squares of its $x$ and $y$ coordinates is 5. Also, its distance from point (3, 5, 0) is 5 units, while its distance from point (5, −1, 2) is 7 units.

**Solution:**

The two given points are A(3, 5, 0) and B(5, −1, 2).

We know that any point lying in the XY-plane is of the form ($x$, $y$, 0). Hence, let the required point be C($x$, $y$, 0).

It is given that the distance between points A and C is 5 units.

On applying the distance formula, we obtain

$$AC=\sqrt{(x-3)^2+(y-5)^2+(0-0)^2}$$
$$=\sqrt{x^2+9-6x+y^2+25-10y}=\sqrt{x^2+y^2-6x-10y+34}$$

Thus,

$$\sqrt{x^2 + y^2 - 6x - 10y + 34} = 5$$
$$\Rightarrow x^2 + y^2 - 6x - 10y + 34 = 25$$
$$\Rightarrow x^2 + y^2 - 6x - 10y + 9 = 0$$

It is given that $x^2 + y^2 = 5$. Hence,

$5 - 6x - 10y + 9 = 0$

$\Rightarrow -6x - 10y + 14 = 0$

$\Rightarrow 3x + 5y - 7 = 0$ ... (1)

It is also given that the distance between points B and C is 7 units.

On applying the distance formula, we obtain

$$BC = \sqrt{(x-5)^2 + (y+1)^2 + (0-2)^2}$$
$$= \sqrt{x^2 + 25 - 10x + y^2 + 1 + 2y + 4} = \sqrt{x^2 + y^2 - 10x + 2y + 30}$$

Thus,

$$\sqrt{x^2 + y^2 - 10x + 2y + 30} = 7$$
$$\Rightarrow x^2 + y^2 - 10x + 2y + 30 = 49$$
$$\Rightarrow x^2 + y^2 - 10x + 2y - 19 = 0$$

It is given that $x^2 + y^2 = 5$. Hence,

$5 - 10x + 2y - 19 = 0$

$\Rightarrow -10x + 2y - 14 = 0$

$\Rightarrow 5x - y + 7 = 0$ ... (2)

On solving equations (1) and (2), we obtain

$x = -1$ and $y = 2$

Thus, the required point is (−1, 2, 0).

**Example 3:**

Show that points (4, 3, −3), (3, 2, −1), and (2, 1, 1) are collinear.

**Solution:**

Let the given points be P(4, 3, −3), Q(3, 2, −1), and R(2, 1, 1).

We know that points are said to be collinear if they lie on a line.

On applying the distance formula, we obtain

$PQ = \sqrt{(3-4)^2 + (2-3)^2 + [-1-(-3)]^2} = \sqrt{(-1)^2 + (-1)^2 + (2)^2} = \sqrt{1+1+4} = \sqrt{6}$

$QR = \sqrt{(2-3)^2 + (1-2)^2 + [1-(-1)]^2} = \sqrt{(-1)^2 + (-1)^2 + (2)^2} = \sqrt{1+1+4} = \sqrt{6}$

$RP = \sqrt{(4-2)^2 + (3-1)^2 + (-3-1)^2} = \sqrt{(2)^2 + (2)^2 + (-4)^2} = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

Thus, PQ + QR = RP.

Hence, the given points i.e., P, Q, and R are collinear.

## Section Formula

- The coordinates of the point that divides the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m: n$ are given by

$\left( \dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}, \dfrac{mz_2 + nz_1}{m+n} \right)$. This formula is known as the section formula.

- The coordinates of a point that divides the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $m: n$ are given by

$\left( \dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}, \dfrac{mz_2 - nz_1}{m-n} \right)$.

- If X is the mid-point of the line segment joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then X divides AB in the ratio 1:1. Hence, by using the section formula, the coordinates of point X will be given by

$$\left( \frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2} \right).$$

- If a point R divides the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $k : 1$, then, by using the section formula, the coordinates of point R will be given by

$$\left( \frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1} \right).$$

- The coordinates of the centroid of a triangle whose vertices are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$ are given by

$$\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right).$$

## Solved Examples

**Example 1:**

The coordinates of the vertices of ΔWXY are W(−12, 5, 6), X(−2, 1, −8) and Y(−1, −6, −7). Does the centroid of ΔWXY lie on the *XZ*-plane?

**Solution:**

It is given that the coordinates of the vertices of ΔWXY are W(−12, 5, 6), X(−2, 1, −8) and Y(−1, −6, −7).

We know that the coordinates of the centroid of the triangle whose vertices are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and $(x_3, y_3, z_3)$ are given by

$$\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right).$$

Thus, the coordinates of the centroid of ΔWXY are given by

$$\left( \frac{(-12)+(-2)+(-1)}{3}, \frac{5+1+(-6)}{3}, \frac{6+(-8)+(-7)}{3} \right)$$

$$= \left( \frac{-12-2-1}{3}, \frac{5+1-6}{3}, \frac{6-8-7}{3} \right)$$

$$= \left( \frac{-15}{3}, 0, \frac{-9}{3} \right)$$

$$= (-5, 0, -3)$$

We also know that any point whose coordinates are of the form (*x*, 0, *z*) lies on the *XZ*-plane. The coordinates of the centroid of ΔWXY are (−5, 0, −3).

Hence, the centroid of ΔWXY lies on the *XZ*-plane.

**Example 2:**

Find the ratio in which the line segment joining the points (23, 16, −19) and (−15, 20, 21) is divided externally by the ZX-plane.

**Solution:**

Let the given points be A(23, 16, −19) and B(−15, 20, 21).

Let point C(*x*, *y*, *z*) divide the line segment AB externally in the ratio *k*: 1.

Accordingly, the coordinates of point C are

$$\left( \frac{k(-15)-1(23)}{k-1}, \frac{k(20)-1(16)}{k-1}, \frac{k(21)-1(-19)}{k-1} \right)$$

$$= \left( \frac{-15k-23}{k-1}, \frac{20k-16}{k-1}, \frac{21k+19}{k-1} \right)$$

Since point C lies on the ZX-plane, its *y*-coordinate is zero.

Therefore,

$$\frac{20k-16}{k-1} = 0$$

$$\Rightarrow 20k-16 = 0$$

$$\Rightarrow k = \frac{16}{20} = \frac{4}{5}$$

Thus, the ZX-plane divides AB externally in the ratio 4: 5.
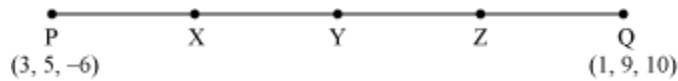
**Example 3:**

Find the coordinates of the points that divide the line segment joining the points P(3, 5, −6) and Q(1, 9, 10) into four equal parts.

**Solution:**

The given points are P(3, 5, −6) and Q(1, 9, 10).

Let points X, Y, and Z divide the line segment PQ. Hence,

PX = XY = YZ = ZQ.



From the figure, we can clearly see that point Y is the mid-point of line segment PQ.

Hence, the coordinates of point Y are given by

$$\left(\frac{3+1}{2}, \frac{5+9}{2}, \frac{-6+10}{2}\right) = \left(\frac{4}{2}, \frac{14}{2}, \frac{4}{2}\right) = (2,7,2)$$

Again, X is the mid-point of line segment PY. Hence, the coordinates of point X are given by

$$\left(\frac{3+2}{2}, \frac{5+7}{2}, \frac{-6+2}{2}\right) = \left(\frac{5}{2}, \frac{12}{2}, \frac{-4}{2}\right) = \left(\frac{5}{2},6,-2\right)$$

Again, Z is the mid-point of line segment YQ. Hence, the coordinates of point Z are given by

$$\left(\frac{2+1}{2}, \frac{7+9}{2}, \frac{2+10}{2}\right) = \left(\frac{3}{2}, \frac{16}{2}, \frac{12}{2}\right) = \left(\frac{3}{2},8,6\right)$$

Thus, the coordinates of the points that divide the line segment joining the given points are (2, 7, 2), $\left(\frac{5}{2},6,-2\right)$, and $\left(\frac{3}{2},8,6\right)$.