Mathematics and Statistics

Model Set - 3

Academic Year: 2020-2021 Marks: 80
Date: April 2021
Duration: 3h

1. The question paper is divided into four sections.
2. **Section A**: Q. No. 1 contains 8 multiple-choice type of questions carrying two marks each.
3. **Section A**: Q. No. 2 contains 4 very short answer type of questions carrying One mark each.
4. **Section B**: Q. No. 3 to Q. No. 14 contains Twelve short answer type of questions carrying Two marks each. **(Attempt any Eight)**.
5. **Section C**: Q. No.15 to Q. No. 26 contains Twelve short answer type of questions carrying Three marks each. **(Attempt any Eight)**.
6. **Section D**: Q.No. 27 to Q. No. 34 contains Five long answer type of questions carrying Four marks each. **(Attempt any Five)**.
7. Use of log table is allowed. Use of calculator is not allowed.
8. Figures to the right indicate full marks.
9. For each MCQ, correct answer must be written along with its alphabet.
   e.g., (a) ..... / (b) .... / (c) .... / (d) ..... Only first attempt will be considered for evaluation.
10. Use of graph paper is not necessary. Only rough sketch of graph is expected:
11. Start answers to each section on a new page.

Q. 1 | Select and write the most appropriate answer from the given alternatives for each sub-question:

1.i Choose the correct alternative :

Which of the following is not a statement?

1. Smoking is injuries to health
2. 2 + 2 = 4
3. 2 is the only even prime number.
4. Come here

1.ii The value of x, y, z for the following system of equations $x + y + z = 6$, $x - y + 2z = 5$, $2x + y - z = 1$ are ______

1. x = 1, y = 2, z = 3
2. x = 2, y = 1, z = 3
3. x = −1, y = 2, z = 3
   a. x = y = z = 3

1.iii

The principal solutions of $\sqrt{3}$ sec x − 2 = 0 are _____

1. $\frac{\pi}{3}, \frac{11\pi}{6}$
2. $\frac{\pi}{6}, \frac{11\pi}{6}$
3. $\frac{\pi}{4}, \frac{11\pi}{4}$
4. $\frac{\pi}{6}, \frac{11\pi}{3}$

1.iv Given that $X \sim B(n = 10, p)$, $E(X) = 8$, then value of p = _____

1. 0.4
2. 0.8
3. 0.6
4. 0.7

1.v If a d.r.v. X has the following probability distribution:

X	1	2	3	4	5	6	7
P(X = x)	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

then k = _____

1. $\frac{1}{7}$
2. $\frac{1}{8}$
3. $\frac{1}{9}$
4. $\frac{1}{10}$

1.vi Select and write the correct alternative from the given option for the question

The order and degree of $\left(\frac{dy}{dx}\right)^3 - \frac{d^3y}{dx^3} + ye^x = 0$ are respectively

Ans.

1. 3, 1

2. 1, 3

3. 3, 3

4. 1, 1

1.vii

$\int \frac{e^x(x-1)}{x^2}\, dx =$ ______

1. $xe^{-x} + c$

2. $\frac{e^x}{x^2} + c$

3. $\left(x - \frac{1}{x}\right)e^x + c$

4. $\frac{e^x}{x} + c$

1.viii Select the correct option from the given alternatives:

If α, β, γ are direction angles of a line and α = 60°, β = 45°, γ = ______

1. 30° or 90°
2. 45° or 60°
3. 90° or 30°
4. 60° or 120°

Q. 2 | Answer the following questions:

2.i State the truth Value of $x^2 = 25$

Ans. '$x^2 = 25$' is an open sentence.

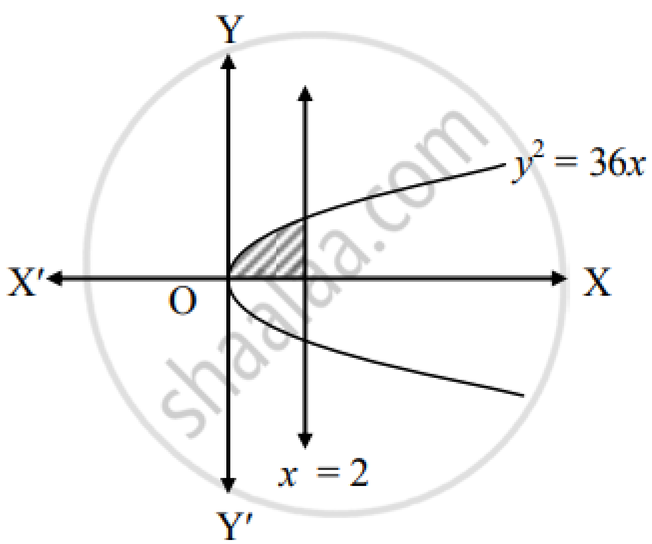
It is not a statement in logic.

2.ii Find the area bounded by the curve $y^2 = 36x$, the line x = 2 in first quadrant

Ans. Given equation of the curve is $y^2 = 36x$

$\therefore y = \pm\sqrt{36x}$

$\therefore y = 6\sqrt{x}$ .....[∵ In first quadrant, y > 0]



Required area = $\int_0^2 y\,dx$

$= \int_0^2 6\sqrt{x}\,dx$

$= 6\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2$

$= 4\left[(2)^{\frac{3}{2}} - 0\right]$

$= 4\left(2\sqrt{2}\right)$

$= 8\sqrt{2}$ sq.units

2.iii

If $y = e^{1+\log x}$ then find $\frac{dy}{dx}$

Ans.

$y = e^{1+\log x}$

$= e \cdot e^{\log x}$

$= e.\,x$

$\therefore \dfrac{dy}{dx} = e.\,1 = e$

OR

$y = e^{1+\log x}$

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(e^{1+\log x}\right)$

$= e^{1+\log x} \cdot \dfrac{d}{dx}(1 + \log x)$

$= e^{1+\log x} \cdot \left(0 + \dfrac{1}{x}\right)$

$= \dfrac{e^{1+\log x}}{x}$

2.iv The displacement of a particle at time t is given by $s = 2t^3 - 5t^2 + 4t - 3$. Find the velocity when $t = 2$ sec

Ans. $s = 2t^3 - 5t^2 + 4t - 3$

$\therefore v = \dfrac{ds}{dt}$

$= 6t^2 - 10t + 4$

$v_{(t=2)} = 6(2)^2 - 10(2) + 4$

$= 24 - 20 + 4$

= 8 units/sec

Q. 3 | Attempt any Eight:

If statements p, q are true and r, s are false, determine the truth values of the following.
(p ∧ ~r) ∧ (~q ∨ s)

Ans. (p ∧ ~r) ∧ (~q ∨ s)

≡ (T ∧ ~F) ∧ (~T ∨ F)

≡ (T ∧ T) ∧ (F ∨ F)

≡ T ∧ F

≡ F

∴ Truth value of (p ∧ ~r) ∧ (~q ∨ s) is F.

Q. 4

Find A^{-1} using adjoint method, where A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Ans. $(A^{-1} B^{-1}) = (BA)^{-1}$ ....[∵ $(AB)^{-1} = B^{-1}A^{-1}$]

$$\therefore BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 4+3 \\ 0+1 & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 7 \\ 1 & 3 \end{bmatrix}$$

$|BA| = 3 \times 3 - 7 \times 1$

$= 2$

Let C = BA.

∴ $C_{11} = (-1)^{1+1} M_{11} = M_{11} = 3$

$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -1$

$C21 = (-1)^{2+1} M_{21} = -M_{21} = -7$

$C22 = (-1)^{2+2} M_{22} = M_{22} = 3$

$$\text{adj (C)} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 7 \\ -1 & 3 \end{bmatrix}$$

$$\therefore C^{-1} = \frac{1}{|C|} \text{adj (C)}$$

$$\text{i.e. } (BA)^{-1} = \frac{1}{|BA|} \text{adj (BA)}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 7 \\ -1 & -3 \end{bmatrix}$$

Q. 5

Find A^{-1} using adjoint method, where $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Ans. $|A| = \cos\theta(\cos\theta) - \sin\theta(-\sin\theta)$

$= \cos^2\theta + \sin^2\theta$

$= 1 \neq 0$

$\therefore A^{-1}$ exists.

$A_{11} = (-1)^{1+1} M_{11} = M_{11} = \cos\theta$

$A_{12} = (-1)^{1+2} M_{12} = -M_{12} = \sin\theta$

$A_{21} = (-1)^{2+1} M_{21} = -M_{21} = -\sin\theta$

$A_{22} = (-1)^{2+2} M_{22} = M_{22} = \cos\theta$

$\therefore$ adj (A) = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$

= $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}$ adj (A)

= $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Q. 6

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then show that $\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 1$

Ans. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

$\therefore \tan^{-1}x + \tan^{-1}y = \pi - \tan^{-1}z$

$\therefore \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \pi - \tan^{-1}z$

$\therefore \frac{x+y}{1-xy} = \tan(\pi - \tan^{-1}z)$

$\therefore \frac{x+y}{1-xy} = -\tan(\tan^{-1}z)$

$\therefore \frac{x+y}{1-xy} = -z$

$\therefore x + y = -z + xyz$

$\therefore x + y + z = xyz$

$\therefore \frac{1}{yz} + \frac{1}{xz} + \frac{1}{xy} = 1$, i.e., $\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 1$

Q. 7

With usual notations, prove that $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

Ans.

Consider L.H.S. = $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$

= $\frac{1}{a}(\cos A)+\frac{1}{b}(\cos B)+\frac{1}{c}(\cos C)$

= $\frac{1}{a}\left(\frac{b^2+c^2-a^2}{2bc}\right)+\frac{1}{b}\left(\frac{a^2+c^2-b^2}{2ac}\right)+\frac{1}{c}\left(\frac{a^2+b^2-c^2}{2ab}\right)$ ....[By consine rule]

= $\frac{b^2+c^2-a^2}{2abc}+\frac{a^2+c^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}$

= $\frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc}$

= $\frac{a^2+b^2+c^2}{2abc}$

= R.H.S.

$\therefore \frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

Q. 8

Evaluate: $\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{(1+\cos x)^2}dx$

Ans.

Let I = $\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{(1+\cos x)^2}dx$

Put $\tan\left(\frac{x}{2}\right)$ = t

∴ x = $2\tan^{-1}t$

∴ dx = $\frac{2dt}{1+t^2}$, sin x $\frac{2t}{1+t^2}$ and x = $\frac{1-t^2}{1+t^2}$

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = 1

$$\therefore I = \int_0^1 \frac{\left(\frac{2t}{1+t^2}\right)^2}{\left(1+\frac{1-t^2}{1+t^2}\right)^2} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{\frac{4t^2}{(1+t^2)^2}}{\frac{4}{(1+t^2)^2}} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{t^2}{1+t^2}\, dt$$

$$= 2\int_0^1 \left(\frac{1+t^2-1}{1+t^2}\right) dt$$

$$= 2\int_0^1 \left(1+\frac{1}{1+t^2}\right) dt$$

$$= 2\left[t - \tan^{-1} t\right]_0^1$$

$$= 2[(1 - \tan^{-1}1) - (0 - \tan^{-1}0)]$$

$$= 2\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{4-\pi}{2}$$

Q. 9

Verify y = log x + c is the solution of differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Ans. y = log x + c

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore x\frac{dy}{dx} = 1$$

Again, differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 = 0$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$\therefore$ y = log x + c is the solution of $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Q. 10 Find the derivative of the inverse of function $y = 2x^3 - 6x$ and calculate its value at x = −2

Ans. $y = 2x^3 - 6x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 6x)$$

$= 2(3x^2) - 6$

$= 6x^2 - 6$

$= 6(x^2 - 1)$

$$\therefore \left(\frac{dy}{dx}\right)_{x=-2} = 6\left[(-2)^2 - 1\right]$$

= 6(3)

= 18

$$\therefore \left(\frac{dx}{dy}\right)_{x=-2} = \frac{1}{\left(\frac{dy}{dx}\right)_{x=-2}}$$

$$= \frac{1}{18}$$

Q. 11 Find the values of x, for which the function $f(x) = x^3 + 12x^2 + 36x + 6$ is monotonically decreasing

Ans. $f(x) = x^3 + 12x^2 + 36x + 6$

$\therefore f'(x) = 3x^2 + 24x + 36$

$= 3(x^2 + 8x + 12)$

$= 3(x + 2)(x + 6)$

f(x) is monotonically decreasing, if $f'(x) < 0$

$\therefore 3(x + 2)(x + 6) < 0$

$\therefore (x + 2)(x + 6) < 0$

$ab < 0 \Leftrightarrow a > 0$ and $b < 0$ or $a < 0$ and $b > 0$

$\therefore$ Either $x + 2 > 0$ and $x + 6 < 0$

or

$x + 2 < 0$ and $x + 6 > 0$

Case I: $x + 2 > 0$ and $x + 6 < 0$

$\therefore x > -2$ and $x < -6$,

which is not possible.

Case II: $x + 2 < 0$ and $x + 6 > 0$

$\therefore x < -2$ and $x > -6$

Thus, f(x) is monotonically decreasing for $x \in (-6, -2)$.

Q. 12 A car is moving in such a way that the distance it covers, is given by the equation $s = 4t^2 + 3t$, where s is in meters and t is in seconds. What would be the velocity and the acceleration of the car at time t = 20 seconds?

Ans. Let v be the velocity and a be the acceleration of the car.

Distance travelled by the car is given by

$\therefore$ Velocity = v = $\frac{ds}{dt}$

= $\frac{d}{dt}\left(4t^2 + 3t\right)$

= 8t + 3 ....(i)

Acceleration = a = $\frac{dv}{dt}$

= $\frac{d}{dt}(8t + 3)$

= 8 ....(ii)

Velocity of the car at t = 20 seconds is

$V_{(t=20)}$ = 8(20) + 3 ....[From (i)]

= 163 m/sec.

Acceleration of the car at t = 20 seconds is

$a_{(t=20)}$ = 8 m/sec^2 ....[From (ii)]

Q. 13

$$\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\, dx$$

Ans.

Let I = $\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x}\, dx$

$$= \int \left(\frac{\cos^2 x - \sin^2 x}{\sin^2 x \cdot \cos^2 x} \right) dx$$ .....[∵ cos 2θ = $\cos^2\theta - \sin^2\theta$]

$$= \int \left(\frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x} - \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} \right) dx$$

$$= \int \frac{1}{\sin^2 x}\, dx - \int \frac{1}{\cos^2 x}\, dx$$

$$= \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx$$

∴ I = − cot x − tan x + c

Q. 14

Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $-5\hat{i} + 2\hat{j} - 5\hat{k}$ in the ratio 3:2

(i) internally

(ii) externally

Ans.

Let $\overline{p}, \overline{q}$ and $\overline{r}$ be the position vectors of points P, Q and R respectively.

$\therefore \overline{p} = 2\hat{i} - \hat{j} + 3\hat{k}, \overline{q} = -5\hat{i} + 2\hat{j} - 5\hat{k}$, m:n = 3:2

(i) R divides the line PQ internally in the ratio 3:2

∴ By using section formula for internal division,

$$\overline{r} = \frac{m\overline{q} + n\overline{p}}{m + n}$$

$$= \frac{3\left(-5\hat{i} + 2\hat{j} - 5\hat{k}\right) + 2\left(2\hat{i} - \hat{j} + 3\hat{k}\right)}{3 + 2}$$

$$= \frac{-15\hat{i} + 6\hat{j} - 15\hat{k} + 4\hat{i} - 2\hat{j} + 6\hat{k}}{5}$$

$= \frac{-11}{5}\hat{i} + \frac{4}{5}\hat{j} - \frac{9}{5}\hat{k}$

(ii) R divides the line PQ externally in ratio 3:2

∴ By using section formula for external division,

$\bar{r} = \frac{m\bar{q} - n\bar{p}}{m - n}$

$= \frac{3\left(-5\hat{i} + 2\hat{j} - 5\hat{k}\right) - 2\left(2\hat{i} - \hat{j} + 3\hat{k}\right)}{3 - 2}$

$= \frac{-15\hat{i} + 6\hat{j} - 15\hat{k} - 4\hat{i} + 2\hat{j} - 6\hat{k}}{1}$

$\therefore \bar{r} = -19\hat{i} + 8\hat{j} - 21\hat{k}$

Q. 15 | Attempt any Eight:

Q. 15. (A) Use quantifiers to convert the following open sentences defined on N, into a true statement.
$n^2 \geq 1$

Q. 15. (B) Use quantifiers to convert the given open sentence defined on N into a true statement
$3x - 4 < 9$

Q. 15. (C) Use quantifiers to convert the given open sentence defined on N into a true statement
$Y + 4 > 6$

Ans. (A) $n^2 \geq 1$

$\forall\, n \in N, n^2 \geq 1$

Since, square of all natural numbers is either 1 or greater than 1.

∴ The statement is true.

Ans. (B) $3x - 4 < 9$

$\exists\, x \in N$ such that $3x - 4 < 9$, is a true statement, since $x = 2 \in N$ satisfies $3x - 4 < 9$

Ans. (C) $Y + 4 > 6$

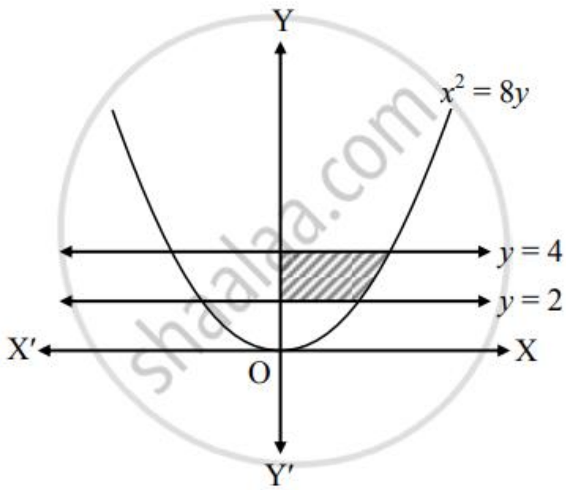
∃ Y ∈ N such that Y + 4 > 6, is a true statement, since Y = 3 ∈ N satisfies Y + 4 > 6.

Q. 16 Find the area of the region bounded by the curves $x^2 = 8y$, y = 2, y = 4 and the Y-axis, lying in the first quadrant

Ans. Given equation of the parabola is $x^2 = 8y$

$\therefore x = \pm 2\sqrt{2y}$

$\therefore x = 2\sqrt{2y}$ .....[∵ In first quadrant, x > 0]



$\therefore$ Required area = $\int_2^4 x\, dy$

$= \int_2^4 2\sqrt{2y}\, dy$

$= 2\sqrt{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$

$= \frac{4\sqrt{2}}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$

$= \frac{4\sqrt{2}}{3}\left(8 - 2\sqrt{2}\right)$

$= \frac{8\sqrt{2}}{3}\left(4 - \sqrt{2}\right)$ sq.units

Q. 17 The probability that certain kind of component will survive a check test is 0.6. Find the probability that exactly 2 of the next 4 tested components survive

Ans. Let X denote the number of tested components survive.

P(component survive the check test) = p = 0.6 ....[Given]

∴ q = 1 – p

= 1 – 0.6

= 0.4

Given, n = 4

∴ X ~ B (4, 0.6)

The p.m.f. of X is given by

$$P(X = x) = {}^{4}C_{x}(0.6)^{x}(0.4)^{4-x}, x = 0, 1, ..., 4$$

∴ P(exactly 2 components tested survive)

= P(X = 2)

= $^{4}C_{2}$ $(0.6)^{2}$ $(0.4)^{2}$

= 6(0.36)(0.16)

= 0.3456

Q. 18

Let the p.m.f. of r.v. X be P(x) $\begin{cases} \frac{3-x}{10}, & \text{for } x = -1, \quad 0, \quad 1, \quad 2 \\ 0, & \text{otherwise} \end{cases}$ Calculate E(X) and Var(X)

Ans.

$$E(X) = \sum_{i=1}^{4} x_i P(x_i)$$

$$= -1 \times \left(\frac{3-(-1)}{10}\right) + 0 \times \left(\frac{3-0}{10}\right) + 1 \times \left(\frac{3-1}{10}\right) + 2 \times \left(\frac{3-2}{10}\right)$$

$$= \frac{-4+0+2+2}{10}$$

= 0

$$E(X^2) = \sum_{i=1}^{4} x_i^2 P(x_i)$$

$$= (-1)^2 \times \frac{4}{10} + (0)^2 \times \frac{3}{10} + (1)^2 \times \frac{2}{10} + (2)^2 \times \frac{1}{10}$$

$$= \frac{4+0+2+4}{10}$$

= 1

= 1

$Var(X) = E(X^2) - [E(X)]^2$

$= 1 - (0)^2$

= 1

Q. 19 Find the combined equation of the following pair of line passing through (−1, 2), one is parallel to x + 3y − 1 = 0 and other is perpendicular to 2x − 3y − 1 = 0

Ans.

Let L_1 be the line passing through the point (−1, 2) and parallel to the line x + 3y − 1 = 0 whose slope is $-\frac{1}{3}$.

∴ slope of the line L_1 is $-\frac{1}{3}$

∴ equation of the line L_1 is

$$y - 2 = -\frac{1}{3}(x + 1)$$

$\therefore$ 3y − 6 = − x − 1

$\therefore$ x + 3y − 5 = 0

Let L_1 be the line passing through (−1, 2) and perpendicular to the line 2x − 3y − 1 = 0 whose

slope is $\frac{-2}{-3} = \frac{2}{3}$

$\therefore$ slope of the line L_2 is $-\frac{3}{2}$

$\therefore$ equation of the line L_2 is

$y - 2 = -\frac{3}{2}(x + 1)$

$\therefore$ 2y − 4 = − 3x − 3

$\therefore$ 3x + 2y − 1 = 0

Hence, the equations of the required lines are

x + 3y − 5 = 0 and 3x + 2y − 1 = 0

$\therefore$ their combined equation is

(x + 3y − 5)(3x + 2y − 1) = 0

$\therefore 3x^2 + 2xy - x + 9xy + 6y^2 - 3y - 15x - 10y + 5 = 0$

$\therefore 3x^2 + 11xy + 6y^2 - 16x - 13y + 5 = 0$

Q. 20

Find the measure of the acute angle between the line represented by $3x^2 - 4\sqrt{3}xy + 3y^2 = 0$

Ans.

Given equation of the lines is $3x^2 - 4\sqrt{3}xy + 3y^2 = 0$

Comparing with $ax^2 + 2hxy + by^2 = 0$,

We get, a = 3, h = $-2\sqrt{3}$ and b = 3

Let θ be the acute angle between the lines.

$\therefore \tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a + b}\right|$

$= \left|\dfrac{2\sqrt{\left(-2\sqrt{3}\right)^2 - 3(3)}}{3+3}\right|$

$= \left|\dfrac{2\sqrt{12-9}}{6}\right|$

$= \left|\dfrac{\sqrt{3}}{3}\right|$

$\therefore \tan\theta = \dfrac{1}{\sqrt{3}}$

$= \theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$\therefore \theta = 30°$

Q. 21

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\ldots\infty}}}$, show that $\dfrac{dy}{dx} = \dfrac{\sin x}{1-2y}$

Ans.

$y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}}$

$\therefore y^2 = \cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}$

$\therefore y^2 = \cos x + y$

Differentiating w. r. t. x, we get

$$2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}(1-2y) = \sin x$$

$$\therefore \frac{dy}{dx} = \frac{\sin x}{1-2y}$$

Q. 22

If x sin(a + y) + sin a cos(a + y) = 0 then show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Ans. x sin(a + y) + sin a cos(a + y) = 0 ....(i)

$$x.\cos(a+y)\cdot\frac{d}{dx}(a+y) + \sin(a+y)\cdot\frac{d}{dx}(x) + \sin a[-\sin(a+y)]\cdot\frac{d}{dx}(a+y) = 0$$

$$\therefore x\cos(a+y)\frac{dy}{dx} + \sin(a+y)(1) - \sin(a+y)\frac{dy}{dx} = 0$$

$$\therefore [x\cos(a+y) - \sin a\sin(a+y)]\frac{dy}{dx} = -\sin(a+y) \quad ....(ii)$$

From (i), we get

$$x = \frac{-\sin a\cos(a+y)}{\sin(a+y)}$$

Substituting the value of x in (ii), we get

$$\left[\frac{-\sin a\cos(a+y)}{\sin(a+y)}\cdot\cos(a+y)-\sin a\sin(a+y)\right]\frac{dy}{dx} = -\sin(a+y)$$

$$\therefore -\sin a\left[\frac{\cos^2(a+y)}{\sin(a+y)}+\sin(a+y)\right]\frac{dy}{dx} = -\sin(a+y)$$

$$\therefore \frac{-\sin a\left[\cos^2(a+y)+\sin^2(a+y)\right]}{\sin(a+y)}\frac{dy}{dx} - \sin(a+y)$$

$$\therefore -\frac{\sin a(1)}{\sin(a+y)}\cdot\frac{dy}{dx} = -\sin(a+y)$$

$$\therefore \frac{dy}{dx} = \sin(a+y)\left[\frac{\sin(a+y)}{\sin a}\right]$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Q. 23

$$\int \sec^2 x\sqrt{\tan^2 x+\tan x-7}\,dx$$

Ans.

Let $I = \int \sec^2 x\sqrt{\tan^2 x+\tan x-7}\,dx$

Put tan x = t

$\therefore \sec^2 x\,dx = dt$

$$\therefore I = \int\sqrt{t^2+t-7}\,dt$$

$$= \int\sqrt{t^2+t+\frac{1}{4}-\frac{1}{4}-7}\,dt$$

$$= \int\sqrt{\left(t+\frac{1}{2}\right)^2-\frac{29}{4}}\,dt$$

$$= \int \sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{29}}{2}\right)^2}\, dt$$

$$= \frac{t + \frac{1}{2}}{2}\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{29}}{2}\right)^2}$$

$$= -\frac{\left(\frac{\sqrt{29}}{2}\right)^2}{2}\log\left|t + \frac{1}{2} + \sqrt{t^2 + t - 7}\right| + c$$

$$= \frac{2t + 1}{4}\sqrt{t^2 + t - 7} - \frac{29}{8}\log\left|t + \frac{1}{2} + \sqrt{t^2 + t - 7}\right| + c$$

$$\therefore I = \frac{(2\tan x + 1)}{4}\sqrt{\tan^2 x + \tan x - 7} - \frac{29}{8}\log\left|\tan x + \frac{1}{2} + \sqrt{\tan^2 x + \tan x - 7}\right| + c$$

Q. 24

$$\int \frac{(x^2 + 2)}{x^2 + 1} a^{x + \tan^{-1} x}\, dx$$

Ans.

$$\text{Let } I = \int \left(\frac{x^2 + 2}{x^2 + 1}\right) a^{x + \tan^{-1} x}\, dx$$

Put $x + \tan^{-1}x = t$

Differentiating w.r.t. x, we get

$$\left(1 + \frac{1}{1 + x^2}\right) dx = dt$$

$$\therefore \left(\frac{x^2 + 2}{x^2 + 1}\right) dx = dt$$

$$\therefore I = \int a^1\, dt$$

$$= \frac{a^1}{\log a} + c$$

$$\therefore I = \frac{a^{x+\tan^{-1}x}}{\log a} + c$$

Q. 25 Find the Cartesian equation of the line passing through (−1, −1, 2) and parallel to the line 2x − 2 = 3y + 1 = 6z − 2

Ans. 2x − 2 = 3y + 1 = 6z − 2 ....[Given]

$$\therefore 2(x-1) = 3\left(y + \frac{1}{3}\right)$$

$$= 6\left(z - \frac{2}{3}\right)$$

$$\therefore \frac{x-1}{\frac{1}{2}} = \frac{y-\left(-\frac{1}{3}\right)}{\frac{1}{3}}$$

$$= \frac{z-\frac{1}{3}}{\frac{1}{6}}$$

Direction ratios of given line are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

Since the required line is parallel to the given line, direction ratios of the required line will be $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

Equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios (a, b, c) is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\therefore \frac{x-(-1)}{\frac{1}{2}} = \frac{y-(-1)}{\frac{1}{3}} = \frac{z-2}{\frac{1}{6}}$$

$$\therefore \frac{x+1}{\frac{1}{2}\times 6} = \frac{y+1}{\frac{1}{3}\times 6} = \frac{z-2}{\frac{1}{6}\times 6}$$

$$\therefore \frac{x+1}{3} = \frac{y+1}{2} = \frac{z-2}{1},$$

which is the required cartesian equation of the line.

Q. 26

Find acute angle between the lines $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{2}$ and $\frac{x-1}{2}=\frac{y-1}{1}=\frac{z-3}{1}$

Ans.

Given equations of lines are $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{2}$ and $\frac{x-1}{2}=\frac{y-1}{1}=\frac{z-3}{1}$

Direction ratios of above lines are

$a_1 = 1$, $b_1 = -1$, $c_1 = 2$ and $a_2 = 2$, $b_2 = 1$, $c_2 = 1$

Angle between two lines is

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$\therefore \cos\theta = \left|\frac{(1)(2)+(-1)(1)+(2)(1)}{\sqrt{1^2+(-1)2+2^2\sqrt{2^2+1^2+1^2}}}\right|$$

$$\therefore \cos\theta = \left|\frac{2-1+2}{\sqrt{6}\sqrt{6}}\right|$$

$$\therefore \cos\theta = \left|\frac{3}{6}\right|$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \theta = 60°$$

Q. 27 | Attempt any Five:

In ΔABC, prove that $\frac{b^2-c^2}{a}\cos A+\frac{c^2-a^2}{b}\cos B+\frac{a^2-b^2}{c}\cos C = 0$

Ans. In ΔABC by cosine rule, we have

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\therefore$ L.H.S. $= \dfrac{b^2 - c^2}{a}\cos A + \dfrac{c^2 - a^2}{b}\cos B + \dfrac{a^2 - b^2}{c}\cos C$

$= \dfrac{b^2 - c^2}{a} \times \dfrac{b^2 + c^2 - a^2}{2bc} + \dfrac{c^2 - a^2}{b} \times \dfrac{a^2 + c^2 - b^2}{2ac} + \dfrac{a^2 - b^2}{c} \times \dfrac{a^2 + b^2 - c^2}{2ab}$

$=$

$\dfrac{1}{2abc}\left(b^4 + b^2c^2 - a^2b^2 - b^2c^2 - c^4 + a^2c^2 + a^2c^2 + c^4 - b^2c^2 - a^4 - a^2c^2 + a^2b^2 + a^4 + a^2b\right.$

= 0

= R.H.S

Q. 28 Maximize z = −x + 2y subjected to constraints x + y ≥ 5, x ≥ 3, x + 2y ≤ 6, y ≥ 0 is this LPP solvable? Justify your answer

Ans. To draw the feasible region, construct table as follows:

Inequality	x + y ≥ 5	x ≥ 3	x + 2y ≥ 6
Corresponding equation (of line)	x + y = 5	x = 3	x + 2y = 6
Intersection of line with X-axis	(5, 0)	(3, 0)	(6, 0)
Intersection of line with Y-axis	(0, 5)	–	(0, 3)
Region	Non-origin side	Non-origin side	Non-origin side

Shaded portion XABC is the feasible region, whose vertices are A (6, 0), B and C

B is the point of intersection of the lines x + y = 5 and x + 2y = 6

Solving the above equations, we get

B ≡ (4, 1)

C is the point of intersection of the lines x = 3 and x + y = 5

Putting x = 3 in x + y = 5, we get

3 + y = 5

∴ y = 2

∴ C ≡ (3, 2)

Here the objective function is

Z = −x + 2y

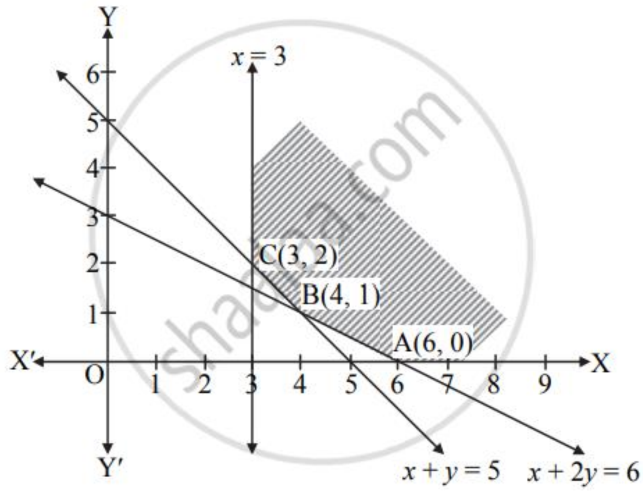
∴ Z at A(6, 0) = −6 + 2(0) = −6

Z at B(4, 1) = −4 + 2(1) = −2

Z at C(3, 2) = −3 + 2(2) = 1

Here, the feasible region is unbounded.

So, the objective function does not have finite maximum value i.e. value of objective function Z increases indefinitely and hence the L.P.P. has unbounded solution.



Q. 29

Prove that: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ If $f(x)$ is even function

$= 0,$ if $f(x)$ is odd function

Ans. L.H.S becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad ....(i)$$

Consider $\int_{-a}^{0} f(x)\,dx$

Put x = – t

∴ dx = – dt

When x = – a, t = a

and when x = 0, t = 0

$$\therefore \int_{-a}^{0} f(x)\, dx = \int_{a}^{0} f(-t)(-dt)$$

$$= -\int_{a}^{0} f(-t)\, dt$$

$$= \int_{0}^{a} f(-t)\, dt \quad ...... \left[\because \int_{a}^{b} f(x)\, dx = -\int_{b}^{a} f(x)\, dx\right]$$

$$= \int_{0}^{a} f(-x)\, dx \quad ...... \left[\because \int_{a}^{b} f(x)\, dx = \int_{a}^{b} f(t)\, dt\right]$$

Equation (i) becomes

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(-x)\, dx + \int_{0}^{a} f(x)\, dx$$

$$= \int_{0}^{a} [f(-x) + f(x)]\, dx \quad ......(ii)$$

Case I:

If f(x) is an even function, then f (– x) = f(x).

Equation (ii) becomes

$$\int_{-a}^{a} f(x)\, dx = 2 \cdot \int_{0}^{a} f(x)\, dx$$

Case II:

If f(x) is an odd function, then f (– x) = – f(x).

Equation (ii) becomes

$$\int_{-a}^{a} f(x)\, dx = 0$$

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} = 2 \cdot \int_{0}^{a} f(x)\, dx, & \text{If } f(x) \text{ is even function} \\ = 0, & \text{if } f(x) \text{ is odd function} \end{cases}$$

Q. 30 If the population of a town increases at a rate proportional to the population at that time. If the population increases from 40

thousand to 60 thousand in 40 years, what will be the population in another 20 years? $\left(\text{Given}\sqrt{\frac{3}{2}} = 1.2247\right)$

Ans. Let 'x' be the population at time 't'.

$$\therefore \frac{dx}{dt} \propto x$$

$$\therefore \frac{dx}{dt} = kx,$$

where k is the constant of proportionality.

$$\frac{dx}{dt} = k\,dt$$

Integrating on both sides, we get

$$\int \frac{dx}{x} = k \int 1 dt$$

∴ log x = kt + c

∴ x = a.e^{kt}, ...(i)

where a = e^c

When t = 0, x = 40,000

∴ 40000 = a.e^0

∴ a = 40000

∴ x = 40000.e^{kt} ......(ii) ......[From (i)]

When t = 40, P = 60000

∴ 60000 = 40000.e^{40k}

$$\therefore e^{40k} = \frac{60000}{40000} = \frac{3}{2} \quad ......(iii)$$

Now, we have to find x when t = 40 + 20

= 60 years

$\therefore$ P = $40000.e^{60k}$ .....[From (iii)]

$= 40000\left(e^{40k}\right)^{\frac{3}{2}}$

$= 40000\left(\frac{3}{2}\right)^{\frac{3}{2}}$ ......[From (iii)]

$= 40000\left(\frac{3}{2}\right)\sqrt{\frac{3}{2}}$

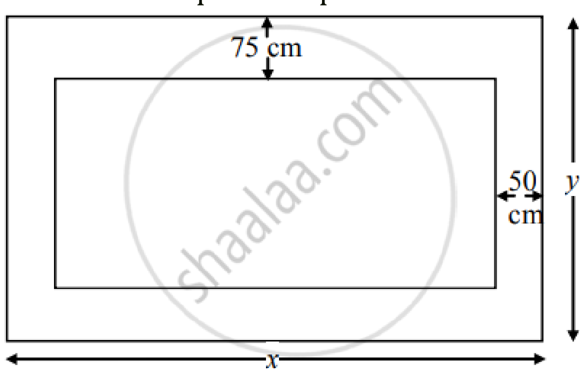
= 60000(1.2247)

= 73482

$\therefore$ The required population will be 73482.

Q. 31 A rectangular sheet of paper has it area 24 sq. Meters. The margin at the top and the bottom are 75 cm each and the sides 50 cm each. What are the dimensions of the paper if the area of the printed space is maximum?

Ans. Let x and y denote the length and breadth in metres of the sheet of paper and A denote the area of the printed space.



Then, area of the sheet of paper = length × breadth

= xy

= 24

$\therefore y = \frac{24}{x}$ ....(i)

Also, length of the printed space = (x − 1) metres and its breadth = (y − 1.5) metres.

$\therefore$ Area of the printed space,

A = (x − 1)(y − 1.5)

$= (x - 1)\left(\frac{24}{x} - 1.5\right)$ ....[From (i)]

$= 24 - 1.5x - \frac{24}{x} + 1.5$

$= 25.5 - 1.5x - \frac{24}{x}$

$\therefore \frac{dA}{dx} = 0 - 1.5 + \frac{24}{x^2}$

$= -\frac{3}{2} + \frac{24}{x^2}$

$\therefore \frac{dA}{dx^2} = 0 + 24(-2x^{-3})$

$= -\frac{48}{x^3}$

Now, A is maximum, if $\frac{dA}{dx} = 0$

$\therefore -\frac{3}{2} + \frac{24}{x^2} = 0$

$\therefore \frac{24}{x^2} = \frac{3}{2}$

$\therefore x^2 = 24 \times \frac{2}{3}$

= 16

$\therefore x = 4$ ....[$\because x > 0$]

For x = 4,

$$\left(\frac{d^2A}{dx^2}\right)_{(x=4)} = -\frac{48}{x^3}$$

$$= -\frac{48}{4^3}$$

$$= -\frac{3}{4} < 0$$

Thus, A is maximum when x = 4.

From (i), we get

$$y = \frac{24}{x}$$

$$= \frac{24}{4}$$

$$= 6$$

Thus, the area of printed space is maximum when length and breadth of the sheet are 4 metres and 6 metres respectively.

Q. 32 Find the Cartesian and vector equation of the plane which makes intercepts 1, 1, 1 on the coordinate axes

Ans. The plane makes intercepts 1, 1, 1 on the co-ordinate axes.

Let A(1, 0, 0), B(0, 1, 0), C(0, 0, 1) be the intersecting point of a plane with co-ordinate axes.

Vector equation of a plane passing through non-collinear points

$A(\bar{a}), B(\bar{b})$ and $C(\bar{c})$ is $(\bar{r} - \bar{a}) \cdot (\bar{b} - \bar{a}) \times (\bar{c} - \bar{a}) = 0$

$$\therefore \left(\bar{b}-\bar{a}\right)\times\left(\bar{c}-\bar{a}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 0\\ -1 & 0 & 1\end{vmatrix}$$

$= \hat{i}+\hat{j}+\hat{k}$

$\therefore \left(\bar{r}-\hat{i}\right)\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) = 0$

$\therefore \bar{r}\left(\hat{i}+ht\hat{j}+\hat{k}\right)-\hat{i}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) = 0$

$\therefore \bar{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)-1 = 0$ ......$\left[\because \hat{i}\cdot\hat{i}=1, \hat{i}\cdot\hat{j}=0, \hat{i}\cdot\hat{k}=0\right]$

$\therefore \bar{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) = 1$ ......(i)

Putting $\bar{r}=\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$ in (i), we get

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) = 1$

$\therefore$ x + y + z = 1, which is the required Cartesian equation

Q. 33

If four points $A\left(\bar{a}\right), B\left(\bar{b}\right), C\left(\bar{c}\right)$ and $D\left(\bar{d}\right)$ are coplanar, then show that

$\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]+\left[\bar{b}\ \ \bar{c}\ \ \bar{d}\right]+\left[\bar{c}\ \ \bar{a}\ \ \bar{d}\right]=\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right].$

Ans.

If points $A\left(\bar{a}\right), B\left(\bar{b}\right), C\left(\bar{c}\right)$ and $D\left(\bar{d}\right)$ are coplanar, then $\overline{AB}, \overline{AC}, \overline{AD}$ are also coplanar.

$\therefore \overline{AB}\cdot\left(\overline{AC}\times\overline{AD}\right) = 0$ ....(i)

Here, $\overline{AB}=\bar{b}-\bar{a}$

$\overline{AC}=\bar{c}-\bar{a}$

$\overline{AD}=\bar{d}-\bar{a}$

From (i), we get

$$\left(\bar{b}-\bar{a}\right).\left[\left(\bar{c}-\bar{a}\right)\times\left(\bar{d}-\bar{a}\right)\right] = 0$$

$$\therefore \left(\bar{b}-\bar{a}\right).\left[\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d}+\bar{a}\times\bar{a}\right] = 0$$

$$\therefore \left(\bar{b}-\bar{a}\right)\cdot\left[\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d}+\bar{a}\times\bar{0}\right] = 0$$

$$\therefore \left(\bar{b}-\bar{a}\right)\cdot\left[\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d}\right] = 0$$

$$\therefore \bar{b}\cdot\left(\bar{c}\times\bar{d}\right)-\bar{b}\cdot\left(\bar{c}\times\bar{a}\right)-\bar{b}\cdot\left(\bar{a}\times\bar{d}\right)-\bar{a}\cdot\left(\bar{c}\times\bar{d}\right)+\bar{a}\cdot\left(\bar{c}\times\bar{a}\right)+\bar{a}\cdot\left(\bar{a}\times\bar{d}\right) = 0$$

$$\therefore \left[\bar{b}\ \ \bar{c}\ \ \bar{d}\right]-\left[\bar{b}\ \ \bar{c}\ \ \bar{a}\right]-\left[\bar{b}\ \ \bar{a}\ \ \bar{d}\right]-\left[\bar{a}\ \ \bar{c}\ \ \bar{d}\right]+\left[\bar{a}\ \ \bar{c}\ \ \bar{a}\right]-\left[\bar{a}\ \ \bar{a}\ \ \bar{d}\right] = 0$$

$$\therefore \left[\bar{b}\ \ \bar{c}\ \ \bar{d}\right]-\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]+\left[\bar{a}\ \ \bar{b}\ \ \bar{d}\right]+\left[\bar{c}\ \ \bar{a}\ \ \bar{d}\right] + 0 + 0 = 0$$

$$\therefore \left[\bar{a}\ \ \bar{b}\ \ \bar{d}\right]+\left[\bar{b}\ \ \bar{c}\ \ \bar{d}\right]+\left[\bar{c}\ \ \bar{a}\ \ \bar{d}\right]=\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right]$$

Q. 34 If Q is the foot of the perpendicular from P(2, 4, 3) on the line joining the point A(1, 2, 4) and B(3, 4, 5), find coordinates of Q

Ans. Let PQ be the perpendicular drawn from point P(2, 4, 3) to the line joining the points A(1, 2, 4) and B (3, 4, 5).

Let Q divides AB internally in the ratio λ:1

$$\therefore Q \equiv \left(\frac{3\lambda+1}{\lambda+1}, \frac{44\lambda+2}{\lambda+1}, \frac{5\lambda+4}{\lambda+1}\right) \quad ....(i)$$

Direction ratios of PQ are

$$\frac{3\lambda+1}{\lambda+1}-2, \frac{4\lambda+2}{\lambda+1}-4, \frac{5\lambda+4}{\lambda+1}-3$$

$$\text{i.e., } \frac{\lambda-1}{\lambda+1}, \frac{-2}{\lambda+1}, \frac{2\lambda+1}{\lambda+1}$$

Now, direction ratios of AB are, 3 – 1, 4 – 2, 5 – 4 i.e., 2, 2, 1.

Since PQ is perpendicular to AB,

$$2\left(\frac{\lambda-1}{\lambda+1}\right)+\frac{2(-2)}{\lambda+1}+1\left(\frac{2\lambda+1}{\lambda+1}\right) = 0$$

$\therefore \frac{2\lambda - 2 - 4 + 2\lambda + 1}{\lambda + 1} = 0$

$\therefore 4\lambda - 5 = 0$

$\therefore 4\lambda = 5$

$\therefore \lambda = \frac{5}{4}$

Putting $\lambda = \frac{5}{4}$ in (i),

Coordinates of Q are,

$$\frac{3\left(\frac{5}{4}\right) + 1}{\left(\frac{5}{4}\right) + 1} = \frac{19}{19}$$

$$\frac{\left(\frac{5}{4}\right) + 2}{\left(\frac{5}{4}\right) + 1} = \frac{28}{9}$$

$$\frac{5\left(\frac{5}{4}\right) + 4}{\left(\frac{5}{4}\right) + 1} = \frac{41}{9}$$

$\therefore Q \equiv \left(\frac{19}{9}, \frac{28}{9}, \frac{41}{9}\right)$