1. $\int \frac{x^2}{x^4 - x^2 - 12} dx$ <u>ગણતરી માટેનું સૂચન :</u> અહીં $\frac{px + q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ ની મદદથી A થી B મેળવી સંકલન કરો.

➡ $I = \int \frac{x^2}{x^4 - x^2 - 12} dx$

અહીં x^2 ને બદલે t ચલ બદલતાં,

$\therefore \frac{t}{t^2 - t - 12} = \frac{t}{(t-4)(t+3)}$

$\therefore \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$ ....(i)

($\because$ આંશિક અપૂર્ણાંક)

$\therefore t = A(t+3) + B(t-4)$

હવે અચળ A અને B મેળવવા માટે તેમના સહગુણકો શૂન્ય બનાવતાં,

પ્રથમ $t = 4$ મૂકો.

$\therefore 4 = A(7) + 0$

$\therefore A = \frac{4}{7}$

હવે $t = -3$ મૂકો.

$\therefore -3 = A(0) + B(-7)$

$\therefore -3 = -7B$

$\therefore B = \frac{3}{7}$

પરિણામ (i) નીચે મુજબ થાય.

$I = \int \frac{t}{(t+3)(t-4)} dt$

$= \int \left(\frac{\frac{4}{7}}{t-4} + \frac{\frac{3}{7}}{t+3} \right) dt$

$= \frac{4}{7} \int \frac{1}{t-4} dt + \frac{3}{7} \int \frac{1}{t+3} dt$

$= \frac{4}{7} \int \frac{1}{x^2 - 4} dt + \frac{3}{7} \int \frac{1}{x^2 + (\sqrt{3})^2} dt$

$= \frac{4}{7} \int \frac{1}{x^2 - (2)^2} dx + \frac{3}{7} \int \frac{1}{x^2 + (\sqrt{3})^2} dx$

➡ $I = \int \frac{x^2}{x^4 - x^2 - 12} dx$

અહીં x^2 ને બદલે t ચલ બદલતાં,

$\therefore \frac{t}{t^2 - t - 12} = \frac{t}{(t-4)(t+3)}$

$\therefore \dfrac{t}{(t-4)(t+3)} = \dfrac{A}{t-4} + \dfrac{B}{t+3}$ ....(i)

($\because$ આંશિક અપૂર્ણાંક)

$\therefore\ t = A(t+3) + B(t-4)$

હવે અચળ A અને B મેળવવા માટે તેમના સહગુણકો શૂન્ય બનાવતાં,

પ્રથમ $t = 4$ મૂકો.

$\therefore\ 4 = A(7) + 0$

$\therefore A = \dfrac{4}{7}$

હવે $t = -3$ મૂકો.

$\therefore\ -3 = A(0) + B(-7)$

$\therefore\ -3 = -7B$

$\therefore\ B = \dfrac{3}{7}$

પરિણામ (i) નીચે મુજબ થાય.

$$I = \int \frac{t}{(t+3)(t-4)}\, dt$$

$$= \int \left(\frac{\frac{4}{7}}{t-4} + \frac{\frac{3}{7}}{t+3} \right) dt$$

$$= \frac{4}{7} \int \frac{1}{t-4}\, dt + \frac{3}{7} \int \frac{1}{t+3}\, dt$$

$$= \frac{4}{7} \int \frac{1}{x^2 - 4}\, dt + \frac{3}{7} \int \frac{1}{x^2 + (\sqrt{3})^2}\, dt$$

$$= \frac{4}{7} \int \frac{1}{x^2 - (2)^2}\, dx + \frac{3}{7} \int \frac{1}{x^2 + (\sqrt{3})^2}\, dx$$

2. $\displaystyle \int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$

➡ $\displaystyle I = \int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$

અહીં x^2 ને બદલે t મૂકતાં,

$\therefore \dfrac{t}{(t+a^2)(t+b^2)} = \dfrac{A}{t+a^2} + \dfrac{B}{t+b^2}$ ....(i)

($\because$ આંશિક અપૂર્ણાંક)

$\therefore\ t = A(t+b^2) + B(t+a^2)$

હવે A તથા B મેળવવા તેમના સહગુણકો શૂન્ય બનાવતાં,

હવે $t = -a^2$ મૂકો.

$\therefore\ -a^2 = A(-a^2 + b^2) + 0$

$\therefore A = \dfrac{-a^2}{b^2 - a^2} = \dfrac{a^2}{a^2 - b^2}$

તથા $t = -b^2$ મૂકતાં,

$\therefore\ -b^2 = A(0) + B(-b^2 + a^2)$

$\therefore\ -b^2 = B(a^2 - b^2)$

$\therefore\ B = \dfrac{-b^2}{a^2 - b^2}$

$\therefore$ પરિણામ (i) નીચે મુજબ થાય.

$$\frac{t}{(t+a^2)(t+b^2)} = \frac{a^2}{(a^2-b^2)}\left(\frac{1}{t+a^2}\right) - \frac{b^2}{a^2-b^2}\left(\frac{1}{t+b^2}\right)$$

$$\therefore \text{I} = \int \frac{a^2}{a^2-b^2}\left(\frac{1}{t+a^2}\right)dx - \int \frac{b^2}{a^2-b^2}\left(\frac{1}{t+b^2}\right)dx$$

$$= \frac{a^2}{a^2-b^2}\int \frac{1}{x^2+a^2}\,dx - \frac{b^2}{a^2-b^2}\int \frac{1}{x^2+b^2}\,dx$$

$$= \frac{1}{a^2-b^2}\left\{a^2\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right) - b^2\left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right)\right\}$$

$$\therefore \text{I} = \frac{1}{a^2-b^2}\left\{a\tan^{-1}\left(\frac{x}{a}\right) - b\tan^{-1}\left(\frac{x}{b}\right)\right\}$$

3. $$\int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

➡ $$\text{I} = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin x}\,dx - \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$\therefore \text{I} = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx - \text{I}$$

$$\therefore 2\text{I} = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$

$$\therefore 2\text{I} = \pi\int_0^{\pi} \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx$$

$$= \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \pi\int_0^{\pi} (\sec^2 x - \sec x\,\tan x)\,dx$$

$$= \pi\,(\tan x - \sec x)_0^{\pi}$$

$$= \pi\left[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)\right]$$

$$= \pi\left[0 - (-1) - (0 - 1)\right]$$

$$2\text{I} = 2\pi$$

$$\therefore\ \text{I} = \pi$$

4. $$\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx$$

➡ $$\text{I} = \int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx$$

$\therefore \dfrac{2x-1}{(x-1)(x+2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$ – (I)

$\therefore 2x - 1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

અચળ A, B તથા C મેળવવા તેમના સહગુણકો શૂન્ય બનાવતાં,

સૌ પ્રથમ $x = 1$ મૂકો.

$\therefore 2 - 1 = A(3)\ (-2) + 0 + 0$

$\therefore 1 = -6A$

$\therefore A = -\dfrac{1}{6}$ ....(i)

હવે $x = -2$ મૂકો.

$\therefore -4 - 1 = A(0) + B(-3)(-5) + B(0)$

$\therefore -5 = 15B$

$\therefore -1 = 3B$

$\therefore B = -\dfrac{1}{3}$ ....(ii)

હવે $x = 3$ મૂકો.

$\therefore 6 - 1 = A(0) + B(0) + C(2)(5)$

$\therefore 5 = 10C$

$\therefore C = \dfrac{1}{2}$ ....(iii)

આ મૂલ્યો પરિણામ (i) માં મૂકી સંકલિત મેળવતાં,

$\therefore I = -\dfrac{1}{6}\int\dfrac{1}{x-1}\,dx - \dfrac{1}{3}\int\dfrac{1}{x+2}\,dx + \dfrac{1}{2}\int\dfrac{1}{x-3}\,dx$

$= \dfrac{1}{2}\log(x-3) - \dfrac{1}{6}\log(x-1) - \dfrac{1}{3}\log(x+2) + C$

$= \log(\sqrt{x-3}) - \left\{\log(x-1)^{\frac{1}{6}} + \log(x+2)^{\frac{1}{2}}\right\} + C$

$\therefore I = \log\left|\dfrac{\sqrt{x-3}}{(x-1)^{\frac{1}{6}}\cdot(x+2)^{\frac{1}{2}}}\right| + C$

5. $\displaystyle\int\frac{e^{\tan^{-1}}}{1+x^2}(1+x+x^2)\,dx$ **= ...... + c**

➡ $I = \displaystyle\int\frac{e^{\tan^{-1}x}}{1+x^2}(1+x+x^2)\,dx$

અહીં $\tan^{-1}x = t$ આદેશ લેતાં,

$\therefore \dfrac{1}{1+x^2}\,dx = dt,\ x = \tan t$

$\therefore I = \int e^t\,(1 + \tan t + \tan^2 t)\cdot dt$

$= \int e^t\,(\tan t + \sec^2 t)\,dt$

$= \displaystyle\int e^t\left[\tan t + \frac{d}{dt}(\tan t)\right]dt$

$= e^t\cdot\tan t + c$

$\therefore I = (e^{\tan^{-1}x})\ x + c$

6. $\displaystyle\int\sin^{-1}\left(\sqrt{\frac{x}{x+a}}\right)dx$

➥ ધારો કે, $x = a\tan^2\theta$

$\therefore\ dx = 2a\tan\theta\cdot\sec^2\theta\cdot d\theta$

$$I = \int \sin^{-1}\sqrt{\frac{x}{x+a}}\ dx$$

$$= \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a\tan^2\theta + a}}\cdot 2a\tan\theta\cdot\sec^2\theta\cdot d\theta$$

$$= \int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\cdot 2a\tan\theta\cdot\sec^2\theta\cdot d\theta$$

$$= \int \sin^{-1}(\sin\theta)\cdot 2a\tan\theta\cdot\sec^2\theta\cdot d\theta$$

$$= 2a\int\theta\cdot\tan\theta\ \sec^2\theta\cdot d\theta$$

$u = \theta$ અને $v = \tan\theta\cdot\sec^2\theta$ લઈને ખંડશઃ સંકલનના નિયમનો ઉપયોગ કરતાં

$$I = 2a\left[\theta\int\tan\theta\cdot\sec^2\theta\cdot d\theta - \int\left[\frac{d}{d\theta}(\theta)\cdot\int\tan\theta\sec^2\theta\cdot d\theta\right]d\theta\right]$$

$$= 2a\left[\theta\cdot\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}\ d\theta\right]$$

$$= 2a\left[\theta\cdot\frac{\tan^2\theta}{2} - \frac{1}{2}\int(\sec^2\theta - 1)\,d\theta\right]$$

$$= a\theta\cdot\tan^2\theta - a\,[\tan\theta - \theta] + c$$

$$= a\theta\cdot\tan^2\theta - a\tan\theta + a\theta + c$$

$$= a\cdot\left(\frac{x}{a}\right)\tan^{-1}\sqrt{\frac{x}{a}} - a\cdot\left(\sqrt{\frac{x}{a}}\right) + a\tan^{-1}\left(\sqrt{\frac{x}{a}}\right) + c$$

$$= x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

$$\therefore\ I = (x + a)\tan^{-1}\left(\sqrt{\frac{x}{a}}\right) - \sqrt{ax} + c$$

7. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{5}{2}}}\ dx$

➥ $$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{5}{2}}}\ dx$$

અહીં $1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$ અને

$1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$ સૂત્ર મુજબ ગણતરી કરતાં,

$$\therefore I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\left(2\cos^2\frac{x}{2}\right)^{\frac{1}{2}}}{\left(2\sin^2\frac{x}{2}\right)^{\frac{5}{2}}}\ dx$$

$$= \frac{2^{\frac{1}{2}}}{2^{\frac{5}{2}}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

$$= \left(\frac{1}{2^2}\right) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

$$= \frac{1}{4} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

હવે $\sin\left(\frac{x}{2}\right) = t$ આદેશ મૂકો.

$\therefore \frac{1}{2} \cos\left(\frac{x}{2}\right) dx = dt \Rightarrow \cos\left(\frac{x}{2}\right) dx = 2\, dt$

તથા $x = \frac{\pi}{3}$ તો $t = \sin\left(\frac{\pi}{6}\right)$

$\therefore t = \frac{1}{2}$

અને $x = \frac{\pi}{2}$ તો $t = \sin\left(\frac{\pi}{4}\right)$

$\therefore t = \frac{1}{\sqrt{2}}$

➥ $$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{\frac{5}{2}}}\, dx$$

અહીં $1 + \cos x = 2 \cos^2\left(\frac{x}{2}\right)$ અને

$1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right)$ સૂત્ર મુજબ ગણતરી કરતાં,

$$\therefore I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\left(2 \cos^2 \frac{x}{2}\right)^{\frac{1}{2}}}{\left(2 \sin^2 \frac{x}{2}\right)^{\frac{5}{2}}}\, dx$$

$$= \frac{2^{\frac{1}{2}}}{2^{\frac{5}{2}}} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

$$= \left(\frac{1}{2^2}\right) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

$$= \frac{1}{4} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

હવે $\sin\left(\frac{x}{2}\right) = t$ આદેશ મૂકો.

$\therefore \frac{1}{2} \cos\left(\frac{x}{2}\right) dx = dt \Rightarrow \cos\left(\frac{x}{2}\right) dx = 2\, dt$

તથા $x = \frac{\pi}{3}$ તો $t = \sin\left(\frac{\pi}{6}\right)$

$\therefore t = \frac{1}{2}$

અને $x = \frac{\pi}{2}$ તો $t = \sin\left(\frac{\pi}{4}\right)$

$\therefore t = \frac{1}{\sqrt{2}}$

8. $\int e^{-3x} \cos^3 x \, dx$ <u>ગણતરી માટેનું સૂચન :</u> $\int e^{ax} \cos(bx)\, dx = \frac{e^{ax}}{a^2 + b^2} \{a \cos dx + b \sin bx\} + C$ સૂત્રથી ગણતરી કરો.

➡ અહીં $\cos(3x) = 4\cos^3 x - 3\cos x$

$\therefore 4\cos^3 x = \cos(3x) + 3\cos x$

$\therefore \cos^3 x = \frac{\cos(3x) + 3\cos x}{4}$

$I = \int e^{-3x} \cos^3 x \, dx$

$= \int e^{-3x} \left(\frac{\cos 3x + 3\cos x}{4}\right) dx$

$= \frac{1}{4} \int e^{-3x} \cos 3x \, dx + \frac{3}{4} \int e^{-3x} \cos x \, dx$

$= \frac{1}{4} \left\{\frac{e^{-3x}}{9+9} (-3\cos 3x + 3\sin 3x)\right\} + \frac{3}{4} \left\{\frac{e^{-3x}}{9+1} (-3\cos 3x + \sin x)\right\} + C$

$= \frac{1}{4} \left(\frac{e^{-3x}}{18}\right) [3(\sin 3x - \cos 3x)] + \frac{3}{4} \left(\frac{e^{-3x}}{10}\right) (\sin x - 3\cos 3x) + C$

$\therefore I = \frac{e^{-3x}}{24} \{\sin 3x - \cos 3x\} + \frac{3e^{-3x}}{40} (\sin x - 3\cos x) + C$

9. $\int \sqrt{\tan x} \, dx$

➡ $I = \int \sqrt{\tan x} \, dx$

અહીં $\sqrt{\tan x} = t$ આદેશ લેતાં

$\therefore \tan x = t^2$

$\therefore \sec^2 x = dx = 2t \, dt$

$\therefore (1 + \tan^2 x)\, dx = 2t \, dt$

$\therefore (1 + t^4)\, dx = 2t \, dt$

$\therefore dx = \frac{2t}{1+t^4} dt$

$\therefore I = \int t \cdot \frac{2t}{1+t^4} dt$

$= \int \frac{2t^2}{1+t^4} dt$

$= \int \frac{t^2 + 1 + t^2 - 1}{1+t^4} dt$

$= \int \frac{t^2+1}{1+t^4} + \int \frac{t^2-1}{1+t^4} dt$

હવે અંશ તથા છેદના દરેક પદને t^2 વડે ભાગતાં

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \, dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \, dt$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} \, dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 + 2} \, dt$$

$\therefore I = I_1 + I_2$ .....(1)

હવે $I_1 = \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} \, dt$

અહીં $t - \frac{1}{t} = u$ આદેશ મૂકો.

➥ $I = \int \sqrt{\tan x} \, dx$

અહીં $\sqrt{\tan x} = t$ આદેશ લેતાં

$\therefore \tan x = t^2$

$\therefore \sec^2 x = dx = 2t \, dt$

$\therefore (1 + \tan^2 x) \, dx = 2t \, dt$

$\therefore (1 + t^4) \, dx = 2t \, dt$

$\therefore dx = \frac{2t}{1+t^4} \, dt$

$\therefore I = \int t \cdot \frac{2t}{1 + t^4} \, dt$

$$= \int \frac{2t^2}{1 + t^4} \, dt$$

$$= \int \frac{t^2 + 1 + t^2 - 1}{1 + t^4} \, dt$$

$$= \int \frac{t^2 + 1}{1 + t^4} + \int \frac{t^2 - 1}{1 + t^4} \, dt$$

હવે અંશ તથા છેદના દરેક પદને t^2 વડે ભાગતાં

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \, dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \, dt$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} \, dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 + 2} \, dt$$

$\therefore I = I_1 + I_2$ .....(1)

હવે $I_1 = \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} \, dt$

અહીં $t - \frac{1}{t} = u$ આદેશ મૂકો.

➥ $I = \int \sqrt{\tan x}\, dx$

અહીં $\sqrt{\tan x} = t$ આદેશ લેતાં

$\therefore \tan x = t^2$

$\therefore \sec^2 x = dx = 2t\, dt$

$\therefore (1 + \tan^2 x)\, dx = 2t\, dt$

$\therefore (1 + t^4)\, dx = 2t\, dt$

$\therefore dx = \frac{2t}{1+t^4}\, dt$

$\therefore I = \int t \cdot \frac{2t}{1 + t^4}\, dt$

$= \int \frac{2t^2}{1 + t^4}\, dt$

$= \int \frac{t^2 + 1 + t^2 - 1}{1 + t^4}\, dt$

$= \int \frac{t^2 + 1}{1 + t^4} + \int \frac{t^2 - 1}{1 + t^4}\, dt$

હવે અંશ તથા છેદના દરેક પદને t^2 વડે ભાગતાં

$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}\, dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}\, dt$

$= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2}\, dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 + 2}\, dt$

$\therefore I = I_1 + I_2$ .....(1)

હવે $I_1 = \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2}\, dt$

અહીં $t - \frac{1}{t} = u$ આદેશ મૂકો.

➥ $I = \int \sqrt{\tan x}\, dx$

અહીં $\sqrt{\tan x} = t$ આદેશ લેતાં

$\therefore \tan x = t^2$

$\therefore \sec^2 x = dx = 2t\, dt$

$\therefore (1 + \tan^2 x)\, dx = 2t\, dt$

$\therefore (1 + t^4)\, dx = 2t\, dt$

$\therefore dx = \frac{2t}{1+t^4}\, dt$

$\therefore I = \int t \cdot \frac{2t}{1 + t^4}\, dt$

$$= \int \frac{2t^2}{1 + t^4} \, dt$$

$$= \int \frac{t^2 + 1 + t^2 - 1}{1 + t^4} \, dt$$

$$= \int \frac{t^2 + 1}{1 + t^4} + \int \frac{t^2 - 1}{1 + t^4} \, dt$$

હવે અંશ તથા છેદના દરેક પદને t^2 વડે ભાગતાં

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \, dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \, dt$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} \, dt + \int \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 + 2} \, dt$$

$\therefore$ $I = I_1 + I_2$ .....(1)

હવે $I_1 = \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} \, dt$

અહીં $t - \frac{1}{t} = u$ આદેશ મૂકો.

10. $\displaystyle \int_0^{\frac{\pi}{2}} \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \, dx$

➡ $I = \displaystyle \int_0^{\frac{\pi}{2}} \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \, dx$

અંશ તથા છેદના દરેક પદને $\cos^2 x$ વડે ભાગતાં,

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^4 x}{(a^2 + b^2 \tan^2 x)^2} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1 + \tan^2 x) \sec^2 x}{(a^2 + b^2 \tan^2 x)^2} \, dx$$

હવે $\tan x = t$ આદેશ મૂકો.

$\therefore$ $\sec^2 x \, dx = dt$

તથા $x = 0$ તો $t = \tan 0 \Rightarrow t = 0$

અને $x = \frac{\pi}{2}$ અને $t = \tan \frac{\pi}{2} \Rightarrow t = \infty$

$$I = \int_0^{\infty} \frac{(1 + t^2)}{(a^2 + b^2 t^2)} \, dt$$

હવે t^2 ના સ્થાને y મૂકીને આંશિક અપૂર્ણાંક લેતાં,

$$\therefore \frac{1 + y}{(a^2 + b^2 y)^2} = \frac{A}{a^2 + b^2 y} + \frac{B}{(a^2 + b^2 y)^2}$$

$\therefore\ 1 + y = A(a^2 + b^2y) + B$

$\therefore\ 1 + y = Ab^2y + Aa^2 + B$

$\therefore$ બંને બાજુ y ના સહગુણક તથા અચળપદ સરખાવતાં,

$\therefore\ Ab^2 = 1, \qquad Aa^2 + B = 1$

$\therefore\ A = \frac{1}{b^2}, \qquad \therefore \frac{a^2}{b^2} + B = 1$

$\therefore B = 1 - \frac{a^2}{b^2}$

$\therefore B = \frac{b^2 - a^2}{b^2}$

➥ $I = \int_0^{\frac{\pi}{2}} \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}\, dx$

અંશ તથા છેદના દરેક પદને $\cos^2 x$ વડે ભાગતાં,

$I = \int_0^{\frac{\pi}{2}} \frac{\sec^4 x}{(a^2 + b^2 \tan^2 x)^2}\, dx$

$= \int_0^{\frac{\pi}{2}} \frac{(1 + \tan^2 x) \sec^2 x}{(a^2 + b^2 \tan^2 x)^2}\, dx$

હવે $\tan x = t$ આદેશ મૂકો.

$\therefore\ \sec^2 x\, dx = dt$

તથા $x = 0$ તો $t = \tan 0 \Rightarrow t = 0$

અને $x = \frac{\pi}{2}$ અને $t = \tan\frac{\pi}{2} \Rightarrow t = \infty$

$I = \int_0^{\infty} \frac{(1 + t^2)}{(a^2 + b^2t^2)}\, dt$

હવે t^2 ના સ્થાને y મૂકીને આંશિક અપૂર્ણાંક લેતાં,

$\therefore \frac{1 + y}{(a^2 + b^2y)^2} = \frac{A}{a^2 + b^2y} + \frac{B}{(a^2 + b^2y)^2}$

$\therefore\ 1 + y = A(a^2 + b^2y) + B$

$\therefore\ 1 + y = Ab^2y + Aa^2 + B$

$\therefore$ બંને બાજુ y ના સહગુણક તથા અચળપદ સરખાવતાં,

$\therefore\ Ab^2 = 1, \qquad Aa^2 + B = 1$

$\therefore\ A = \frac{1}{b^2}, \qquad \therefore \frac{a^2}{b^2} + B = 1$

$\therefore B = 1 - \frac{a^2}{b^2}$

$\therefore B = \frac{b^2 - a^2}{b^2}$

11. $\int_0^1 x \log(1 + 2x)\, dx$

➥ $I = \int^1 x \log(1 + 2x)\, dx$

0

$u = \log(1 + 2x)$ અને $v = x$ લઈને ખંડશઃ સંકલન કરતાં,

$\therefore u' = \dfrac{2}{1 + 2x}, \int v \, dx = \int x \, dx = \dfrac{x^2}{2}$

$\therefore \text{I} = u \int v \, dx - \int (u' \int v \, dx) \, dx$

(ખંડશઃ સંકલનનો નિયમ)

$$= \left[\frac{x^2}{2} \log(1 + 2x) - \int \frac{2}{1 + 2x} \cdot \frac{x^2}{2} \, dx\right]_0^1$$

$$= \left[\frac{x^2 \log(1 + 2x)}{2} - \int \frac{x^2}{1 + 2x} \, dx\right]_0^1$$

$$= \left[\frac{x^2 \log(1 + 2x)}{2}\right]_0^1 - \frac{1}{2} \int_0^1 \frac{2x^2}{1 + 2x} \, dx$$

$$= \left[\frac{x^2 \log(1 + 2x)}{2}\right]_0^1 - \frac{1}{2} \int_0^1 \frac{x + 2x^2 - x}{2(1 + 2x)} \, dx$$

$$= \left[\frac{x^2 \log(1+2x)}{2}\right]_0^1 - \int_0^1 \frac{x(1+2x)}{2(1+2x)} - \frac{x}{2(1+2x)} \, dx$$

$$= \left[\frac{x^2 \log(1+2x)}{2}\right]_0^1 - \int_0^1 \frac{x}{2} \, dx + \int_0^1 \frac{x}{2(1+2x)} \, dx$$

$$= \left[\frac{1 \log(3)}{2} - 0\right] - \int_0^1 \frac{x}{2} \, dx + \frac{1}{4} \int_0^1 \frac{1+2x-1}{(1+2x)} \, dx$$

$$= \frac{1}{2} \log 3 - \left(\frac{x^2}{4}\right)_0^1 + \frac{1}{4} \int_0^1 \left(1 - \frac{1}{1 + 2x}\right) dx$$

$$= \frac{1}{2} \log 3 - \frac{1}{4} + \frac{1}{4} \left(x - \frac{1}{2} \log(1 + 2x)\right)_0^1$$

➡ $\text{I} = \int_0^1 x \log(1 + 2x) \, dx$

$u = \log(1 + 2x)$ અને $v = x$ લઈને ખંડશઃ સંકલન કરતાં,

$\therefore u' = \dfrac{2}{1 + 2x}, \int v \, dx = \int x \, dx = \dfrac{x^2}{2}$

$\therefore \text{I} = u \int v \, dx - \int (u' \int v \, dx) \, dx$

(ખંડશઃ સંકલનનો નિયમ)

$$= \left[\frac{x^2}{2} \log(1 + 2x) - \int \frac{2}{1 + 2x} \cdot \frac{x^2}{2} \, dx\right]_0^1$$

$$= \left[\frac{x^2 \log(1 + 2x)}{2} - \int \frac{x^2}{1 + 2x} \, dx\right]_0^1$$

$$= \left[\frac{x^2 \log(1 + 2x)}{2}\right]_0^1 - \frac{1}{2} \int_0^1 \frac{2x^2}{1 + 2x} \, dx$$

$$= \left[\frac{x^2 \log(1 + 2x)}{2}\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x + 2x^2 - x}{2(1 + 2x)}\, dx$$

$$= \left[\frac{x^2 \log(1+2x)}{2}\right]_0^1 - \int_0^1 \frac{x(1+2x)}{2(1+2x)} - \frac{x}{2(1+2x)}\, dx$$

$$= \left[\frac{x^2 \log(1+2x)}{2}\right]_0^1 - \int_0^1 \frac{x}{2}\, dx + \int_0^1 \frac{x}{2(1+2x)}\, dx$$

$$= \left[\frac{1 \log(3)}{2} - 0\right] - \int_0^1 \frac{x}{2}\, dx + \frac{1}{4}\int_0^1 \frac{1+2x-1}{(1+2x)}\, dx$$

$$= \frac{1}{2}\log 3 - \left(\frac{x^2}{4}\right)_0^1 + \frac{1}{4}\int_0^1 \left(1 - \frac{1}{1 + 2x}\right) dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}\left(x - \frac{1}{2}\log(1 + 2x)\right)_0^1$$

12. $\int_0^{\pi} x \log(\sin x)\, dx$

➥ $I = \int_0^{\pi} x \log(\sin x)\, dx$ ....(i)

અહીં $a = \pi$ છે.

$\therefore$ x ના સ્થાને $\pi - x$ લેતાં,

$$\therefore I = \int_0^{\pi} (\pi - x) \log\big(\sin(\pi - x)\big)\, dx$$

$$\left(\because \int_0^{a} f(x)dx = \int_0^{a} f(a - x)dx\right)$$

$$= \int_0^{\pi} (\pi - x) \log \sin x\, dx \quad \text{.......(ii)}$$

$$= \pi\int_0^{\pi} \log \sin x\, dx - \int_0^{\pi} x \log \sin x\, dx$$

$$I = \pi\int_0^{\pi} \log \sin x\, dx - I$$

$$\therefore 2I = \pi\int_0^{\pi} \log \sin x\, dx \quad \text{.......(iii)}$$

$$\therefore 2I = 2\pi\int_0^{\frac{\pi}{2}} \log (\sin x)\, dx$$

$$\left(\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx \text{ છે.}\right)$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} \log \sin x \, dx \qquad \text{.......(iv)}$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

➡ $$I = \int_0^{\pi} x \log(\sin x) \, dx \qquad \text{.......(i)}$$

અહીં $a = \pi$ છે.

∴ xના સ્થાને $\pi - x$ લેતાં,

$$\therefore I = \int_0^{\pi} (\pi - x) \log(\sin(\pi - x)) \, dx$$

$$\left(\because \int_0^{a} f(x)dx = \int_0^{a} f(a - x)dx\right)$$

$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx \qquad \text{.......(ii)}$$

$$= \pi \int_0^{\pi} \log \sin x \, dx - \int_0^{\pi} x \log \sin x \, dx$$

$$I = \pi \int_0^{\pi} \log \sin x \, dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} \log \sin x \, dx \qquad \text{.......(iii)}$$

$$\therefore 2I = 2\pi \int_0^{\frac{\pi}{2}} \log (\sin x) \, dx$$

$$\left(\because \int_0^{2a} f(x) \, dx = 2\int_0^{a} f(x) \, dx \text{ છે.}\right)$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} \log \sin x \, dx \qquad \text{.......(iv)}$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

➡ $$I = \int_0^{\pi} x \log(\sin x) \, dx \qquad \text{.......(i)}$$

અહીં $a = \pi$ છે.

∴ xના સ્થાને $\pi - x$ લેતાં,

$$\therefore I = \int_0^{\pi} (\pi - x) \log(\sin(\pi - x)) \, dx$$

$$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx \qquad \text{.......(ii)}$$

$$= \pi\int_0^{\pi} \log \sin x \, dx - \int_0^{\pi} x \log \sin x \, dx$$

$$I = \pi\int_0^{\pi} \log \sin x \, dx - I$$

$$\therefore 2I = \pi\int_0^{\pi} \log \sin x \, dx \qquad \text{.......(iii)}$$

$$\therefore 2I = 2\pi\int_0^{\frac{\pi}{2}} \log (\sin x) \, dx$$

$$\left(\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)\,dx \text{ છે.}\right)$$

$$\therefore I = \pi\int_0^{\frac{\pi}{2}} \log \sin x \, dx \qquad \text{.......(iv)}$$

$$\therefore I = \pi\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

13. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log (\sin x + \cos x) \, dx$$

➡ $$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log (\sin x + \cos x) \, dx \qquad \text{.........(i)}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\sin\left(-\frac{\pi}{4} + \frac{\pi}{4} - x\right) + \cos\left(-\frac{\pi}{4} + \frac{\pi}{4} - x\right)\right) dx$$

$$\left(\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \text{ છે.}\right)$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log \left(\sin(-x) + \cos(-x)\right) dx$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log \left(\cos x - \sin x\right) dx \qquad \text{.........(ii)}$$

હવે (i) + (ii) લેતાં,

$\frac{\pi}{}$ $\frac{\pi}{}$

$$2I = \int_{-\frac{\pi}{4}}^{4} \log(\cos x + \sin x)\,dx + \int_{-\frac{\pi}{4}}^{4} \log(\cos x - \sin x)\,dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\cos^2 x - \sin^2 x\right) dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos 2x)\,dx$$

હવે $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ જો $f(x)$ યુગ્મ અર્થાત્

$f(-x) = -f(x)$ થાય.

➡ $$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sin x + \cos x)\,dx \quad \text{.........(i)}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\sin\left(-\frac{\pi}{4}+\frac{\pi}{4}-x\right)+\cos\left(-\frac{\pi}{4}+\frac{\pi}{4}-x\right)\right)dx$$

$$\left(\because \int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx \text{ છે.}\right)$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sin(-x) + \cos(-x))\,dx$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos x - \sin x)\,dx \quad \text{.........(ii)}$$

હવે (i) + (ii) લેતાં,

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos x + \sin x)\,dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos x - \sin x)\,dx$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\cos^2 x - \sin^2 x\right) dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos 2x)\,dx$$

હવે $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ જો $f(x)$ યુગ્મ અર્થાત્

$f(-x) = -f(x)$ થાય.

➥ $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sin x + \cos x)\, dx$ ....(i)

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\sin\left(-\frac{\pi}{4}+\frac{\pi}{4}-x\right)+\cos\left(-\frac{\pi}{4}+\frac{\pi}{4}-x\right)\right)dx$

$\left(\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \text{ છે.}\right)$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sin(-x) + \cos(-x))\, dx$

$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos x - \sin x)\, dx$ ....(ii)

હવે (i) + (ii) લેતાં,

$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos x + \sin x)\, dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos x - \sin x)\, dx$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\cos^2 x - \sin^2 x\right) dx$

$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\cos 2x)\, dx$

હવે $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$ જો $f(x)$ યુગ્મ અર્થાત્
$f(-x) = -f(x)$ થાય.