Chapter 3

z-Transform

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Sampling
- Sampling theorem
- The *z*-transform
- Region of convergence
- *z*-transform properties
- Inverse *z*-transform
- Analysis of discrete-time LTI systems
- Causal and stable systems
- The unilateral *z*-transform
- Relationship between *z* and Laplace transform
- Direct form-I structure
- Direct form-II structure of IIR system

Sampling

Let $x(t)$ be any continuous signal periodically sampled at equal sample time period T_s to get discrete time signal represented by $x[n]$

$x[n] = x(nT_s)$
$n = 0, \pm 1, \pm 2, \pm 3 \ldots \infty$
T_s = Sampling time period

Sampling Theorem

Let $x(t)$ is strictly band limited signal with Fourier spectrum.

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_m$$

Then $x(t)$ can be recovered without loss of information from its sample signal $x[n] = x(nT_s)$, if the sampling frequency ω_s is greater than twice of the maximum frequency ω_m (of $x(t)$)

$$\omega_s > 2\omega_m.$$

Minimum sampling frequency $\omega_s = 2\omega_m$ is called Nyquist sampling rate.

In case of sampling frequency $\omega_s < 2\omega_m$.

$X_p(j\omega)$ which is spectrum of $x_p(t) = x(nT_s)$ no longer contains all information of $X(j\omega)$; hence $x(t)$ is not fully recoverable from sampler $x_p(t)$ and there is definite loss of original signal $x(t)$.

The loss of information in such sampling case ($\omega_s < 2\omega_m$) is due to super imposition of high frequency components onto the less-frequency component. This phenomena is called frequency folding or aliasing.

Due to aliasing, high frequency noise components get mixed with signal frequencies between 0 to $\omega_s/2$.

To prevent high frequency noise components having frequency higher than $\omega_s/2$ an analog filter is used before sampler to attenuate signals with frequency higher than $\omega_s/2$. Such filters are called anti-aliasing filters.

The z-Transform

A discrete time LTI system with impulse response $h[n]$, the output $y[n]$ of the system to the complex exponential input of the form z^n is $y[n] = T\{z^n\}$

$$= h[n] * z^n$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$$= H(z)\, z^n$$

where, $H(z) = \sum_{-\infty}^{\infty} h[n] z^{-n}$

The variable z is generally complex valued.

When $|z|$ is not restricted to unity, the summation is referred to as the z-transform of $h[n]$.

Definition

For general discrete time signal $x[n]$, the z transfer function $X(z)$ is defined as $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$.

The variable is generally complex valued and is expressed in polar form as $z = re^{j|}$.

where 'r' is the magnitude of z and 'Ω' is the angle of 'z'

$x[n]$ and $X(z)$ are said to form a z transform pair as $x[n] \leftrightarrow X(z)$

Region of Convergence

The range of values of the complex variable z for which the z-transform converges is called the region of convergence.

Consider the sequence $x[n] = a^n u[n]$

$$\text{z-transform of } x[n],\ X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

For the convergence this summation should be finite.

This ROC is the range of values of z for which $|az^{-1}| < 1$, or $|z| > |a|$ then,

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}},\ |z|>|a| = \frac{z}{z-a}\ |z| > |a|.$$

There is one zero, at $z = 0$, and one pole at $z = a$, ROC and the pole zero plot for the examples are shown in figure.

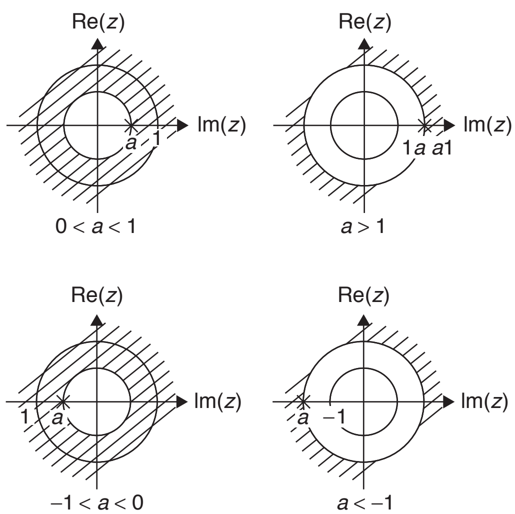


Figure 1 ROC of the form $|z| > |a|$

Now let $x[n] = -a^n u[-n-1]$, then z-transform

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n.$$

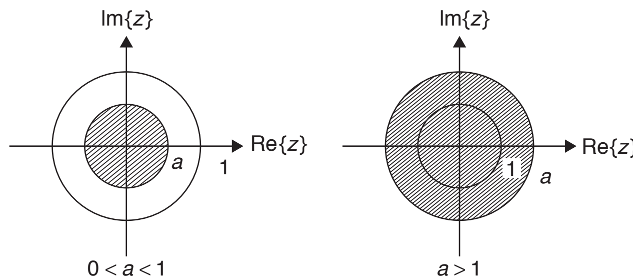
By taking $n = -n$,

$$= 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

If $|a^{-1} z| < 1$, then the above summation converges, i.e., for $|z| < |a|$

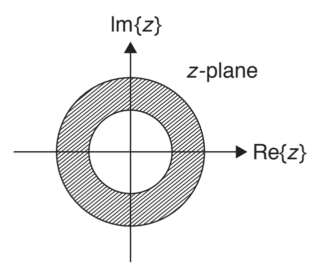
$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a} = |z| < |a|$$

The pole zero plots are



Properties of the ROC

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin. In some cases, the inner boundary can extend inward to the origin, in which case the ROC becomes a disc. In other case, the outer boundary can extend outward to infinity.



2. ROC does not contain any poles; at poles, $X(z)$ is infinite and therefore, by definition it does not converge.
3. If $x[n]$ is a finite sequence, then $X(z)$ converges for every value of z, then ROC is the entire z-plane except possibly $z = 0$ and/ or $z = \infty$.
4. If $x[n]$ is a right-handed sequence, ($x[n] = 0$, for $n < N_1 < \infty$), then ROC is of the form $|z| > r_{max}$ or $\infty > |z| > r_{max}$ where r_{max} is the largest magnitude of any of the poles of $X(z)$. Thus ROC is the exterior of the circle $|z| = r_{max}$.
5. If $x[n]$ is a left-handed sequence ($x[n] = 0$, for $n > N_2 > -\infty$), then ROC is of the form $|z| < r_{min}$ or $0 < |z| < r_{min}$ where r_{min} is the smallest magnitude of any of the poles of $X(z)$, thus ROC is interior of the circle $|z| = r_{min}$.
6. If $x[n]$ is two-sided sequence, ($x[n]$ is infinite duration sequence). Then the ROC is of the form $r_1 < |z| < r_2$ where r_1, r_2 are the magnitudes of the two poles of $X(z)$. It forms like a ring.

7. Relationship between the ROC and the time extent of a signal

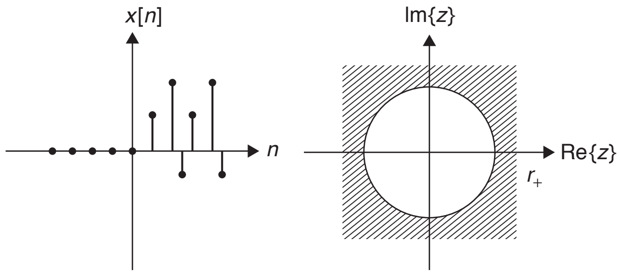


Figure 2 A right sided signal has an ROC of form $|z| > r_+$

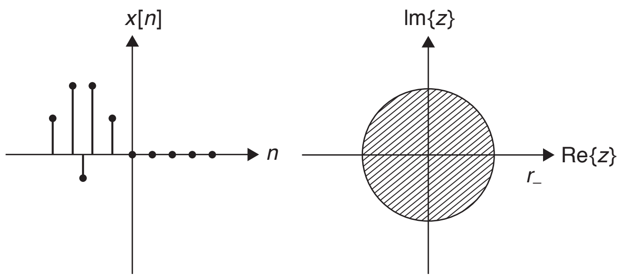


Figure 3 A left sided signal has an ROC of form $|z| < r_-$

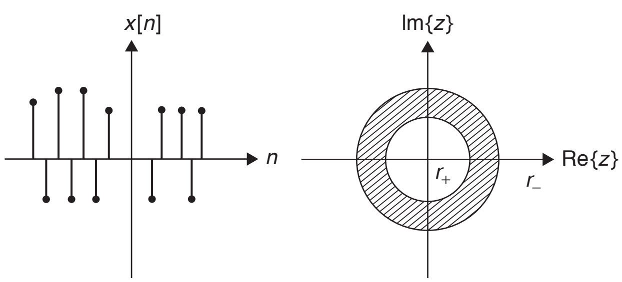


Figure 4 A two sided signal has an ROC of form $r_+ < |z| < r_-$

Z-TRANSFORMS OF THE SOME COMMON SEQUENCES

For unit impulse signal $\delta[n]$, its z-transform is given by

$$\delta[n] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

ROC is entire z-plane including $z = 0$, and $z = \infty$.
For delayed unit impulse

$$\delta[n-n_0] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = z^{-n_0}$$

ROC is entire z-plane, expect $z = 0$.
For advanced unit impulse

$$\delta[n+n_0] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n+n_0] z^{-n} = z^{n_0}$$

ROC is entire z-plane, except $z = \infty$

***X*(*n*)**	***X*(*z*)**	**ROC**
$\delta(n)$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 ($m > 0$) or ∞($m < 0$)
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$(\cos \Omega_0 n)u[n]$	$\frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\lvert z\rvert > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\lvert z\rvert > 1$
$(r^n \text{Cos}\, \Omega_0 n)u[n]$	$\frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\lvert z\rvert > r$
$(r^n \sin \Omega_0 n)u[n]$	$\frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\lvert z\rvert > r$
$a^n \quad 0 < n < N-1$ $0 \quad$ Otherwise	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Example 1: What are the ROC, and z-transforms of the signal $x[n] = 5\left(\frac{1}{4}\right)^n u[n] - 4\left(\frac{1}{3}\right)^n u[n]$?

Solution: $X(z) = \sum_{n=-\infty}^{\infty} \left\{5\left(\frac{1}{4}\right)^n u[n] - 4\left(\frac{1}{3}\right)^n u[n]\right\} z^{-n}$

$$= 5\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n z^{-n} - 4\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n}$$

$$= 5\sum_{n=0}^{\infty}\left(\frac{1}{4}z^{-1}\right)^n - 4\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n \quad (1)$$

$$= \frac{5}{1-\frac{1}{4}z^{-1}} - \frac{4}{1-\frac{1}{3}z^{-1}} = \frac{1-\frac{5}{3}z^{-1}+z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$= \frac{1-\frac{2}{3}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{z\left(z-\frac{2}{3}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}$$

For convergence of $X(z)$, both sums in the equation (1) should converge, which requires $\left|\frac{1}{4}z^{-1}\right| < 1$, and $\left|\frac{1}{3}z^{-1}\right| < 1$ or equivalently $|z| > \frac{1}{4}$ and $|z| > \frac{1}{3}$.

By combining these two regions, the ROC will be $|z| > \frac{1}{3}$.

Example 2: What is the z-transform and ROC of the signal

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1, a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution: $X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$

$$= \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-(az^{-1})} = \frac{1}{z^{N-1}} \cdot \frac{z^N - a^N}{z-a}$$

$$= \frac{1}{z^{N-1}} \frac{(z-a)(z^{N-1} + K_1 z^{N-2} + \cdots + a^{N-1})}{z-a}$$

$$= \frac{1}{z^{N-1}} (z^{N-1} + K_1 z^{N-2} + \cdots + a^{N-1})$$

There is a pole of order $N-1$ at $z = 0$, and ROC is all values of z except $z = 0$

Example 3: What is the z-transform of $x[n] = b^{|n|}$ $b > 0$

Solution: $x[n] = b^n u[n] + b^{-n} u[-n-1]$

$$b^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-bz^{-1}}, \ |z| > b,$$

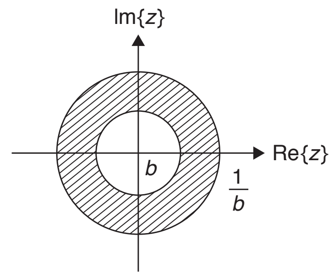
$$\left(\frac{1}{b}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{-1}{1-b^{-1}z^{-1}}, \ |z| < \frac{1}{b}$$

When $b > 1$, there is no common ROC for the above functions, but for $0 < b < 1$, the ROCs of above two equations will overlap and thus the z-transform for composite sequence is

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}, \ b < |z| < \frac{1}{b}$$

$$= \frac{z}{z-b} - \frac{z}{z-b^{-1}}, \ b < |z| < \frac{1}{b}$$

$$= \frac{b^2-1}{b} \frac{z}{(z-b)(z-b^{-1})}, \ b < |z| < \frac{1}{b}$$



ROC for $0 < b < 1$

z-Transform Properties

1. Linearity
   $x_1[n] \leftrightarrow X_1[z]$, ROC $= R_1$
   $x_2[n] \leftrightarrow X_2[z]$, ROC $= R_2$
   $a_1x_1[n] + a_2x_2[n] \leftrightarrow a_1X_1(z) + a_2X_2(z)$
   $R' \supset R_1 \quad R_2$
2. Time shifting
   $x_1[n] \leftrightarrow X(z)$, ROC $= R$
   $x[n-n_0] \leftrightarrow z^{-n_0}X(z)$, $R^1 = R \cap \{0 < |z| < \infty\}$
3. Multiplication by z_0^n
   $z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$, $R' = |z_0| R$
   $a^n x[n] \leftrightarrow X(a^{-1}z)$, $R' = |a| R$
   $e^{+j\Omega_0 n} x[n] \leftrightarrow X(e^{-j\Omega_0 n})$, $R' = R$
4. Time reversal
   $x[-n] \leftrightarrow X\left(\frac{1}{z}\right)$, $R' = \frac{1}{R}$
5. Multiplication by 'n'
   $nx[n] \leftrightarrow z - \frac{dx(z)}{dz}$, $R' = R$
6. Accumulation
   $\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{1}{1-z^{-1}} X(z)$, $R' \supset R \cap \{|z| > 1\}$
7. Convolution
   $x_1[n] * x_2[n] \leftrightarrow X_1(z)\,X_2(z)$, $R' \supset R_1 \cap R_2$
8. Conjugation
   $x^*[n] \leftrightarrow X^*[z^*]$ with ROC $= R$
9. First difference
   $x[n] - x[n-1] \leftrightarrow (1 - z^{-1})\,X(z)$,
   $R' \supset R \cap \{|z| > 0\}$
10. Initial value theorem
    if $x[n] = 0$, for $n < 0$, then $x[0] = \operatorname{Lt}_{z\to\infty} X(z)$
11. Final value theorem
    $\operatorname{Lt}_{n\to\infty} x[n] = \operatorname{Lt}_{z\to 1}(z-1)X(z)$, Poles of $(z-1)\,X(z)$ must lie inside the unit circle.
12. Time Expansion
    $x_k[n] = x[n/k] \leftrightarrow X(z^k)$, $R' = R^{1/k}$, for n, k, $\frac{n}{k}$ are integers.

Inverse z-Transform

Inversion of the z-transform to find the sequence $x[n]$ from its z-transform $X(z)$ is called inverse z-transform.

$$x[n] = z^{-1}\{X(z)\} \text{ or}$$

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Where C is a counter clockwise contour of integration enclosing the origin.

Example 4: Consider the z-transform $X(z)$

$$= \frac{3-\frac{11}{15}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}.$$

Find the inverse z-transform for different ROCs

(i) $|z| > \frac{1}{3}$

(ii) $\frac{1}{5} < |z| < \frac{1}{3}$

(iii) $|z| < \frac{1}{5}$

Solution: $X(z) = \dfrac{3-\frac{11}{15}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}$

$$X(z) = \frac{z^{-1}\left(3z-\frac{11}{15}\right)}{z^{-2}\cdot\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)} = \frac{1}{z^{-1}}\left[\frac{2}{z-\frac{1}{3}}+\frac{1}{z-\frac{1}{5}}\right]$$

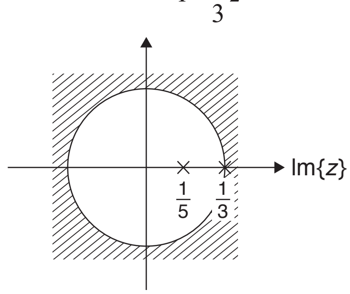
$$X(z) = \frac{2}{1-\frac{1}{3}z^{-1}}+\frac{1}{1-\frac{1}{5}z^{-1}}$$

(i) When ROC is $|z| > \frac{1}{3}$

There are two poles $z = \frac{1}{3}$ and $\frac{1}{5}$, and ROC lies outside outermost pole, so the inverse z-transform is a right sided sequence

$$\left(\frac{1}{5}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{5}z^{-1}}, |z| > \frac{1}{5}$$

$$\left(\frac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{3}z^{-1}}, |z| > \frac{1}{3}$$



For the above two functions ROC can be combined as $|z| > \frac{1}{3}$

So, $x[n] = 2\left(\frac{1}{3}\right)^n u[n]+\left(\frac{1}{5}\right)^n u[n]$.

(ii) When ROC is $\frac{1}{5} < |z| < \frac{1}{3}$

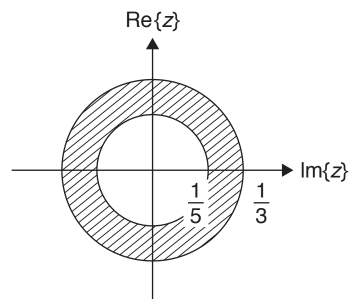
ROC is $|z| < \frac{1}{3}$ and $|z| > \frac{1}{5}$

So, the functions

$$\left(\frac{1}{5}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{5}z^{-1}} |z| > \frac{1}{5}$$

$$-\left(\frac{1}{3}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{3}z^{-1}} |z| < \frac{1}{3}.$$

By combining above two functions



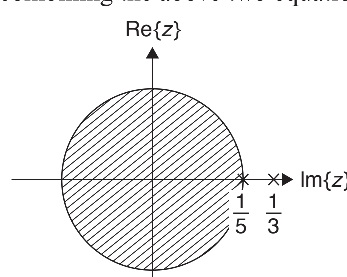
we get $x[n] = \left(\frac{1}{5}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$

(iii) When ROC is $|z| < \frac{1}{5}$; in this case the ROC is inside the both poles

$$-\left(\frac{1}{5}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{5}z^{-1}} |z| < \frac{1}{5}$$

$$-\left(\frac{1}{3}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{3}z^{-1}} |z| < \frac{1}{3}$$

By combining the above two equations



we get $x[n] = -2\left(\frac{1}{3}\right)^n u[-n-1] - \left(\frac{1}{5}\right)^n u[-n-1]$

Example 5: What is the inverse z-transform of

$$X(z) = \log(z + a) - \log z,\ |z| > |a|$$

Solution: $X(z) = \log(z + a) - \log z,\ |z| > |a|$

$$= \log\left(\frac{z+a}{z}\right), |z| > |a|$$

$$X(z) = \log(1 + az^{-1}),\ |z| > |a|$$

By using multiplication by n property,

$$n \cdot x[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} \tag{2}$$

$$\frac{dX(z)}{dz} = \frac{a}{1+az^{-1}} \cdot (-z^{-2})$$

$$-z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}\ |z| > |a| \tag{3}$$

$$(-a)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1+az^{-1}}, |z| > |a|$$

$$a(-a)^n u[n] \overset{z}{\longleftrightarrow} \frac{a}{1+az^{-1}}, |z| > |a|$$

By using time shifting property

$$a(-a)^{n-1}u[n-1] \overset{z}{\longleftrightarrow} z^{-1}\frac{a}{1+az^{-1}},$$

$$|z| > |a| \tag{4}$$

So from equations (2), (3) and (4)

$$nx[n] = a(-a)^{n-1}\, u[n-1]$$

or $x[n] = \dfrac{-(-a)^n u[n-1]}{n}$

Example 6: What is the inverse z-transform of $X(z) = 5z^4 + 3z^2 + 2 + z^{-1} + 4z^{-3}$ $0 < |z| < \infty$

Solution: $\delta[n] \overset{z}{\longleftrightarrow} 1$

for any $x[n]$

$x[n - n_0] \leftrightarrow z^{-n_0} X(z)$

So $\delta[n + 4] \leftrightarrow z^4$

$\delta[n + 2] \leftrightarrow z^2$

$\delta[n - 1] \leftrightarrow z^{-1}$

$\delta[n - 3] \leftrightarrow z^{-3}$

$X(z) = 5z^4 + 3z^2 + 2 + z^{-1} + 4z^{-3}$

Then $x[n] = 5\delta[n + 4] + 3\delta[n + 2] + 2$

$\delta[n] + \delta[n - 1] + 4\,\delta[n - 3]$

Example 7: Inverse z-transform for $X(z) = \dfrac{z^{-1}}{(1-az^{-1})^2}, |z| > |a|$

Solution: $a^n u[n] \overset{Z}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}\ |z| > |a|$

Consider $x[n] = a^n\, u[n]$

$$n\,x[n] \overset{Z}{\longleftrightarrow} -z\frac{d}{dz}\{X(z)\}$$

$$-z\frac{d}{dz}\left\{\frac{1}{1-az^{-1}}\right\} = -z\frac{-az^{-2}}{(1-az^{-1})^2} = \frac{+az^{-1}}{(1-az^{-1})^2}$$

$$\frac{n}{a}x[n] \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1-az^{-1})^2}$$

So the inverse z-transform of $\dfrac{z^{-1}}{(1-az^{-1})^2}$

is $\dfrac{n}{a}x[n] = \dfrac{n}{a} \cdot a^n u[n] = n \cdot a^{n-1} u[n]$

Analysis of Discrete-time LTI Systems

The system function of a discrete-time LTI systems: The output $y[n]$ of a discrete-time LTI system equals the convolution of the input $x[n]$ with the impulse response $h[n]$ that is $y[n] = x[n] * h[n]$

By applying z-transform $Y(z) = X(z)\, H(z)$

$$H(z) = \frac{Y(z)}{X(z)}$$

The transform $H(z)$ of $h[n]$ is referred to as the system function (or transfer function) of the system

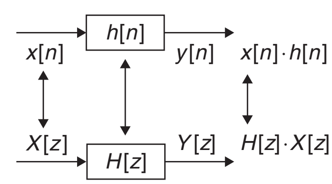


Figure 5 Impulse response and system function

Characterization of Discrete-time LTI Systems

Causality

For causal discrete-time LTI system we have $h[n] = 0$, $n < 0$.

Since $h[n]$ is right-sided signal, the corresponding requirement on $H(z)$ is that the ROC of $H(z)$ must be of the form $|z| > r_{max}$, exterior of circle, including infinity.

Similarly if system is anti-causal, ($h[n] = 0$, $n \geq 0$), the ROC of $H(z)$ must be of the form $|z| < r_{min}$

Stability

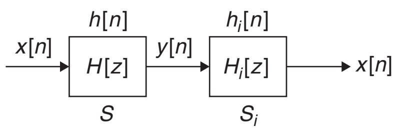
A discrete-time LTI system is BIBO stable if and only if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ corresponding requirement on $H(z)$ is that ROC of $H(z)$ contains the unit circle (i.e., $|z| = 1$)

Causal and stable systems: If the system is both causal and stable, then all of the poles of $H(z)$ must lie inside the unit circle of the z-plane because the ROC is of the form $|z| > r_{max}$ and since unit circle is included in ROC we must have $r_{max} < 1$.

Invertibility

If $H(z)$ is the transfer function of a system S, the S_i, its inverse system has a transfer function $H_i(z)\dfrac{1}{H(z)}\cdot$

The inverse system S_i undoes the operation of S, hence if $H(z)$ is placed in cascade with $H_i(z)$, the transfer function of the composite system is unity.



$H(z) \cdot H_1(z) = 1$, i.e., $h[n] * h_i[n] = \delta[n]$. The zeros of $H(z)$ are the poles of $H_i(z)$ and the poles of $H(z)$ are the zeros of $H_i(z)$ as $H_i(z)\dfrac{1}{H(z)}\cdot$ If $H(z)$ is both stable and causal, then all poles of $H(z)$ will lie inside the unit circle. If $H_i(z)$ is to be both stable and causal, then all poles of $H_i(z)$ have to lie inside the unit circle, the poles of $H_i(z)$ = zeros of $H(z)$. So for the systems $H(z)$ and $H_i(z)$ to be stable and causal, all the poles and zero of $H(z)$ must lie inside the unit circle. A system with all poles and zeros inside the unit circle is termed as minimum phase system.

System Function of LTI System Described by Linear Constant Coefficient Difference Equations

A distinct time LTI system for which input $x[n]$ and output $y[n]$ satisfy the general linear constant coefficient difference equation is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

By applying z-transform using time shift property

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

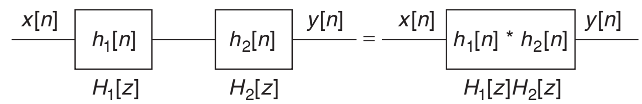
So, $H(z)$ is always rational.

The difference equation by itself does not provide information about which ROC to associate with the algebraic expression $H(z)$. An additional constraint, such as the causality or stability of the system, however serves to specify the region of convergence. For example if the system is causal, the ROC will be outside the outermost pole. If the system is stable, the ROC must include the unit circle.

Systems Interconnections

For two LTI system ($h_1[n]$, $h_2[n]$) in cascade, the overall impulse response $h[n]$ is given by $h[n] = h_1[n] * h_2[n]$

Corresponding system function $H(z) = H_1(z) \cdot H_2(z)$

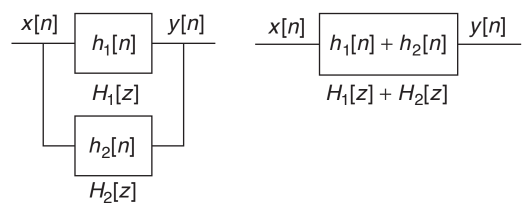


The impulse response of a parallel combination of two LTI systems is given by

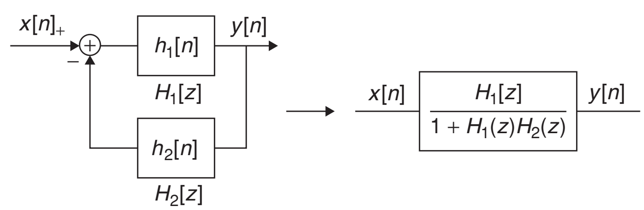
$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

When two systems are connected in feedback inter-connection, as shown in figure, then the overall system function for the feedback system



$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$



Examples 8: Consider the LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equations

$$y[n] - \frac{1}{3} y[n-1] = x[n] + \frac{1}{5} x[n-1]$$

Then what is the impulse response of the system when (i) system is stable, (ii) system is anti causal?

Solution: Given difference equations is

$$y[n] - \frac{1}{3} y[n-1] = x[n] + \frac{1}{5} x[n-1]$$

By considering z-transform

$$Y(z) - \frac{1}{3} z^{-1} Y(z) = X(z) + \frac{1}{5} z^{-1} X(z)$$

$$Y(z)\left(1-\frac{1}{3}z^{-1}\right)=X(z)\left(1+\frac{1}{5}z^{-1}\right)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\left(1+\frac{1}{5}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)}$$

For ROC we need to consider the system conditions

1. When the system is stable, it includes unit circle $|z|>\frac{1}{3}$ (and the system is causal too)

$$H(z)=\frac{1}{1-\frac{1}{3}z^{-1}}+\frac{1}{5}\frac{z^{-1}}{1-\frac{1}{3}z^{-1}}|z|>\frac{1}{3}$$

By considering inverse z-transform

$h[n]=\left(\frac{1}{3}\right)^n u[n]+\frac{1}{5}\left(\frac{1}{3}\right)^{n-1}u[n-1]$ is the impulse response, when the system is causal (or) stable.

2. When the system is anti causal, $|z|<\frac{1}{3}$

$$H(z)=\frac{1}{1-\frac{1}{3}z^{-1}}+\frac{1}{5}\frac{z^{-1}}{1-\frac{1}{3}z^{-1}},\ |z|<\frac{1}{3}$$

By considering inverse z-transform

$$h[n]=-\left(\frac{1}{3}\right)^n u[-n-1]-\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^{n-1}u[-(n-1)-1]$$

$$h[n]=-\left(\frac{1}{3}\right)^n u[-n-1]-\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)^{n-1}u[-n]$$

is the impulse response when the system is anti causal

THE UNILATERAL z-TRANSFORM

The unilateral (or one-sided) z-transform $X_I(z)$ of a sequence $x[n]$ is defined as $X_I(z)=\sum_{n=0}^{\infty}x[n]z^{-n}$ and differs from the bilateral transform in that the summation is carried over only $n \geq 0$.

ROC of $X_I(z)$ is always outside a circle in the z-plane. Time shifting property of unilateral z-transform if $x[n] \leftrightarrow X_I[z]$ then for $m \geq 0$

$$x[n-m] \leftrightarrow z^{-m}X_I(z)+z^{-m+1}x[-1]+z^{-m+2}x[-2]+\cdots+x[-m]$$

$$x[n+m] \leftrightarrow z^{m}X_I(z)-z^{m}x[0]-z^{m-1}x[1] \supset -zx[m-1]$$

The unilateral z-transform, that is particularly useful in analyzing causal systems specified by linear constant coefficient difference equations with non zero initial conditions

$$\text{Time delay } x[n-1]\xleftrightarrow{\text{(unilateralz)}}z^{-1}X(z)+x[-1]$$

$$\text{Time delay } x[n+1]\xleftrightarrow{\text{(unilateralz)}}zX(z)-zx[0]$$

RELATIONSHIP BETWEEN Z AND LAPLACE TRANSFORM

For a general signal $x(t)$ is sampled at sampling rate $\frac{1}{T}$ to get discrete value $x(kT)$ which has z-transform

$$X(z)=\sum_{k=-\infty}^{\infty}x(kT)z^{-k}$$

The same general signal $x(t)$ can be considered as the impulse sampled at the same rate $\frac{1}{T}$ and may be represented as

$$x(t)=\sum_{k=-\infty}^{\infty}x(kT)\delta(t-kT)$$

Laplace transform of above equation is

$$X(s)=\sum_{k=-\infty}^{\infty}x(kT)e^{-ksT}$$

If $e^{sT}=z$, we can write as

$$X(s)=\sum_{k=-\infty}^{\infty}x(k)z^{-k}=X(z)$$

Thus, $z=e^{sT}$

$\ln z=sT$

$$s=\frac{1}{T}\ln z$$

$$X(s)=X(z)\big|_{z=e^{Ts}}$$

STRUCTURES FOR REALIZATION OF IIR SYSTEMS

In time domain the representation Nth order IIR system is

$$y(n)=-\sum_{k=1}^{N}a_k y(n-k)+\sum_{k=0}^{M}b_k x(n-k) \qquad (5)$$

And the z-domain representation of an Nth order IIR system is,

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

In the above representation the value of M gives the number of zeros and N gives the number of poles of the system.

Types of structures for realizing the IIR systems are

1. **Direct form-I structure:** Consider the difference equation governing an IIR system.

   Consider the equation (1)

   $y(n)+a_1y(n-1)+a_2y(n-2)+\cdots a_Ny(n-N) = b_0x(n)+b_1x(n-1)+b_2x(n-2)+\cdots b_Mx(n-M)$

   On taking z-transform both sides, we get,

   $Y(z)=-a_1z^{-1}Y(z) - a_2z^{-2}Y(z) - \cdots - a_Nz^{-N}Y(z)+b_0X(z)+b_1z^{-1}X(z)+\cdots+b_Mz^{-M}X(z)$

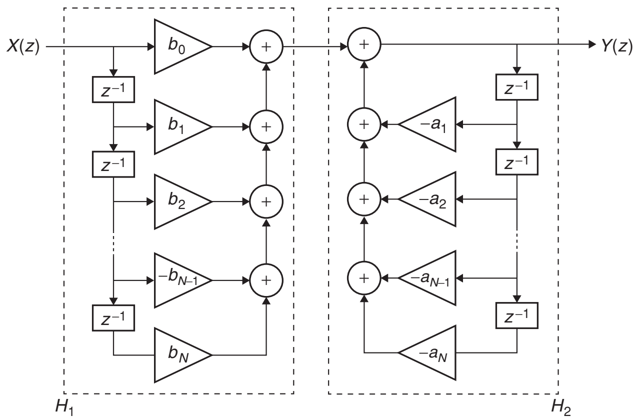


Figure 6 Direct form-I structure of IIR system

This provides a direct relation between time domain and z-domain equations.

⇒ This required more memory elements (z^{-1}).

2. **Direct form-II structure of IIR system:** This structure can be realized which uses less number of delay elements or memory elements than the direct form-I. Consider the general difference equation governing an IIR system.

$$Y(n) - \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

On taking z-transform of the above equation.
We can obtained,
$Y(z)\ [1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}] = X(z)\ [b_0 + b_1 z^{-1} + b_2 z^{-2} \cdots + b_M z^{-M}]$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Let $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)}\dfrac{Y(z)}{W(z)}$ (6)

Where, $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_N z^{-N}}$ (7)

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M} \quad (8)$$

On cross multiplying equation (6) and (7) we get,

$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \cdots - a_N z^{-N} W(z)$ and (9)

$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + \supset + b_M z^{-M} W(z)$ (10)

The equation (9) and (10) represents the IIR system in z-domain and it can be realized by a direct structure called direct form-II.

In direct form-II the number of delays is equal to order of the system.

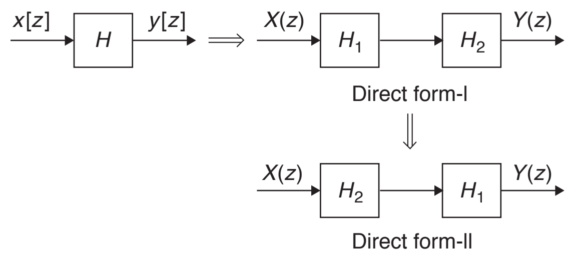


Figure 7 Conversion of Direct form-I to Direct form-II.

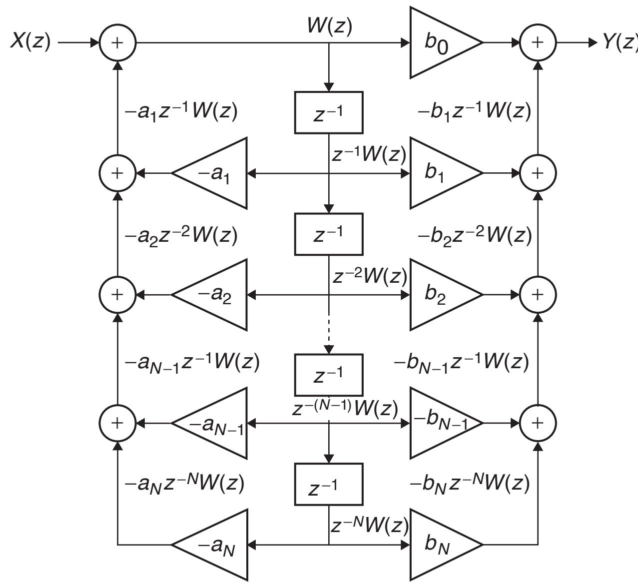
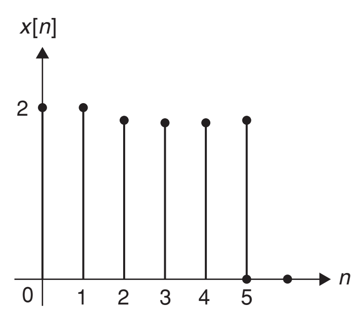


Figure 8 Direct form-II structure of IIR system

Solved Examples

Example 1: What is the z-transform of the signal shown in figure?



Solution: Here $x[0] = x[1] = \cdots = x[5] = 2$, and $x[6] = x[7] = \cdots = 0$

Now, $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$$= 2 + \frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \frac{2}{z^4} + \frac{2}{z^5}$$

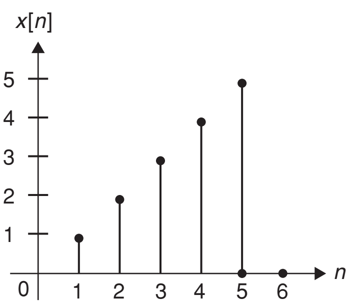
$$= \frac{2}{z^5}(z^5 + z^4 + z^3 + z^2 + z^4 + 1) \text{ for all } z \neq 0$$

Or $X(z) = 2\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5}\right)$

This expression in geometric progression, the summation of n terms with initial value a is $S_n = \dfrac{a(r^n - 1)}{r - 1}$

So $X(z) = 2\left(\dfrac{1\left(\left(\frac{1}{z}\right)^6 - 1\right)}{\left(\frac{1}{z} - 1\right)}\right) = \dfrac{2z}{z-1}(1 - z^{-6})$

Example 2: Find the z-transform of the signal $x[n]$ depicted in the figure



Soluton: The signal $x[n]$ can be expressed as

$x[n] = n$ for $0 \le n \le 5$

Now, $x[n] = n\{u[n] - u[n-6]\}$

$= nu[n] - nu[n-6]$

$= nu[n] - (n - 6 + 6)u[n-6]$

$= nu[n] - (n-6)u[n-6] - 6u[n-6]$

we have $\quad u[n] \leftrightarrow \dfrac{1}{1 - Z^{-1}}$

$$nu[n] \leftrightarrow \frac{z^{-1}}{(1 - z^{-1})^2}$$

$x[n-k] \leftrightarrow z^{-k} X(z)$

So applying z-transform

$$X(z) = \frac{z^{-1}}{(1 - z^{-1})^2} - \frac{1}{z^6}\frac{z^{-1}}{(1 - z^{-1})^2} - \frac{1}{z^6}\frac{6}{(1 - z^{-1})}, |z| > 1$$

$$X(z) = \frac{z^{-1} - z^{-7} - 6z^{-6}(1 - z^{-1})}{(1 - z^{-1})^2} = \frac{5z^{-7} - 6z^{-6} + z^{-1}}{(1 - z^{-1})^2}$$

$$= \frac{z^6 - 6z + 5}{z^5(z-1)^2}$$

Example 3: A discrete LTI system has the difference equation $y[n+1] - 2\,y[n] = x[n+1]$, if the initial conditions are $y[-1] = -1$, and the input $x[n] = \left(\frac{1}{3}\right)^n u[n]$ then find output $y[n]$.

Solution: Given $y[n+1] - 2y[n] = x[n+1]$

$\Rightarrow y[n] - 2y[n-1] = x[n]$

$\Rightarrow y[n]\,u[n] \leftrightarrow Y(z)$

$$y[n-1]u[n] \leftrightarrow \frac{1}{z}Y(z) + y[-1] = \frac{1}{z}y(z) + (-1)$$

$x[-1] = x[-2] \supset = x[-n] = 0$, $x[n]$ is causal input

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$\Rightarrow X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}$$

By taking the z-transform for the difference equation

$$Y(z) - 2\left[\frac{1}{2}Y(z) - 1\right] = X(z)$$

$$Y(z)\left[1 - \frac{2}{z}\right] + 2 = X(z)$$

Now, $Y(z) = \dfrac{X(z) - 2}{1 - 2z^{-1}} = \dfrac{\left(\dfrac{1}{1 - \frac{1}{3}z^{-1}}\right) - 2}{1 - 2z^{-1}}$

$$= \frac{1 - 2 + \frac{2}{3}z^{-1}}{(1 - 2z^{-1})\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{-1 + \frac{2}{3}z^{-1}}{(1 - 2z^{-1})\left(1 - \frac{1}{3}z^{-1}\right)}$$

Now we have, $Y(z) = \dfrac{z(-3z + 2)}{(z-2)(3z-1)} = z\left[\dfrac{A}{z-2} + \dfrac{B}{3z-1}\right]$

$$= z\left[\frac{-4}{5}\frac{1}{z-2} - \frac{3}{5}\frac{1}{(3z-1)}\right] = \frac{-1}{5}\left[\frac{4}{1 - 2z^{-1}} + \frac{1}{1 - \frac{1}{3}z^{-1}}\right]$$

$$\Rightarrow y[n] = \frac{-1}{5}\left[4.2^n u[n] + \left(\frac{1}{3}\right)^n u[n]\right]$$

$$= -0.2\left[2^{n+2} + \left(\frac{1}{3}\right)^n\right]u[n]$$

Example 4: For a causal system specified by the transfer function $H(z) = \dfrac{z}{z - 0.4}$, find the zero-state response to input

$$x[n] = (0.5)^n u[n] + 2\,2^n u[-n-1]$$

Soluiton: Zero state response means, $y[n]$ with zero initial conditions

$$X(z)\frac{1}{1 - 0.5z^{-1}} - \frac{2}{1 - 2z^{-1}} = \frac{z(z - 2 - 2z + 1)}{(z - 0.5)(z - 2)}$$

$$= \frac{-z(z+1)}{(z - 0.5)(z - 2)}$$

The ROC of $X(z)$ is for the first term $|z| > 0.5$ and for the second term $|z| < 2$.

The ROC for $X(z)$ is common region given by $0.5 < |z| < 2$ and $X|z| = \dfrac{-z(z+1)}{(z-0.5)(z-2)}$

Then $Y(z) = X(z)H(z) = \dfrac{-z^2(z+1)}{(z-0.4)(z-0.5)(z-2)}$

As $H(z)$ is causal system ROC is $|z| > 0.4$, yhe common region of convergence of $X(z)$ and $H(z)$ is $0.5 < |z| < 2$, so

$$Y(z) = \frac{-z^2(z+1)}{(z-0.4)(z-0.5)(z-2)}$$

Now, $$\frac{Y(z)}{z} = \frac{-z(z+1)}{(z-0.4)(z-0.5)(z-2)}$$

$$= \frac{-7}{2}\frac{1}{(z-0.4)} + \frac{5}{(z-0.5)} - \frac{5}{2}\frac{1}{z-2}$$

So, $Y(z) = \dfrac{-7}{2}\dfrac{z}{z-0.4} + 5.\dfrac{z}{z-0.5} - \dfrac{5}{2}\dfrac{z}{z-2}$,

$0.5 < |z| < 2$

$$y[n] = \left[\frac{-7}{2}(0.4)^n + 5(0.5)^n\right]u[n] + \frac{5}{2}2^n u[-n-1]$$

Example 5: Find the z-transform of the signal $x[n]$

$$= \left\{n\left(\frac{-1}{3}\right)^n u[n]\right\} * (5)^n u[-n]$$

Solution: We can write $x[n] = w[n] * y[n]$

$$w[n] = n\left(\frac{-1}{3}\right)^n u[n]$$

$$\left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1+\frac{1}{3}z^{-1}} \text{ with ROC}$$

$$|z| > \frac{1}{3}$$

$$w[n] = n\left(\frac{-1}{3}\right)^n u[n] \xleftrightarrow{z} W(z)$$

$$= -z\frac{d}{dz}\left\{\frac{1}{1+\frac{1}{3}z^{-1}}\right\}$$

$$w(z) = -z\frac{-1}{\left(1+\frac{1}{3}z^{-1}\right)^2}\left(\frac{-1}{3}z^{-2}\right)$$

$$= \frac{-1}{3}\frac{z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)^2} \text{ with ROC } |z| > \frac{1}{3}$$

$$y[n] = (5)^n u[-n] = \left(\frac{1}{5}\right)^{-n} u[-n]$$

$$\left(\frac{1}{5}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z-\frac{1}{5}}, \text{ ROC } |z| > \frac{1}{5}$$

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{2}\right), \text{with ROC } \frac{1}{R_x}$$

So, $Y(z) = \dfrac{\frac{1}{z}}{\frac{1}{z}-\frac{1}{5}}$ ROC $\dfrac{1}{|z|} > \dfrac{1}{5}$

$$= \frac{-z^{-1}}{\frac{1}{5}-z^{-1}} |z| < 5$$

By applying convolution $y[n] * w[n]$, and z-transform we have $X(z) = Y(z)\, W(z)$

$$= \frac{-z^{-1}}{\frac{1}{5}-z^{-1}} \cdot \frac{-1}{3} \frac{z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)^2}$$

The common ROC is $|z| > \frac{1}{3}$, and $|z| < 5$

Also, $X(z) = \dfrac{5}{z-5} \cdot \dfrac{1}{3} \dfrac{z}{\left(z+\frac{1}{3}\right)^2}$

$$= \frac{5}{3}\frac{z}{(z-5)\left(z+\frac{1}{3}\right)^2} \text{ with ROC } \frac{1}{3} < |z| < 5$$

Example 6: Find the transfer function and impulse response of a causal LTI system, if the input to the system is $x[n] = u[n]$ and output is $y[n] = \delta[n]$

Soluton: The z-transform of inputs and outputs

$$X(z) = \frac{1}{1-z^{-1}}, \text{ ROC } |z| > 1,$$

$Y(z) = 1$, ROC for all z

Then transfer function $H(z)$

$$= \frac{Y(z)}{X(z)} = \frac{1}{1/(1-z^{-1})}$$

$= 1 - z^{-1}$, with ROC all z except at $z = 0$

By taking inverse z-transform,

$h[n] = \delta[n] - \delta[n-1]$ is the impulse response.

Example 7: An LTI system is described by the difference equation

$$y[n] - \frac{1}{2}y[n-1] + \frac{1}{18}y[n-2] = x[n] + \frac{1}{20}x[n-1] - \frac{1}{20}x[n-2].$$

Does a stable and causal LTI inverse system exists?

Solution: $y[n]-\frac{1}{2}y[n-1]+\frac{1}{18}y[n-2]=x[n]+\frac{1}{20}x[n-1]-\frac{1}{20}x[n-2]$

By considering z-transform

$$Y(z)-\frac{1}{2}z^{-1}Y(z)+\frac{1}{18}z^{-2}Y(z)=X(z)+\frac{z^{-1}}{20}X(z)-\frac{1}{20}z^{-2}X(z)$$

Now, $H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{20}z^{-1}-\frac{1}{20}z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{18}z^{-2}}=\frac{z^2+\frac{1}{20}z-\frac{1}{20}}{z^2-\frac{1}{2}z+\frac{1}{18}}$

$$=\frac{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{6}\right)}=\frac{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{6}\right)}$$

Then, $H_i(z)=\frac{1}{H(z)}=\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{6}\right)}{\left(z+\frac{1}{4}\right)\left(z-\frac{1}{4}\right)}$

Is the transfer function of inverse system. All the poles and zeros of $H(z)$ lie inside the unit circle, so stable and causal inverse system exists and this is a minimum phase system.

Example 8: A discrete time LTI model for two path communication channel is $y[n] = x[n] + ax[n-1]$, find the difference equation description of the inverse system.

Solution: $y[n] = x[n] + ax[n-1]$, by taking z-transform $Y(z) = X(z) + az^{-1}X(z)$

Transfer function $H(z)=\frac{Y(z)}{X(z)}=1+az^{-1}$

The inverse system has transfer function

$$H_i(z)=\frac{1}{H(z)}=\frac{1}{1+az^{-1}}=\frac{Y_i(z)}{X_i(z)}$$

Now, $Y_i(z) + az^{-1}Y_i(z) = X_i(z)$,

By taking inverse z-transform

$$y[n] + ay[n-1] = x[n]$$

The inverse system is both stable and causal if $|a| < 1$.

Exercises

Practice Problems I

Directions for questions 1 to 22: Select the correct alternative from the given choices.

Common Data for Questions 1 to 3:

1. $x[n] = a^n u[n]$ where a is real. z-transform of $x[n]$ is
   (A) $\frac{z}{z-a}, |z|<|a|$ (B) $\frac{z}{z-a}, |z|>|a|$
   (C) $\frac{z}{z+a}, |z|>|a|$ (D) $\frac{z}{z+a}, |z|<|a|$
2. z-transform of $a^{n+1}u[n+1]$ is
   (A) $\frac{z^2}{z-a}, |a|<|z|<\infty$ (B) $\frac{z^2}{z+a}, |a|<|z|<\infty$
   (C) $\frac{z}{z-a}, |a|<|z|<\infty$ (D) $\frac{z^2}{z-a}, |a|>|z|<\infty$
3. z-transform of $n\,a^n u[n]$ is
   (A) $\frac{-az}{(z-a)^2}, |z|>|a|$ (B) $\frac{az}{(z-a)}, |z|>|a|$
   (C) $\frac{az}{(z-a)^2}, |z|>|a|$ (D) $\frac{az}{(z-a)^2}, |z|<|a|$
4. Inverse z-transform of $\log(1-az^{-1})$, $|z|>|a|$ is
   (A) $\frac{1}{n}a^n u[n-1]$ (B) $\frac{-1}{n}a^n u[n-1]$
   (C) $\frac{-1}{n}a^n u[n+1]$ (D) $\frac{1}{n}a^n u[n+1]$
5. Find the initial and final values of z-domain signal $\frac{2z^{-1}}{1-1.8z^{-1}+0.8z^{-2}}$
   (A) 10, 1 (B) 1, 10
   (C) 0, 10 (D) 0, ∞
6. The transfer function of a system is $H(z) = 1-3z^{-1}$, the system response for input $x[n]$ is
   (A) $x[n]-0.3\,x[n-1]$ (B) $x[n]+0.3x[n-1]$
   (C) $x[n]-3x[n+1]$ (D) $x[n]-3x[n-1]$
7. Unilateral z-transform of $\left(\frac{1}{3}\right)^n u(n)+\left(\frac{1}{5}\right)^n u(n)$ and ROC are respectively?
   (A) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/5)}; |z|>\frac{1}{5}$
   (B) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/5)}; |z|>\frac{1}{3}$
   (C) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/3)}; \frac{1}{5}>|z|>\frac{1}{3}$
   (D) $\frac{z}{(z-1/3)}+\frac{z}{(z-1/3)}; \frac{1}{5}<|z|<\frac{1}{3}$
8. Find $f(0)$, $f(1)$, $f(2)$ and $f(3)$, if

   $$F(z)=\frac{z^2+2}{z^3-3z^2+3z-1}$$

   (A) 0, 9, 31, 16 (B) 0, 1, 3, 8
   (C) 1, 3, 3, 1 (D) 0, 1, 2, 4

9. The minimum Nyquist sampling rate $\frac{\sin t}{\pi t}$ is

(A) $2/\pi$ (B) $1/\pi$
(C) π (D) 2π

10. Match the following:

List I	List II
$x(n)$	ROC
(1) $\delta(n)$	(a) All z
(2) $u(n)$	(b) $\lvert z\rvert > 1$
(3) $-u(-n-1)$	(c) All z except 0
(4) $\delta(n-m)$	(d) $\lvert z\rvert < 1$

1 2 3 4   1 2 3 4
(A) a b c d (B) b a c d
(C) a b d c (D) b a d c

11. $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = X(n)$ where $x(n)$ and $y(n)$ are the input and output of the systems then determine system function.

(A) $\frac{z^2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} |z| > \frac{1}{2}$

(B) $\frac{z^2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}, |z| < \frac{1}{2}$

(C) $\frac{z^2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}, |z| > \frac{1}{2}$

(D) $\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}, |z| > \frac{1}{2}$

12. For a z-transform $X(z) = \frac{z(2z-5/6)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$. Find out the sequences when the region of convergence is $|z| > 1/2$

(A) $\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n u(n)\right]$

(B) $\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(-n-1)$

(C) $\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] u(n)$

(D) $\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(-n-1)$

13. Find the sequence when region of convergence $\left|\frac{1}{3}\right| < |z| < \frac{1}{2}$ for above $X(z)$,

(A) $-\left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{3}\right)^n u(n)$

(B) $\left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{3}\right)^n u(n)$

(C) $\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(-n-1)$

(D) $\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(-n-1)$

14. Impulse response for the cascade of LTI systems having impulse response $h_1[n] = \left(\frac{1}{2}\right)^n u[n]$ and $h_2[n] = \left(\frac{1}{4}\right)^n u[n]$ is?

(A) $\left(\frac{1}{4}\right)^n (2n-1)u[n]$ (B) $\left(\frac{1}{4}\right)^n (2^{n+1}-1)u[n]$

(C) $\left(\frac{1}{4}\right)^n (2^{n+1}+1)u[n]$ (D) $\left(\frac{1}{4}\right)^n (2^n+1)u[n]$

15. What is the Nyquist rate for the signal $x(t) = \cos 2000\pi t + 6\sin 9000\pi t$

(A) 9 kHz (B) 4 kHz
(C) 2 kHz (D) 11 kHz

16. Which of the following is the inverse z-transform of

$$X(z) = \frac{z}{(z-2)(z-3)} |z| < 2$$

(A) $[2^n - 3^n]\, u(-n-1)$ (B) $[2^n - 3^n]\, u(n+1)$
(C) $[3^n - 2^n]\, u(-n-1)$ (D) $[2^n - 3^n]\, u(n)$

17. The impulse-response of a relaxed linear time invariant system is $h(n) = \alpha^n u(n)$ with $|\alpha| < 1$. Determine the value of the step response as $n \rightarrow \infty$

(A) $\left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u(n)$ (B) $\left[\frac{1-\alpha^{n-1}}{1-\alpha}\right]u(n)$

(C) $\left[\frac{1+\alpha^{n+1}}{1-\alpha}\right]u(n)$ (D) $\left[\frac{1+\alpha^{n+1}}{1-\alpha}\right]u(n)$

18. Consider z-transform $X(z) = 5z^2 + 4z^{-1} + 3$; $0 < |z| < \infty$, the inverse z-transform $x[n]$ is?

(A) $5\delta[n+1]$
(B) $3\delta[n]$
(C) $4\delta[n+2]$
(D) $5\delta[n+2] + 4\delta[n-1] + 3\delta[n]$

19. The transfer function of DT, LTI system is given by

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Consider the following statements

S1: System is stable and causal for

ROC$|z| > \frac{1}{2}$

S2: System is stable but not causal for

ROC: $|z| < \frac{1}{4}$

S3: System is neither stable nor causal for

ROC: $\frac{1}{4} < |z| < \frac{1}{2}$

Which one of the following statements is valid?

(A) Both $S1$ and $S2$ are true
(B) Both $S2$ and $S3$ are true
(C) Both $S1$ and $S3$ are true
(D) All are true

20. ROC of z-transform of the discrete time sequence $x[n] = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u[-n-1]$ is

(A) $|z| > \frac{1}{3}$ (B) $|z| < \frac{1}{2}$
(C) $|z|$ for all (D) $\frac{1}{3} < |z| < \frac{1}{2}$

21. A causal LTI system is described by the difference equation $2y[n] = \alpha y[n-1] - 2x[n] + \beta x[n-1]$. The system is stable for which values of α, β?

(A) $|\alpha| > 2, |\beta| < 1$
(B) $|\alpha| < 2$, any β
(C) $|\alpha| < 3, |\beta| > 1$
(D) $|\alpha| > 2$, any β

22. A sequence $x[n]$ with z-transform $X(z) = z^4 + z^2 - 2z + 2 - 3z^{-4}$ is applied as an input to a linear, time-invariant system with impulse response $h[n] = 2\delta[n-3]$, the output at $n = 4$ is

(A) 2 (B) 0
(C) −2 (D) −6

Practice Problems 2

Directions for questions 1 to 21: Select the correct alternative from the given choices.

1. Two DT systems with impulse response $h_1[n] = \delta[n-1]$, $h_2[n] = \delta[n-2]$ are connected in cascade, the overall impulse response of the cascaded system is?

(A) $\delta[n-1]$ (B) $\delta[n-2]$
(C) $\delta[n-3]$ (D) $\delta[n-4]$

2. A system with transfer function $H(z)$ has impulse response $h[n]$, defined as $h[2] = 1$, $h[3] = -1$, and $h[n] = 0$ otherwise, consider following statements

S1: $H(z)$ is a low pass filter

S2: $H(z)$ is an FIR filter

Which one of the following is correct?

(A) Only S2 is true
(B) Both S1 and S2 are false
(C) Both S1 and S2 are true, and S1 is reason for S2
(D) Both S1 and S2 are true, and S2 is a reason for S1

3. z-transform $X(Z)$ of a sequence $x[n]$ is given by $X(Z) = \frac{0.5}{1-2z^{-1}}$, it is given that the ROC of $X(Z)$ includes unit circle, the value of $x[0]$ is

(A) 0 (B) 0.5
(C) 1 (D) ∞

4. If ROC of $x_1[n] + x_2[n]$ is $\frac{1}{3} < |z| < \frac{2}{3}$, then the ROC of $x_1[n] - x_2[n]$ includes

(A) $\frac{1}{3} < |z| < 3$ (B) $\frac{2}{3} < |z| < 3$
(C) $\frac{3}{2} < |z| < 3$ (D) $\frac{1}{3} < |z| < \frac{2}{3}$

5. $y(n) = \sum_{k=0}^{n} x[k]$

Which one of the following correctly relate the z-transform of the input and output denoted by $X(z)$ and $Y(z)$ respectively?

(A) $Y(z) = \frac{d}{dz}X(z)$ (B) $Y(z) = \frac{X(z)}{1-z^{-1}}$
(C) $Y(z) = X(z)z^{-1}$ (D) $Y(z) = (1 - z^{-1})X(z)$

6. $x(n) = \frac{1}{7}\left(\frac{1}{2}\right)^n u(n) - \frac{6}{7}(3)^n u(-n-1)$. Find out ROC of z-transform?

(A) $1/2 > |z| < 3$ (B) $|z| > 1/2$
(C) $|z| < 3$ (D) $1/2 < |z| < 3$

7. The range of values a and b for which the z-transform with impulse response $h(n) = a^n, n \geq 0 = b^n, n < 0$ will be stable

(A) $|a| < 1, |b| < 1$ (B) $|a| < 1, |b| > 1$
(C) $|a| > 1, |b| > 1$ (D) $|a| > 1, |b| < 1$

8. The z-transform of $a^{-n}\, u[-n-1]$ is

(A) $\frac{az}{1+az}, |z| < \frac{1}{|a|}$ (B) $\frac{az}{1-az}, |z| < \frac{1}{|a|}$
(C) $\frac{-az}{1-az}, |z| < \frac{1}{|a|}$ (D) $\frac{az}{1-az}, |z| > \frac{1}{|a|}$

9. z-transform of $x[n] = \{3, -2, \underset{\uparrow}{4}, 1, 0, -3\}$ is
(A) $3z^2 + 2z + 4 + z^{-1} + 3z^{-2}$, $2 < |z| < 4$
(B) Does not exist
(C) $3z^2 - 2z + 4 + z^{-1} - 3z^{-3}$, $0 < |z| < \infty$
(D) $3z^2 - 2z + 4 + z^{-1} - 3z^{-3}$, $|z| < 0$, $|z| < \infty$

10. z-transform of $u[n + 1]$ is
(A) $\frac{z}{z-1}$ (B) $\frac{z^2}{z-1}$
(C) $\frac{z^{-1}}{1-z^{-1}}$ (D) $\frac{1}{z^{-1}+z^{-2}}$

11. Inverse z-transform of $\frac{2}{z-2}$, ($|z| > 2$) is
(A) $2^{n-1}\,u[n + 1]$ (B) $2^{n+1}\,u[n - 1]$
(C) $2^n\,u[n + 1]$ (D) $2^n\,u[n - 1]$

12. z-transform and Laplace transfrom are related by
(A) $s = \ln z$ (B) $s = \frac{\ln z}{T}$
(C) $s = T \ln z$ (D) Not related

13. The final value of $x[n]$ is
(A) $\underset{z\to\infty}{\text{Lt}}\, zX(z)$ (B) $\underset{z\to 0}{\text{Lt}}\, zX(z)$
(C) $\underset{z\to\infty}{\text{Lt}}\,(1-z^{-1})X(z)$ (D) $\underset{z\to 1}{\text{Lt}}\,(1-z^{-1})X(z)$

14. The initial value of $x[n]$, if $X(z) = \frac{3z^2}{(z+3)(z-3)}$
(A) 3 (B) 6
(C) ∞ (D) 0

15. An LTI system defined as $y[n] + 0.5y[n - 1] = x[n] + 0.3x[n - 1]$. The transfer function in z-plane is
(A) $\frac{z+0.3}{z+0.5}$ (B) $\frac{z-0.3}{z-0.5}$
(C) $\frac{z+0.3}{z-0.5}$ (D) $\frac{z-0.3}{z+0.5}$

16. A signal $x[n] = \{\underset{\uparrow}{1}, -1, 2, 1, 3, 2\}$. The ROC z-transform $X(z)$ is
(A) $|z| > 3$
(B) $-1 < |z| < 3$
(C) $|z| < 3$
(D) the entire z-plane except at $z = 0$

17. The minimum sampling frequency of $x(t) = 5\cos 50\pi t + 2\cos 200\pi t - 10\cos 100\pi t$ is
(A) 100 Hz (B) 200 Hz
(C) 50 Hz (D) Cannot be sampled

18. The minimum number of samples required to exactly describe $x(t) = 10\cos 4\pi t + 4\sin 8\pi t$ is
(A) 4 samples per second.
(B) 2 samples per second.
(C) 8 samples per second.
(D) 16 samples per second.

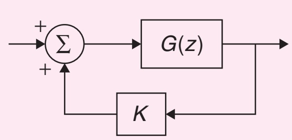
19. Aliasing occurs when sampling frequency ω_s is (ω_m – band limited signal frequency)
(A) $\omega_s = 2\omega_m$ (B) $\omega_s < 2\omega_m$
(C) $\omega_s > 2\omega_m$ (D) 0

20. Interpolation is the process of
(A) Inserting $N - 1$ unity sequence values to $x[n]$
(B) Inserting $N - 1$ zero sequence values to $x[n]$
(C) Deleting $N - 1$ unity sequence values from $x[n]$
(D) Deleting $N - 1$ zero sequence values from $x[n]$

21. Given signal $m(t) = \sin 5\pi t + 5\sin 10\pi t$ is sampled instantaneously, the maximum interval between samples is
(A) 0.1 seconds.
(B) 0.2 seconds.
(C) 10 seconds.
(D) None

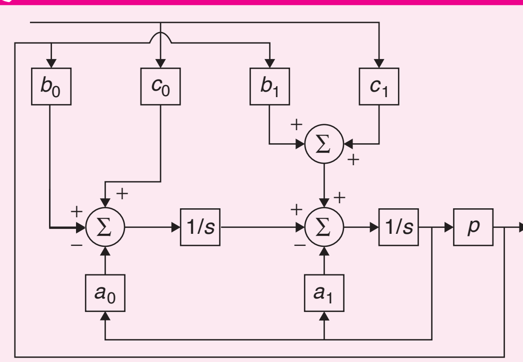
PREVIOUS YEARS' QUESTIONS

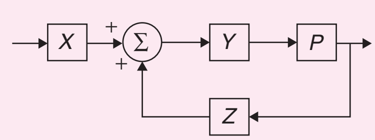
1. Consider the discrete-time system shown in the figure where the impulse response of $G(z)$ is $g(0) = 0$, $g(1) = g(2) = 1$, $g(3) = g(4) = 0$.



This system is stable for range of values of K **[2007]**
(A) $[-1, 1/2]$ (B) $[-1, 1]$
(C) $[-1/2, 1]$ (D) $[-1/2, 2]$

2. The system shown in figure below can be reduced to the form with





[2007]

(A) $X = c_0 s + c_1, Y = \dfrac{1}{(s^2 + a_0 s + a_1)}, Z = b_0 s + b_1$

(B) $X = 1, Y = \dfrac{(c_0 s + c_1)}{(s^2 + a_0 s + a_1)}, Z = b_0 s + b_1$

(C) $X = c_1 s + c_0, Y = \dfrac{(b_1 s + b_0)}{(s^2 + a_1 s + a_0)}, Z = 1$

(D) $X = c_1 s + c_0, Y = \dfrac{1}{(s^2 + a_1 s + a_0)}, Z = b_1 s + b_0$

3. $X(z) = 1 - 3z^{-1}$, $Y(z) = 1 + 2x^{-2}$ are z-transforms of two signals $x[n]$, $y[n]$ respectively. A linear time invariant system has the impulse response $h[n]$ defined by these two signal as

$H[n] = x[n - 1] * y[n]$

Where * denotes discrete time convolution. Then the output of the system for the input $\delta[n - 1]$ **[2007]**

(A) has z-transform $z^{-1} X(z) Y(z)$
(B) equals $\delta[n-2] - 3\delta[n-3] + 2\delta[n-4] - 6\delta[n-5]$
(C) has z-transform $1 - 3z^{-1} + 2z^{-2} - 6z^{-3}$
(D) does not satisfy any of the above three.

Common Data for Questions 4 and 5:

4. A signal is processed by a causal filter with transfer function $G(s)$. For a distortion free output signal waveform, $G(s)$ must **[2007]**

(A) Provide zero phase shift for all frequency
(B) Provide constant phase shift for all frequency
(C) Provide linear phase shift that is proportional to frequency
(D) Provide a phase shift that is inversely proportional to frequency

5. $G(s) = \alpha z^{-1} + \beta z^{-1}$ is a low-pass digital filter with a phase characteristics same as that of the above question if **[2007]**

(A) $\alpha = \beta$ (B) $\alpha = -\beta$
(C) $\alpha = \beta^{(t/3)}$ (D) $\alpha = \beta^{-(t/3)}$

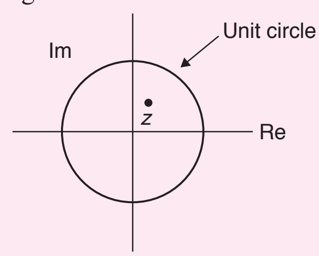
6. Given a sequence $x[n]$, to generate the sequence $y[n] = x[3 - 4n]$, which one of the following procedures would be correct? **[2008]**

(A) First delay $x[n]$ by 3 sample to generate $z_1[n]$, then pick every 4th sample of $z_1[n]$ to generate $z_2[n]$, and then finally time reverse $z_2[n]$ to obtain $y[n]$
(B) First advance $x[n]$ by 3 samples to generate $z_1[n]$, then pick every 4th sample of $z_1[n]$ to generate $z_2[n]$, and then finally time reverse $z_2[n]$ to obtain $y[n]$
(C) First pick every fourth sample of $x[n]$ to generate $v_1[n]$, time-reverse $v_1[n]$ to obtain $v_2[n]$, and finally advance $v_2[n]$ by 3 sample to obtain $y[n]$
(D) First pick every fourth sample of $x[n]$ to generate $v_1[n]$, time-reverse $v_1[n]$ to obtain $v_2[n]$, and finally delay $v_2[n]$ by 3 samples to obtain $y[n]$

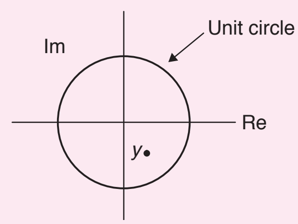
7. $H(z)$ is a transfer function of a real system. When a signal $x[n] = (1 + j)^n$ is the input to such a system, the output is zero. Further, the region of convergence (ROC) of $\left[1 - \frac{1}{2}z^{-1}\right] H(z)$ is the entire z-plane (except $z = 0$). It can then be inferred than $H(z)$ can have a minimum of **[2008]**

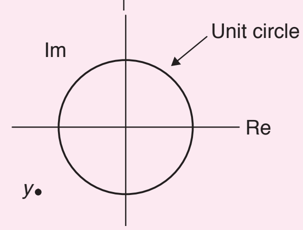
(A) One pole and one zero
(B) One pole and two zeros
(C) Two poles and one zero
(D) Two poles and two zeros

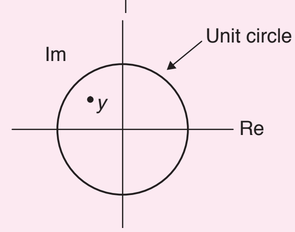
8. A point z has been plotted in the complex plane, as shown in figure below.

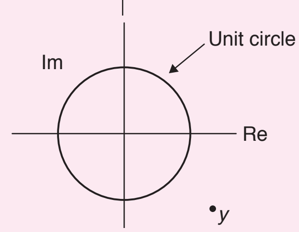


The plot of the complex number $y = \dfrac{1}{z}$ is **[2011]**

(A)


(B)


(C)


(D)


9. Let S be the set of points in the complex plane corresponding to the unit circle. (That is $S = \{z : |z| = 1\}$). Consider the function $f(z) = zz^*$ where z^* denotes the complex conjugate of z. The $f(z)$ maps S to which one of the following in the complex plane **[2014]**
(A) Unit circle
(B) Horizontal axis line segment from origin to (1, 0)
(C) The point (1, 0)
(D) The entire horizontal axis

10. Let $X(z) = \dfrac{1}{1-z^{-3}}$ be the z-transform of a causal signal $x[n]$. Then, the values of $x[2]$ and $x[3]$ are **[2014]**
(A) 0 and 0
(B) 0 and 1
(C) 1 and 0
(D) 1 and 1

11. An input signal $x(t) = 2 + 5\sin(100\pi t)$ is sampled frequency of 400 Hz and applied to the system whose transfer function is represented by

$$\frac{Y(z)}{X(z)} = \frac{1}{N}\left(\frac{1 - z^{-N}}{1 - z^{-1}}\right)$$

Where, N represents the number of samples per cycle. The output $y(n)$ of the system under steady state is **[2014]**
(A) 0 (B) 1
(C) 2 (D) 5

12. Consider a discrete time signal given by
$x[n] = (-0.25)^n u[n] + (0.5)^n u[-n-1]$
The region of convergence of its Z-transform would be **[2015]**
(A) The region inside the circle of radius 0.5 and centered at origin.
(B) The region outside the circle of radius 0.25 and centered at origin.
(C) The annular region between the two circles, both centered origin and having radii 0.25 and 0.5.
(D) The entire Z plane.

13. The z-Transform of a sequence $x[n]$ is given as $X(z) = 2z + 4 - 4/z + 3/z^2$. If $y[n]$ is the first difference of $x[n]$, then $Y(z)$ is given by **[2015]**
(A) $2z + 2 - 8/z + 7/z^2 - 3/z^3$
(B) $-2z + 2 - 6/z + 1/z^2 + 3/z^3$
(C) $-2z - 2 + 8/z - 7/z^2 + 3/z^3$
(D) $4z - 2 - 8/z - 1/z^2 + 3/z^3$

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** A	**3.** C	**4.** B	**5.** C	**6.** D	**7.** A	**8.** B	**9.** B	**10.** C
11. D	**12.** C	**13.** A	**14.** B	**15.** A	**16.** A	**17.** A	**18.** D	**19.** C	**20.** D
21. B	**22.** B								

Practice Problems 2

1. C	**2.** A	**3.** B	**4.** D	**5.** B	**6.** D	**7.** B	**8.** B	**9.** C	**10.** B
11. D	**12.** B	**13.** D	**14.** A	**15.** A	**16.** D	**17.** B	**18.** C	**19.** B	**20.** B
21. A									

Previous Years' Questions

1. A	**2.** D	**3.** B	**4.** C	**5.** A	**6.** D	**7.** A	**8.** D	**9.** C	**10.** B
11. C	**12.** C	**13.** A							