CBSE Test Paper 05

Chapter 8 Application of Integrals

1. AOB is the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in which OA = a and OB = b. The area between the arc AB and chord AB of the ellipse is

   a. $\frac{1}{4}ab(\pi-4)$
   b. $\frac{1}{4}ab(\pi-2)$
   c. none of these
   d. $\frac{1}{2}ab(\pi+2)$

2. Let f (x) = x^4 – 4x, then

   a. none of these
   b. f is decreasing in $[1,\infty)$
   c. f is increasing in $[1,\infty)$
   d. f is increasing in $[-\infty,1)$

3. The area enclosed by the parabola y^2 = 2x and its tangents through the point (-2 , 0) is

   a. 3
   b. $\frac{8}{3}$
   c. none of these
   d. 4

4. The area bounded by the curve y^2= x, line y = 4 and y – axis is equal to

   a. $\frac{16}{3}$
   b. none of these
   c. $\frac{64}{3}$
   d. $7\sqrt{2}$

5. Area bounded by the curves satisfying the conditions $\frac{x^2}{25}+\frac{y^2}{36}\leqslant 1\leqslant\frac{x}{5}+\frac{y}{6}$ is given by

a. 30($\frac{\pi}{2}$ - 1) sq. units
b. 15($\frac{\pi}{2}$ - 1) sq. units
c. $\frac{15}{4}$ ($\frac{\pi}{2}$ - 1) sq. units
d. $\frac{15}{2}$ ($\frac{\pi}{2}$ - 1) sq. units

6. Find the area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$

7. If the ordinate at x=a divides the area bounded by the X-axis,part of the curve $y=1+\frac{8}{x^2}$ and the ordinates at x=2 and x=4 into two equal parts, then find value of a.

8. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

9. Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, $a > 0$.

10. Sketch the graph of y = |x + 3| and evaluate $\int_{-6}^{0} |x+3|\, dx$.

11. Find the area of the region bounded by the parabola $y = x^2$ and y = |x|.

12. Determine the area bounded by the curve $y = x(1-x)^2$, the Y-axis and the line x = 2.

13. Find value of $\lim_{m\to\infty} \sum_{r=1}^{m} \frac{r}{m^2} \cdot sec^2 \frac{r^2}{m^2}$.

14. If S be the area of the region enclosed by $y = e^{-x^2}$, y = 0 and x = 1, then show that $1 - \frac{1}{e} \leq S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$.

15. Using integration, find the area of the circle $x^2 + y^2 = 16$, which is exterior to the parabola $y^2 = 6x$.

Solution

1. b. $\frac{1}{4}ab(\pi - 2)$

   Explanation: Required area := $\frac{1}{4}$ area of ellipse – area of right angled triangle AOB. =$\frac{1}{4}$ πab – $\frac{1}{2}$ ab = $\frac{ab}{4}$ (π - 2).

2. c. f is increasing in $[1, \infty)$

   Explanation: f (x) = x^4 – 4x

   f'(x) = 4(x^3) - 4 = 4(x^{3}- 1) = 4 {(x^2) + x + 1)} (x-1)

   = f '(x) > 0,if,(x - 1) > 0,

   and f '(x) <0, if (x -1) < 0.

   So,f is decreasing on (∞, 1] and f is increasing on [1,∞)

3. b. $\frac{8}{3}$

   Explanation: The tangents are $y = mx + \frac{a}{m}, y = mx + \frac{1}{2m}$

   since $a = \frac{1}{2}$.

   It passes through (-2 , 0).

   $\therefore 4m^2 = 1 \Rightarrow m = \pm\frac{1}{2}$

   The tangents are:

   $y = \frac{x}{2} + 1, y = -\frac{x}{2} - 1$

   Required area:

   $= 2\int\limits_0^3 \left(\frac{y^2}{2} - 2y + 2\right) dy$

   $= 2\left[\frac{4}{3} - 4 + 4\right]$

   $= \frac{8}{3} sq. units$

4. a. $\frac{16}{3}$

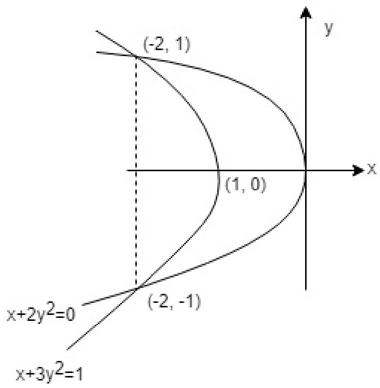
   Explanation: Required area : $\int\limits_0^4 (y^2 - 4)dy$= $\left[\frac{y^3}{3} - 4y\right]_0^4$ =$\frac{16}{3}$ sq.units

5. b. 15($\frac{\pi}{2}$ - 1) sq. units

Explanation: Required area : $=\int\limits_0^5\left(\frac{6}{5}\sqrt{5^2-x^2}-\frac{6}{5}(5-x)\right)dx$

$=\frac{6}{5}\left[\frac{x\sqrt{5^2-x^2}}{2}+\frac{5^2}{2}\sin^{-1}\frac{x}{5}-5x+\frac{x^2}{2}\right]_0^5$
$=\frac{6}{5}\left[0+\frac{5^2}{2}\sin^{-1}1-5^2+\frac{5^2}{2}\right]$
$-\frac{6}{5}\left[0+\frac{5^2}{2}\sin^{-1}0-0+0\right]$
$=\frac{6}{5}\left[\frac{5^2}{2}.\frac{\pi}{2}-\frac{5^2}{2}-0\right]=15\left(\frac{\pi}{2}-1\right)$

6. Solving the equations, we get the points of intersection as (-2,1) and (-2,-1)
   Required area =$2\int_0^1(1-3y^2)-(-2y^2)dy$
   $=2\int_0^1(1-y^2)dy=2\left[y-\frac{y^3}{3}\right]_0^1$
   $=2\times\frac{2}{3}=\frac{4}{3}$

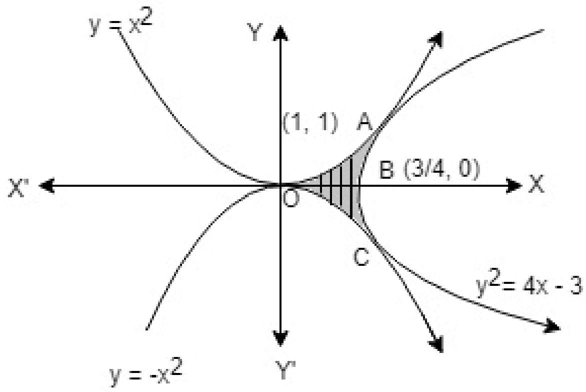


7. According to he question, $\int_2^a(1+\frac{8}{x^2})dx=\int_a^4(1+\frac{8}{x^2})dx$
   $\left[x-\frac{8}{x}\right]_2^a=\left[x-\frac{8}{x}\right]_a^4$
   $\left(a-\frac{8}{a}-(2-4\right)=(4-2)-\left(a-\frac{8}{a}\right)$
   $\left(a-\frac{8}{a}\right)+2=2-a+\frac{8}{a}$
   $2a-\frac{16}{a}=0$
   $2(a^2-8)=0$
   $a=\pm2\sqrt{2}$
   $a=2\sqrt{2}$ [neglecting negative sign]
   $\therefore$ If the ordinate at x=a divides the area bounded by the X-axis,part of the curve y=1+ $\frac{8}{x^2}$ and the ordinates at x=2 and x=4 into two equal parts, then $a=2\sqrt{2}$.

8. The region bounded by the curves y = x^2, y = $-x^2$ and y^2 = 4x - 3 is symmetrical about X-

axis and, y=4x-3 meets at (1,1).

$\therefore$ Area of region (OABCO)=$2\left[\int_0^1 x^2dx - \int_{3/4}^1 \sqrt{4x-3}dx\right]$



=$2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{(4x-3)^{3/2}}{3.4/2}\right)_{3/4}^1\right]$

=$2\left(\frac{1}{3}-\frac{1}{6}\right) = 2\times\frac{1}{6} = \frac{1}{3}\ sq\ units$

The area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$ is $\frac{1}{3}$ sq units

9. $y^2 = 4ax$

   $x^2 = 4ay$

   On solving

   $x = 4a$, $y = 4a$

   Area $= \int_0^{4a}\sqrt{4ax}dx - \int_0^{4a}\frac{x^2}{4a}dx$

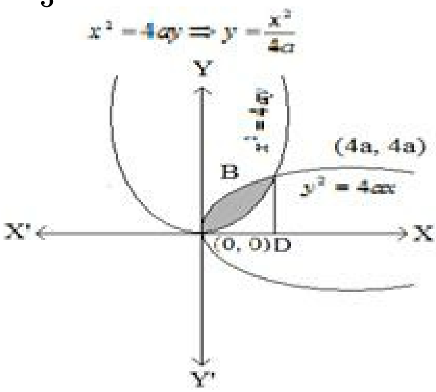
   $= \int_0^{4a}[2\sqrt{ax} - \frac{x^2}{4a}]dx$

   $= \left[\frac{4\sqrt{a}}{3}x^{3/2} - \frac{x^3}{12a}\right]_0^{4a}$

   $= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a}$

   $= \frac{32a^2}{3} - \frac{16a^2}{3}$

   $= \frac{16a^2}{3}$ sq. units



10. y = |x + 3|

$\Rightarrow y = (x+3), if\ x \geq -3$

y = -(x + 3), if x < -3

$\int_{-6}^{0} |x+3|\, dx = ?$

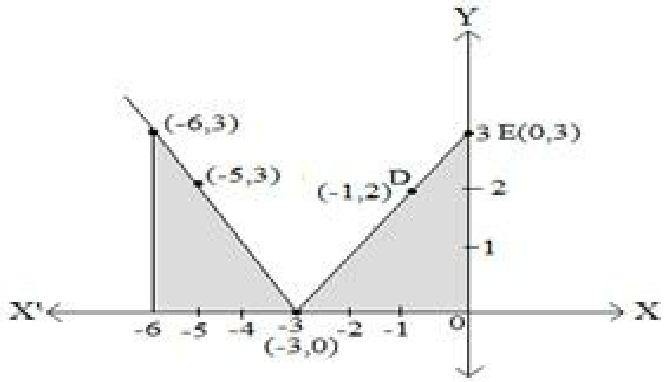
Area $= \int_{-6}^{-3} -(x+3)\, dx + \int_{-3}^{0} (x+3)\, dx$

$= \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$

$= \left[(-\frac{9}{2}+9) - (-\frac{36}{2}+18)\right] + \left[(0+0) - (\frac{9}{2}-9)\right]$

$= \left[(\frac{9}{2}+0) + (0+\frac{9}{2})\right]$

= 9 sq units



11. y = x^2

y = |x|

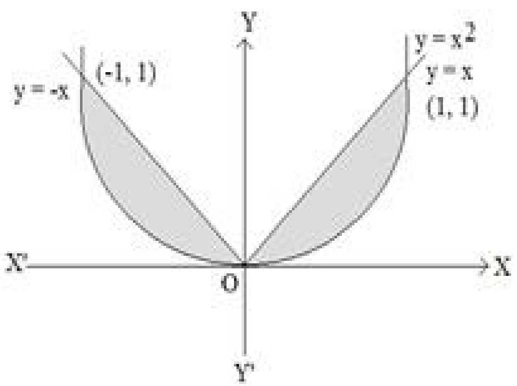
$\Rightarrow$ y = x

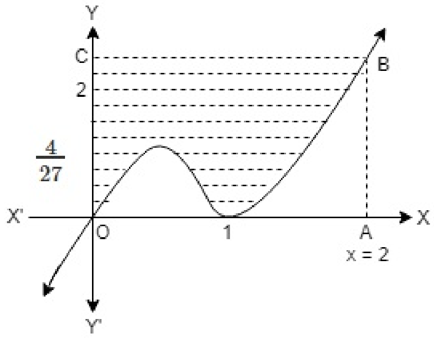
y = -x

$= 2\int_{0}^{1} \left(x - x^2\right) dx$

$= 2\left[(\frac{x^2}{2})_0^1 - (\frac{x^3}{3})_0^1\right]$

$= 2(\frac{1}{2} - \frac{1}{3})$=$\frac{1}{3}$ sq.units



12. 

Required Area = the area bounded by the curve $y=x(1-x)^2$, the Y-axis and the line x = 2.

$y=x(1-x)^2$.......(1)

Required area =Area of a square OABC - $\int_0^2 y\,dx$

$=2\times 2-\int_0^2 x(1-x)^2 dx$ (from equation (1))

$=4-\int_0^2 x(x^2+1-2x)dx$

$=4-\int_0^2 (x^3+x-2x^2)dx$

$=4-\left[\frac{x^4}{4}+\frac{x^2}{2}-2\frac{x^3}{3}\right]_0^2$

$=4-\left[\frac{2^4}{4}+\frac{2^2}{2}-2\times\frac{2^3}{3}\right]$

$=4-\left[\frac{16}{4}+\frac{4}{2}-\frac{16}{3}\right]$

$=4-\left[\frac{16+8}{4}-\frac{16}{3}\right]$

$=4-\left[\frac{12}{2}-\frac{16}{3}\right]$

$=4-\left[6-\frac{16}{3}\right]$

$=4-\frac{2}{3}$

$=\frac{10}{3}$ sq.units

Required Area = the area bounded by the curve $y=x(1-x)^2$, the Y-axis and the line (x = 2) = $\frac{10}{3}$ sq. units

13. $\lim_{m\to\infty}\sum_{r=1}^{m}\frac{r}{m^2}.\,sec^2\,\frac{r^2}{m^2}$

$=\lim_{m\to\infty}\sum_{r=1}^{m}\left(\frac{r}{m}.\,sec^2\,\frac{r^2}{m^2}\right)\frac{1}{m}$

$=\int_0^1 xsec^2\ x^2\ dx$

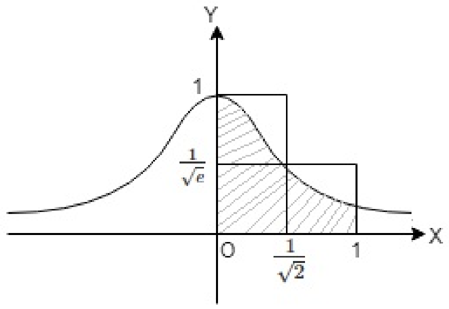
$[putting\ x^2=t\Rightarrow\ 2xdx=dt]$

$=\frac{1}{2}\int_0^1 sec^2 t\ dt$

$=\frac{1}{2}[tan\ t]_0^1=\frac{1}{2}\ (tan\ 1-tan0)$

$=\frac{1}{2}\ tan\ 1$

14. As, $y=e^{-x^2}$



since, $x^2 \leq x$ when x$\in$ $[0,1]$

$\Rightarrow \; -x^2 \geq -x$

$\Rightarrow \; e^{-x^2} \geq e^{-x}$

$\therefore \; \int_0^1 e^{-x^2} dx \geq \int_0^1 e^{-x} dx$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} S \geq -(e^{-x})_0^1 = 1-\frac{1}{e} \; \ldots\ldots(1)$

Also, $\int_0^1 e^{-x^2} dx \leq$Area of the two rectangles

$\leqslant \left(1\times\frac{1}{\sqrt{2}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)\times\frac{1}{\sqrt{e}}$

$\leqslant \left(1\times\frac{1}{\sqrt{2}}\right)+\left(1-\frac{1}{\sqrt{2}}\right)\times\frac{1}{\sqrt{e}}$

$\leq \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)\ldots\ldots\ldots(2)$

From (1) and (2), we conclude that

$1-\frac{1}{e} \leq S \leq \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$

If S be the area of the region enclosed by $y=e^{-x^2}$, y = 0 and x = 1, then

$1-\frac{1}{e} \leq S \leq \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$

15. According to the Given question,

equation of circle is

$x^2+y^2=16$...(i)

and equation of parabola is

$y^2=6x$...(ii)

The given circle has centre (0, 0) and

Radius = 4 units

The given parabola has vertex (0, 0) and

Axis of parabola lies parallel to X-axis.

On substituting $y^2 = 6x$ in Eq. (i), we get

$x^2 + 6x - 16 = 0$

(x + 8)(x - 2) = 0

x = - 8 or 2

from Eq. (ii),

when x = -8, then y^2 = - 48, which is not possible , because square root of negative terms does not exist.

So, x $\neq$ -8

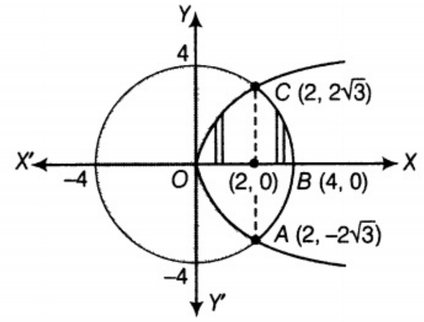
when x = 2 , then

y^2 = 12

y = ± $2\sqrt{3}$

Thus, the points of intersection are (2 , $2\sqrt{3}$) and (2, - $2\sqrt{3}$)

Now, let us sketch the graph of given curves.



Required area= Area of shaded region

= Area of circle - Area of region OABCO

= $\pi(4)^2$ - 2(Area of region OBCO)

$$= 16\pi - 2\left[\int_0^2 y_{\text{(parabola)}}\,dx + \int_2^4 y_{(\text{ circle })}dx\right]$$
$$= 16\pi - 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16 - x^2}dx\right]$$
$$= 16\pi - 2\left[\sqrt{6}\int_0^2 x^{1/2}dx + \int_2^4 \sqrt{4^2 - x^2}dx\right]$$
$$= 16\pi - 2\left\{\sqrt{6} \times \tfrac{2}{3}\left[x^{3/2}\right]_0^2 + \left[\tfrac{x}{2}\sqrt{4^2 - x^2} + \tfrac{16}{2}\sin^{-1}\left(\tfrac{x}{4}\right)\right]_2^4\right\}$$
$$= 16\pi - 2\left\{\tfrac{2\sqrt{6}}{3} \times 2\sqrt{2} + \left[8\sin^{-1}(1) - \left(\sqrt{4^2 - 2^2} + 8\sin^{-1}\left(\tfrac{1}{2}\right)\right)\right]\right\}$$
$$= 16\pi - 2\left\{\tfrac{8\sqrt{3}}{3} + \left[8 \cdot \tfrac{\pi}{2} - \left(2\sqrt{3} + 8 \cdot \tfrac{\pi}{6}\right)\right]\right\}$$
$$= 16\pi - \tfrac{16\sqrt{3}}{3} - 8\pi + 4\sqrt{3} + \tfrac{8\pi}{3}$$
$$= 8\pi + \tfrac{8\pi}{3} - \tfrac{4\sqrt{3}}{3}$$
$$= \tfrac{32\pi}{3} - \tfrac{4\sqrt{3}}{3}$$
$$= \tfrac{4}{3}(8\pi - \sqrt{3})\,\text{sq.units.}$$