Chapter 3

Differentiation

Contents

Issac Newton

In the history of mathematics two names are prominent to share the credit for inventing calculus, **Issac Newton** (1642-1727) *and* **G.W. Leibinitz** (1646-1717). *Both of them independently invented calculus around the seventeenth century. After the advent of calculus many mathematicians contributed for further development of calculus. The rigorous concept is mainly attributed to the great mathematicians,* **A.L. Cauchy, J.L., Lagrange** *and* **Karl Weierstrass.**

Before 1900, *it was thought that calculus is quite difficult to teach. So calculus became beyond the reach of youngsters. But just in* 1900, *John Perry and others in England started propagating the view that essential ideas and methods of calculus were simple and could be taught even in schools.* **F.L. Griffin**, *pioneered the teaching of calculus to first year students. This was regarded as one of the most daring act in those days.*

Today not only the mathematics but many other subjects such as Physics, Chemistry, Economics and Biological Sciences are enjoying the fruits of calculus.

Differentiation

Introduction

The rate of change of one quantity with respect to some another quantity has a great importance. For example, the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity is called its acceleration.

The rate of change of a quantity '*y*' with respect to another quantity '*x*' is called the derivative or differential coefficient of *y* with respect to *x*.

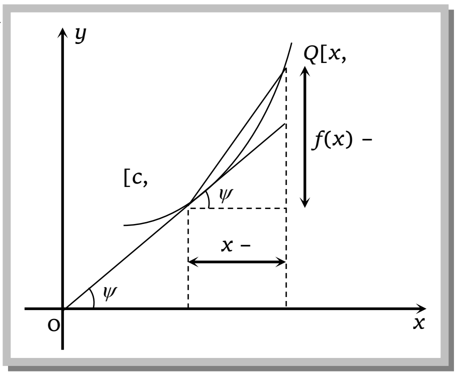
3.1 Derivative at a Point

The derivative of a function at a point $x = a$ is defined by $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ (provided the limit exists and is finite)

The above definition of derivative is also called derivative by first principle.

(1) **Geometrical meaning of derivatives at a point:** Consider the curve $y = f(x)$. Let $f(x)$ be differentiable at $x = c$. Let $P(c, f(c))$ be a point on the curve and $Q(x, f(x))$ be a neighbouring point on the curve. Then,

Slope of the chord $PQ = \frac{f(x) - f(c)}{x - c}$. Taking limit as $Q \to P$, *i.e.*, $x \to$



we get $\lim_{Q \to P} (\text{slope of the chord } PQ) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ ......(i)

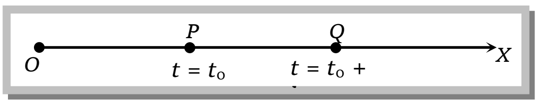
As $Q \to P$, chord *PQ* becomes tangent at *P*.

Therefore from (i), we have

Slope of the tangent at $P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left(\frac{df(x)}{dx}\right)_{x = c}$.

Note : ❑ Thus, the derivatives of a function at a point $x = c$ is the slope of the tangent to curve, $y = f(x)$ at point $(c, f(c))$.

(2) **Physical interpretation at a point :** Let a particle moves in a straight line *OX* starting from *O* towards *X*. Clearly, the position of the particle at any instant would depend upon the time elapsed. In other words, the distance of the particle from *O* will be some function *f* of time *t*.



Let at any time $t = t_0$, the particle be at P and after a further time h, it is at Q so that OP = $f(t_0)$ and $OQ = f(t_0 + h)$. Hence, the average speed of the particle during the journey from P to Q is $\frac{PQ}{h}$, *i.e.,* $\frac{f(t_0+h)-f(t_0)}{h} = f(t_0, h)$. Taking the limit of $f(t_0, h)$ as $h \to 0$, we get its instantaneous speed to be $\lim_{h\to 0}\frac{f(t_0+h)-f(t_0)}{h}$, which is simply $f'(t_0)$. Thus, if $f(t)$ gives the distance of a moving particle at time t, then the derivative of f at $t = t_0$ represents the instantaneous speed of the particle at the point P, *i.e.,* at time $t = t_0$.

Important Tips

- ☞ *$\frac{dy}{dx}$ is $\frac{d}{dx}(y)$ in which $\frac{d}{dx}$ is simply a symbol of operation and not 'd' divided by dx.*
- ☞ *If $f'(x_0) = \infty$, the function is said to have an infinite derivative at the point x_0. In this case the line tangent to the curve of $y = f(x)$ at the point x_0 is perpendicular to the x-axis*

Example: 1 If $f(2) = 4,\ \ f'(2) = 1,$ then $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2} =$ **[Rajasthan PET 1995, 2000]**

(a) 1 (b) 2 (c) 3 (d) – 2

Solution: (b) Given $f(2) = 4, f'(2) = 1$

$$\therefore\ \lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2} = \lim_{x\to 2}\frac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2} = \lim_{x\to 2}\frac{(x-2)f(2)}{x-2} - \lim_{x\to 2}\frac{2f(x)-2f(2)}{x-2}$$

$$= f(2) - 2\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = f(2) - 2f'(2) = 4 - 2(1) = 4 - 2 = 2$$

Trick : Applying L-Hospital rule, we get $\lim_{x\to 2}\frac{f(2)-2f'(2)}{1} = 2$.

Example: 2 If $f(x+y) = f(x).f(y)$ for all x and y and $f(5) = 2,\ \ f'(0) = 3,$ then $f'(5)$ will be

[IIT 1981; Karnataka CET 2000; UPSEAT 2002; MP PET 2002; AIEEE 2002]

(a) 2 (b) 4 (c) 6 (d) 8

Solution: (c) Let $x = 5, y = 0 \Rightarrow f(5+0) = f(5).f(0)$

$\Rightarrow f(5) = f(5)f(0) \Rightarrow f(0) = 1$

Therefore, $f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0}\frac{f(5)f(h)-f(5)}{h} = \lim_{h\to 0} 2\left[\frac{f(h)-1}{h}\right]$ $\{\because f(5) = 2\}$

$= 2\lim_{h\to 0}\left[\frac{f(h)-f(0)}{h}\right] = 2\times f'(0) = 2\times 3 = 6$.

Example: 3 If $f(a) = 3, f'(a) = -2, g(a) = -1, g'(a) = 4,$ then $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a} =$ **[MP PET 1997]**

(a) – 5 (b) 10 (c) – 10 (d) 5

Solution: (b) $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$. We add and subtract $g(a)f(a)$ in numerator

$= \lim_{x\to a}\frac{g(x)f(a)-g(a)f(a)+g(a)f(a)-g(a)f(x)}{x-a} = \lim_{x\to a} f(a)\left[\frac{g(x)-g(a)}{x-a}\right]-\lim_{x\to a} g(a)\left[\frac{f(x)-f(a)}{x-a}\right]$

$= f(a)\lim_{x\to a}\left[\frac{g(x)-g(a)}{x-a}\right]-g(a)\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right] = f(a)g'(a)-g(a)f'(a)$ [by using first principle formula]

= 3.4 – (–1)(–2) = 12 – 2 = 10

Trick : $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$

Using L–Hospital's rule, Limit = $\lim_{x\to a}\frac{g'(x)f(a)-g(a)f'(x)}{1}$;

Limit = $g'(a)\ f(a)-g(a)f'(a) = (4)(3)-(-1)(-2)$ = 12 – 2 = 10.

Example: 4 If $5f(x)+3f\left(\frac{1}{x}\right)=x+2$ and $y=xf(x)$ then $\left(\frac{dy}{dx}\right)_{x=1}$ is equal to

(a) 14 (b) $\frac{7}{8}$ (c) 1 (d) None of these

Solution: (b) $\because 5f(x)+3f\left(\frac{1}{x}\right)=x+2$ ......(i)

Replacing x by $\frac{1}{x}$ in (i), $5f\left(\frac{1}{x}\right)+3f(x)=\frac{1}{x}+2$ ......(ii)

On solving equation (i) and (ii), we get, $16f(x)=5x-\frac{3}{x}+4$, $\therefore\ 16f'(x)=5+\frac{3}{x^2}$

$\because\ y=xf(x) \Rightarrow \frac{dy}{dx}=f(x)+xf'(x)=\frac{1}{16}(5x-\frac{3}{x}+4)+x.\frac{1}{16}(5+\frac{3}{x^2})$

at $x=1$, $\frac{dy}{dx}=\frac{1}{16}(5-3+4)+\frac{1}{16}(5+3) = \frac{7}{8}$.

3.2 Some Standard Differentiation

(1) **Differentiation of algebraic functions**

(i) $\frac{d}{dx}x^n = nx^{n-1}, x\in R, n\in R, x>0$ (ii) $\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$ (iii) $\frac{d}{dx}\left(\frac{1}{x^n}\right)=-\frac{n}{x^{n+1}}$

(2) **Differentiation of trigonometric functions :** The following formulae can be applied directly while differentiating trigonometric functions

(i) $\frac{d}{dx}\sin x=\cos x$ (ii) $\frac{d}{dx}\cos x=-\sin x$ (iii) $\frac{d}{dx}\tan x=\sec^2 x$

(iv) $\frac{d}{dx}\sec x=\sec x\tan x$ (v) $\frac{d}{dx}\text{cosec}\, x=-\text{cosec}\, x\cot x$ (vi) $\frac{d}{dx}\cot x=-\text{cosec}^2 x$

(3) **Differentiation of logarithmic and exponential functions :** The following formulae can be applied directly when differentiating logarithmic and exponential functions

(i) $\frac{d}{dx}\log x=\frac{1}{x}$, for $x > 0$ (ii) $\frac{d}{dx}e^x=e^x$

(iii) $\frac{d}{dx} a^x = a^x \log a$, for $a > 0$ (iv) $\frac{d}{dx} \log_a x = \frac{1}{x \log a}$, for $x > 0$, $a > 0$, $a \neq 1$

(4) **Differentiation of inverse trigonometrical functions :** The following formulae can be applied directly while differentiating inverse trigonometrical functions

(i) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$ (ii) $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

(iii) $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x| \sqrt{x^2-1}}$, for $|x| > 1$ (iv) $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{|x| \sqrt{x^2-1}}$, for $|x| > 1$

(v) $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$, for $x \in R$ (vi) $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$, for $x \in R$

(5) **Differentiation of hyperbolic functions :**

(i) $\frac{d}{dx} \sin h\, x = \cos h\, x$ (ii) $\frac{d}{dx} \cos h\, x = \sin h\, x$

(iii) $\frac{d}{dx} \tan h\, x = \sec h^2 x$ (iv) $\frac{d}{dx} \cot h\, x = -\operatorname{cosec} h^2 x$

(v) $\frac{d}{dx} \sec h\, x = -\sec h\, x \tan h\, x$ (vi) $\frac{d}{dx} \operatorname{cosec} h\, x = -\operatorname{cosec} h\, x \cot h\, x$

(vii) $\frac{d}{dx} \sin h^{-1} x = 1/\sqrt{(1+x^2)}$ (viii) $\frac{d}{dx} \cos h^{-1} x = 1/\sqrt{(x^2-1)}$

(ix) $\frac{d}{dx} \tan h^{-1} x = 1/(x^2-1)$ (x) $\frac{d}{dx} \cot h^{-1} x = 1/(1-x^2)$

(xi) $\frac{d}{dx} \sec h^{-1} x = -1/x\sqrt{(1-x^2)}$ (xii) $\frac{d}{dx} \operatorname{cosec} h^{-1} x = -1/x\sqrt{(1+x^2)}$

(6) **Differentiation by inverse trigonometrical substitution:** For trigonometrical substitutions following formulae and substitution should be remembered

(i) $\sin^{-1} x + \cos^{-1} x = \pi/2$ (ii) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(iii) $\sec^{-1} x + co\sec^{-1} x = \pi/2$ (iv) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(v) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right]$ (vi) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left[\frac{x \pm y}{1 \mp xy}\right]$

(vii) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$ (viii) $2\cos^{-1} x = \cos^{-1}(2x^2-1)$

(ix) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(x) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ (xi) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

(xii) $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$ (xiii) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(xiv) $\sin^{-1}(-x) = -\sin^{-1} x$

(xv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(xvi) $\tan^{-1}(-x) = -\tan^{-1} x$ or $\pi - \tan^{-1} x$

(xvii) $\frac{\pi}{4} - \tan^{-1} x = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

(7) **Some suitable substitutions**

S. N.	Function	Substitution	S. N.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$	(ii)	$\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$	(iv)	$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
(v)	$\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$	$x^2 = a^2\cos 2\theta$	(vi)	$\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$	(viii)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^2\theta$
(ix)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$	(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

3.3 Theorems for Differentiation

Let $f(x), g(x)$ and $u(x)$ be differentiable functions

(1) If at all points of a certain interval. $f'(x) = 0$, then the function $f(x)$ has a constant value within this interval.

(2) **Chain rule**

(i) **Case I :** If y is a function of u and u is a function of x, then derivative of y with respect to x is $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or $y = f(u) \Rightarrow \frac{dy}{dx} = f'(u)\frac{du}{dx}$

(ii) **Case II :** If y and x both are expressed in terms of t, y and x both are differentiable with respect to t then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

(3) **Sum and difference rule :** Using linear property $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

(4) **Product rule :** (i) $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$ (ii) $\frac{d}{dx}(u.v.w.) = u.v.\frac{dw}{dx} + v.w.\frac{du}{dx} + u.w.\frac{dv}{dx}$

(5) **Scalar multiple rule :** $\frac{d}{dx}(k\,f(x)) = k\frac{d}{dx}f(x)$

(6) **Quotient rule :** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$, provided $g(x) \neq 0$

Example: 5 The derivative of $f(x) = |x|^3$ at $x = 0$ is **[Rajasthan PET 2001; Haryana CEE 2002]**

(a) 0 (b) 1 (c) –1 (d) Not defined

Solution: (a) $f(x)=\begin{cases} x^3 & , x\ge 0 \\ -x^3 & , x<0\end{cases}$ and $f'(x)=\begin{cases} 3x^2 & , x\ge 0 \\ -3x^2 & , x<0\end{cases}$

$f'(0^+)=f'(0^-)=0$

Example: 6 The first derivative of the function $(\sin 2x\cos 2x\cos 3x+\log_2 2^{x+3})$ with respect to x at $x=\pi$ is

(a) 2 (b) –1 (c) $-2+2^{\pi}\log_e 2$ (d) $-2+\log_e 2$

Solution: (b) $f(x)=\sin 2x.\cos 2x.\cos 3x+\log_2 2^{x+3}$, $f(x)=\frac{1}{2}\sin 4x\cos 3x+(x+3)\log_2 2$, $f(x)=\frac{1}{4}[\sin 7x+\sin x]+x+3$

Differentiate *w.r.t.* x,

$f'(x)=\frac{1}{4}[7\cos 7x+\cos x]+1$, $f'(x)=\frac{1}{4}7\cos 7x+\frac{1}{4}\cos x+1$, $f'(\pi)=-2+1=-1$.

Example: 7 If $y=|\cos x|+|\sin x|$ then $\frac{dy}{dx}$ at $x=\frac{2\pi}{3}$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0 (c) $\frac{1}{2}(\sqrt{3}-1)$ (d) None of these

Solution: (c) Around $x=\frac{2\pi}{3}$, $|\cos x|=-\cos x$ and $|\sin x|=\sin x$

$\therefore\ y=-\cos x+\sin x\ \therefore\ \frac{dy}{dx}=\sin x+\cos x$

At $x=\frac{2\pi}{3}$, $\frac{dy}{dx}=\sin\frac{2\pi}{3}+\cos\frac{2\pi}{3}=\frac{\sqrt{3}}{2}-\frac{1}{2}=\frac{1}{2}(\sqrt{3}-1)$.

Example: 8 If $f(x)=\log_x(\log x)$, then $f'(x)$ at $x=e$ is **[IIT 1985; Rajasthan PET 2000; MP PET 2000; Karnataka CET 2002]**

(a) e (b) $1/e$ (c) 1 (d) None of these

Solution: (b) $f(x)=\log_x(\log x)=\frac{\log(\log x)}{\log x}\Rightarrow f'(x)=\frac{\frac{1}{x}-\frac{1}{x}\log(\log x)}{(\log x)^2}\Rightarrow f'(e)=\frac{\frac{1}{e}-0}{1}=\frac{1}{e}$

Example: 9 If $f(x)=|\log x|$, then for $x\ne 1, f'(x)$ equals

(a) $\frac{1}{x}$ (b) $\frac{1}{|x|}$ (c) $\frac{-1}{x}$ (d) None of these

Solution: (d) $f(x)=|\log x|=\begin{cases}-\log x, & \text{if } 0<x<1\\ \log x, & \text{if } x\ge 1\end{cases}\Rightarrow f'(x)=\begin{cases}-\frac{1}{x}, & \text{if } 0<x<1\\ \frac{1}{x}, & \text{if } x>1\end{cases}$.

Clearly $f'(1^-)=-1$ and $f'(1^+)=1$, $\therefore$ $f'(x)$ does not exist at $x=1$

Example: 10 $\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ equals to

(a) 1 (b) $\frac{x^2+1}{x^2-4}$ (c) $\frac{x^2-1}{x^2-4}$ (d) $e^x\frac{x^2-1}{x^2-4}$

Solution: (c) Let $y = \left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right] = \log e^x + \log\left(\frac{x-2}{x+2}\right)^{3/4}$

$\Rightarrow y = x + \frac{3}{4}[\log(x-2) - \log(x+2)] \Rightarrow \frac{dy}{dx} = 1 + \frac{3}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right] = 1 + \frac{3}{(x^2-4)}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2-1}{x^2-4}$.

Example: 11 If $x = \exp\left\{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)\right\}$ then $\frac{dy}{dx}$ equals **[MP PET 2002]**

(a) $2x[1+\tan(\log x)] + x\sec^2(\log x)$ (b) $x[1+\tan(\log x)] + \sec^2(\log x)$

(c) $2x[1+\tan(\log x)] + x^2 \sec^2(\log x)$ (d) $2x[1+\tan(\log x)] + \sec^2(\log x)$

Solution: (a) $x = \exp\left\{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)\right\} \Rightarrow \log x = \tan^{-1}\left(\frac{y-x^2}{x^2}\right)$

$\Rightarrow \frac{y-x^2}{x^2} = \tan(\log x) \Rightarrow y = x^2\tan(\log x) + x^2 \Rightarrow \frac{dy}{dx} = 2x.\tan(\log x) + x^2.\frac{\sec^2(\log x)}{x} + 2x$

$\Rightarrow \frac{dy}{dx} = 2x\tan(\log x) + x\sec^2(\log x) + 2x \Rightarrow \frac{dy}{dx} = 2x[1+\tan(\log x)] + x\sec^2(\log x)$.

Example: 12 If $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx} =$ **[UPSEAT 1999; AMU 2002]**

(a) 0 (b) $\frac{1}{\sqrt{x}+1}$ (c) 1 (d) None of these

Solution: (a) $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 0$ $\left\{\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right\}$

Example: 13 $\frac{d}{dx}\tan^{-1}\left[\frac{\cos x - \sin x}{\cos x + \sin x}\right]$ **[AISSE 1985, 87; DSSE 1982, 84; MNR 1985; Karnataka CET 2002; Rajasthan PET 2002, 03]**

(a) $\frac{1}{2(1+x^2)}$ (b) $\frac{1}{1+x^2}$ (c) 1 (d) – 1

Solution: (d) $\frac{d}{dx}\tan^{-1}\left[\frac{\cos x - \sin x}{\cos x + \sin x}\right] = \frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right] = -1$.

Example: 14 $\frac{d}{dx}\left[\sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$ equals **[MP PET 2002; EAMCET 1996]**

(a) -1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

Solution: (b) Let $y = \sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$\Rightarrow y = \sin^2\cot^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\} = \sin^2\cot^{-1}\left(\tan\frac{\theta}{2}\right) \Rightarrow y = \sin^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta) = \frac{1}{2}(1+x)$

$\therefore \frac{dy}{dx} = \frac{1}{2}$

Example: 15 If $y = \cos^{-1}\left(\frac{5\cos x - 12\sin x}{13}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) 1 (b) – 1 (c) 0 (d) None of these

Solution: (a) Let $\cos\alpha = \frac{5}{13}$. Then $\sin\alpha = \frac{12}{13}$. So, $y = \cos^{-1}\{\cos\alpha . \cos x - \sin\alpha . \sin x\}$

$\therefore y = \cos^{-1}\{\cos(x+\alpha)\} = x + \alpha$ ($\because$ $x + \alpha$ is in the first or the second quadrant)

$\therefore \frac{dy}{dx} = 1$.

Example: 16 $\frac{d}{dx}\cosh^{-1}(\sec x) =$ **[Rajasthan PET 1997]**

(a) $\sec x$ (b) $\sin x$ (c) $\tan x$ (d) $\text{cosec} x$

Solution: (a) We know that $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx}\cosh^{-1}(\sec x) = \frac{1}{\sqrt{\sec^2 x - 1}}\sec x \tan x = \frac{\sec x \tan x}{\tan x} = \sec x$.

Example: 17 $\frac{d}{dx}\left[\left(\frac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x}\right)\cot 3x\right]$ **[AMU 2000]**

(a) $\tan 2x \tan x$ (b) $\tan 3x \tan x$ (c) $\sec^2 x$ (d) $\sec x \tan x$

Solution: (c) Let $y = \frac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x} = \frac{(\tan 2x - \tan x)}{(1 + \tan 2x \tan x)}\frac{(\tan 2x + \tan x)}{(1 - \tan 2x \tan x)} = \tan(2x - x)\tan(2x + x) = \tan x \tan 3x$.

$\therefore \frac{d}{dx}[y . \cot 3x] = \frac{d}{dx}[\tan x] = \sec^2 x$.

Example: 18 If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ is equal to **[Rajasthan PET 2000]**

(a) – 1 (b) 1 (c) $\log 2$ (d) $-\log 2$

Solution: (a) $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$

Put $x^x = \tan\theta$, $\therefore$ $y = f(x) = \cot^{-1}\left(\frac{\tan^2\theta - 1}{2\tan\theta}\right) = \cot^{-1}(-\cot 2\theta) = \pi - \cot^{-1}(\cot 2\theta)$

$\Rightarrow y = \pi - 2\theta = \pi - 2\tan^{-1}(x^x) \Rightarrow \frac{dy}{dx} = \frac{-2}{1 + x^{2x}} . x^x(1 + \log x) \Rightarrow f'(1) = -1$.

Example: 19 If $y = (1+x)(1+x^2)(1+x^4).......(1+x^{2^n})$ then $\frac{dy}{dx}$ at $x = 0$ is

(a) 1 (b) – 1 (c) 0 (d) None of these

Solution: (a) $y = \frac{(1-x)(1+x)(1+x^2).....(1+x^{2^n})}{1-x} = \frac{1 - x^{2^{n+1}}}{1-x}$

$\therefore \frac{dy}{dx} = \frac{-2^{n+1}.x^{2^{n+1}-1}(1-x) + 1 - x^{2^{n+1}}}{(1-x)^2}$, $\therefore$ At $x = 0$, $\frac{dy}{dx} = \frac{-2^{n+1}\, 0.1 + 1 - 0}{1^2} = 1$.

Example: 20 If $f(x) = \cos x . \cos 2x . \cos 4x . \cos 8x . \cos 16x$ then $f'\left(\frac{\pi}{4}\right)$ is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) None of these

Solution: (a) $f(x) = \frac{2\sin x.\cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x}{2\sin x} = \frac{\sin 32x}{2^5 \sin x}$

$\therefore\ f'(x) = \frac{1}{32}.\frac{32\cos 32x.\sin x - \cos x.\sin 32x}{\sin^2 x}$

$\therefore\ f'\left(\frac{\pi}{4}\right) = \frac{32.\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}.0}{32.\left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2}$.

3.4 Relation between *dy*/*dx* and *dx*/*dy*

Let x and y be two variables connected by a relation of the form $f(x, y) = 0$. Let Δx be a small change in x and let Δy be the corresponding change in y. Then $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ and $\frac{dx}{dy} = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y}$.

Now, $\frac{\Delta y}{\Delta x}.\frac{\Delta x}{\Delta y} = 1 \Rightarrow \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}.\frac{\Delta x}{\Delta y}\right) = 1$

$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = 1$ $[\because \Delta x \to 0 \Leftrightarrow \Delta y \to 0] \Rightarrow \frac{dy}{dx}.\frac{dx}{dy} = 1$. So, $\frac{dy}{dx} = \frac{1}{dx/dy}$.

3.5 Methods of Differentiation

(1) **Differentiation of implicit functions :** If y is expressed entirely in terms of x, then we say that y is an explicit function of x. For example $y = \sin x$, $y = e^x$, $y = x^2 + x + 1$ etc. If y is related to x but can not be conveniently expressed in the form of $y = f(x)$ but can be expressed in the form $f(x, y) = 0$, then we say that y is an implicit function of x.

(i) *Working rule* 1 : (a) Differentiate each term of $f(x, y) = 0$ with respect to x.

(b) Collect the terms containing dy/dx on one side and the terms not involving dy/dx on the other side.

(c) Express dy/dx as a function of x or y or both.

Note : ❑ In case of implicit differentiation, dy/dx may contain both x and y.

(ii) *Working rule* 2 : If $f(x, y)$= constant, then $\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$ are partial differential coefficients of $f(x,y)$ with respect to x and y respectively.

Note : ❑ Partial differential coefficient of $f(x,y)$ with respect to x means the ordinary differential coefficient of $f(x,y)$ with respect to x keeping y constant.

Example: 21 If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} =$ **[IIT 1996]**

(a) – 1 (b) – 2 (c) 1 (d) 2

Solution: (c) We are given that $xe^{xy} = y + \sin^2 x$

When $x = 0$, we get $y = 0$

Differentiating both sides *w.r.t.* x, we get, $e^{xy} + xe^{xy}\left[x\frac{dy}{dx} + y\right] = \frac{dy}{dx} + 2\sin x\cos x$

Putting, $x = 0$, $y = 0$, we get $\frac{dy}{dx} = 1$.

Example: 22 If $\sin(x+y) = \log(x+y)$, then $\frac{dy}{dx} =$ **[Karnataka CET 1993; Rajasthan PET 1989, 1992; Roorkee 2000]**

(a) 2 (b) – 2 (c) 1 (d) – 1

Solution: (d) $\sin(x+y) = \log(x+y)$

Differentiating with respect to x, $\cos(x+y)\left[1 + \frac{dy}{dx}\right] = \frac{1}{x+y}\left[1 + \frac{dy}{dx}\right]$

$\left[\cos(x+y) - \frac{1}{x+y}\right]\left[1 + \frac{dy}{dx}\right] = 0$

$\because$ $\cos(x+y) \neq \frac{1}{x+y}$ for any x and y. So, $1 + \frac{dy}{dx} = 0$, $\frac{dy}{dx} = -1$.

Trick: It is an implicit function, so $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\cos(x+y) - \frac{1}{x+y}}{\cos(x+y) - \frac{1}{x+y}} = -1$.

Example: 23 If $\ln(x+y) = 2xy$, then $y'(0) =$ **[IIT Screening 2004]**

(a) 1 (b) – 1 (c) 2 (d) 0

Solution: (a) $\ln(x+y) = 2xy \Rightarrow \frac{(1 + dy/dx)}{(x+y)} = 2\left(x\frac{dy}{dx} + y\right) \Rightarrow \frac{dy}{dx} = \frac{1 - 2xy - 2y^2}{2x^2 + 2xy - 1} \Rightarrow y'(0) = \frac{1-2}{-1} = 1$, at $x = 0$, $y = 1$.

(2) **Logarithmic differentiation :** If differentiation of an expression or an equation is done after taking log on both sides, then it is called logarithmic differentiation. This method is useful for the function having following forms.

(i) $y = [f(x)]^{g(x)}$

(ii) $y = \frac{f_1(x).f_2(x)........}{g_1(x).g_2(x)........}$ where $g_i(x) \neq 0$ (where i = 1, 2, 3,.....), $f_i(x)$ and $g_i(x)$ both are differentiable

(i) **Case I :** $y = [f(x)]^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x. To find the derivative of this type of functions we proceed as follows:

Let $y = [f(x)]^{g(x)}$. Taking logarithm of both the sides, we have $\log y = g(x).\log f(x)$

Differentiating with respect to x, we get $\frac{1}{y}\frac{dy}{dx} = g(x).\frac{1}{f(x)}\frac{df(x)}{dx} + \log\{f(x)\}.\frac{dg(x)}{dx}$

$$\therefore \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}.\frac{df(x)}{dx} + \log[f(x)].\frac{dg(x)}{dx}\right] = [f(x)^{g(x)}\left[\frac{g(x)}{f(x)}\frac{df(x)}{dx} + \log[f(x)\frac{dg(x)}{dx}\right]$$

(ii) **Case II :** $y = \frac{f_1(x).f_2(x)}{g_1(x).g_2(x)}$

Taking logarithm of both the sides, we have $\log y = \log[f_1(x)] + \log[f_2(x)] - \log[g_1(x)] - \log[g_2(x)]$

Differentiating with respect to x, we get $\frac{1}{y}\frac{dy}{dx} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} - \frac{g_1'(x)}{g_2(x)} - \frac{g_2'(x)}{g_2(x)}$

$$\frac{dy}{dx} = y\left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} - \frac{g_1'(x)}{g_1(x)} - \frac{g_2'(x)}{g_2(x)}\right] = \frac{f_1(x).f_2(x)}{g_1(x).g_2(x)}\left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} - \frac{g_1'(x)}{g_1(x)} - \frac{g_2'(x)}{g_2(x)}\right]$$

Working rule : (a) To take logarithm of the function (b) To differentiate the function

Example: 24 If $x^m y^n = 2(x+y)^{m+n}$, the value of $\frac{dy}{dx}$ is **[MP PET 2003]**

(a) $x+y$ (b) $\frac{x}{y}$ (c) $\frac{y}{x}$ (d) $x-y$

Solution: (c) $x^m y^n = 2(x+y)^{m+n} \Rightarrow m\log x + n\log y = \log 2 + (m+n)\log(x+y)$

Differentiating *w.r.t.* x both sides

$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left[1 + \frac{dy}{dx}\right] \Rightarrow \frac{dy}{dx} = \frac{y}{x}$.

Example: 25 If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to **[IIT 1994; Rajasthan PET 1996]**

(a) $(\sin x)^{\tan x}.(1 + \sec^2 x.\log\sin x)$ (b) $\tan x.(\sin x)^{\tan x - 1}.\cos x$

(c) $(\sin x)^{\tan x}.,\sec^2 x\log\sin x$ (d) $\tan x.(\sin x)^{\tan x - 1}$

Solution: (a) Given $y = (\sin x)^{\tan x}$

$\log y = \tan x.\log\sin x$

Differentiating *w.r.t.* x, $\frac{1}{y}.\frac{dy}{dx} = \tan x.\cot x + \log\sin x.\sec^2 x$

$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + \log\sin x.\sec^2 x]$.

(3) **Differentiation of parametric functions :** Sometimes x and y are given as functions of a single variable, *e.g.*, $x = \phi(t)$, $y = \psi(t)$ are two functions and t is a variable. In such a case x and y are called parametric functions or parametric equations and t is called the parameter. To find $\frac{dy}{dx}$ in case of parametric functions, we first obtain the relationship between x and y by eliminating the parameter t and then we differentiate it with respect to x. But every time it is

not convenient to eliminate the parameter. Therefore $\frac{dy}{dx}$ can also be obtained by the following formula $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

To prove it, let Δx and Δy be the changes in x and y respectively corresponding to a small change Δt in t.

Since $\frac{\Delta y}{\Delta x}=\frac{\Delta y/\Delta t}{\Delta x/\Delta t}$, $\quad\therefore \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}}{\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\Psi'(t)}{\phi'(t)}$

Example: 26 If $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, $\frac{dy}{dx}=$ **[DCE 1999]**

(a) $\cos\theta$ (b) $\tan\theta$ (c) $\sec\theta$ (d) cosecθ

Solution: (b) $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a[\cos\theta-\theta(-\sin\theta)-\cos\theta]}{a[-\sin\theta+\theta\cos\theta+\sin\theta]}=\frac{\theta\sin\theta}{\theta\cos\theta}=\tan\theta$.

Example: 27 If $\cos x=\frac{1}{\sqrt{1+t^2}}$ and $\sin y=\frac{t}{\sqrt{1+t^2}}$, then $\frac{dy}{dx}=$ **[MP PET 1994]**

(a) – 1 (b) $\frac{1-t}{1+t^2}$ (c) $\frac{1}{1+t^2}$ (d) 1

Solution: (d) Obviously $x=\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y=\sin^{-1}\frac{t}{\sqrt{1+t^2}}$

$\Rightarrow x=\tan^{-1}t$ and $y=\tan^{-1}t \Rightarrow y=x \Rightarrow \frac{dy}{dx}=1$.

Example: 28 If $x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$, then $\frac{dy}{dx}=$ **[Karnataka CET 2000]**

(a) $\frac{-y}{x}$ (b) $\frac{y}{x}$ (c) $\frac{-x}{y}$ (d) $\frac{x}{y}$

Solution: (c) $x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$

Put $t=\tan\theta$ in both the equations, we get $x=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$ and $y=\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$.

Differentiating both the equations, we get $\frac{dx}{d\theta}=-2\sin 2\theta$ and $\frac{dy}{d\theta}=2\cos 2\theta$.

Therefore $\frac{dy}{dx}=-\frac{\cos 2\theta}{\sin 2\theta}=-\frac{x}{y}$.

(4) **Differentiation of infinite series :** If y is given in the form of infinite series of x and we have to find out $\frac{dy}{dx}$ then we remove one or more terms, it does not affect the series

(i) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+.......\infty}}}$, then $y=\sqrt{f(x)+y}\Rightarrow y^2=f(x)+y$

$$2y\frac{dy}{dx}=f'(x)+\frac{dy}{dx},\quad \therefore \frac{dy}{dx}=\frac{f'(x)}{2y-1}$$

(ii) If $y = f(x)^{f(x)^{f(x)^{f(x)\ldots\infty}}}$ then $y = f(x)^y$

$\therefore\ \log y = y \log f(x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y.f'(x)}{f(x)} + \log f(x).\frac{dy}{dx},\quad \therefore\ \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$$

(iii) If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \ldots\infty}}$ then $\dfrac{dy}{dx} = \dfrac{yf'(x)}{2y - f(x)}$

Example: 29 If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\ldots \text{ to } \infty}}}$ then $\dfrac{dy}{dx} =$ **[Rajasthan PET 2002]**

(a) $\dfrac{x}{2y-1}$ (b) $\dfrac{2}{2y-1}$ (c) $\dfrac{-1}{2y-1}$ (d) $\dfrac{1}{2y-1}$

Solution: (d) $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\ldots \text{ to } \infty}}} \Rightarrow y = \sqrt{x+y} \Rightarrow y^2 = x + y \Rightarrow 2y\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx}(2y-1) = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2y-1}$

Example: 30 If $y = x^{x^{x\ldots\infty}}$, then $x(1 - y\log_e x)\dfrac{dy}{dx}$ is **[DCE 2000]**

(a) x^2 (b) y^2 (c) xy^2 (d) None of these

Solution: (b) $y = x^{x^{x\ldots\ldots\infty}} \Rightarrow y = x^y \Rightarrow \log_e y = y\log_e x \Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{y}{x} + \log_e x\dfrac{dy}{dx} \Rightarrow \left(\dfrac{1}{y} - \log_e x\right)\dfrac{dy}{dx} = \dfrac{y}{x} \Rightarrow$
$x(1 - y\log_e x)\dfrac{dy}{dx} = y^2$

Example: 31 If $y = x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \ldots\ldots\infty}}}$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{2xy}{2y - x^2}$ (b) $\dfrac{xy}{y + x^2}$ (c) $\dfrac{xy}{y - x^2}$ (d) $\dfrac{2x}{2 + \dfrac{x^2}{y}}$

Solution: (a) $y = x^2 + \dfrac{1}{y} \Rightarrow y^2 = x^2 y + 1 \Rightarrow 2y\dfrac{dy}{dx} = y.2x + x^2\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{2xy}{2y - x^2}$

Example: 32 If $x = e^{y + e^{y + \ldots\ldots \text{to } \infty}}$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{1+x}{x}$ (b) $\dfrac{1}{x}$ (c) $\dfrac{1-x}{x}$ (d) $\dfrac{x}{1+x}$

Solution: (c) $x = e^{y+x}$

Taking log both sides, $\log x = (y + x)\log e = y + x \Rightarrow y + x = \log x \Rightarrow \dfrac{dy}{dx} + 1 = \dfrac{1}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$

(5) **Differentiation of composite function :** Suppose function is given in form of $fog(x)$ or $f[g(x)]$

Working rule : Differentiate applying chain rule $\frac{d}{dx} f[g(x)] = f'[g(x)].g'(x)$

Example: 33 If $f(x) = |x-2|$ and $g(x) = f(f(x))$, then for $x>20$, $g'(x)$ equals

(a) –1 (b) 1 (c) 0 (d) None of these

Solution: (b) For $x > 20$, we have

$f(x) = |x-2| = x-2$ and, $g(x) = f(f(x)) = f(x-2) = x-2-2 = x-4$

$\therefore\ g'(x) = 1$

Example: 34 If g is inverse of f and $f'(x) = \frac{1}{1+x^n}$, then $g'(x)$ equals

(a) $1+x^n$ (b) $1+[f(x)]^n$ (c) $1+[g(x)]^n$ (d) None of these

Solution: (c) Since g is inverse of f. Therefore,

$fog(x) = x$ for all $x \Rightarrow \frac{d}{dx}\{fog(x)\} = 1$ for all x

$\Rightarrow f'(g(x)).g'(x) = 1 \Rightarrow f'\{g(x)\} = \frac{1}{g'(x)} \Rightarrow \frac{1}{1+[g(x)]^n} = \frac{1}{g'(x)}$ $\left[\because f'(x) = \frac{1}{1+x^n}\right]$

$\Rightarrow g'(x) = 1+[g(x)]^n$

3.6 Differentiation of a Function with Respect to Another Function

In this section we will discuss derivative of a function with respect to another function. Let $u = f(x)$ and $v = g(x)$ be two functions of x. Then, to find the derivative of $f(x)$ *w.r.t.* $g(x)$ i.e., to find $\frac{du}{dv}$ we use the following formula $\frac{du}{dv} = \frac{du/dx}{dv/dx}$

Thus, to find the derivative of $f(x)$ *w.r.t.* $g(x)$ we first differentiate both *w.r.t.* x and then divide the derivative of $f(x)$ *w.r.t.* x by the derivative of $g(x)$ *w.r.t.* x.

Example: 35 The differential coefficient of $\tan^{-1}\frac{2x}{1-x^2}$ *w.r.t.* $\sin^{-1}\frac{2x}{1+x^2}$ is

[Roorkee 1966; BIT Mesra 1996; Karnataka CET 1994; MP PET 1999; UPSEAT 1999, 2001]

(a) 1 (b) –1 (c) 0 (d) None of these

Solution: (a) Let $y_1 = \tan^{-1}\frac{2x}{1-x^2}$ and $y_2 = \sin^{-1}\frac{2x}{1+x^2}$

Putting $x = \tan\theta$

$\therefore\ y_1 = \tan^{-1}\tan 2\theta = 2\theta = 2\tan^{-1}x$ and $y_2 = \sin^{-1}\sin 2\theta = 2\tan^{-1}x$

Again $\frac{dy_1}{dx} = \frac{d}{dx}[2\tan^{-1}x] = \frac{2}{1+x^2}$ ....(i)

and $\frac{dy_2}{dx} = \frac{d}{dx}[2\tan^{-1}x] = \frac{2}{1+x^2}$ ....(ii)

Hence $\frac{dy_1}{dy_2} = 1$

Example: 36 The first derivative of the function $\left[\cos^{-1}\left(\sin\frac{\sqrt{1+x}}{2}\right) + x^x\right]$ with respect to x at $x = 1$ is

(a) $\frac{3}{4}$ (b) 0 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Solution: (a) $f(x) = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - \sqrt{\frac{1+x}{2}}\right)\right] + x^x = \frac{\pi}{2} - \sqrt{\frac{1+x}{2}} + x^x$

$\therefore \; f'(x) = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{1+x}} + x^x(1 + \log x) \Rightarrow f'(1) = -\frac{1}{4} + 1 = \frac{3}{4}$

3.7 Successive Differentiation or Higher Order Derivatives

(1) **Definition and notation :** If y is a function of x and is differentiable with respect to x, then its derivative $\frac{dy}{dx}$ can be found which is known as derivative of first order. If the first derivative $\frac{dy}{dx}$ is also a differentiable. function, then it can be further differentiated with respect to x and this derivative is denoted by d^2y/dx^2 which is called the second derivative of y with respect to x further if $\frac{d^2y}{dx^2}$ is also differentiable then its derivative is called third derivative of y which is denoted by $\frac{d^3y}{dx^3}$. Similarly n^{th} derivative of y is denoted by $\frac{d^ny}{dx^n}$. All these derivatives are called as successive derivative and this process is known as successive differentiation. We also use the following symbols for the successive derivatives of $y = f(x)$:

$y_1, \; y_2, \; y_3, ..., y_n, ...$ $\qquad$ $y^I, \; y^{II}, \; y^{III}........., y^n, ...$

$Dy, \; D^2y, \; D^3y........., D^ny, ...$ (where $D = \frac{d}{dx}$) $\qquad$ $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ..., \frac{d^ny}{dx^n}, ...$

$f'(x), \; f''(x), \; f'''(x), ..., f^n(x), ...$

If $y = f(x)$, then the value of the n^{th} order derivative at $x = a$ is usually denoted by

$\left(\frac{d^ny}{dx^n}\right)_{x=a}$ or $(y_n)_{x=a}$ or $(y^n)_{x=a}$ or $f^n(a)$

(2) n^{th} Derivatives of some standard functions :

(i) (a) $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$ $\qquad$ (b) $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$, where $m > n$

Particular cases :

(i) (a) When $m = n$

$D^n\{(ax+b)^n\} = a^n.n!$

(b) When $m < n, D^n\{(ax+b)^m\} = 0$

(ii) When $a = 1, b = 0$, then $y = x^n$

$\therefore \; D^n(x^m) = m(m-1).......(m-n+1)x^{m-n} = \frac{m!}{(m-n)!}x^{m-n}$

(iii) When $a = 1$, $b = 0$ and $m = n$,

(iv) When $m = -1$, $y = \frac{1}{(ax+b)}$

then $y = x^n$

$\therefore D^n(x^n) = n!$

$D^n(y) = a^n(-1)(-2)(-3)........(-n)(ax+b)^{-1-n}$

$= a^n(-1)^n(1.2.3......n)(ax+b)^{-1-n} = \dfrac{a^n(-1)^n n!}{(ax+b)^{n+1}}$

(3) $\dfrac{d^n}{dx^n}\log(ax+b) = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

(4) $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(5) $\dfrac{d^n(a^x)}{dx^n} = a^x(\log a)^n$

(6) (i) $\dfrac{d^n}{dx^n} e^{ax}\sin(bx+c) = r^n e^{ax}\sin(bx+c+n\phi)$

where $r = \sqrt{a^2+b^2}$; $\phi = \tan^{-1}\dfrac{b}{a}$,

$y = e^{ax}\sin(bx+c)$

(ii) $\dfrac{d^n}{dx^n} e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\phi)$

Example: 37 If $y = \left(x+\sqrt{1+x^2}\right)^n$, then $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$ is **[AIEEE 2002]**

(a) n^2y (b) $-n^2y$ (c) $-y$ (d) $2x^2y$

Solution: (a) $y = (x+\sqrt{1+x^2})^n \Rightarrow \dfrac{dy}{dx} = n(x+\sqrt{1+x^2})^{n-1}\left(1+\dfrac{x}{\sqrt{1+x^2}}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}} \Rightarrow (\sqrt{1+x^2})\dfrac{dy}{dx} = n\left(x+\sqrt{1+x^2}\right)^n$

$\Rightarrow \dfrac{d^2y}{dx^2}.\sqrt{1+x^2} + \dfrac{dy}{dx}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = n^2\left(x+\sqrt{1+x^2}\right)^{n-1}\left(1+\dfrac{x}{\sqrt{1+x^2}}\right)$

$\Rightarrow (1+x^2).\dfrac{d^2y}{dx^2} + x.\dfrac{dy}{dx} = n^2(x+\sqrt{1+x^2})^n \Rightarrow (1+x^2)\dfrac{d^2y}{dx^2} + x.\dfrac{dy}{dx} = n^2y$.

Example: 38 If $f(x) = x^n$, then the value of $f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} - \dfrac{f'''(1)}{3!} + ... + \dfrac{(-1)^n f^n(1)}{n!}$ is **[AIEEE 2003]**

(a) 2^n (b) 2^{n-1} (c) 0 (d) 1

Solution: (c) $f(x) = x^n \Rightarrow f(1) = 1$, $f'(x) = nx^{n-1} \Rightarrow f'(1) = n$

$f''(x) = n(n-1)x^{n-2} \Rightarrow f''(1) = n(n-1)$ .....

$f^n(x) = n! \Rightarrow f^n(1) = n!$, $\therefore f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} ...... + \dfrac{(-1)^n f^n(1)}{n!}$

$= 1 - \dfrac{n}{1!} + \dfrac{n(n-1)}{2!} - \dfrac{n(n-1)(n-2)}{3!} + ... + (-1)^n\dfrac{n!}{n!} = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\ {}^nC_n = 0$.

Example: 39 If $f(x) = \tan^{-1}\left\{\dfrac{\log\left(\dfrac{e}{x^2}\right)}{\log(ex^2)}\right\} + \tan^{-1}\left(\dfrac{3+2\log x}{1-6\log x}\right)$, then $\dfrac{d^ny}{dx^n}$ is $(n \ge 1)$

(a) $\tan^{-1}\{(\log x)^n\}$ (b) 0 (c) 1/2 (d) None of these

Solution: (b) We have $y = \tan^{-1}\left(\dfrac{\log e - \log x^2}{\log e + \log x^2}\right) + \tan^{-1}\left(\dfrac{3+2\log x}{1-6\log x}\right) = \tan^{-1}\left(\dfrac{1-2\log x}{1+2\log x}\right) + \tan^{-1}\left(\dfrac{3+2\log x}{1-6\log x}\right)$

$= \tan^{-1}1 - \tan^{-1}(2\log x) + \tan^{-1}3 + \tan^{-1}(2\log x) \Rightarrow y = \tan^{-1}1 + \tan^{-1}3 \Rightarrow \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{d^ny}{dx^n} = 0$.

Example: 40 If $f(x) = (\cos x + i\sin x)(\cos 3x + i\sin 3x).....\ (\cos(2n-1)x + i\sin(2n-1)x)$, then $f''(x)$ is equal to

(a) $n^2 f(x)$ (b) $-n^4 f(x)$ (c) $-n^2 f(x)$ (d) $n^4 f(x)$

Solution: (b) We have, $f(x) = \cos(x + 3x + ... + (2n-1)x) + i\sin(x + 3x + 5x + ... + (2n-1)x) = \cos n^2 x + i\sin n^2 x$

$\Rightarrow f'(x) = -n^2(\sin n^2 x) + n^2(i\cos n^2 x) \Rightarrow f''(x) = -n^4 \cos n^2 x - n^4 i\sin n^2 x$

$\Rightarrow f''(x) = -n^4(\cos n^2 x + i\sin n^2 x) \Rightarrow f''(x) = -n^4 f(x)$

3.8 n^{th} Derivative using Partial fractions

For finding n^{th} derivative of fractional expressions whose numerator and denominator are rational algebraic expression, firstly we resolve them into partial fractions and then we find n^{th} derivative by using the formula giving the n^{th} derivative of $\frac{1}{ax+b}$.

Example: 41 If $y = \frac{x^4}{x^2 - 3x + 2}$, then for $n > 2$ the value of y_n is equal to

(a) $(-1)^n n![16(x-2)^{-n-1} - (x-1)^{-n-1}]$ (b) $(-1)^n n![16(x-2)^{-n-1} + (x-1)^{-n-1}]$

(c) $n![16(x-2)^{-n-1} + (x-1)^{-n-1}]$ (d) None of these

Solution: (a) $y = \frac{x^4}{x^2 - 3x + 2} = x^2 + 3x + 7 + \frac{15x - 14}{(x-1)(x-2)} = x^2 + 3x + 7 - \frac{1}{(x-1)} + \frac{16}{(x-2)}$

$\therefore y_n = D^n(x^2) + D^n(3x) + D^n(7) - D^n[(x-1)^{-1}] + 16 D^n[(x-2)^{-1}]$

$= (-1)^n n![-(x-1)^{-n-1} + 16(x-2)^{-n-1}] = (-1)^n n![16(x-2)^{-n-1} - (x-1)^{-n-1}]$.

3.9 Differentiation of Determinants

Let $\Delta(x) = \begin{vmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{vmatrix}$. Then $\Delta'(x) = \begin{vmatrix} a_1'(x) & b_1'(x) \\ a_2(x) & b_2(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & b_1(x) \\ a_2'(x) & b_2'(x) \end{vmatrix}$

If we write $\Delta(x) = |C_1 C_2 C_3|$. Then $\Delta'(x) = |C_1' C_2 C_3| + |C_1 C_2' C_3| + |C_1 C_2 C_3'|$

Similarly, if $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} R_1' \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3' \end{vmatrix}$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

Example: 42 If $f_r(x), g_r(x), h_r(x), r = 1,2,3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a), r = 1,2,3$ and

$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$, then find $F'(x)$ at $x = a$ **[IIT 1985]**

(a) 0 (b) $f_1(a) g_2(a) h_3(a)$ (c) 1 (d) None of these

Solution: (a) $F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$

$$\therefore\ F'(a) = \begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$$

$= 0+0+0=0$ $\quad [\because f_r(a) = g_r(a) = h_r(a), r = 1, 2, 3]$

Example: 43 Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then $\frac{d^3}{dx^3}[f(x)]$ at $x=0$ is **[IIT 1997]**

(a) p (b) $p + p^2$ (c) $p + p^3$ (d) Independent of p

Solution: (d) Given $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$, 2nd and 3rd rows are constant, so only 1st row will take part in differentiation

$$\therefore \frac{d^3}{dx^3} f(x) = \begin{vmatrix} \frac{d^3}{dx^3}x^3 & \frac{d^3}{dx^3}\sin x & \frac{d^3}{dx^3}\cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$$

We know that $\frac{d^n}{dx^n}x^n = n!, \frac{d^n}{dx^n}\sin x = \sin(x + \frac{n\pi}{2})$ and $\frac{d^n}{dx^n}\cos x = \cos(x + \frac{n\pi}{2})$

Using these results, $\frac{d^3}{dx^3} f(x) = \begin{vmatrix} 3! & \sin\left(x + \frac{3\pi}{2}\right) & \cos\left(x + \frac{3\pi}{2}\right) \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$

$\frac{d^3}{dx^3} f(x)\Big|_{\text{at } x=0} = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$ = 0 *i.e.*, independent of p.

3.10 Differentiation of Integral Function

If $g_1(x)$ and $g_2(x)$ both functions are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$ and $f(t)$ is continuous for $g_1(a) \le f(t) \le g_2(b)$

Then $\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f[g_2(x)]g_2'(x) - f[g_1(x)]g_1'(x) = f[g_2(x)]\frac{d}{dx}g_2(x) - f[g_1(x)]\frac{d}{dx}g_1(x)$.

Example: 44 If $F(x) = \int_{x^2}^{x^3} \log t\, dt\ (x > 0)$, then $F'(x) =$ **[MP PET 2001]**

(a) $(9x^2 - 4x)\log x$ (b) $(4x - 9x^2)\log x$ (c) $(9x^2 + 4x)\log x$ (d) None of these

Solution: (a) Applying formula we get $F'(x) = (\log x^3)3x^2 - (\log x^2)2x$

$= (3\log x)3x^2 - 2x(2\log x) = 9x^2\log x - 4x\log x = (9x^2 - 4x)\log x$.

Example: 45 If $x = \int_0^y \frac{1}{\sqrt{1+4t^2}}dt$, then $\frac{d^2y}{dx^2}$ is

(a) $2y$ (b) $4y$ (c) $8y$ (d) $6y$

Solution: (b) $x = \int_0^y \frac{1}{\sqrt{1+4t^2}}dt \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+4y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{4y}{\sqrt{1+4y^2}}\frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{4y}{\sqrt{1+4y^2}}\cdot\sqrt{1+4y^2} = 4y$

3.11 Leibnitz's Theorem

G.W. Leibnitz, a German mathematician gave a method for evaluating the nth differential coefficient of the product of two functions. This method is known as Leibnitz's theorem.

Statement of the theorem – If u and v are two functions of x such that their nth derivative exist then $D^n(u.v.) = {}^nC_0(D^nu)v + {}^nC_1D^{n-1}u.Dv + {}^nC_2D^{n-2}u.D^2v + ... + {}^nC_rD^{n-r}u.D^rv + ... + u.(D^nv)$.

Note : ❐ The success in finding the nth derivative by this theorem lies in the proper selection of first and second function. Here first function should be selected whose nth derivative can be found by standard formulae. Second function should be such that on successive differentiation, at some stage, it becomes zero so that we need not to write further terms.

Example: 46 If $y = x^2e^x$, then value of y_n is

(a) $\{x^2 - 2nx + n(n-1)\}e^x$ (b) $\{x^2 + 2nx + n(n-1)\}e^x$

(c) $\{x^2 + 2nx - n(n-1)\}e^x$ (d) None of these

Solution: (b) Applying Leibnitz's theorem by taking x^2 as second function. We get , $D^ny = D^n(e^x.x^2)$

$= {}^nC_0D^n(e^x)x^2 + {}^nC_1D^{n-1}(e^x).D(x^2) + {}^nC_2D^{n-2}(e^x).D^2(x^2) + ... = e^x.x^2 + ne^x.2x + \frac{n(n-1)}{2!}e^x.2 + 0 + 0 + ...$

$y_n = \{x^2 + 2nx + n(n-1)\}e^x$.

Example: 47 If $y = x^2\log x$, then value of y_n is

(a) $\frac{(-1)^{n-1}(n-3)!}{x^{n-2}}$ (b) $\frac{(-1)^{n-1}(n-3)!}{x^{n-2}}.2$ (c) $\frac{(-1)^{n-1}(n-2)!}{x^{n-2}}$ (d) None of these

Solution: (b) Applying Leibnitz's theorem by taking x^2 as second function, we get, $D^ny = D^n(\log x.x^2)$

$= {}^nC_0D^n(\log x).x^2 + {}^nC_1D^{n-1}(\log x).D(x^2) + {}^nC_2D^{n-2}(\log x)D^2(x^2) + ...$.

$= \frac{(-1)^{n-1}(n-1)!}{x^n}.x^2 + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.2x + \frac{n(n-1)}{2!}\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}.2 + 0 + 0.........$

$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + \frac{2n(-1)^{n-2}(n-2)!}{x^{n-2}} + \frac{n(n-1)(-1)^{n-3}(n-3)!}{x^{n-2}}$

$= \frac{(-1)^{n-1}(n-3)!}{x^{n-2}} \times \{(n-1)(n-2) - 2n(n-2) + n(n-1)\} = .\ \frac{(-1)^{n-1}(n-3)!}{x^{n-2}}.2$