4 Applications of Derivatives in Commerce and Economics

INTRODUCTION

Quantitative techniques and mathematical models are now being increasingly used in business and economic problems. Differential calculus is used while determining the rate of change of a given function (dependent variable) due to change in one of the independent variables. Integration is the inverse of differentiation and it involves finding a function whose rate of change is given.

In this chapter, we shall start with the a few basic concepts of economics—fixed and variable cost, average cost, revenue, profit etc., and then go on to marginal functions (marginal cost and marginal revenue) using first derivative. We shall use second derivatives to find minimum costs and maximum revenue or maximum profit.

4.1 FIXED AND VARIABLE COST; AVERAGE COST

The cost of production of a commodity depends upon a number of factors—size of plant, level of output, prices of raw materials, technology etc.

$$C = f(S, O, P, T, \ldots)$$

However, in this chapter, we will study cost as a function of level of output only. Thus, the total cost of producing x units of a commodity is written as

$$TC = f(x)$$

The total cost consists of two components—fixed costs and variable costs.

Fixed Costs are those which are incurred regardless of the level of production—like interest, rent, wages of permanent staff etc. Thus total fixed cost,

$$TFC = TC \text{ when } x = 0$$

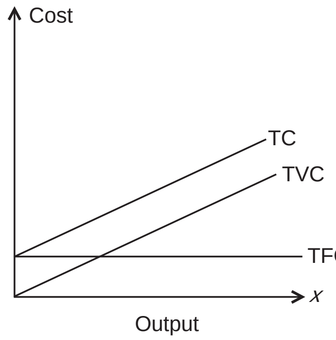
Fixed cost does not change whether there is any increase or decrease in level of production.

Variable costs are those which vary with output, for example, raw materials and wages of casual labour. Thus,

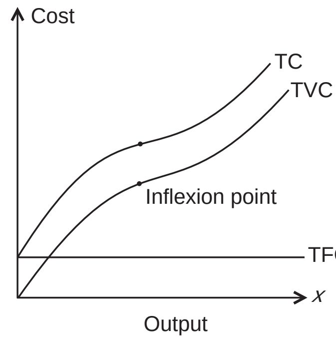
$$\text{Total Cost} = \text{Total Fixed Cost} + \text{Total Variable Cost}$$

$$TC = TFC + TVC$$

Note that TFC is constant while TVC starts from zero at zero output and goes up. TC is sum of TFC and TVC. In actual practice, the TVC is usually not a straight line. First it increases at a decreasing rate, goes through an inflexion point and then increases at an increasing rate. This point will become clearer when we study marginal concepts.



(i)



(ii)

Average cost is obtained by dividing the cost by level of output.

Thus, $C = f(x) \Rightarrow$ Average cost $AC = \dfrac{C}{x} = \dfrac{f(x)}{x}$

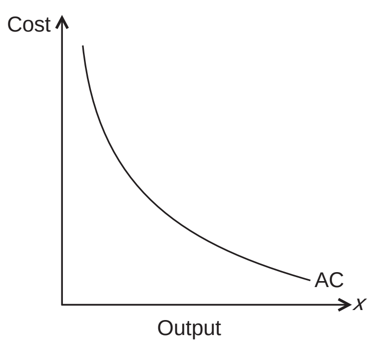
Also $TC = TFC + TVC$

$\Rightarrow \quad \dfrac{TC}{x} = \dfrac{TFC}{x} + \dfrac{TVC}{x}$

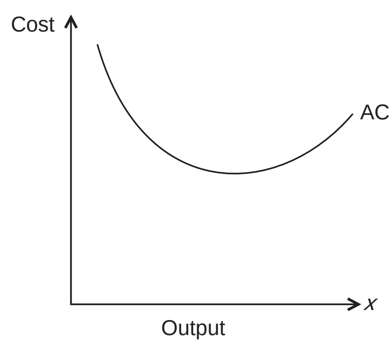
$\Rightarrow \quad AC = AFC + AVC$

Thus, Average Cost = Average Fixed Cost + Average Variable Cost

In case of constant variable cost, the average cost curve is shown in figure (*i*) below. However, usually in real life, the economics of scale work only up to a point and the cost of producing each new unit again starts rising. In this case, the average cost curve is as shown in figure (*ii*) below.



(i)



(i)

ILLUSTRATIVE EXAMPLES

Example 1. *For manufacturing a certain item, the fixed cost is ₹ 6000 and the cost of producing each unit is ₹ 20.*

(i) What is the cost function ?

(ii) What is the total cost and average cost of producing 15 units ?

(iii) What is the total cost and average cost of producing 100 units ?

Solution. (*i*) Total fixed cost = ₹ 6000,

Variable cost of producing one unit = ₹ 20

$\therefore$ Total cost, TC = ₹ (6000 + 20x), where x is the number of units produced.

(*ii*) Total cost of producing 15 units = $TC|_{x=15}$

= ₹ (6000 + 20.15) = ₹ 6300

$\therefore$ Average cost of producing 15 units = ₹ $\dfrac{6300}{15}$ = ₹ 420.

(*iii*) Total cost of producing 100 units = ₹ (6000 + 20.100) = ₹ 8000

∴ Average cost of producing 100 units = ₹ $\frac{8000}{100}$ = ₹ 80.

Example 2. *The cost function for a certain commodity is*

$$C(x) = 3 + 2x - \frac{1}{4}x^2$$

Write the various cost components (TC, TFC, TVC, AC, AFC, AVC) when 4 items are produced. Verify your result.

Solution. Total cost TC or C (x) = $3 + 2x - \frac{1}{4}x^2$

Total fixed cost, TFC = $C(x)|_{x=0} = 3$

Total variable cost, TVC = $2x - \frac{1}{4}x^2$

Average cost, AC = $\frac{C(x)}{x} = \frac{3}{x} + 2 - \frac{1}{4}x$

Average fixed cost, AFC = $\frac{TFC}{x} = \frac{3}{x}$

Average variable cost, AVC = $\frac{TVC}{x} = 2 - \frac{1}{4}x$

When $x = 4$, we get

Total Cost = C(4) = $3 + 2.4 - \frac{1}{4}.4^2 = 7$

Total fixed cost TFC = 3

Total variable cost TVC = $2.4 - \frac{1}{4}.4^2 = 4$

We see that TC = TFC + TVC

Average Cost AC = $\frac{3}{4} + 2 - \frac{1}{4}.4 = 1\frac{3}{4}$

Average fixed cost AFC = $\frac{3}{4}$

Average variable cost AVC = $2 - \frac{1}{4}.4 = 1$

We see that AC = AFC + AVC.

Example 3. *It is known that cost of producing 100 units of a commodity is ₹ 250 and cost of producing 200 units is ₹ 300. Assuming that AVC is constant, find the cost function.*

Solution. Let the total fixed cost TFC be a.

Let AVC = constant = b

Then TVC = (AVC) $x = bx$

∴ TC = TFC + TVC = $a + bx$

Given that when $x = 100$, TC = ₹ 250 and when $x = 200$, TC = ₹ 300

∴ $a + 100b = 250$ ...(*i*)

$a + 200b = 300$ ...(*ii*)

Subtracting (*i*) from (*ii*),

$$100b = 50 \Rightarrow b = \frac{1}{2}$$

Putting this value of b in (*i*),

$$a + 100.\frac{1}{2} = 250 \Rightarrow a = 200$$

Hence, the cost function is C (x) = $200 + \frac{1}{2}x$.

EXERCISE 4.1

1. For manufacturing a certain item, the fixed cost is ₹ 6500 and the cost of producing each unit is ₹ 12·50.

(*i*) What is the cost function ? Draw a graph, clearly indicating TC, TFC and TVC.

(*ii*) What is the total cost of producing 75 items ?

(*iii*) What is the average cost of producing 400 items ?

2. The cost function for a certain commodity is

$$C(x) = 12 + 3x - \frac{1}{3}x^2$$

Write down the total cost, fixed cost, variable cost and average cost when 3 units are produced.

3. Cost of producing 75 units of a commodity is ₹ 275 and cost of producing 150 units is ₹ 300. Assuming that TVC is linear, find the

(*i*) cost function.

(*ii*) average total cost of producing 75, 150, 225 units respectively.

(*iii*) average fixed cost of producing 75, 150, 225 units respectively.

(*iv*) average variable cost of producing 75, 150, 225 units respectively.

4. The total cost of producing and marketing x units of commodity is given by

$$C = 2x + e^x + 5e$$

Find (*i*) the fixed cost (*ii*) the variable cost

(*iii*) total cost of producing 5 units

(*iv*) average cost of producing 5 units.

4.2 DEMAND FUNCTION; REVENUE AND PROFIT FUNCTIONS

4.2.1 Demand function

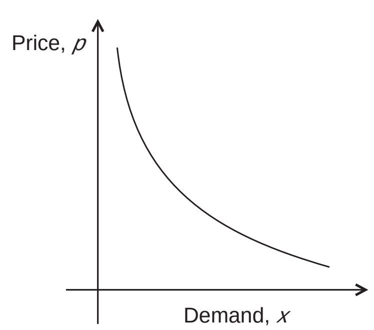
Various economic studies show that the quantity demanded of a commodity depends upon many factors, viz., price of the commodity, consumer's income, taste of the consumer, price of other related commodities etc. To simplify things, we will consider the relationship between demand and price of the commodity only, assuming that all other factors remain constant.

The price per unit and the quantity demanded at that price are usually related inversely—higher the price, lesser is the quantity demanded—or, higher the quantity demanded, lower is the price.

Demand function $x = f(p)$

However, by tradition, price p is shown against demand x.

Thus, $p = f(x)$



Remarks

1. If the demand of the item depends upon the price of the item, then the demand function is given by $x = f(p)$. For example, $x = 70 + 3p$ where p is the price per unit and the demand of the item is x units.

2. If the price of the item depends upon the demand of the item, then the demand function is given by $p = f(x)$. For example, $p = 30 - 5x + 2x^2$ where x units is the demand of the item and p is the price per unit.

3. From above remarks, it follows that any relation between x and p is called a demand function. We can make x as the subject of the relation or p as the subject of the relation.

4.2.2 Revenue function

Revenue means the amount received by a company by selling a certain number of units of a commodity. Let p be the price per unit and x be the number of units sold. Then total revenue

$$\text{R or R}(x) = p \,.\, x$$

If price p is constant, then $R(x)$ is obviously a straight line. If price p varies with demand x, then $p = f(x)$ so that

$$\text{R or R}(x) = (\text{demand}) \,.\, (\text{price}) = x\, f(x)$$

The **average revenue** or revenue per unit is given by

$$\text{AR} = \frac{\text{R}}{x} = \frac{px}{x} = p$$

Thus, **average revenue is the same as price per unit.**

4.2.3 Profit function

The **profit function** $P(x)$ or $\pi(x)$ of producing, marketing and selling x units of a commodity is

$$\text{P}(x) \text{ or } \pi(x) = \text{R}(x) - \text{C}(x)$$

where $R(x)$ is the revenue function and $C(x)$ is the cost function.

The **average profit** is $\dfrac{P(x)}{x} = \dfrac{R(x)}{x} - \dfrac{C(x)}{x}$

Thus, **average profit = average revenue – average cost.**

Imposition of taxes reduces profit margins; subsidies reduce costs and hence increase profit margins.

4.3 BREAKEVEN ANALYSIS

Usually, as the companies incur capital costs (fixed costs), they are in loss when the production/sale is low. However, as the production/sale increases, the average cost comes down, and beyond a certain point, the company starts making profit. This leads us to breakeven point analysis.

The **breakeven point** is the level of production where the revenue from sales is equal to the total cost of production. At this point, the company makes neither profits nor losses.

Thus, at breakeven point,

$$R(x) = C(x)$$

i.e. $R(x) - C(x) = 0$

i.e. $P(x) = 0$

Any of these forms may be used to determine the breakeven point. There may be zero, one or more than one breakeven points.

ILLUSTRATIVE EXAMPLES

Example 1. *(i) A chocolate bar sells for ₹ 20. What is the total revenue and average revenue by selling 30 bars ?*

(ii) The demand function for T.V. sets is $p = 20000 - 100x$ (rupees). Determine the total revenue and average revenue by selling 20 sets.

Solution. (*i*) Price p = ₹ 20

Total revenue by selling 30 bars = ₹ (30 × 20) = ₹ 600

Average revenue by selling 30 bars = $\dfrac{R}{x}$ = ₹ $\dfrac{600}{30}$ = ₹ 20 (*ii*) Price p = ₹ $(20000 - 100x)$ per item

∴ Total revenue by selling x sets $= px =$ ₹ $(20000 - 100x)\,x$

∴ Total revenue by selling 20 sets $=$ ₹ $(20000 - 100.20) \times 20$

$=$ ₹ $(18000 \times 20) =$ ₹ 360000

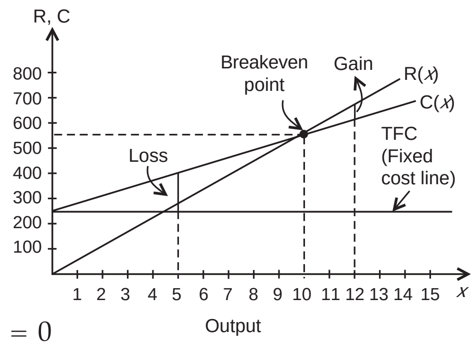
Average revenue by selling 20 items $=$ ₹ $\dfrac{360000}{20} =$ ₹ 18000.

Example 2. *The price of a commodity is fixed at ₹ 55 and its cost function is C(x) = 30x + 250.*

(i) Determine the breakeven point.

(ii) What is the profit when 12 items are sold ?

(iii) What is the profit when 5 items are sold ?



Solution. (*i*) Here revenue $R(x)$

$=$ (price) (demand) $= 55x$,

Cost $C(x) = 30x + 250$

∴ Profit function

$P(x) = R(x) - C(x)$

$= 55x - (30x + 250)$

$= 25x - 250$

To determine the breakeven point, we have $P(x) = 0$

$\Rightarrow \quad 25x - 250 = 0 \Rightarrow x = 10$

Hence, the breakeven point is $x = 10$. At this level of production,

revenue $=$ cost $=$ ₹ 550 and profit $= 0$.

(*ii*) When 12 items are produced,

Profit $=$ ₹ $(25.12 - 250) =$ ₹ $(300 - 250) =$ ₹ 50

Thus, there is a profit of ₹ 50 when 12 items are produced and sold.

(*iii*) When 5 items are sold,

Profit $=$ ₹ $(25.5 - 250) =$ ₹ $(125 - 250) = -$ ₹ 125

The minus sign shows that there is a loss.

Thus, there is a loss of ₹ 125 when only 5 items are produced and sold.

Example 3. *A manufacturer finds that selling price of a product is ₹ 25 while cost is C(x) = 30x + 120. What would you advise him ?*

Solution. Here revenue $R(x)$ = (price) (demand) $= 25x$

Cost $C(x) = 30x + 120$

∴ Profit $P(x) = R(x) - C(x) = 25x - (30x + 120) = -5x - 120$

We can see that profit is always negative, regardless of the level of production. ***Alternatively,*** to determine breakeven point, we have $P(x) = 0$

$\Rightarrow \quad -5x - 120 = 0 \quad \Rightarrow x = -24$

As the number of units ≥ 0, we see that there is no breakeven point.

As the project is doomed from the start, the businessmen should not take up this project.

Example 4. *The fixed cost of a new product is ₹ 18000 and the variable cost per unit is ₹ 550. If the demand function is p (x) = 4000 – 150x, find the breakeven values.* (*I.S.C. 2007*)

Solution. Let x units of the product be produced and sold.

As the variable cost per units is ₹ 550,

∴ the variable cost of producing x units $=$ ₹ $550x$.

As the fixed cost is ₹ 18000,

∴ total cost of producing x units, $C(x) =$ ₹ $(18000 + 550x)$.

Given demand function is $p(x) = 4000 - 150x$ *i.e.* the selling price per unit is

₹ $(4000 - 150x)$.

∴ Total revenue on selling x units,

$$R(x) = \text{(price per unit) (number of units sold)}$$
$$= ₹\ (4000 - 150x)\ x.$$

At breakeven values, $C(x) = R(x)$

⇒ $18000 + 550x = (4000 - 150x)\ x$

⇒ $150x^2 - 4000x + 550x + 18000 = 0$

⇒ $150x^2 - 3450x + 18000 = 0$

⇒ $x^2 - 23x + 120 = 0$

⇒ $(x - 8)(x - 15) = 0$

⇒ $x = 8, 15.$

Hence, the breakeven values are $x = 8$ and $x = 15$.

Example 5. *The total cost and the total revenue of a company that produces and sells x units of particular product are respectively*

$$C(x) = 5x + 350 \text{ and } R(x) = 50x - x^2.$$

Find (i) the breakeven values (ii) the values of x that produce a profit (iii) the values of x that result in a loss.

Solution. (*i*) At breakeven values, $R(x) = C(x)$

⇒ $50x - x^2 = 5x + 350$ ⇒ $x^2 - 45x + 350 = 0$

⇒ $(x - 10)(x - 35) = 0$ ⇒ $x = 10$ or 35

Hence, the breakeven values are $x = 10$ and $x = 35$.

(*ii*) For profit, $R(x) > C(x)$ ⇒ $50x - x^2 > 5x + 350$

⇒ $x^2 - 45x + 350 < 0$ ⇒ $(x - 10)(x - 35) < 0$

⇒ $10 < x < 35$

(*iii*) From (*i*) and (*ii*) we see that losses occur when $x < 10$ or $x > 35$.

Alternatively, losses occur when $R(x) < C(x)$

⇒ $50x - x^2 < 5x + 350$ ⇒ $x^2 - 45x + 350 > 0$

⇒ $(x - 10)(x - 35) > 0$ ⇒ $x < 10$ or $x > 35$.

Example 6. *A company produces a commodity with ₹ 24000 fixed cost. The variable cost is estimated to be 25% of the total revenue recovered on selling the product at a rate of ₹ 8 per unit. Find the following :*

(i) Cost function (ii) Revenue function (iii) Break-even point. (*I.S.C. 2013*)

Solution. Let x units of the product be produced and sold. As the selling price of one unit is ₹ 8, so the total revenue on selling x units = ₹ $8x$.

(*i*) Since the variable cost is 25% of total revenue recovered,

so the variable cost = 25% of ₹ $8x$ = ₹ $\left(\frac{25}{100} \times 8x\right)$ = ₹ $2x$.

Fixed cost of the company is ₹ 24000.

∴ Cost function (in ₹) = $C(x) = 2x + 24000$.

(*ii*) Revenue function (in ₹) = $R(x) = 8x$.

(*iii*) At break-even points, $R(x) = C(x)$

⇒ $8x = 2x + 24000$ ⇒ $6x = 24000$ ⇒ $x = 4000$.

Hence, break-even point is $x = 4000$.

Example 7. *A company is selling a certain product. The demand function for the product is linear. The company can sell 2000 units when the price is ₹ 8 per unit and it can sell 3000 units when the price is ₹ 4 per unit. Determine :*

(i) the demand function (ii) the total revenue function. (*I.S.C. 2005*)

ANSWERS

EXERCISE 4.1

1. (*i*) $C(x) = 6500 + 12{\cdot}50x$ (*ii*) ₹ 7437·50 (*iii*) ₹ 28·75.

2. TC = ₹ 18, AC = ₹ 6.

3. (*i*) $C(x) = 250 + \frac{1}{3}x$ (*ii*) ₹ 3·67, ₹ 2, ₹ 1·44

(*iii*) ₹ 3·33, ₹ 1·67, ₹ 1·11 (*iv*) ₹ 0·33, ₹ 0·33, ₹ 0·33.

4. (*i*) FC = $5e$ (*ii*) VC = $2x + e^x$

(*iii*) $TC|_{x=5} = 10 + e^5 + 5e$ (*iv*) $AC|_{x=5} = 2 + \frac{e^5}{5} + e$.

EXERCISE 4.2

1. (*i*) $P(x) = 10x - 150$; breakeven point is $x = 15$

(*ii*) $P(x) = -5x - 150$; no breakeven point

(*iii*) $P(x) = 56x - 4x^2 - 180$; breakeven at $x = 5, 9$

(*iv*) $P(x) = 650x - 5x^2 - 20000$; breakeven at $x = 50, 80$.

2. (*i*) $C(x) = 15000 + 30x$ (*ii*) $R(x) = 45x$ (*iii*) $x = 1000$.

3. $x = 5, 9$.

4. (*i*) $C(x) = 150000 + 150x$ (*ii*) $R(x) = 350x$

(*iii*) $P(x) = 200x - 150000$ (*iv*) $x = 750$.

5. $P(x) = 67x - 40200$, breakeven point is 600.

6. (*i*) $R(x) = 6x$ (*ii*) $C(x) = 4500 + 1{\cdot}5x$

(*iii*) $P(x) = 4{\cdot}5x - 4500$ (*iv*) $x = 1000$

(*v*) 750 units.

7. (*i*) 579 (*ii*) 440 (*iii*) ₹ 3·15.

8. (*i*) $P(x) = 4500x - 100x^2 - 35000$ (*ii*) $x = 10, 35$

(*iii*) $x < 10$ or $x > 35$.

9. Breakeven points are $x = 40, 320$. Thus, the company should produce minimum 40 units to recover its cost.

10. 400 units; the company will always remain in profit if it produces and sells more than 400 units of the product.

11. Breakeven points are $x = 8, 1250$. Hence, the company must sell at least 8 units to cover its costs.

12. (*i*) $R(x) = 5x$ (*ii*) $C = 3200 + 1{\cdot}25x$

(*iii*) $x = 853{\cdot}33$ (*iv*) 640 units.

13. $P(x) = 5x - 1200$; ₹ 3800 profit; ₹ 1300 profit; ₹ 200 loss.

14. (*i*) $C = \frac{20}{9}(245 - 7p)$ (*ii*) $R = \frac{p}{9}(245 - 7p)$

(*iii*) $P = \frac{1}{9}(p - 20)(245 - 7p)$ (*iv*) $p = 20, x = \frac{105}{9}$.

EXERCISE 4.3

1. (*i*) $AC = \frac{1200}{x} + 20 + x$ (*ii*) $MC = 20 + 2x$

(*iii*) MC = 40 at $x = 10$; it indicates that ₹ 40 are needed to increase the production from 10 to 11 units

(*iv*) ₹ 41; it indicates exact amount needed to increase the production from 10 to 11 units.

2. (*i*) $MC = x^2 + 6x - 16$ (*ii*) $AC = \frac{1}{3}x^2 + 3x - 16 + \frac{2}{x}$.

3. (*i*) 16·5 (*ii*) 36.

4. (*i*) $x^2 + 6x - 7$ (*ii*) $\frac{x^2}{3} + 3x - 7 + \frac{16}{x}$ **5.** (*i*) $x^2 + 6x - 7$

8. (*i*) Total cost = $x^2 + 5x + 6$, marginal cost = $2x + 5$ (*ii*) $x > 6$.

9. (*i*) $MC = 3x^2 - 48x + 600$, $AC = x^2 - 24x + 600$