Exercise 17.3

Chapter 17 Second Order Differential Equations 17.3 1E

By Hooke's Law k(0.25) = 25 so k = 100 is the spring constant and the differential equation is 5x'' + 100x = 0. The auxiliary equation is 5r2 + 100 = 0 with roots r = ±2√5 i, so the general solution to the differential equation is x(t) = c1 cos(2√5 t) + c2 sin(2√5 t). We are given that x(0) = 0.35 → c1 = 0.35 and x'(0) = 0 → 2√5 c2 = 0 → c2 = 0, so the position of the mass after t seconds is x(t) = 0.35 cos(2√5 t).

Chapter 17 Second Order Differential Equations 17.3 2E

By Hooke's Law k(0.4) = 32 so k = 32/0.4 = 80 is the spring constant and the differential equation is 8x'' + 80x = 0. The general solution is x(t) = c1 cos(√10 t) + c2 sin(√10 t). But

0 = x(0) = c1 and 1 = x'(0) = √10 c2 → c2 = 1/√10, so the position of the mass after t seconds is x(t) = 1/√10 sin(√10 t).

Chapter 17 Second Order Differential Equations 17.3 3E

A spring with mass of 2kg is stretched by 0.5m beyond its natural length by a force of 6N

Then the spring constant is

$$k = \frac{6}{0.5}$$
$$= 12$$

Now the spring has damping constant r = 14

Let x(t) be the position of mass at any time t

Then the equation of motion is

$$m\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + kx = 0$$

i.e. $$2\frac{d^2x}{dt^2} + 14\frac{dx}{dt} + 12x = 0 \qquad \text{------------ (1)}$$

This is a homogeneous differential equation with characteristic equation

$$2r^2 + 14r + 12 = 0$$

Or $r^2 + 7r + 6 = 0$

Or $(r+1)(r+6) = 0$

i.e. $r = -1,\ r = -6$ are the roots.

Then the solution of equation (1) is

$$x(t) = c_1 e^{-t} + c_2 e^{-6t}$$

Where c_1 and c_2 are arbitrary constants

By initial conditions

$$x(0) = 1$$

And $x'(0) = 0$

Now $x'(t) = -c_1 e^{-t} - 6c_2 e^{-6t}$

Then $x'(0) = 0$ implies $c_1 + 6c_2 = 0$

And $x(0) = 1$ implies $c_1 + c_2 = 1$

On solving we find $c_1 = \frac{6}{5},\ c_2 = \frac{-1}{5}$

Hence the position of mass at any time t is

$$\boxed{x(t) = \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t}}$$

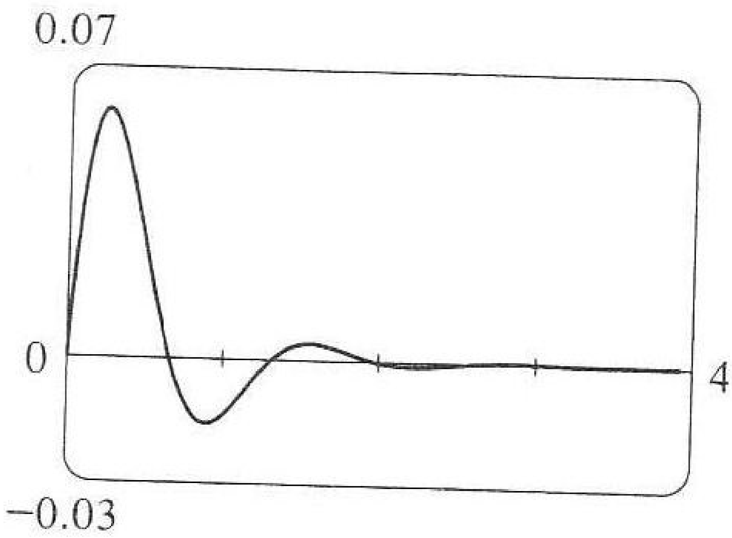
Chapter 17 Second Order Differential Equations 17.3 4E

(a)

k(0.25) = 13 → k = 52, so the differential equation is 2x'' + 8x' + 52x = 0 with general solution x(t) = e-2t[c1 cos(√22 t) + c2 sin(√22 t)].

Then 0 = x(0) = c1 and 0.5 = x'(0) = √22 c2 → c2 = 1/(2√22), so the position is given by x(t) = [1/(2√22)]e-2t sin(√22 t).

(b)



Chapter 17 Second Order Differential Equations 17.3 5E

Let m be the mass that would produce critical damping.
Here the damping constant c = 14

And spring constant $k = \frac{6}{0.5} = 12$

For the critical damping $c^2 - 4mk = 0$

i.e. $(14)^2 - 4m(12) = 0$

i.e. $196 - 489m = 0$

i.e. $m = \frac{196}{48}$

i.e. $\boxed{m = \frac{49}{12} kg}$

Chapter 17 Second Order Differential Equations 17.3 6E

$m = 3kg, \quad k = 123$

We need to find damping constant that would produce critical damping

Let the damping constant be c
We know that for the critical damping

$c^2 - 4mk = 0$

i.e. $c^2 - 4(3)(123) = 0$

i.e. $c^2 = 1467$

i.e. $c = 38.41$

Hence damping constant of 38.41 will produce critical damping

Chapter 17 Second Order Differential Equations 17.3 7E

Here m = 1, k = 100 and damping constant = c

Then the equation of motion is

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

i.e. $$\frac{d^2x}{dt^2}+c\frac{dx}{dt}+100x=0 \quad \text{--------------- (1)}$$

Where x(t) is the position of mass at any time t

This is a homogeneous differential equation

Its characteristic equation is

$$r^2+cr+100=0$$

$$\therefore\ r=\frac{-c\pm\sqrt{c^2-400}}{2}$$

(1)

When c = 10

Then $r=-5\pm5\sqrt{3}i$

And then the solution of equation (1) is

$$x(t)=e^{-5t}\left(c_1\cos5\sqrt{3}t+c_2\sin5\sqrt{3}t\right)$$

Using initial conditions, $x(0)=-0.1,\ x'(0)=0$

We have $c_1=-0.1$ and $c_2=\frac{-0.1}{\sqrt{3}}$

Then $x(t)=e^{-5t}\left(-0.1\cos5\sqrt{3}t-\frac{0.1}{\sqrt{3}}\sin5\sqrt{3}t\right)$

Damping: - under damping

(2)

When c = 15

Then $r=\frac{-15}{2}\pm\frac{5}{2}\sqrt{7}i$

Then the solution of equation (1) is

$$x(t)=e^{-\frac{15}{2}t}\left(c_1\cos\frac{5\sqrt{7}t}{2}+c_2\sin\frac{5\sqrt{7}t}{2}\right)$$

Using initial conditions $x(0)=-0.1,\ x'(0)=0$

We have $c_1=-0.1$ and $c_2=-\frac{0.2}{\sqrt{7}}$

Then $x(t)=e^{\frac{-15}{2}t}\left(-0.1\cos\frac{5\sqrt{7}t}{2}-\frac{0.3}{\sqrt{7}}\sin\frac{5\sqrt{7}t}{2}\right)$

Damping: - under damping

(3)

When c = 20

Then $r=-10,-10$

And then the solution of equation (1) is

$$x(t)=c_1e^{-10t}+c_2te^{-10t}$$

Using initial conditions $x(0)=-0.1,\ x'(0)=0$

We have $c_1=-0.1,\ c_2=-1$

Then $x(t)=-0.1e^{-10t}-te^{-10t}$

Damping: -_critical

(4)

When c = 25

Then $r=-20,-5$

And then the solution of equation (1) is

$$x(t)=c_1e^{-20t}+c_2e^{-5t}$$

Using initial conditions $x(0) = -0.1,\ x'(0) = 0$

We have $c_1 = 0.03,\quad c_2 = 0.133$

Then $\quad x(t) = 0.03e^{-12.5t} - 0.133e^{-2.5t}$

Damping: - over damping

(5)

When c = 30

Then $\quad r = \dfrac{-30 \pm 10\sqrt{5}}{2}$

$= -15 \pm 5\sqrt{5}$

i.e $\quad$ r $= -26.18,\ -3.81$

Then the solution of equation (1) is

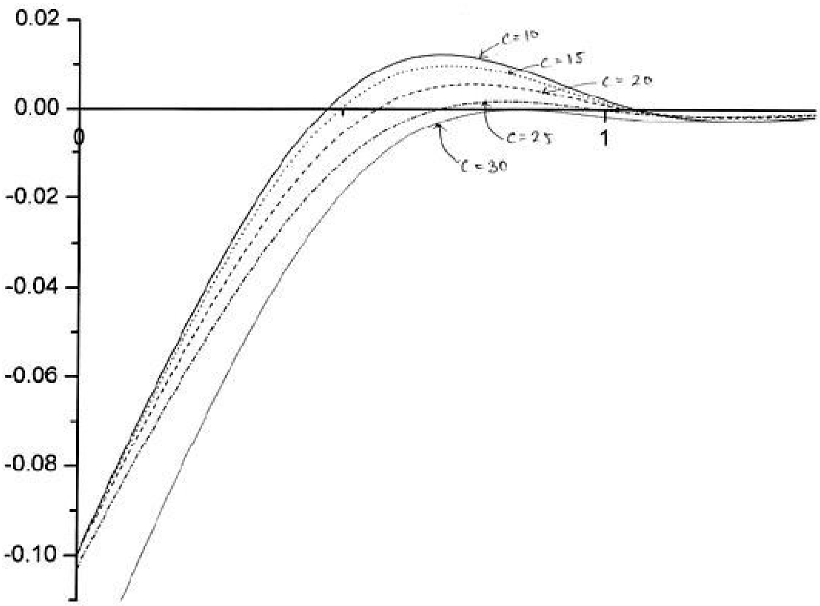
$$x(t) = c_1 e^{-26.18t} + c_2 e^{-3.81t}$$

Using initial conditions $x(0) = -0.1,\quad x'(0) = 0$

We have $c_1 = 0.017$ and $c_2 = -0.117$

Then $\quad x(t) = 0.017e^{-26.18t} - 0.117e^{-3.81t}$

Damping: -_over damping



Chapter 17 Second Order Differential Equations 17.3 8E

Here m = 1kg, spring constant = k

Damping constant = 10

Then the equation of motion is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

i.e. $\quad \dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + kx = 0 \quad$ ------------ (1)

This is a homogeneous differential equation

Its characteristic equation is

$$r^2 + 10r + k = 0$$

Then $\quad r = \dfrac{-10 \pm \sqrt{100 - 4k}}{2}$

(1)

When k = 10

Then $c^2 - 4mk > 0$ and thus there is over damping

And $\quad r = \dfrac{-10 \pm 2\sqrt{15}}{2}$

$= -5 \pm \sqrt{15}$

That is $r = -8.87,\ -1.12$

Then the solution of equation (1) is

$$x(t) = c_1 e^{-8.87t} + c_2 e^{-1.12t}$$

By given initial conditions $x(0)=0, \quad x'(0)=1 m/s$

We have $c_1=-0.13, \quad c_2=+0.13$

Thus $\quad x(t)=0.13e^{-8.87t}+0.13e^{-1.12t}$

(2)

When k = 20

Then $c^2-4mk>0$ and thus there is over damping

And $r=\dfrac{-10\pm 2\sqrt{5}}{2}$

$=-5\pm\sqrt{5}$

i.e. $\quad r=-7.23, \ -2.76$

Then the solution of equation (1) is

$$x(t)=c_1e^{-7.23t}+c_2e^{-2.76t}$$

By using initial conditions $x(0)=0, \ x'(0)=1$

We have $c_1=-0.22, \ c_2=0.22$

Thus $\quad x(t)=-0.22e^{-7.23t}+0.22e^{-2.76t}$

(3)

When k = 25

Then $c^2-4mk=0$ and thus there is critical damping

And $\quad r=-5, -5$

And then the solution of equation (1) is

$$x(t)=c_1e^{-5t}+c_2te^{-5t}$$

On using initial conditions $x(0)=0, \ x'(0)=1$

We have $\quad c_1=0, \quad c_2=1$

Thus $\quad x(t)=te^{-5t}$

(4)

When k = 30

Then $c^2-4mk<0$ and thus there is under damping

And $\quad r=\dfrac{-10\pm 2\sqrt{5}i}{2}$

$=-5\pm\sqrt{5}i$

Then the solution of equation (1) is

$$x(t)=e^{-5t}\left(c_1\cos\sqrt{5}t+c_2\sin\sqrt{5}t\right)$$

On using initial conditions $x(0)=0, \ x'(0)=1$

We have $c_1=0$ and $c_2=\dfrac{1}{\sqrt{5}}$

Then $\quad x(t)=\dfrac{1}{\sqrt{5}}e^{-5t}\sin\sqrt{5}t$

(5)

When k = 40

Then $c^2-4mk<0$ and thus there is under damping

And $\quad r=-5\pm\sqrt{15}i$

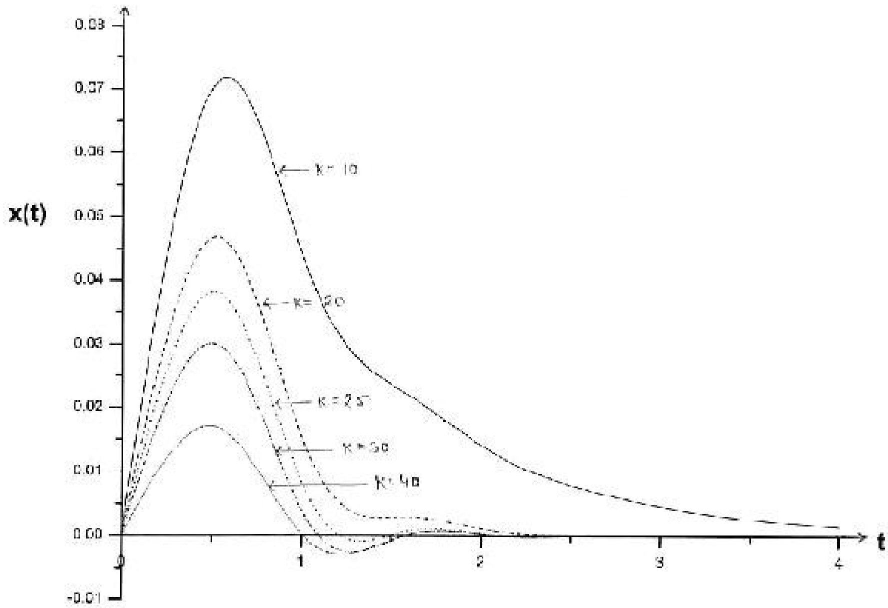
Then the solution of equation (1) is

$$x(t)=e^{-5t}\left(c_1\cos\sqrt{15}t+c_2\sin\sqrt{15}t\right)$$

On using initial conditions $x(0)=0, \quad x'(0)=1$

We have $c_1=0$ and $c_2=\dfrac{1}{\sqrt{15}}$

Thus $x(t)=\dfrac{1}{\sqrt{15}}e^{-5t}\sin\sqrt{15}t$



Chapter 17 Second Order Differential Equations 17.3 9E

When there is no damping that is c = 0 and the external force $F(t) = F_0 \cos w_0 t$ acts on a spring with mass m and spring constant k, then the equation of motion is given as;

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos w_0 t \qquad \text{----------------- (1)}$$

This is a non – homogeneous differential equation

Its corresponding homogeneous equation is

$$m\frac{d^2x}{dt^2} + kx = 0$$

It has characteristic equation $mr^2 + k = 0$

$$\therefore \quad r = \pm\sqrt{\frac{k}{m}}\,i$$

Let $\sqrt{\frac{k}{m}} = w$ then $r = \pm wi$

And thus the homogeneous solution is

$$x_h(t) = c_1 \cos wt + c_2 \sin wt$$

For the particular solution, using method of undetermined coefficients let the particular solution is $x_p = A\cos w_0 t + B\sin w_0 t$

(Since $w \neq w_0$ given)

Then $\quad \dfrac{dx p}{dt} = -Aw_0 \sin w_0 t + Bw_0 \cos w_0 t$

And $\quad \dfrac{d^2x p}{dt^2} = -Aw_0^2 \cos w_0 t - Bw_0^2 \sin w_0 t$

Substituting for x_p and $\dfrac{d^2xp}{dt^2}$ in equation (1)

$$-m\,Aw_0^2 \cos w_0 t - m\,Bw_0^2 \sin w_0 t + kA\cos w_0 t + kB\sin w_0 t = F_0 \cos w_0 t$$

i.e. $\quad \left(-m\,Aw_0^2 + kA\right)\cos w_0 t + \left(-mB\,w_0^2 + kB\right)\sin w_0 t - F_0 \cos w_0 t$

Equating coefficients on both sides

$$A\left(k - mw_0^2\right) = F_0, \text{ and } B\left(k - m\,w_0^2\right) = 0$$

That is $B = 0$ and $A = \dfrac{F_0}{\left(k - m\,w_0^2\right)}$

$$\text{Or} \qquad A = \frac{F_0}{m\left(\frac{k}{m} - w_0^2\right)}$$

$$= \frac{F_0}{m\left(w^2 - w_0^2\right)}$$

Thus the particular solution is

$$x_p(t) = \frac{F_0}{m(w^2 - w_0^2)} \cos w_0 t$$

And hence the required solution of equation (1) is

$$x(t) = x_h(t) + x_p(t)$$

i.e. $$\boxed{x(t) = c_1 \cos wt + c_2 \sin wt + \frac{F_0 \cos w_0 t}{km(w^2 - w_0^2)}}$$

Chapter 17 Second Order Differential Equations 17.3 10E

When there is no damping i.e. $c = 0$ and the external force $F(t) = F_0 \cos wt$ acts on a spring with mass m and spring constant k, then the equation of motion is given by

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos wt \qquad \text{-----------} \ (1)$$

This is a non – homogeneous differential equation

Its corresponding homogeneous equation is

$$m\frac{d^2x}{dt^2} + kx = 0$$

The characteristic equation is

$$mr^2 + k = 0$$

$$\therefore r = \pm\sqrt{k/m}$$

Now $\sqrt{\frac{k}{m}} = w$ then $r = \pm wi$

And thus the homogeneous solution is

$$x_h(t) = c_1 \cos wt + c_2 \sin wt$$

For the particular solution, using method of undetermined coefficients let the particular solution be

$$x_p = (At^2 + Bt)\cos wt + (Ct^2 + Dt)\sin wt$$

Then $$x_p' = \left[C wt^2 + (2A + Dw)t + B\right]\cos wt + \left[-Awt^2 + (2C - Bw)t + D\right]\sin wt$$

And $$x_p'' = \left[-Aw^2t^2 + (4Cw - Bw^2)t + 2(4 + Dw)\right]\cos wt + \left[-Cw^2t^2 - (4Aw + Dw^2)t - 2(C + Bw)\right]\sin wt$$

Substituting for x_p'' and x_p in equation (1)

$$\left[-Amw^2t^2 + 4c\,mwt - Bw^2mt + 2Am + 2Dmw + Akt^2 + Bkt\right]\cos wt$$

$$+\left[mCw^2t^2 - 4Amwt - mDw^2t - 2Cm - 2Bwm + kCt^2 + kDt\right]\sin wt = F_0 \cos wt$$

Equating coefficients on both sides

$$-A(mw^2 - k)t^2 + (d4Cmw - Bmw^2 + Bw)t + 2Am + 2Dwm - F_0$$

And $$-C(mw^2 - k)t^2 - (4Amw - Dw^2m - Dw^2m + Dk)t - 2Cm - 2Bmw = 0$$

i.e. $$-A(mw^2 - k) = 0, \quad 4Cmw - Bmw^2 + Bk = 0$$

$$2Am + 2Dwm = F_0$$

And $$-C(mw^2 - k) = 0, \quad -(4Amw - Dw^2m + Dk) = 0$$

$$-2Cm - 2Bmw = 0$$

i.e. $$A = 0,\ B = 0,\ C = 0,\ D = \frac{F_0}{2mw}$$

Then the particular solution is

$$x_p(t) = \frac{F_0}{2mw} t \sin wt$$

And hence the required solution of equation (1) is

$$x(t) = x_p(t) + x_h(t)$$

i.e. $$\boxed{x(t) = c_1 \cos wt + c_2 \sin wt + \frac{F_0 t}{2mw} \sin wt}$$

Chapter 17 Second Order Differential Equations 17.3 11E

Motion described by equation specified is:

$$x(t) = c_1 \cos wt + c_2 \sin wt + \frac{f_o}{m(w^2 - w_0^2)} \cos w_o t$$

Given $\frac{w}{w_o}$ is a rational number. So it will be of the form $\frac{p}{q}$, where p,q both are integers and $q \neq 0$.

Let

$$\frac{w}{w_o} = \frac{p}{q} \qquad \Rightarrow \qquad p = \frac{w}{w_o} q.$$

Now

$$x\left(t + p.\frac{2\pi}{w}\right) = c_1 \cos w\left(t + p.\frac{2\pi}{w}\right) + c_2 \sin w\left(t + p.\frac{2\pi}{w}\right) + \frac{f_o}{m(w^2 - w_0^2)} \cos w_o \left(t + p.\frac{2\pi}{w}\right)$$

$$= c_1 \cos(wt + 2\pi p) + c_2 \sin(wt + 2\pi p) + \frac{f_o}{m(w^2 - w_0^2)} \cos\left(w_o t + + p w_o \frac{2\pi}{w}\right)$$

$$= c_1 \cos wt + c_2 \sin wt + \frac{f_o}{m(w^2 - w_o^2)} \cos\left(w_o t + \frac{w}{w_o} q w_o \frac{2\pi}{w}\right)$$

$$= c_1 \cos wt + c_2 \sin wt + \frac{f_o}{m(w^2 - w_o^2)} \cos(w_o t + 2\pi q)$$

$$= c_1 \cos wt + c_2 \sin wt + \frac{f_o}{m(w^2 - w_o^2)} \cos w_o t$$

$$= x(t)$$

$\Rightarrow$ Motion described by equation specified is periodic with period $\frac{2\pi p}{w}$.

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Chapter 17 Second Order Differential Equations 17.3 12E

(A) Equation of motion of spring is given by,

$$x = c_1 e^{rt} + c_2 t e^{rt}$$

The graph of x will cross the t-axis where $x = 0$

Therefore, we have,

$$0 = c_1 e^{rt} + c_2 t e^{rt}$$

$$\Rightarrow \quad e^{rt}(c_1 + c_2 t) = 0$$

Since e^{rt} is always positive, therefore

$$c_1 + c_2 t = 0$$

$$\Rightarrow \quad c_1 = -c_2 t$$

Since t is always positive therefore c_1 and c_2 will have opposite signs.

(B) Equation of motion of spring is given as,

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \text{where } r_1 > r_2$$

The graph of x crosses the t-axis where $x = 0$

Therefore, we have,

$$0 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$\Rightarrow \quad c_2 e^{r_2 t} = -c_1 e^{r_1 t}$$

$$\Rightarrow \quad c_2 = -c_1 \frac{e^{r_1 t}}{e^{r_2 t}}$$

$$\Rightarrow \quad c = -c_1 e^{(r_1 - r_2)t}$$

Since $r_1 > r_2 \quad \Rightarrow \qquad r_1 - r_2 > 0$

Also $t > 0$

Therefore $e^{(r_1 - r_2)t} > 1$

Now

$$c_2 = -c_1 e^{(r_1 - r_2)t}$$

$$\Rightarrow \qquad |c_2| = |c_1| e^{(r_1 - r_2)t}$$

$$\Rightarrow \qquad |c_2| > |c_1| \quad \text{as } e^{(r_1 - r_2)t} > 1$$

Hence

The condition on the relative magnitudes of c_1 and c_2 under which the graph of x crosses the t-axis at a positive value of t is $|c_2| > |c_1|$

Chapter 17 Second Order Differential Equations 17.3 13E

We know by Kirchoff's aw

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{c}Q = E(t) \quad \text{----------------- (1)}$$

Here $R = 20\Omega$, $L = 1H$, $C = 0.002F$ and $E(t) = 12V$

Then equation (1) becomes

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 500Q = 12 \quad \text{----------------- (2)}$$

This is a non – homogeneous differential equation

Its corresponding homogeneous equation is

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 500Q = 0$$

The characteristic equation is

$$r^2 + 20r + 500 = 0$$

i.e. $$r = \frac{-20 \pm 40i}{2}$$

$$= -10 \pm 20i$$

Then the homogeneous solution is

$$Q_h = e^{-10t}(c_1 \cos 2t + c_2 \sin 20t)$$

For the particular solution, let $Q_p = At + B$ be the particular solution by the method of undetermined coefficients.

Then $\frac{dQ_p}{dt} = A$ and $\frac{d^2Q_p}{dt^2} = 0$

Substituting $\frac{dQ_p}{dt}$ and $\frac{d^2Q_p}{dt^2}$ in equation (2)

$$20A + 500At + 500B = 12$$

i.e. $500A = 0$ and $20A + 500B = 12$

i.e. $A = 0$ and $B = \frac{12}{500} = \frac{3}{125}$

Then $Q_p = \frac{3}{125}$

And therefore the solution of equation (2) is

$$Q = Q_h + Q_p$$

i.e. $$Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) + 3/125$$

By given initial conditions $Q(0) = 0$ and $Q'(0) = 0$

We have $c_1 = \frac{-3}{125}$ and $c_2 = \frac{-3}{250}$

Hence the required solution is

$$\boxed{Q(t) = \frac{-e^{-10t}}{250}(6\cos 20t + 3\sin 20t) + \frac{3}{125}}$$

Chapter 17 Second Order Differential Equations 17.3 14E

We know by Kirchoff's law:

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)$$

Here $L=2H,\ r=24\Omega,\ C=0.005F,\ E=12$

Then $2\frac{d^2Q}{dt^2}+24\frac{dQ}{dt}+200Q=12$

Or $\frac{d^2Q}{dt^2}+12\frac{dQ}{dt}+100Q=6$ --------------- (1)

This is a non – homogeneous differential equation

Its corresponding homogeneous differential equation is

$$\frac{d^2Q}{dt^2}+12\frac{dQ}{dt}+100Q=0$$

The characteristic equation is

$$r^2+12r+100=0$$

i.e. $r=\frac{-12\pm16i}{2}$

$=-6\pm8i$

Then the homogeneous solution is

$$Q_h(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)$$

For the particular solution, let $Q=At+B$ be the particular solution

Then $\frac{dQ}{dt}=A$ and $\frac{d^2Q}{dt^2}=0$

Substituting for Q, dQ/dt and d^2Q/dt^2 in equation (1)

$$12A+100At+100B=6$$

i.e. $A=0,\ B=\frac{6}{100}$

$=\frac{3}{50}$

Then the particular solution is $Q_p=\frac{3}{50}$

And therefore the solution of equation (1) is

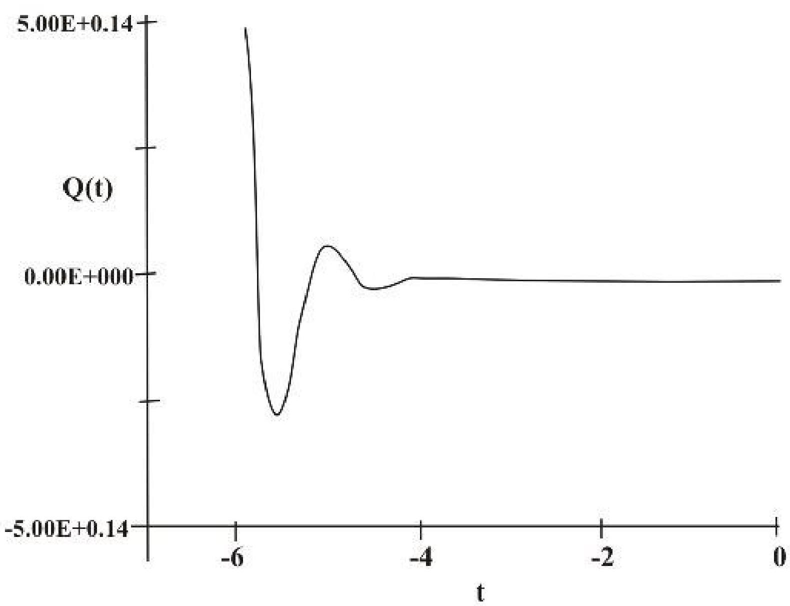
$$Q(t)=e^{-6t}(c_1\cos 8t+c_2\sin 8t)+3/50$$

By the given initial conditions $Q(0)=0.001c$

And $Q'(0)=0$, we have $c_1=\frac{-59}{1000},\ c_2=\frac{-177}{4000}$

Then the particular solution of equation (1) is

$$\boxed{Q(t)=\frac{-59}{4000}e^{-6t}(4\cos 8t+3\sin 8t)+\frac{3}{50}}$$



Chapter 17 Second Order Differential Equations 17.3 15E

We know by Kirchoff's law

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)$$

Here $R=20\,\Omega,\ L=1\,H, C=0.002\,F, E(t)=12\sin 10t$

Then we have $\frac{d^2Q}{dt^2}+20\frac{dQ}{dt}+500Q=12\sin 10t$ ----------- (1)

This is a non – homogeneous differential equation

Its corresponding homogeneous equation is

$$\frac{d^2Q}{dt^2}+20\frac{dQ}{dt}+500Q=0$$

The characteristic equation is

$$r^2+20r+500=0$$

i.e. $r=-10\pm 20i$

Then the homogeneous solution is

$$Q_h(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)$$

For the particular solution by the method of undetermined coefficients
Let $Q=A\cos 10t+B\sin 10t$ be the particular solution

Then $\frac{dQ}{dt}=-10A\sin 10t+10B\cos 10t$

And $\frac{d^2Q}{dt^2}=-100A\cos 10t-100B\sin 10t$

Substituting in equation (1)

$$-100A\cos 10t-100B\sin 10t-200A\sin 10t+200B\cos 10t$$
$$+500A\cos 10t+500B\sin 10t=12\sin 10t$$

Equating coefficients on both sides

$$(-100A+200B+500A)=0$$

And $(-100B-200A+500B)=12$

i.e. $A=\frac{-3}{250},\ B=\frac{6}{250}$

$$=\frac{3}{125}$$

Then the particular solution is

$$Q_p(t)=-\frac{3}{250}\cos 10t+\frac{3}{125}\sin 10t$$

And the solution of equation (1) is $Q(t)=Q_h(t)+Q_p(t)$

i.e. $Q(t)=e^{-10t}(c_1\cos 20t+c_2\sin 20t)-\frac{3}{250}\cos 10t+\frac{3}{125}\sin 10t$

By given initial conditions $Q(0)=0,\ Q'(0)=0$

We have $c_1=\frac{3}{250},\ c_2=\frac{-3}{500}$

Hence the required solution is

$$\boxed{Q(t)=e^{-10t}\left(\frac{3}{250}\cos 20t-\frac{3}{500}\sin 20t\right)-\frac{3}{250}\cos 10t+\frac{3}{125}\sin 10t}$$

We know by Kirchoff's law,

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)$$

Here $L=2H$, $R=24\Omega$, $C=0.005\,F$, $E=12\sin 10t$

Then we have $2\frac{d^2Q}{dt^2}+24\frac{dQ}{dt}+200Q=12\sin 10t$

Or $\frac{d^2Q}{dt^2}+12\frac{dQ}{dt}+100Q=6\sin 10t$ ------------- (1)

This is a non – homogeneous differential equation

Its corresponding homogeneous differential equation is

$$\frac{d^2Q}{dt^2}+12\frac{dQ}{dt}+100Q=0$$

The characteristic equation is

$$r^2+12r+100=0$$

i.e. $r=-6\pm 8i$

Then the homogeneous solution is

$$Q_h(t)=e^{-6t}\left(c_1\cos 8t+c_2\sin 8t\right)$$

For the particular solution, let $Q=A\cos 10t+B\sin 10t$ be the particular solution

Then $\frac{dQ}{dt}=-10A\sin 10t+10B\cos 10t$

And $\frac{d^2Q}{dt^2}=-100A\cos 10t-100B\sin 10t$

Substituting in (1)

$$-100A\cos 10t-100B\sin 10t-120A\sin 10t+120B\cos 10t+100\cos 10t$$
$$+100B\sin 10t=6\sin 10t$$

Equating coefficients on both sides

$$-100A+120B+100A=0$$

And $-100B-120A+100B=6$

i.e. $B=0$ and $A=-\frac{1}{20}$

Then the particular solution is $Q_p=\frac{-1}{10}\cos 10t$

Thus the solution of equation (1) is

$$Q(t)=Q_h(t)+Q_p(t)$$
$$=e^{-6t}\left(c_1\cos 8t+c_2\sin 8t\right)-\frac{1}{20}\cos 10t$$

By given initial conditions $Q(0)=0.001c$, $Q'(0)=0$

Now $Q'(t)=e^{-6t}\left[\left(-6c_1+8c_2\right)\cos 8t+\left(-6c_2-8c_1\right)\sin 8t\right]-\frac{1}{2}\sin 10t$

Then $Q(0)=0.001$, gives $c_1-\frac{1}{20}=0.001$

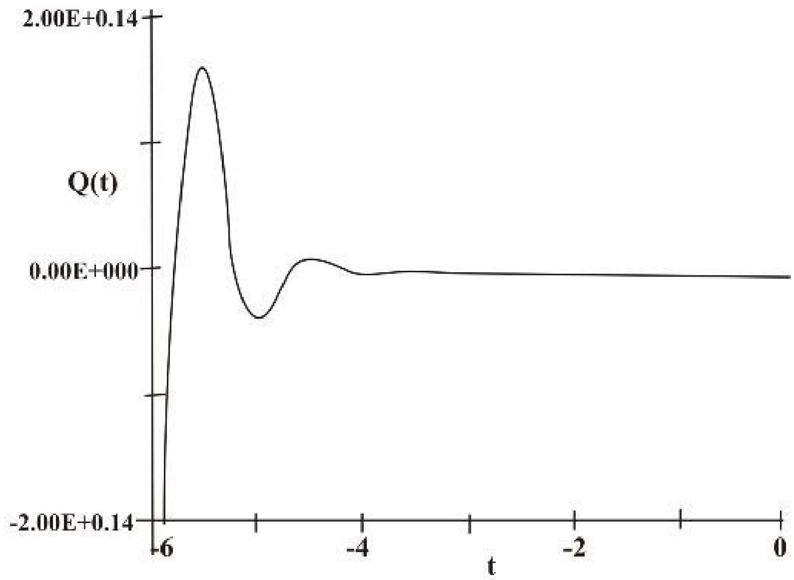
And $Q'(0)=0$ gives $-6c_1+8c_2=0$

i.e. $c_1=0.051$ and $c_2=0.03825$

Hence the charge is given by

$$\boxed{Q(t)=e^{-6t}\left(0.051\cos 8t+0.03825\sin 8t\right)-\frac{1}{20}\cos 10t}$$

(B)



Chapter 17 Second Order Differential Equations 17.3 17E

The given equation is $m\frac{d^2Q}{dt^2}+kx=0$ -------------- (1)

Or $\frac{d^2Q}{dt^2}+\frac{k}{m}x=0$

Its characteristic equation is $r^2+\frac{k}{m}=0$

i.e. $r=\pm\sqrt{\frac{k}{m}}i$

Take $w=\sqrt{\frac{k}{m}}$

Then $r=\pm wi$

Therefore the solution of equation (1) is

$x(t)=c_1\cos wt+c_2\sin wt$

Take $c_1=A\cos\delta,\ c_2=-A\sin\delta$

Where $A=\sqrt{c_1^2+c_2^2}$

And $\delta=\tan^{-1}\left(-\frac{c_2}{c_1}\right)$

Then $x(t)=A\cos wt\cos\delta-A\sin wt\sin\delta$

$=A(\cos wt\cos\delta-\sin wt\sin\delta)$

$=A\cos(wt+\delta)$

Verification: - Now if $x(t)=A\cos(wt+\delta)$ is a solution of equation (1) then it must satisfy this equation. Now $x=A\cos(wt+\delta)$

Then $\frac{dx}{dt}=-Aw\sin(wt+\delta)$

And $\frac{d^2Q}{dt^2}=-Aw^2\cos(wt+\delta)$

Consider $m\frac{d^2Q}{dt^2}+kx$

$=-mAw^2\cos(wt+\delta)+kA\cos(wt+\delta)$

$=A\cos(wt+\delta)\left[-mw^2+k\right]$

$=A\cos(wt+\delta)\left[-m\frac{k}{m}+k\right]$ (As $w=\sqrt{\frac{k}{m}}$)

$=A\cos(wt+\delta)[-k+k]$

$=0$, which is true

Hence the solution of equation (1) can be written as $x(t)=A\cos(wt+\delta)$

Chapter 17 Second Order Differential Equations 17.3 18E

Given $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$

For small values of θ we can use the linear approximation sinθ ˜ θ, therefore we can approximate the differential equation as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

a)

The equation of motion of a pendulum with length 1 m if θ is initially 0.2 rad and the initial angular velocity dθ/dt =1 rad/s

Using L=1 and g=9.8, the differential equation becomes

$$\frac{d^2\theta}{dt^2} + 9.8\theta = 0.$$

The auxiliary equation is r2+9.8=0

We solve for r

$$r = \pm i\sqrt{9.8}$$

so the solution of the complementary equation is

$$\theta(t) = c_1\cos(\sqrt{9.8}\,t) + c_2\sin(\sqrt{9.8}\,t)$$

Imposing the initial condition $\theta(0) = 0.2$ we get

$$\theta(0) = c_1\cos(\sqrt{9.8}\cdot 0) + c_2\sin(\sqrt{9.8}\cdot 0)$$

$$\theta(0) = c_1$$

$$c_1 = 0.2$$

Imposing the initial condition $\frac{d\theta}{dt}|_{t=0} = 1$

$$\theta'(t) = -\sqrt{9.8}\,c_1\sin(\sqrt{9.8}\,t) + \sqrt{9.8}\,c_2\cos(\sqrt{9.8}\,t)$$

$$\theta'(0) = -\sqrt{9.8}\,c_1\sin(\sqrt{9.8}\cdot 0) + \sqrt{9.8}\,c_2\cos(\sqrt{9.8}\cdot 0)$$

$$\theta'(0) = \sqrt{9.81}\,c_2$$

$$\sqrt{9.81}\,c_2 = 1$$

$$c_2 = \frac{1}{\sqrt{9.81}}$$

so the equation is

$$\theta\left(t\right) = 0.2\cos\left(\sqrt{9.8}\,t\right) + \frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right)$$

b) The maximum angle from the vertical

$$\theta'(t) = -0.2\sqrt{9.81}\sin(\sqrt{9.81}\,t) + \cos(\sqrt{9.81}\,t)$$

For the critical numbers we set θ'(t) =0

$$-0.2\sqrt{9.8}\sin(\sqrt{9.8}\,t) + \cos(\sqrt{9.8}\,t) = 0$$

$$0.2\sqrt{9.8}\sin(\sqrt{9.8}\,t) = \cos(\sqrt{9.8}\,t)$$

$$\tan\left(\sqrt{9.8}\,t\right) = \frac{5}{\sqrt{9.8}}$$

We solve for t and we get that the critical numbers are

$t = \frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right) + \frac{n}{\sqrt{9.8}}\pi$ where n is any integer.

The maximum angle from the vertical

$\theta\left(\frac{1}{\sqrt{9.8}}\tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)\right)$ ~0.377 radians

θ ~ 21.7°

c) The period of the pendulum

From the result of part (b) the critical numbers of θ(t) are spaced $\frac{\pi}{\sqrt{9.8}}$ apart.

The time between succssive maximum values is $2\left(\frac{\pi}{\sqrt{9.8}}\right)$.

Thus the period of the pendulum is $T = \frac{2\pi}{\sqrt{9.8}}$ ~2.007 seconds.

d) When will the pendulum first be vertical?

We need the value of t when θ(t)=0

$$0.2\cos\left(\sqrt{9.8}\,t\right) + \frac{1}{\sqrt{9.8}}\sin\left(\sqrt{9.8}\,t\right) = 0$$

$$\tan(\sqrt{9.8}\,t) = -0.2\sqrt{9.8}$$

$$t = \frac{1}{\sqrt{9.8}}\left[\tan^{-1}\left(-0.2\sqrt{9.8}\right) + \pi\right]$$

t ~ 0.825 seconds.

e) the angular velocity when the pendulum is vertical

$$\theta'(t) = -0.2\sqrt{9.81}\sin(\sqrt{9.81}\,t) + \cos(\sqrt{9.81}\,t)$$

$$\theta'(0.825) = -0.2\sqrt{9.81}\sin(\sqrt{9.81}\cdot 0.825) + \cos(\sqrt{9.81}\cdot 0.825)$$

θ'(0.825) ~ -1.180 rad/s.