20. INVERSE TRIGONOMETRIC FUNCTIONS

1. Principal Values & Domains of Inverse Trigonometric/Circular Functions:

	Function		Domain	Range
(i)	$y = \sin^{-1} x$	where	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1} x$	where	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1} x$	where	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv)	$y = \text{cosec}^{-1} x$	where	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(v)	$y = \sec^{-1} x$	where	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi$; $y \ne \frac{\pi}{2}$
(vi)	$y = \cot^{-1} x$	where	$x \in R$	$0 < y < \pi$

P - 2 (i) $\sin^{-1}(\sin x) = x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (ii) $\cos^{-1}(\cos x) = x; \quad 0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ (iv) $\cot^{-1}(\cot x) = x; \quad 0 < x < \pi$

(v) $\sec^{-1}(\sec x) = x; \quad 0 \le x \le \pi, x \ne \frac{\pi}{2}$ (vi) $\text{cosec}^{-1}(\text{cosec } x) = x; \quad x \ne 0, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

P - 3 (i) $\sin^{-1}(-x) = -\sin^{-1} x, \quad -1 \le x \le 1$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x, \quad x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, \quad -1 \le x \le 1$ (iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, \quad x \in R$

P - 5 (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, -1 \le x \le 1$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad x \in R$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}, \quad |x| \ge 1$

2. Identities of Addition and Substraction:

I - 1 (i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], x \ge 0, y \ge 0$ & $(x^2 + y^2) \le 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], x \ge 0, y \ge 0$ & $x^2 + y^2 > 1$

(ii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left[xy - \sqrt{1-x^2}\ \sqrt{1-y^2}\right], \quad x \ge 0, y \ge 0$

(iii) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0$ & $xy < 1$

$= \pi + \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0$ & $xy > 1 \quad = \frac{\pi}{2}, x > 0, y > 0$ & $xy = 1$

I - 2 (i) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$, $x \ge 0,\ y \ge 0$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\ \sqrt{1-y^2}\right]$, $x \ge 0,\ y \ge 0,\ x \le y$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$, $x \ge 0,\ y \ge 0$

I - 3 (i) $$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1}x & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -\left(\pi + 2\sin^{-1}x\right) & \text{if } x < -\frac{1}{\sqrt{2}} \end{cases}$$

(ii) $$\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & \text{if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & \text{if } -1 \le x < 0 \end{cases}$$

(iii) $$\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{cases}$$

(iv) $$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & \text{if } x < -1 \end{cases}$$

(v) $$\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0,\ y > 0,\ z > 0$ & $(xy + yz + zx) < 1$

NOTE:

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(iv) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$