CBSE Test Paper 03

Chapter 5 Continuity and Differentiability

1. If x + |y| = 2y, then y as a function of x is
   a. differentiable for all x
   b. not continuous at x = 0
   c. such that $\frac{dy}{dx} = \frac{1}{3}$ for x < 0
   d. not defined for all real x
2. $\underset{x \to 0}{Lt} \frac{\tan x}{\log(1+x)}$ is equal to
   a. 1
   b. does not exist
   c. 0
   d. none of these
3. $\underset{x \to -2}{Lt} \frac{\sqrt{x^2+5}-3}{x+2}$ is equal to
   a. 0
   b. None of these
   c. $-\frac{2}{3}$
   d. $\frac{2}{3}$
4. $\underset{h \to 0}{Lt} \left(\frac{1}{h\sqrt[3]{8+h}} - \frac{1}{2h}\right)$ is equal to
   a. $-\frac{1}{48}$
   b. $\frac{1}{24}$
   c. None of these
   d. $\frac{1}{48}$
5. If f(x) = $x\left(\sqrt{x} - \sqrt{x+1}\right)$, then
   a. f (x) is not differentiable at x = 0
   b. f (x) is continuous but not differentiable at x = 0
   c. f(x) is differentiable at x = 0
   d. None of these
6. If f(x) = |cos x - sin x|, then $f'\left(\frac{\pi}{3}\right)$ = ________.
7. The derivative of sin x w.r.t. cos x is ________.

8. If $f(x) = \begin{cases} ax+1, \ if\ x \geq 1 \\ x+2, \ if\ x < 1 \end{cases}$ is continuous, then a should be equal to ______.

9. Differentiate the following function with respect to x : $\sin\left(x^2+5\right)$.

10. Differentiate the following function with respect to x: $\frac{e^x}{\sin x}$.

11. Find the value of the constant k so that the function f defined below is continuous at $x=0$, Where $f\left(x\right) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, x \neq 0 \\ k, \ x=0 \end{cases}$.

12. If x and y are connected parametrically by the equations given, without eliminating the parameter, find $\frac{dy}{dx}$. **(2)**

    x = sin t, y = cos 2t

13. If y = $(\tan^{-1}x)^2$ show that $(x_2 + 1)^2y2 + 2x(x^2 + 1)y_1 = 2$.

14. Find $\frac{dy}{dx}$, if $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

15. Find $\frac{dy}{dx}, y = (\sin x - \cos x)^{(\sin x - \cos x)}$.

16. Find the values of k so that the function f is continuous at the indicated point:

    $f\left(x\right) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, \ if\, x \neq \dfrac{\pi}{2} \\ 3, \ if\ x = \dfrac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$

17. Find the values of a and b such that the following function f(x) is a continuous function.

    f(x) = $\begin{cases} 5, & x \leq 2 \\ ax+b, & 2 < x < 10 \\ 21, & x \geq 10 \end{cases}$

18. If $y\sqrt{x^2+1} - \log\left(\sqrt{x^2+1} - x\right) = 0$ prove that $\left(x^2+1\right)\frac{dy}{dx} + xy + 1 = 0$.

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Solution

1. c. such that $\frac{dy}{dx}=\frac{1}{3}$ for x < 0, **Explanation:** $x+|y|=2y \Rightarrow x=2y-|y|$ when $y \geqslant 0\ then\ x=2y-y \Rightarrow x=y\ for\ y \geqslant 0\ i.e. y=x\ for\ x \geqslant 0$and when $y<0, then\ x=2y-(-y) \Rightarrow x=3y$ $for\ y<0\ i.e.\ y=1/3\ x\ for\ \ x<0.$
   We have , $y=\begin{cases} x & , x \geqslant 0 \\ \frac{1}{3}x & , x<0 \end{cases}$
2. a. 1, **Explanation:** $\lim\limits_{x\to 0}\frac{tanx}{log(1+x)}=\lim\limits_{x\to 0}\frac{tanx}{x}\frac{x}{log(1+x)}$
   $=\lim\limits_{x\to 0}\frac{tanx}{x}.\lim\limits_{x\to 0}\frac{1}{\frac{1}{x}log(1+x)}=1.\frac{1}{1}=1$
3. c. $-\frac{2}{3}$, **Explanation:** $\lim\limits_{x\to -2}\frac{\sqrt{x^2+5}-3}{x+2}=\lim\limits_{x\to -2}\frac{x^2+5-9}{(x+2)(\sqrt{x^2+5}+3)}$
   $=\lim\limits_{x\to -2}\frac{x-2}{(\sqrt{x^2+5}+3)}=\frac{-4}{3+3}=-\frac{2}{3}$
4. a. $-\frac{1}{48}$, **Explanation:** $\lim\limits_{h\to 0}\left(\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right)=\lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{2}\left(1+\frac{h}{8}\right)^{-\frac{1}{3}}-\frac{1}{2}\right\}$
   $$=\lim_{h\to 0}\frac{1}{h}.\frac{1}{2}\left\{1-\frac{1}{3}.\frac{h}{8}+\frac{\left(-\frac{1}{3}\right)\left(-\frac{5}{3}\right)}{2}\frac{h^2}{64}+\ldots-1\right\}$$
   $=\lim_{h\to 0}\frac{1}{2h}\left\{-\frac{h}{24}+\frac{5}{18}.\frac{h^2}{64}+\ldots\right\}$
   $=\lim_{h\to 0}\frac{1}{2}\left\{-\frac{1}{24}+\frac{5}{18}.\frac{h}{64}+\ldots\right\}$
   $=-\frac{1}{48}$
5. a. f (x) is not differentiable at x = 0, **Explanation:** f is defined on the left of x = 0, therefore, f is neither continuous nor differentiable at x = 0
6. $\frac{\sqrt{3}+1}{2}$
7. -cot x
8. a = 2

9. Let $y=\sin\left(x^2+5\right)$

$\therefore \frac{dy}{dx}=\cos\left(x^2+5\right)\frac{d}{dx}\left(x^2+5\right)$

$=\cos\left(x^2+5\right)(2x+0)$

$=2x\cos\left(x^2+5\right)$

10. Let $y=\frac{e^x}{\sin x}$

$\therefore \frac{dy}{dx}=\frac{\sin x\frac{d}{dx}e^x-e^x\frac{d}{dx}\sin x}{\sin^2 x}$ [By quotient rule]

$=\frac{\sin x.e^x-e^x\cos x}{\sin^2 x}$

$=e^x\frac{(\sin x-\cos x)}{\sin^2 x}$

11. It is given that the function f is continuous at x = 0.

Therefore, $\lim\limits_{x\to 0}f(x)=f(0)$

$\Rightarrow \lim\limits_{x\to 0}\frac{1-\cos 4x}{8x^2}=k$

$\Rightarrow \lim\limits_{x\to 0}\frac{2\sin^2 2x}{8x^2}=k\ \left(\therefore 1-\cos 2x=2\sin^2 x\right)$

$\Rightarrow \lim\limits_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2=k\left(\lim\limits_{x\to 0}\frac{\sin x}{x}=1\right)$

$\Rightarrow k=1\left(\lim\limits_{x\to 0}\frac{\sin x}{x}=1\right)$

12. Given: x = sin t and y = cos 2t

$\therefore \frac{dx}{dt}=\cos t$ and $\frac{dy}{dt}=-\sin 2t\frac{d}{dt}(2t)=-2\sin 2t$

Now $\frac{dy}{dt}=\frac{dy/dt}{dx/dt}=\frac{-2\sin 2t}{\cos t}=\frac{-2\times 2\sin t\cos t}{\cos t}=-4\sin t$

13. Given: y = $(\tan^{-1}x)^2$ ....(i)

$\therefore y_1=2\left(\tan^{-1}x\right)\frac{d}{dx}\tan^{-1}x\left[\because \frac{d}{dx}\{f(x)\}^n=n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$

And $y_1=2\left(\tan^{-1}x\right)\frac{1}{1+x^2}$

$=\frac{2\tan^{-1}x}{1+x^2}$

$\Rightarrow \left(1+x^2\right)y_1=2\tan^{-1}x$

Again differentiating both sides w.r.t. x,

$\left(1+x^2\right)\frac{d}{dx}y_1+y_1\frac{d}{dx}\left(1+x^2\right)=2.\frac{1}{1+x^2}$

$\Rightarrow \left(1+x^2\right)y_2+y_1.2x=\frac{2}{1+x^2}$

$\Rightarrow \left(1+x^2\right)^2y_2+2x\left(1+x^2\right)y_1=2$ .Hence proved.

14. Put $x=\tan\theta$, where $\frac{-\pi}{6}<\theta<\frac{\pi}{6}$

Therefore, $y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$=\tan^{-1}(\tan 3\theta)$

$= 3\theta \left(\because \frac{-\pi}{2} < 3\theta < \frac{\pi}{2}\right) = 3\tan^{-1}x$

Hence, $\frac{dy}{dx} = \frac{3}{1+x^2}$

15. $y = (\sin x - \cos x)^{\sin x - \cos x}$

Taking log both sides

$\log y = \log (\sin x - \cos x)^{\sin x - \cos x}$

$\log y = (\sin x - \cos x).\log(\sin x - \cos x)$

Differentiate both sides w.r.t. x

$\frac{1}{y}.\frac{dy}{dx} = (\sin x - \cos x).\frac{1}{(\sin x - \cos x)}(\cos x + \sin x)$

$+\log(\sin x - \cos x).(\cos x + \sin x)$

$\frac{dy}{dx} = y\left[((\cos x + \sin x)) + \log((\sin x - \cos x)).(\cos x + \sin x)\right]$

16. Here, $\lim\limits_{x \to \frac{\pi}{2}} f(x) = \lim\limits_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}$

$\because x \to \frac{\pi}{2}$

$\Rightarrow x \neq \frac{\pi}{2}$

Putting $x = \frac{\pi}{2} + h$ where $h \to 0$

$= \lim\limits_{h \to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)} = \lim\limits_{h \to 0} \frac{-k\sinh}{\pi - \pi - 2h}$

$= \lim\limits_{h \to 0} \frac{k\sin\ h}{2h} = \frac{k}{2} \times \lim\limits_{h \to 0} \frac{\sin h}{h}$

$= \frac{k}{2}$ ....(i)

And $f\left(\frac{\pi}{2}\right) = 3$ ....(ii)

$\because f(x) = 3$ when $x = \frac{\pi}{2}$ [Given]

Since f(x) is continuous at $x = \frac{\pi}{2}$

$\therefore \lim\limits_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$\therefore$ From eq. (i) and (ii),

$\frac{k}{2} = 3 \Rightarrow k = 6$

17. According to the question, f(x) = $\begin{cases} 5, & x \leq 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \geq 10 \end{cases}$ is a continuous function.

So, it is continuous at x = 2 and at x = 10.

$\therefore$LHL$_{x=2}$ = RHL$_{x=2}$ = f(2)......................(i)

and LHL$_{x=10}$ = RHL$_{x=10}$ = f(10)..........(ii)

Now, let us calculate LHL and RHL at x = 2.

LHL = $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5 = 5$

and RHL = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$

$= \lim_{h \to 0} \{a(2 + h) + b\} = \lim_{h \to 0} (2a + ah + b)$

= 2a + b

From Equation (i), LHL=RHL

$\Rightarrow$2a + b = 5.........(iii)

Now, we have to find LHL and RHL at x = 10.

LHL = $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b)$

$= \lim_{h \to 0} [a(10 - h) + b]$

$= \lim_{h \to 0} (10a - ah + b)$

$\Rightarrow$ LHL = 10a + b

and RHL =$\lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} 21$ = 21

Now, from Eq. (ii), we have LHL= RHL

$\Rightarrow 10a + b = 21$............(iv)

Subtracting Equation (iv) from Equation (iii),

$\Rightarrow -8a = -16 \Rightarrow a = 2$

Putting a = 2 in Equation (iv),

$\Rightarrow -8a = -16 \Rightarrow b = 1$

$\therefore a = 2 \; and \; b = 1.$

18. $y\sqrt{x^2 + 1} - \log\left(\sqrt{x^2 + 1} - x\right) = 0$

differentiating both sides w.r.t x

$$y. \frac{1}{2\sqrt{x^2+1}}(2x) + \sqrt{x^2 + 1}. \frac{dy}{dx} - \frac{1}{\sqrt{x^2+1} - x}\left[\frac{1(2x)}{2\sqrt{x^2+1}} - 1\right] = 0$$

$$\frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2 + 1}. \frac{dy}{dx} - \frac{1}{\sqrt{x^2+1} - x}\left[\frac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}}\right] = 0$$

$$\frac{xy + (x^2+1)}{\sqrt{x^2+1}} \frac{dy}{dx} = \frac{-\left(\sqrt{x^2+1} - x\right)}{\left(\sqrt{x^2+1} - x\right)\sqrt{x^2+1}}$$

$$xy + (x^2 + 1)\frac{dy}{dx} = -1$$

$$(x^2 + 1)\frac{dy}{dx} + xy + 1 = 0$$