6

Longitudinal Waves

In deriving the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

in Chapter 5, we used the example of a transverse wave and continued to discuss waves of this type on a vibrating string. In this chapter we consider longitudinal waves, waves in which the particle or oscillator motion is in the same direction as the wave propagation. Longitudinal waves propagate as sound waves in all phases of matter, plasmas, gases, liquids and solids, but we shall concentrate on gases and solids. In the case of gases, limitations of thermodynamic interest are imposed; in solids the propagation will depend on the dimensions of the medium. Neither a gas nor a liquid can sustain the transverse shear necessary for transverse waves, but a solid can maintain both longitudinal and transverse oscillations.

Sound Waves in Gases

Let us consider a fixed mass of gas, which at a pressure P_0 occupies a volume V_0 with a density ρ_0. These values define the equilibrium state of the gas which is disturbed, or deformed, by the compressions and rarefactions of the sound waves. Under the influence of the sound waves

$$\text{the pressure } P_0 \text{ becomes } P = P_0 + p$$
$$\text{the volume } V_0 \text{ becomes } V = V_0 + v$$

and

$$\text{the density } \rho_0 \text{ becomes } \rho = \rho_0 + \rho_d.$$

The excess pressure p_m is the maximum pressure amplitude of the sound wave and p is an alternating component superimposed on the equilibrium gas pressure P_0.

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The fractional change in volume is called the *dilatation*, written $v/V_0 = \delta$, and the fractional change of density is called the *condensation*, written $\rho_d/\rho_0 = s$. The values of δ and s are $\approx 10^{-3}$ for ordinary sound waves, and a value of $p_m = 2 \times 10^{-5}\,\mathrm{N\,m^{-2}}$ (about 10^{-10} of an atmosphere) gives a sound wave which is still audible at 1000 Hz. Thus, the changes in the medium due to sound waves are of an extremely small order and define limitations within which the wave equation is appropriate.

The fixed mass of gas is equal to

$$\rho_0 V_0 = \rho V = \rho_0 V_0 (1 + \delta)(1 + s)$$

so that $(1 + \delta)(1 + s) = 1$, giving $s = -\delta$ to a very close approximation. The elastic property of the gas, a measure of its compressibility, is defined in terms of its *bulk modulus*

$$B = -\frac{\mathrm{d}P}{\mathrm{d}V/V} = -V\frac{\mathrm{d}P}{\mathrm{d}V}$$

the difference in pressure for a fractional change in volume, a volume increase with fall in pressure giving the negative sign. The value of B depends on whether the changes in the gas arising from the wave motion are adiabatic or isothermal. They must be thermodynamically reversible in order to avoid the energy loss mechanisms of diffusion, viscosity and thermal conductivity. The complete absence of these random, entropy generating processes defines an adiabatic process, a thermodynamic cycle with a 100% efficiency in the sense that none of the energy in the wave, potential or kinetic, is lost. In a sound wave such thermodynamic concepts restrict the excess pressure amplitude; too great an amplitude raises the local temperature in the gas at the amplitude peaks and thermal conductivity removes energy from the wave system. Local particle velocity gradients will also develop, leading to diffusion and viscosity.

Using a constant value of the adiabatic bulk modulus limits sound waves to small oscillations since the total pressure $P = P_0 + p$ is taken as constant; larger amplitudes lead to non-linear effects and shock waves, which we shall discuss separately in Chapter 15.

All adiabatic changes in the gas obey the relation $\mathrm{PV}^\gamma =$ constant, where γ is the ratio of the specific heats at constant pressure and volume, respectively.

Differentiation gives

$$V^\gamma\,\mathrm{d}P + \gamma P V^{\gamma-1}\,\mathrm{d}V = 0$$

or

$$-V\frac{\mathrm{d}P}{\mathrm{d}V} = \gamma P = B_a \text{ (where the subscript } a \text{ denotes adiabatic)}$$

so that the elastic property of the gas is γP, considered to be constant. Since $P = P_0 + p$, then $\mathrm{d}P = p$, the excess pressure, giving

$$B_a = -\frac{p}{v/V_0} \quad \text{or} \quad p = -B_a\delta = B_a s$$

In a sound wave the particle displacements and velocities are along the x-axis and we choose the co-ordinate η to define the displacement where $\eta(x, t)$.

In obtaining the wave equation we consider the motion of an element of the gas of thickness Δx and unit cross section. Under the influence of the sound wave the behaviour

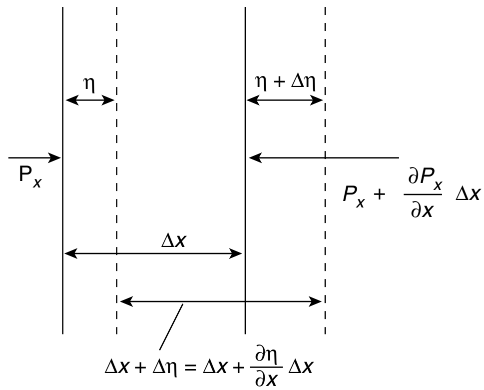


Figure 6.1 Thin element of gas of unit cross-section and thickness Δx displaced an amount η and expanded by an amount $(\delta\eta/\partial x)\Delta x$ under the influence of a pressure differene $-(\partial P_x/\partial x)\Delta x$

of this element is shown in Figure 6.1. The particles in the layer x are displaced a distance η and those at $x + \Delta x$ are displaced a distance $\eta + \Delta\eta$, so that the increase in the thickness Δx of the element of unit cross section (which therefore measures the increase in volume) is

$$\Delta\eta = \frac{\partial\eta}{\partial x}\Delta x$$

and

$$\delta = \frac{v}{V_0} = \left(\frac{\partial\eta}{\partial x}\right)\Delta x/\Delta x = \frac{\partial\eta}{\partial x} = -s$$

where $\partial\eta/\delta x$ is called the *strain.*

The medium is deformed because the pressures along the x-axis on either side of the thin element are not in balance (Figure 6.1). The net force acting on the element is given by

$$\begin{aligned} P_x - P_{x+\Delta x} &= \left[P_x - \left(P_x + \frac{\partial P_x}{\partial x}\Delta x\right)\right] \\ &= -\frac{\partial P_x}{\partial x}\Delta x = -\frac{\partial}{\partial x}(P_0 + p)\Delta x = -\frac{\partial p}{\partial x}\Delta x \end{aligned}$$

The mass of the element is $\rho_0\Delta x$ and its acceleration is given, to a close approxmation, by $\partial^2\eta/\mathrm{d}t^2$.

From Newton's Law we have

$$-\frac{\partial p}{\partial x}\Delta x = \rho_0\Delta x\frac{\partial^2\eta}{\partial t^2}$$

where

$$p = -B_a\delta = -B_a\frac{\partial\eta}{\partial x}$$

so that

$$-\frac{\partial p}{\partial x} = B_a\frac{\partial^2\eta}{\partial x^2}, \quad \text{giving} \quad B_a\frac{\partial^2\eta}{\partial x^2} = \rho_0\frac{\partial^2\eta}{\partial t^2}$$

But $B_a/\rho_0 = \gamma P/\rho_0$ is the ratio of the elasticity to the inertia or density of the gas, and this ratio has the dimensions

$$\frac{\text{force}}{\text{area}}\cdot\frac{\text{volume}}{\text{mass}} = (\text{velocity})^2, \quad \text{so} \quad \frac{\gamma P}{\rho_0} = c^2$$

where c is the sound wave velocity.

Thus

$$\frac{\partial^2\eta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2}$$

is the wave equation. Writing η_m as the maximum amplitude of displacement we have the following expressions for a wave in the *positive x-direction*:

$$\eta = \eta_m\,\mathrm{e}^{\mathrm{i}(\omega t - kx)} \quad \dot\eta = \frac{\partial\eta}{\partial t} = \mathrm{i}\omega\eta$$

$$\delta = \frac{\partial\eta}{\partial x} = -\mathrm{i}k\eta = -s \quad (\text{so } s = \mathrm{i}k\eta)$$

$$p = B_a s = \mathrm{i}B_a k\eta$$

The phase relationships between these parameters (Figure 6.2a) show that when the wave is in the positive x-direction, the excess pressure p, the fractional density increase s and the particle velocity $\dot\eta$ are all $\pi/2$ rad in phase ahead of the displacement η, whilst the volume change (π rad out of phase with the density change) is $\pi/2$ rad behind the displacement. These relationships no longer hold when the wave direction is reversed (Figure 6.2b); for a *wave in the negative x-direction*

$$\eta = \eta_m\,\mathrm{e}^{\mathrm{i}(\omega t + kx)} \quad \dot\eta = \frac{\partial\eta}{\partial t} = \mathrm{i}\omega\eta$$

$$\delta = \frac{\partial\eta}{\partial x} = -\mathrm{i}k\eta = -s \quad (\text{so } s = \mathrm{i}k\eta)$$

$$p = B_a s = -\mathrm{i}B_a k\eta$$

In both waves the particle displacement η is measured in the positive x-direction and the thin element Δx of the gas oscillates about the value $\eta = 0$, which defines its central position. For a wave in the positive x-direction the value $\eta = 0$, with $\dot\eta$ a maximum in the positive x-direction, gives a *maximum positive excess pressure* (*compression*) with a maximum condensation s_m (maximum density) and a minimum volume. For a wave in the

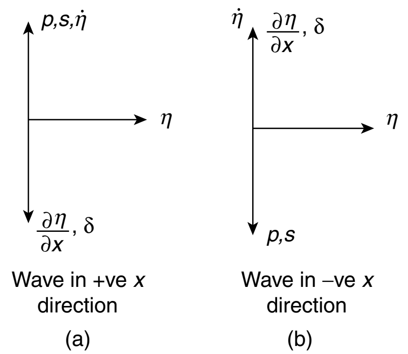


Figure 6.2 Phase relationships between the particle displacement η, particle velocity $\dot{\eta}$, excess pressure p and condensation $s = -\delta$ (the dilatation) for waves travelling in the positive and negative x directions. The displacement η is taken in the positive x direction for both waves

negative x-direction, the same value $\eta = 0$, with $\dot{\eta}$ a maximum in the positive x-direction, gives a *maximum negative excess pressure* (*rarefaction*), a maximum volume and a minimum density. To produce a compression in a wave moving in the negative x-direction the particle velocity $\dot{\eta}$ must be a maximum in the negative x-direction at $\eta = 0$. This distinction is significant when we are defining the impedance of the medium to the waves. A change of sign is involved with a change of direction—a convention we shall also have to follow when discussing the waves of Chapters 7 and 8.

Energy Distribution in Sound Waves

The kinetic energy in the sound wave is found by considering the motion of the individual gas elements of thickness Δx.

Each element will have a kinetic energy per unit cross section

$$\Delta E_{\text{kin}} = \tfrac{1}{2}\rho_0\,\Delta x\,\dot{\eta}^2$$

where $\dot{\eta}$ will depend upon the position x of the element. The average value of the kinetic energy density is found by taking the value of $\dot{\eta}^2$ averaged over a region of n wavelengths.

Now

$$\dot{\eta} = \dot{\eta}_m \sin\frac{2\pi}{\lambda}(ct - x)$$

so that

$$\overline{\dot{\eta}^2} = \frac{\dot{\eta}_m^2 \int_0^{n\lambda} \sin^2 2\pi(ct - x)/\lambda\,\Delta x}{n\lambda} = \tfrac{1}{2}\dot{\eta}_m^2$$

so that the *average kinetic energy density* in the medium is

$$\overline{\Delta E}_{\text{kin}} = \tfrac{1}{4}\rho_0\dot{\eta}_m^2 = \tfrac{1}{4}\rho_0\omega^2\eta_m^2$$

(a simple harmonic oscillator of maximum amplitude a has an average kinetic energy over one cycle of $\frac{1}{4}m\omega^2a^2$).

The potential energy density is found by considering the work $P\,\mathrm{d}V$ done on the fixed mass of gas of volume V_0 during the adiabatic changes in the sound wave. This work is expressed for the complete cycle as

$$\Delta E_{\text{pot}} = -\int P\mathrm{d}V = -\frac{-1}{2\pi}\int_0^{2\pi} pv\mathrm{d}(\omega t) = \frac{p_m v_m}{2} : \left[\frac{p}{p_m} = \frac{-v}{v_m} = \sin(\omega t - kx)\right]$$

The negative sign shows that the potential energy change is positive in both a compression (p positive, $\mathrm{d}V$ negative) and a rarefaction (p negative, $\mathrm{d}V$ positive) Figure 6.3.

The condensation

$$s = \frac{-\int \mathrm{d}v}{V_0} = \frac{-v}{V_0} = -\delta$$

we write

$$\frac{s}{s_m} = \frac{-\delta}{\delta_m} = \sin(\omega t - kx) \text{ and } -v = V_0 s$$

which, with

$$p = B_a s$$

gives

$$\Delta E_{\text{pot}} = \frac{-1}{2\pi}\int_0^{2\pi} pv\mathrm{d}(\omega t) = \frac{B_a V_0}{2\pi}\int_0^{2\pi} s^2\mathrm{d}(\omega t)$$

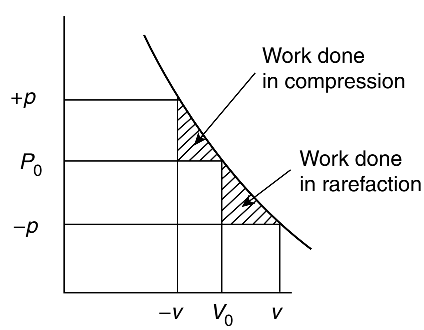


Figure 6.3 Shaded triangles show that potential energy $\frac{pv}{2} = \frac{p_m v_m}{4}$ gained by gas in compression equals that gained in rarefaction when both p and v change sign

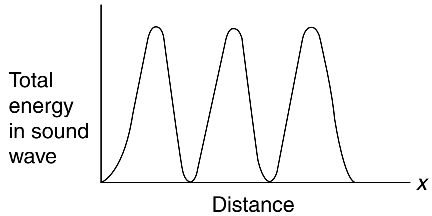


Figure 6.4 Energy distribution in space for a sound wave in a gas. Both potential and kinetic energies are at a maximum when the particle velocity $\dot{\eta}$ is a maximum and zero at $\dot{\eta} = 0$

where $s = -\delta$ and the thickness Δx of the element of unit cross section represents its volume V_0.

Now

$$\eta = \eta_m \, \mathrm{e}^{\mathrm{i}(\omega t \pm kx)}$$

so that

$$\delta = \frac{\partial \eta}{\partial x} = \pm \frac{1}{c} \frac{\partial \eta}{\partial t}, \quad \text{where} \quad c = \frac{\omega}{k}$$

Thus

$$\Delta E_{\text{pot}} = \frac{1}{2} \frac{B_a}{c^2} \dot{\eta}^2 \Delta x = \frac{1}{2} \rho_0 \dot{\eta}^2 \Delta x$$

and its average value over $n\lambda$ gives the potential energy density

$$\overline{\Delta E}_{\text{pot}} = \tfrac{1}{4} \rho_0 \dot{\eta}_m^2$$

We see that the average values of the kinetic and potential energy density in the sound wave are equal, but more important, since the value of each for the element Δx is $\frac{1}{2}\rho_0 \dot{\eta}^2 \Delta x$, we observe that the element possesses maximum (or minimum) potential and kinetic energy at the same time. A compression or rarefaction produces a maximum in the energy of the element since the value $\dot{\eta}$ governs the energy content. Thus, the energy in the wave is distributed in the wave system with distance as shown in Figure 6.4. Note that this distribution is non-uniform with distance unlike that for a transverse wave.

Intensity of Sound Waves

This is a measure of the energy flux, the rate at which energy crosses unit area, so that it is the product of the energy density (kinetic plus potential) and the wave velocity c. Normal sound waves range in intensity between 10^{-12} and 1 W m^{-2}, extremely low levels which testify to the sensitivity of the ear. The roar of a large football crowd greeting a goal will just about heat a cup of coffee.

The intensity may be written

$$I = \tfrac{1}{2}\rho_0 c\dot{\eta}_m^2 = \tfrac{1}{2}\rho_0 c\omega^2\eta_m^2 = \rho_0 c\dot{\eta}_{\text{rms}}^2 = p_{\text{rms}}^2/\rho_0 c = p_{\text{rms}}\dot{\eta}_{\text{rms}}$$

A commonly used standard of sound intensity is given by

$$I_0 = 10^{-2}\,\text{W m}^{-2}$$

which is about the level of the average conversational tone between two people standing next to each other. Shouting at this range raises the intensity by a factor of 100 and in the range 100 I_0 to 1000 I_0 (10 W m^{-2}) the sound is painful.

Whenever the sound intensity increases by a factor of 10 it is said to have increased by 1 B so the dynamic range of the ear is about 12 B. An intensity increase by a factor of

$$10^{0.1} = 1 \cdot 26$$

increases the intensity by 1 dB, a change of loudness which is just detected by a person with good hearing. dB is a decibel.

We see that the product $\rho_0 c$ appears in most of the expressions for the intensity; its significance becomes apparent when we define the impedance of the medium to the waves as the

$$\textit{Specific Acoustic Impedance} = \frac{\text{excess pressure}}{\text{particle velocity}} = \frac{p}{\dot{\eta}}$$

(the ratio of a force per unit area to a velocity).

Now, for a wave in the positive x-direction.

$$p = B_a s = \mathrm{i}B_a k\eta \quad \text{and} \quad \dot{\eta} = \mathrm{i}\omega\eta$$

so that,

$$\frac{p}{\dot{\eta}} = \frac{B_a k}{\omega} = \frac{B_a}{c} = \rho_o c$$

Thus, the acoustic impedance presented by the medium to these waves, as in the case of the transverse waves on the string, is given by the product of the density and the wave velocity and is governed by the elasticity and inertia of the medium. For a wave in the negative x-direction, the specific acoustic impedance

$$\frac{p}{\dot{\eta}} = -\frac{\mathrm{i}B_a k\eta}{\mathrm{i}\omega\eta} = -\rho_0 c$$

with a change of sign because of the changed phase relationship.

The units of $\rho_0 c$ are normally stated as kg m^{-2} s^{-1} in books on practical acoustics; in these units air has a specific acoustic impedance value of 400, water a value of 1.45×10^6 and steel a value of 3.9×10^7. These values will become more significant when we use them later in examples on the reflection and transmission of sound waves.

Although the specific acoustic impedance $\rho_0 c$ is a real quantity for plane sound waves, it has an added reactive component $\mathrm{i}k/r$ for spherical waves, where r is the distance travelled by the wavefront. This component tends to zero with increasing r as the spherical wave becomes effectively plane.

(Problems 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8)

Longitudinal Waves in a Solid

The velocity of longitudinal waves in a solid depends upon the dimensions of the specimen in which the waves are travelling. If the solid is a thin bar of finite cross section the analysis for longitudinal waves in a gas is equally valid, except that the bulk modulus B_a is replaced by Young's modulus Y, the ratio of the longitudinal stress in the bar to its longitudinal strain.

The wave equation is then

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2}, \quad \text{with} \quad c^2 = \frac{Y}{\rho}$$

A longitudinal wave in a medium compresses the medium and distorts it laterally. Because a solid can develop a shear force in any direction, such a lateral distortion is accompanied by a transverse shear. The effect of this upon the wave motion in solids of finite cross section is quite complicated and has been ignored in the very thin specimen above. In bulk solids, however, the longitudinal and transverse modes may be considered separately.

We have seen that the longitudinal compression produces a strain $\partial\eta/\partial x$; the accompanying lateral distortion produces a strain $\partial\beta/\partial y$ (of opposite sign to $\partial\eta/\partial x$ and perpendicular to the x-direction).

Here β is the displacement in the y-direction and is a function of both x and y. The ratio of these strains

$$-\frac{\partial\beta}{\partial y}\bigg/\frac{\partial\eta}{\partial x} = \sigma$$

is known as Poisson's ratio and is expressed in terms of Lamé's elastic constants λ and μ for a solid as

$$\sigma = \frac{\lambda}{2(\lambda+\mu)} \quad \text{where} \quad \lambda = \frac{\sigma Y}{(1+\sigma)(1-2\sigma)}$$

These constants are always positive, so that $\sigma < \frac{1}{2}$, and is commonly $\approx \frac{1}{3}$. In terms of these constants Young's modulus becomes

$$Y = (\lambda + 2\mu - 2\lambda\sigma)$$

The constant μ is the transverse coefficient of rigidity; that is, the ratio of the transverse stress to the transverse strain. It plays the role of the elasticity in the propagation of pure

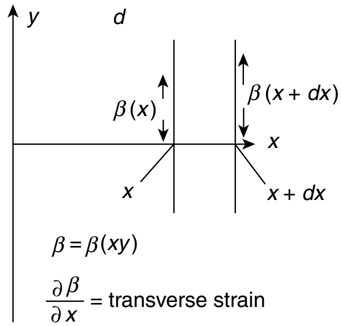


Figure 6.5 Shear in a bulk solid producing a transverse wave. The transverse shear strain is $\partial\beta/\partial x$ and the transverse shear stress is $\mu\,\partial\beta/\partial x$, where μ is the shear modulus of rigidity

transverse waves in a bulk solid which Young's modulus plays for longitudinal waves in a thin specimen. Figure 6.5 illustrates the shear in a transverse plane wave, where the transverse strain is defined by $\partial\beta/\partial x$. The transverse stress at x is therefore $T_x = \mu\,\partial\beta/\partial x$. The equation of transverse motion of the thin element $\mathrm{d}x$ is then given by

$$T_{x+\mathrm{d}x} - T_{\mathrm{d}x} = \rho\,\mathrm{d}x\ddot{y}$$

where ρ is the density, or

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial\beta}{\partial x}\right) = \rho\ddot{y}$$

but $\ddot{y} = \partial^2\beta/\partial t^2$, hence

$$\frac{\partial^2\beta}{\partial x^2} = \frac{\rho}{\mu}\frac{\partial^2\beta}{\partial t^2}$$

the wave equation with a velocity given by $c^2 = \mu/\rho$.

The effect of the transverse rigidity μ is to stiffen the solid and increase the elastic constant governing the propagation of longitudinal waves. In a bulk solid the velocity of these waves is no longer given by $c^2 = Y/\rho$, but becomes

$$c^2 = \frac{\lambda + 2\mu}{\rho}$$

Since Young's modulus $Y = \lambda + 2\mu - 2\lambda\sigma$, the elasticity is increased by the amount $2\lambda\sigma \approx \lambda$, so that longitudinal waves in a bulk solid have a higher velocity than the same waves along a thin specimen.

In an isotropic solid, where the velocity of propagation is the same in all directions, the concept of a bulk modulus, used in the discussion on waves in gases, holds equally well. Expressed in terms of Lamé's elastic constants the bulk modulus for a solid is written

$$B = \lambda + \tfrac{2}{3}\mu = Y[3(1 - 2\sigma)]^{-1}$$

the longitudinal wave velocity for a bulk solid becomes

$$c_L = \left(\frac{B + (4/3)\mu}{\rho}\right)^{1/2}$$

whilst the transverse velocity remains as

$$c_T = \left(\frac{\mu}{\rho}\right)^{1/2}$$

Application to Earthquakes

The values of these velocities are well known for seismic waves generated by earthquakes. Near the surface of the earth the longitudinal waves have a velocity of 8 km s^{-1} and the transverse waves travel at 4.45 km s^{-1}. The velocity of the longitudinal waves increases with depth until, at a depth of about 1800 miles, no waves are transmitted because of a discontinuity and severe mismatch of impedances associated with the fluid core.

At the surface of the earth the transverse wave velocity is affected by the fact that stress components directed through the surface are zero there and these waves, known as Rayleigh Waves, travel with a velocity given by

$$c = f(\sigma)\left(\frac{\mu}{\rho}\right)^{1/2}$$

where

$$f(\sigma) = 0.9194 \quad \text{when} \quad \sigma = 0 \cdot 25$$

and

$$f(\sigma) = 0.9553 \quad \text{when} \quad \sigma = 0 \cdot 5$$

The energy of the Rayleigh Waves is confined to two dimensions; their amplitude is often much higher than that of the three dimensional longitudinal waves and therefore they are potentially more damaging.

In an earthquake the arrival of the fast longitudinal waves is followed by the Rayleigh Waves and then by a complicated pattern of reflected waves including those affected by the stratification of the earth's structure, known as Love Waves.

(Problem 6.9)

Longitudinal Waves in a Periodic Structure

Lamé's elastic constants, λ and μ, which are used to define such macroscopic quantities as Young's modulus and the bulk modulus, are themselves determined by forces which operate over interatomic distances. The discussion on transverse waves in a periodic structure has already shown that in a one-dimensional array representing a crystal lattice a stiffness $s = T/a$ dyn cm^{-1} can exist between two atoms separated by a distance a.

When the waves along such a lattice are longitudinal the atomic displacements from equilibrium are represented by η (Figure 6.6). An increase in the separation between two atoms from a to $a + \eta$ gives a strain $\varepsilon = \eta/a$, and a stress normal to the face area a^2 of a unit cell in a crystal equal to $s\eta/a^2 = s\varepsilon/a$, a force per unit area.

Now Young's modulus is the ratio of this longitudinal stress to the longitudinal strain, so that $Y = s\varepsilon/\varepsilon a$ or $s = Ya$. The longitudinal vibration frequency of the atoms of mass m connected by stiffness constants s is given, very approximately by

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} \approx \frac{1}{2\pi a}\sqrt{\frac{Y}{\rho}} \approx \frac{c_0}{2\pi a}$$

where $m = \rho a^3$ and c_0 is the velocity of sound in a solid. The value of $c_0 \approx 5 \times 10^3$ m s^{-1}, and $a \approx 2 \times 10^{-10}$ m, so that $\nu \approx 3 \times 10^{12}$ Hz, which is almost the same value as the frequency of the transverse wave in the infrared region of the electromagnetic spectrum. The highest ultrasonic frequency generated so far is about a factor of 10 lower than $\nu = c_0/2\pi a$. At frequencies $\approx 5 \times 10^{12}$ to 10^{13} Hz many interesting experimental results must be expected. A more precise mathematical treatment yields the same equation of motion for the r th particle as in the transverse wave; namely

$$m\ddot{\eta}_r = s(\eta_{r+1} + \eta_{r-1} - 2\eta_r)$$

where $s = T/a$ and

$$\eta_r = \eta_{\max}\,\mathrm{e}^{\mathrm{i}(\omega t - kra)}$$

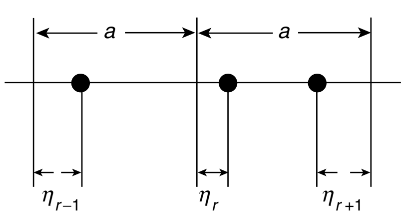


Figure 6.6 Displacement of atoms in a linear array due to a longitudinal wave in a crysal structure

The results are precisely the same as in the case of transverse waves and the shape of the dispersion curve is also similar. The maximum value of the cut-off frequency ω_m is, however, higher for the longitudinal than for the transverse waves. This is because the longitudinal elastic constant Y is greater than the transverse constant μ; that is, the force required for a given displacement in the longitudinal direction is greater than that for the same displacement in the transverse direction.

Reflection and Transmission of Sound Waves at Boundaries

When a sound wave meets a boundary separating two media of different acoustic impedances two boundary conditions must be met in considering the reflection and transmission of the wave. They are that

(i) the particle velocity $\dot{\eta}$

and

(ii) the acoustic excess pressure p

are both continuous across the boundary. Physically this ensures that the two media are in complete contact everywhere across the boundary.

Figure 6.7 shows that we are considering a plane sound wave travelling in a medium of specific acoustic impedance $Z_1 = \rho_1 c_1$ and meeting, at normal incidence, an infinite plane boundary separating the first medium from another of specific acoustic impedance $Z_2 = \rho_2 c_2$. If the subscripts i, r and t denote incident, reflected and transmitted respectively, then the boundary conditions give

$$\eta_{\mathrm{i}} + \dot{\eta}_{\mathrm{r}} = \dot{\eta}_{\mathrm{t}} \tag{6.1}$$

and

$$p_{\mathrm{i}} + p_{\mathrm{r}} = p_{\mathrm{t}} \tag{6.2}$$

For the incident wave $p_{\mathrm{i}} = \rho_1 c_1 \dot{\eta}_{\mathrm{i}}$ and for the reflected wave $p_{\mathrm{r}} = -\rho_1 c_1 \dot{\eta}_{\mathrm{r}}$, so equation (6.2) becomes

$$\rho_1 c_1 \dot{\eta}_{\mathrm{i}} - \rho_1 c_1 \dot{\eta}_{\mathrm{r}} = \rho_2 c_2 \dot{\eta}_{\mathrm{t}}$$

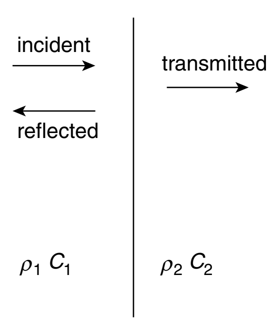


Figure 6.7 Incident, reflected and transmitted sound waves at a plane boundary between media of specific acoustic impedances $\rho_1 c_1$ and $\rho_2 c_2$

or

$$Z_1\dot{\eta}_\mathrm{i} - Z_1\dot{\eta}_\mathrm{r} = Z_2\dot{\eta}_\mathrm{t} \tag{6.3}$$

Eliminating $\dot{\eta}_\mathrm{t}$ from (6.1) and (6.3) gives

$$\frac{\dot{\eta}_\mathrm{r}}{\dot{\eta}_\mathrm{i}} = \frac{\omega\eta_\mathrm{r}}{\omega\eta_\mathrm{i}} = \frac{\eta_\mathrm{r}}{\eta_\mathrm{i}} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

Eliminatiing $\dot{\eta}_\mathrm{r}$ from (6.1) and (6.3) gives

$$\frac{\dot{\eta}_\mathrm{t}}{\dot{\eta}_\mathrm{i}} = \frac{\eta_\mathrm{t}}{\eta_\mathrm{i}} = \frac{2Z_1}{Z_1 + Z_2}$$

Now

$$\frac{p_\mathrm{r}}{p_\mathrm{i}} = -\frac{Z_1\dot{\eta}_\mathrm{r}}{Z_1\dot{\eta}_\mathrm{i}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} = -\frac{\dot{\eta}_\mathrm{r}}{\dot{\eta}_\mathrm{i}}$$

and

$$\frac{p_\mathrm{t}}{p_\mathrm{i}} = \frac{Z_2\dot{\eta}_\mathrm{t}}{Z_1\dot{\eta}_\mathrm{i}} = \frac{2Z_2}{Z_1 + Z_2}$$

We see that if $Z_1 > Z_2$ the incident and reflected particle velocities are in phase, whilst the incident and reflected acoustic pressures are out of phase. The superposition of incident and reflected velocities which are in phase leads to a cancellation of pressure (a pressure node in a standing wave system). If $Z_1 < Z_2$ the pressures are in phase and the velocities are out of phase.

The transmitted particle velocity and acoustic pressure are always in phase with their incident counterparts.

At a rigid wall, where Z_2 is infinite, the velocity $\dot{\eta} = 0 = \dot{\eta}_\mathrm{i} + \dot{\eta}_\mathrm{r}$, which leads to a doubling of pressure at the boundary. (See Summary on p. 546.)

Reflection and Transmission of Sound Intensity

The intensity coefficients of reflection and transmission are given by

$$\frac{I_\mathrm{r}}{I_\mathrm{i}} = \frac{Z_1(\dot{\eta}_\mathrm{r}^2)_\mathrm{rms}}{Z_1(\dot{\eta}_\mathrm{i}^2)_\mathrm{rms}} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

and

$$\frac{I_\mathrm{t}}{I_\mathrm{i}} = \frac{Z_2(\dot{\eta}_\mathrm{t}^2)_\mathrm{rms}}{Z_1(\dot{\eta}_\mathrm{i}^2)_\mathrm{rms}} = \frac{Z_2}{Z_1}\left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2}$$

The conservation of energy gives

$$\frac{I_\mathrm{r}}{I_\mathrm{i}} + \frac{I_\mathrm{t}}{I_\mathrm{i}} = 1 \quad \text{or} \quad I_\mathrm{i} = I_\mathrm{t} + I_\mathrm{r}$$

The great disparity between the specific acoustic impedance of air on the one hand and water or steel on the other leads to an extreme mismatch of impedances when the transmission of acoustic energy between these media is attempted.

There is an almost total reflection of sound wave energy at an air-water interface, independent of the side from which the wave approaches the boundary. Only 14% of acoustic energy can be transmitted at a steel-water interface, a limitation which has severe implications for underwater transmission and detection devices which rely on acoustics.

(Problems 6.10, 6.11, 6.12, 6.13, 6.14, 6.15, 6.16, 6.17)

Problem 6.1
Show that in a gas at temperature T the average thermal velocity of a molecule is approximatley equal to the velocity of sound.

Problem 6.2
The velocity of sound in air of density 1.29 kg m^{-3} may be taken as 330 m s^{-1}. Show that the acoustic pressure for the painful sound of 10 W $\mathrm{m}^{-2} \approx 6.5 \times 10^{-4}$ of an atmosphere.

Problem 6.3
Show that the displacement amplitude of an air molecule at a painful sound level of 10 W m^{-2} at 500 Hz $\approx 6.9 \times 10^{-5}$ m.

Problem 6.4
Barely audible sound in air has an intensity of $10^{-10} I_0$. Show that the displacement amplitude of an air molecule for sound at this level at 500 Hz is $\approx 10^{-10}$ m; that is, about the size of the molecular diameter.

Problem 6.5
Hi-fi equipment is played very loudly at an intensity of $100 I_0$ in a small room of cross section 3 m $\times$ 3 m. Show that this audio output is about 10 W.

Problem 6.6
Two sound waves, one in water and one in air, have the same intensity. Show that the ratio of their pressure amplitudes (p water/p air) is about 60. When the pressure amplitudes are equal show that the intensity ratio is $\approx 3 \times 10^{-2}$.

Problem 6.7
A spring of mass m, stiffness s and length L is stretched to a length $L + l$. When longitudinal waves propagate along the spring the equation of motion of a length $\mathrm{d}x$ may be written

$$\rho\,\mathrm{d}x \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial F}{\partial x}\,\mathrm{d}x$$

where ρ is the mass per unit length of the spring, η is the longitudinal displacement and F is the restoring force. Derive the wave equation to show that the wave velocity v is given by

$$v^2 = s(L + l)/\rho$$

Problem 6.8

In Problem 1.10 we showed that a mass M suspended by a spring of stiffness s and mass m oscillated simple harmonically at a frequency given by

$$\omega^2 = \frac{s}{M + m/3}$$

We may consider the same problem in terms of standing waves along the vertical spring with displacement

$$\eta = (A \cos kx + B \sin kx) \sin \omega t$$

where $k = \omega/v$ is the wave number. The boundary conditions are that $\eta = 0$ at $x = 0$ (the top of the spring) and

$$M \frac{\partial^2 \eta}{\partial t^2} = -sL \frac{\partial \eta}{\partial x} \quad \text{at } x = L$$

(the bottom of the spring). Show that these lead to the expression

$$kL \tan kL = \frac{m}{M}$$

and expand tan kL in powers of kL to show that, in the second order approximation

$$\omega^2 = \frac{s}{M + m/3}$$

The value of v is given in Problem 6.7.

Problem 6.9

A solid has a Poissons ratio $\sigma = 0.25$. Show that the ratio of the longitudinal wave velocity to the transverse wave velocity is $\sqrt{3}$. Use the values of these velocities given in the text to derive an appropriate value of σ for the earth.

Problem 6.10

Show that when sound waves are normally incident on a plane steel water interface 86% of the energy is reflected. If the waves are travelling in water and are normally incident on a plane water-ice interface show that 82.3% of the energy is transmitted.

(ρc values in kg m^{-2} s^{-1})

$$\text{water} = 1.43 \times 10^6$$
$$\text{ice} = 3.49 \times 10^6$$
$$\text{steel} = 3.9 \times 10^7$$

Problem 6.11

Use the boundary conditions for standing acoustic waves in a tube to confirm the following:

	Particle displacement		Pressure	
	closed end	open end	closed end	open end
Phase change on reflection	180°	0	0	180°
	node	antinode	antinode	node

Problem 6.12

Standing acoustic waves are formed in a tube of length l with (a) both ends open and (b) one end open and the other closed. If the particle displacement

$$\eta = (A \cos kx + B \sin kx) \sin \omega t$$

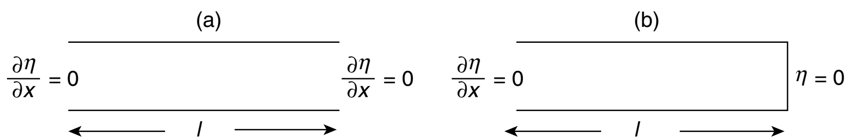
and the boundary conditions are as shown in the diagrams, show that for

$$\text{(a)} \quad \eta = A \cos kx \sin \omega t \quad \text{with} \quad \lambda = 2l/n$$

and for

$$\text{(b)} \quad \eta = A \cos kx \sin \omega t \quad \text{with} \quad \lambda = 4l/(2n+1)$$

Sketch the first three harmonics for each case.



Problem 6.13

On p. 121 we discussed the problem of matching two strings of impedances Z_1 and Z_3 by the insertion of a quarter wave element of impedance

$$Z_2 = (Z_1 Z_3)^{1/2}$$

Repeat this problem for the acoustic case where the expressions for the string displacements

$$y_\mathrm{i}, y_\mathrm{r}, y_\mathrm{t}$$

now represent the appropriate acoustic pressures p_i, p_r and p_t.

Show that the boundary condition for pressure continuity at $x = 0$ is

$$A_1 + B_1 = A_2 + B_2$$

and that for continuity of particle velocity is

$$Z_2(A_1 - B_1) = Z_1(A_2 - B_2)$$

Similarly, at $x = l$, show that the boundary conditions are

$$A_2 \,\mathrm{e}^{-\mathrm{i}k_2 l} + B_2 \,\mathrm{e}^{\mathrm{i}k_2 l} = A_3$$

and

$$Z_3(A_2\,e^{-ik_2l} - B_2\,e^{ik_2l}) = Z_2A_3$$

Hence prove that the coefficient of sound transmission

$$\frac{Z_1}{Z_3}\frac{A_3^2}{A_1^2} = 1$$

when

$$Z_2^2 = Z_1Z_3 \quad \text{and} \quad l = \frac{\lambda_2}{4}$$

(Note that the expressions for both boundary conditions and transmission coefficient differ from those in the case of the string.)

Problem 6.14

For sound waves of high amplitude the adiabatic bulk modulus may no longer be considered as a constant. Use the adiabatic condition that

$$\frac{P}{P_0} = \left[\frac{V_0}{V_0(1+\delta)}\right]^{\gamma}$$

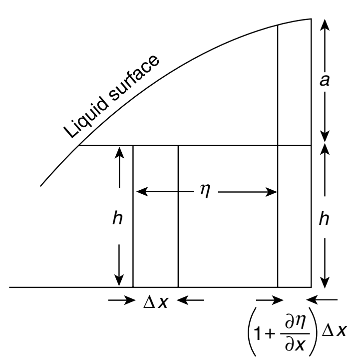
in deriving the wave equation to show that each part of the high amplitude wave has its own sound velocity $c_0(1+s)^{(\gamma+1)/2}$, where $c_0^2 = \gamma P_0/\rho_0$, δ is the dilatation, s the condensation and γ the ratio of the specific heats at constant pressure and volume.

Problem 6.15

Some longitudinal waves in a plasma exhibit a combination of electrical and acoustical phenomena. They obey a dispersion relation at temperature T of $\omega^2 = \omega_e^2 + 3aTk^2$, where ω_e is the constant electron plasma frequency (see Problem 5.18) and the Boltzmann constant is written as a to avoid confusion with the wave number k. Show that the product of the phase and group velocities is related to the average thermal energy of an electron (found from $pV = RT$).

Problem 6.16

It is possible to obtain the wave equation for tidal waves (long waves in shallow water) by the method used in deriving the acoustic wave equation. In the figure a constant mass of fluid in an element of unit width, height h and length Δx moves a distance η and assumes



a new height $h + \alpha$ and length $(1 + \partial\eta\partial x)\Delta x$, but retains unit width. Show that, to a first approximation,

$$\alpha = -h\frac{\partial\eta}{\partial x}$$

Neglecting surface tension, the force on the element face of height $h + \alpha$ arises from the product of the height and the mean hydrostatic pressure. Show, if $\rho gh \ll P_0$ (i.e. $h \ll 10$ m) and $\alpha \ll h$, that the net force on the liquid element is given by

$$-\frac{\partial F}{\partial x}\Delta x = -\rho gh\frac{\partial\alpha}{\partial x}\Delta x$$

Continue the derivation using the acoustic case as a model to show that these waves are non-dispersive with a phase velocity given by $v^2 = gh$.

Problem 6.17

Waves near the surface of a non-viscous incompressible liquid of density ρ have a phase velocity given by

$$v^2(k) = \left[\frac{g}{k} + \frac{Tk}{\rho}\right]\tanh kh$$

where g is the acceleration due to gravity, T is the surface tension, k is the wave number and h is the liquid depth. When $h \ll \lambda$ the liquid is shallow; when $h \gg \lambda$ the liquid is deep.

(a) Show that, when gravity and surface tension are equally important and $h \gg \lambda$, the wave velocity is a minimum at $v^4 = 4gT/\rho$, and show that this occurs for a 'critical' wavelength $\lambda_c = 2\pi(T/\rho g)^{1/2}$.
(b) The condition $\lambda \gg \lambda_c$ defines a *gravity* wave, and surface tension is negligible. Show that gravity waves in a shallow liquid are non-dispersive with a velocity $v = \sqrt{gh}$ (see Problem 6.16).
(c) Show that gravity waves in a deep liquid have a phase velocity $v = \sqrt{g/k}$ and a group velocity of half this value.
(d) The condition $\lambda < \lambda_c$ defines a ripple (dominated by surface tension). Show that short ripples in a deep liquid have a phase velocity $v = \sqrt{Tk/\rho}$ and a group velocity of $\frac{3}{2}v$. (Note the anomalous dispersion).

Summary of Important Results

Wave Velocity

$$c^2 = \frac{\text{Bulk Modulus}}{\rho} = \frac{\gamma P}{\rho}$$

Specific Acoustic Impedance

$$Z = \frac{\text{acoustic pressure}}{\text{particle velocity}}$$

$$\begin{aligned} Z &= \rho c \text{ (for right-going wave)} \\ &= -\rho c \text{ (for left-going wave because pressure} \\ &\qquad \text{and particle velocity become anti-phase)} \end{aligned}$$

$$\textit{Intensity} = \tfrac{1}{2}\rho c \dot{\eta}_m^2 = \frac{p_{\text{rms}}^2}{\rho c} = p_{\text{rms}}\dot{\eta}_{\text{rms}}$$

Reflection and Transmission Coefficients

$$\frac{\text{Reflected Amplitude}}{\text{Incident Amplitude}}\begin{Bmatrix}\text{displacement}\\ \text{and velocity}\end{Bmatrix} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = -\frac{\text{Reflected pressure}}{\text{Incident pressure}}$$

$$\frac{\text{Transmitted Amplitude}}{\text{Incident Amplitude}}\begin{Bmatrix}\text{displacement}\\ \text{and velocity}\end{Bmatrix} = \frac{2Z_1}{Z_1 + Z_2} = \frac{Z_1}{Z_2} \times \frac{\text{Transmitted pressure}}{\text{Incident pressure}}$$

$$\frac{\text{Reflected Intensity}}{\text{Incident Intensity}}(\text{energy}) = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

$$\frac{\text{Transmitted Intensity}}{\text{Incident Intensity}}(\text{energy}) = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2}$$