Chapter 5

UNIPLANAR MOTION WHEN THE ACCELERATION IS CENTRAL AND VARYING AS THE INVERSE SQUARE OF THE DISTANCE

End of Art 74
EXAMPLES

1. $v = \frac{h}{SY} = \frac{h}{b^2} \cdot HZ$, *i.e.* the velocity is perpendicular to and proportional to, HZ, and hence is equivalent to two components perpendicular and proportional to CH and CZ, *i.e.* to $\frac{h}{b^2} \cdot ae$ and $\frac{h}{b^2} \cdot a$ perpendicular to the axis and CZ. Now $h^2 = \mu \frac{b^2}{a}$, so that these velocities are $\frac{\mu e}{h}$ and $\frac{\mu}{h}$. Also CZ and SP are parallel.

2. $HP^2 \cdot \omega = vp = v \cdot HZ = \frac{h}{SY} \cdot HZ$, where ω is the required angular velocity about the second focus H.

$$\therefore \ \omega = \frac{h}{SY \cdot HP} \cdot \frac{HZ}{HP} = \frac{h}{SY \cdot HP} \cdot \frac{SY}{SP} = \frac{h}{CD^2}.$$

Now $PG : CD :: b : a$. $\therefore \ \omega \propto \frac{1}{PG^2}$.

3. In a hyperbola $V^2 = \mu \left(\frac{2}{R} + \frac{1}{a} \right)$. Also, if α is the angle of projection, $\frac{h}{R \sin \alpha} = \frac{h}{p} = V$. If the hyperbola is rectangular, we have $b = a$ and $h^2 = \mu \frac{b^2}{a} = \mu a$. On eliminating a and h, we have the given result.

4. $V^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$, so that $\frac{1}{a} = \frac{2}{r} - \frac{V^2}{\mu}$.

Hence the periodic time $= \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} =$ etc.

5. The velocity v of the Earth at its mean distance $a = \sqrt{\frac{\mu}{a}}$. The velocity v_1 at this distance in a parabolic path $= \sqrt{\frac{2\mu}{a}}$. Hence

$$\frac{v_1}{v} = \sqrt{2} = 1{\cdot}41 = \text{nearly } 1\tfrac{1}{2}.$$

If R is the radius of the Earth, and μ_1 the constant of attraction for the Earth, then $\frac{\mu_1}{R^2} = g$.

The least velocity of projection for a parabola about the centre of the Earth as focus $= \sqrt{\frac{2\mu_1}{R}} = \sqrt{2gR} = \sqrt{64 \times 4000 \times 1760 \times 3}$ feet per sec.

$$= \frac{40}{33}\sqrt{33} \text{ miles per sec.} = \text{about 7 miles per sec.}$$

Also the velocity of the Earth in its orbit is about 18·5 miles per second. Hence, by what we have just shown, the required velocity for it to describe a parabola $= \sqrt{2} \times 18{\cdot}5 =$ about 26·1 miles per second, *i.e.* its velocity must be increased by about 7·6 miles per second.

Hence a particle projected from the Earth in the direction of the Earth's velocity with a velocity relative to the Earth of about 7·6 miles per second would describe a parabola about the Sun.

6. $\frac{\mu}{a^2} = g$ and $v^2 = \mu\left(\frac{2}{a} - \frac{1}{a_1}\right)$, where $2a_1$ is the required major axis. Hence, etc.

7. $v^2 = ga^2\left(\frac{2}{a} - \frac{1}{\infty}\right) = 2ga.$

Also $V^2 = \frac{2\mu_1}{R}$, where $\frac{V_1^2}{R_1} = \frac{\mu_1}{R_1^2}$, R being the radius of the Sun, R_1 the radius of the Earth's orbit, and V_1 the Earth's velocity.

Hence $\frac{V^2}{V_1^2} = \frac{2}{R} \cdot R_1$. Hence, etc.

8. $\frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} = T$, and (Art. 31) $T_1 = \sqrt{\frac{a}{2\mu}}\left[\sqrt{ax - x^2} + a\cos^{-1}\sqrt{\frac{x}{a}}\right]_a^0$

$$= \frac{1}{a}\frac{1}{\sqrt{2}} \cdot \frac{T}{2\pi}\left[\frac{\pi a}{2}\right] = \frac{T}{4\sqrt{2}} = \frac{\sqrt{2}}{8} \cdot T.$$

9. $\frac{\text{Required time}}{\text{365 days}} = \frac{\text{Elliptic area } SBA'B'}{\text{Area of Ellipse}} = \frac{\frac{1}{2}\pi ab + aeb}{\pi ab} = \frac{1}{2} + \frac{1}{60\pi}.$

$$\therefore \text{ time} = \frac{\text{365 days}}{2} + 2 \text{ days approx.}$$

10. By Kepler's Law, the periodic time of Mars

$$=1 \text{ yr.} \times (1{\cdot}524)^{\frac{3}{2}} = 1{\cdot}8814 \text{ years} = 686{\cdot}7 \text{ days.}$$

11. By Art. 73, $\dfrac{S+P}{P+p} = \dfrac{D^3}{d^3} \div \left(\dfrac{T}{t}\right)^2 = \left(\dfrac{1415 \times 10^6}{146 \times 10^3}\right)^3 \div \left(\dfrac{687}{1\frac{21}{80}}\right)^2$

$$= \text{about } 3{,}074{,}000 \text{ on reduction.}$$

Hence $\dfrac{S}{P}$ is as stated, since p is small compared with P and S.

12. $\dfrac{S+P}{P+p} = \left(\dfrac{483 \times 10^6}{261 \times 10^3}\right)^3 \div \left(\dfrac{11{\cdot}86 \times 365}{1\frac{3}{4}}\right)^2 = \left(\dfrac{483}{261}\right)^3 \times 10^9 \div (2474)^2$ nearly

$$= \frac{(1{\cdot}85)^3 \times 10^9}{(2500)^2} \text{ nearly} = \frac{633 \times 10^7}{625 \times 10^4} = \text{a little greater than } 1000.$$

Hence, etc.

13. We have $\qquad \left(\dfrac{16\frac{2}{3} \text{ days}}{12 \text{ hours}}\right)^2 = \left(\dfrac{26\frac{1}{2} \,.\, a}{x \,.\, a}\right)^3,$

so that $\qquad x = 26\tfrac{1}{2} \div (33\tfrac{1}{3})^{\frac{2}{3}} = 26\tfrac{1}{2} \div 10{\cdot}34 = 2{\cdot}56$ approx.

Again $\quad 27\tfrac{1}{3} = \dfrac{2\pi}{\sqrt{\gamma(E+m)}}(60a)^{\frac{3}{2}}$ and $16\tfrac{2}{3} = \dfrac{2\pi}{\sqrt{\gamma(P+p)}}(26\tfrac{1}{2}b)^{\frac{3}{2}}$.

$$\therefore \frac{82}{50} = \sqrt{\frac{P}{E}}\left(\frac{60a}{26\frac{1}{2}b}\right)^{\frac{3}{2}} = \sqrt{\frac{\rho_1}{\rho}} \times \left(\frac{120}{53}\right)^{\frac{3}{2}},$$

where ρ_1 and ρ are the densities of Jupiter and the Earth.

$$\therefore \frac{\rho_1}{\rho} = \left(\frac{41}{25}\right)^2 \times \left(\frac{53}{120}\right)^3 = {\cdot}23 \text{ nearly.}$$

14. $\sin^2\phi = \dfrac{SY.HZ}{SP.HP} = \dfrac{b^2}{r(2a-r)}$.

$$\therefore \text{ velocity along } SP = V\cos\phi = \sqrt{\mu}\sqrt{\frac{2a-r}{ar}}\sqrt{1-\frac{b^2}{r(2a-r)}}$$

$$= \sqrt{\frac{\mu}{a}}\sqrt{\left(\frac{a^2}{b^2}-1\right)-\left(\frac{b}{r}-\frac{a}{b}\right)^2}.$$

This is a maximum when $\dfrac{b}{r}-\dfrac{a}{b}=0$, *i.e.* $r=\dfrac{b^2}{a}$, so that the particle is then at the end of the latus rectum.

The velocity then along the radius vector $= \sqrt{\dfrac{\mu}{a}} \cdot \dfrac{ae}{b} = \dfrac{2\pi}{T}\dfrac{ae}{\sqrt{1-e^2}}$.

End of Art 86
EXAMPLES

1. Taking the formula (5) of Art. 75, putting $\theta=\frac{\pi}{2}$ and $e=\frac{1}{60}$, the time of describing the lesser half of the orbit bounded by the latus rectum

$$=\frac{2a^{\frac{3}{2}}}{\sqrt{\mu}}\left[2\tan^{-1}\sqrt{\frac{1-\frac{1}{60}}{1+\frac{1}{60}}}-\frac{1}{60}\sqrt{1-\frac{1}{60^2}}\right]$$

$$=\frac{365}{\pi}\left[2\tan^{-1}\left(1-\frac{1}{60}\right)-\frac{1}{60}\right],$$ on neglecting squares of $\frac{1}{60}$.

If $\tan^{-1}\left(1-\frac{1}{60}\right)=\frac{\pi}{4}-\psi$, where ψ is small, we have

$$1-\frac{1}{60}=\frac{1-\tan\psi}{1+\tan\psi}=1-2\psi,$$

on neglecting squares, so that $\psi=\frac{1}{120}$.

$\therefore$ this time $=\frac{365}{\pi}\left[\frac{\pi}{2}-\frac{1}{60}-\frac{1}{60}\right]=\frac{365}{2}\left(1-\frac{1}{15\pi}\right)$,

and the other time $=365-\frac{365}{2}\left(1-\frac{1}{15\pi}\right)=\frac{365}{2}\left(1+\frac{1}{15\pi}\right)$.

3. Taking the result of the previous question, we have

$$\frac{(n+2)^2}{n^2}\cdot\frac{n-1}{n}=1+\frac{3}{n}-\frac{4}{n^3}.$$

This is a maximum when $-\frac{3}{n^2}+\frac{12}{n^4}=0$, *i.e.* when $n=2$. The result then gives $\frac{2}{3\pi}$ of a year as the maximum.

4. The velocity just after the impact $=\frac{M-m}{M+m}v=\left(1-\frac{2m}{M}\right)v$ approx.

Hence $\mu\left[\frac{2}{a(1+e)}-\frac{1}{a'}\right]=\left(1-\frac{2m}{M}\right)^2\cdot\mu\cdot\left[\frac{2}{a(1+e)}-\frac{1}{a}\right]$,

where $2a'$ is the new major axis.

$$\therefore\ a-a'=\frac{4m}{M}\frac{1-e}{1+e}a \text{ approx.}$$

Also $\frac{vT}{\pi}=\sqrt{\frac{\mu}{a}\cdot\frac{1-e}{1+e}}\cdot\frac{2}{\sqrt{\mu}}a^{\frac{3}{2}}=2\sqrt{\frac{1-e}{1+e}}\,a$. Hence, etc.

5. $v^2 = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a}\right] = \frac{\mu}{a}\frac{1-e}{1+e}$. As in Art. 84, $\delta a = \frac{2v\,\delta v\,a^2}{\mu}$,

and $$\delta e = \frac{\delta v}{v}\,\frac{1-e^2}{e}\,\frac{av^2-\mu}{\mu} = \frac{2\delta v}{v}(1-e).$$

Now the shortest distance $x = a - ae$.

$$\therefore\ \delta x = (1-e)\,\delta a - a\,\delta e = \frac{2\delta v}{v}\,a(1-e)\left[\frac{v^2 a}{\mu} + 1\right] = 4\,\frac{\delta v}{v}\,\frac{a(1-e)}{1+e} = \text{as stated.}$$

6. $\frac{\mu+\delta\mu}{\mu} = \frac{M+m}{M}$, so that $\frac{\delta\mu}{\mu} = \frac{m}{M}$, and $v^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a}$.

Hence, by Art. 86,

$$\delta a = -\frac{v^2a^2}{\mu}\cdot\frac{\delta\mu}{\mu} = -\frac{m}{M}a,$$

$$\delta T = -\frac{1}{2}T\frac{\delta\mu}{\mu}\left[1 + \frac{3av^2}{\mu}\right] = -\frac{2m}{M}T,$$

and $$\tan\psi = \frac{HH'\sin H}{SH} = \frac{2\delta a}{2ae}\cdot\frac{b}{a} = -\frac{m}{M}\cdot\frac{b}{ae}.$$

7. $v^2 = \frac{\mu}{a}$. As in Art. 86, we then have

$$\frac{\mu}{a} = \frac{\mu}{n}\left(\frac{2}{a} - \frac{1}{a'}\right),\ \text{and}\ \mu a = h^2 = \frac{\mu}{n}a'(1-e'^2).$$

Hence $$a' = \frac{a}{2-n},\ \text{and}\ e' = n-1.$$

If $n > 2$, the path is an hyperbola, and then, as in Art. 66, we have

$$\frac{\mu}{a} = \frac{\mu}{n}\left(\frac{2}{a} + \frac{1}{a'}\right),\ \text{and}\ \mu a = \frac{\mu}{n}a'(e'^2 - 1).$$

Hence $$a' = \frac{a}{n-2},\ \text{and}\ e' = n-1.$$

8. $n^2\cdot\frac{2\mu}{l} = n^2V^2 = \mu\left(\frac{2}{l} - \frac{1}{a}\right)$, and $nV\frac{l}{\sqrt{2}} = h' = \sqrt{\mu a(1-e^2)}$.

$$\therefore\ 2a = \frac{l}{1-n^2},\ \text{and}\ e^2 = 1 - 2n^2 + 2n^4.$$

9. $\mu\left(\frac{2}{r} - \frac{1}{a}\right) = V^2 = \mu\left(\frac{2}{r} - \frac{1}{a'}\right)$, so that $a' = a$.

$\sqrt{\mu a'(1-e'^2)} = h' = Vr\cos\psi$, where $\sin^2\psi = \frac{SY.\,HZ}{SP.\,HP} = \frac{b^2}{r(2a-r)}$.

$$\therefore\ a^2(1-e'^2) = 2ar - b^2 - r^2,\ \text{so that}\ a^2e'^2 = b^2 + (a-r)^2.$$

Now $2CP^2 + 2a^2e^2 = r^2 + (2a-r)^2$, so that $CP^2 = b^2 + (a-r)^2$. $\therefore\ e' = \frac{CP}{a}$.

10. $V_1^2 = V_2^2 = \frac{2\mu}{R}$.

$$\therefore\ \mu\left(\frac{2}{R} - \frac{1}{a'}\right) = V^2 = \frac{m_1^2V_1^2 + m_2^2V_2^2}{(m_1+m_2)^2} = \frac{m_1^2+m_2^2}{(m_1+m_2)^2}\cdot\frac{2\mu}{R}.$$

$$\therefore\ \frac{1}{a'} = \frac{4m_1m_2}{R(m_1+m_2)^2},\ i.e.\ 2a' = R\,\frac{(m_1+m_2)^2}{2m_1m_2}.$$

11. $V^2=\mu\left(\frac{2}{a}-\frac{1}{a}\right)=\frac{\mu}{a}$, and for a circle at the same distance we have

$$V_1^2=\frac{\mu}{a}.$$

$$\therefore \text{ tangential blow}=m\,[V-V_1 \sin SBT]=m\sqrt{\frac{\mu}{a}}\left(1-\frac{b}{a}\right),$$

and $$\text{normal blow}=mV_1 \cos SBT=m\sqrt{\frac{\mu}{a}}\cdot e. \quad \text{Hence, etc.}$$

12. Let $\angle BSH=\alpha$, so that $\cos\alpha=e$.

$$V^2=\mu\left(\frac{2}{a}-\frac{1}{a}\right)=\frac{\mu}{a}, \text{ and } h=V.b.$$

Then $$V^2+u^2+2Vu\cos\alpha=\mu\left(\frac{2}{a}-\frac{1}{a'}\right)=V^2\left[2-\frac{a}{a'}\right].$$

$$\therefore \frac{a'}{a}=\left(1-\frac{2u\cos\alpha}{V}\right)^{-1}=1+\frac{2u\cos\alpha}{V}, \text{ on neglecting squares of } u.$$

$$\therefore 2a'-2a=\frac{4au\cos\alpha}{V}=\frac{4abue}{h}.$$

Also $\sqrt{\mu a(1-e^2)}$ is constant, since h is unaltered.

$$\therefore \delta a(1-e^2)-2ea\delta e=0. \quad \therefore \delta e=\delta a\frac{1-e^2}{2ae}=\frac{bu}{h}(1-e^2)=\frac{au}{h}(1-e^2)^{\frac{3}{2}}.$$

Also $$1-e\cos\alpha=\frac{l}{SB}=\frac{h^2}{\mu.SB}. \quad \therefore de\cos\alpha-e\sin\alpha\, d\alpha=0.$$

$$\therefore d\alpha=\frac{de}{e}\frac{\cos\alpha}{\sin\alpha}=\frac{au}{eh}(1-e^2)^{\frac{3}{2}}\cdot\frac{e}{\sqrt{1-e^2}}=\frac{au}{h}(1-e^2).$$

13. $SP=r$, so that $V^2=\mu\cdot\frac{2}{r}$.

At the end of the time τ the particle will be at the point Q on the tangent at P, such that $PQ=V.\tau$.

$$\therefore SQ^2=V^2\tau^2+r^2-2V\tau.r\sqrt{1-\frac{SY^2}{SP^2}}=V^2\tau^2+r^2-2V\tau\sqrt{r^2-ar}.$$

The new path is therefore an ellipse, parabola or hyperbola

$$\text{according as } \frac{2\mu}{r} \left(i.e.\ V^2\right) \lesseqgtr \frac{2\mu}{SQ}, \text{ i.e. according as } SQ^2 \lesseqgtr r^2,$$

$$\text{i.e. according as } V^2\tau^2 \lesseqgtr 4(r^2-ar), \text{ i.e. as } \tau \lesseqgtr 2r\sqrt{\frac{r-a}{2\mu}}.$$

14. By Art. 79, the furthest point, X, reached on the tangent at P is reached by a path whose foci are S and a point H on PX, such that $PH=h$. Hence, if $XP=x$, we have

$$R+h=SP+PH=SX+XH=x-h+\sqrt{x^2+R^2}.$$

Hence x.

15. If X be the furthest point on the Earth that can be reached, the path to it has foci S and a point H on PX, such that $PH=h$. But, since SP and SX are equal radii of this path, its major axis must bisect $\angle PSX$, and hence H must bisect PX, and so $PX=2h$. The arc, PX, of the Earth

$$=R.\angle PSX=2R\sin^{-1}\frac{PH}{SP}=2R\sin^{-1}\frac{h}{R}.$$

If $2a$ and $2b$ are the axes of this path, and e its eccentricity, then

$$2a=SP+PH=R+h, \text{ and } 2ae=SH=\sqrt{R^2-h^2}.$$

Hence $$b^2=a^2-a^2e^2=\frac{h}{2}(R+h).$$

If the tangent to the path at P makes an angle ψ with SP produced, and ϕ with the tangent to the Earth at P, then

$$\sin^2\psi=\frac{SY.HZ}{SP.HP}=\frac{b^2}{Rh}=\frac{R+h}{2R},$$

and $$\cos 2\phi=\cos[\pi-2\psi]=-\cos 2\psi=-1+\frac{R+h}{R}=\frac{h}{R}.$$

Hence the angle of elevation necessary $=\frac{1}{2}\cos^{-1}\frac{h}{R}$.

16. Here $$2\sin^{-1}\frac{h}{R}=\frac{\pi}{2}, \text{ so that } h=\frac{R}{\sqrt{2}}.$$

Now, as in Art. 79, $V^2=2\mu\left(\frac{1}{R}-\frac{1}{R+h}\right)=2gR\left[1-\frac{\sqrt{2}}{\sqrt{2}+1}\right]=2gR(\sqrt{2}-1)$.

$\therefore V=\sqrt{64(\sqrt{2}-1).4000.1760.3}$ ft. per sec.

$=\sqrt{\frac{1600}{33}\times \cdot 4142}$ miles per sec. $=\sqrt{20\cdot 08}=4\frac{1}{2}$ miles per sec. approx.

Also $$\phi=\frac{1}{2}\cos^{-1}\frac{1}{\sqrt{2}}=\frac{1}{2}.\frac{\pi}{4}=22\tfrac{1}{2}^\circ, \text{ and } \psi=67\tfrac{1}{2}^\circ.$$

Chapter 5

UNIPLANAR MOTION WHEN THE ACCELERATION IS CENTRAL AND VARYING AS THE INVERSE SQUARE OF THE DISTANCE

65. In the present chapter we shall consider the motion when the central acceleration follows the Newtonian Law of Attraction.

This law may be expressed as follows; between every two particles, of masses m_1 and m_2 placed at a distance r apart, the mutual attraction is

$$\gamma\frac{m_1 m_2}{r^2}$$

units of force, where γ is a constant, depending on the units of mass and length employed, and known as the constant of gravitation.

If the masses be measured in grammes, and the length in centimeters (C.G.S. System) the value of γ is 6.66×10^{-8} approximately, and the attraction is expressed in dynes.

If the masses be measured in kilogrammes, and the length in metres (M.K.S. System) the value of γ is 6.66×10^{-11} approximately, and the attraction is expressed in newtowns.

66. *A particle moves in a path so that its acceleration is always directed to a fixed point and is equal to* $\frac{\mu}{(distance)^2}$*; to show that its path is a conic section and to distinguish between the three cases that arise.*

When $P=\dfrac{\mu}{r^2}$, the equation (5) of Art. 53 becomes

$$\frac{h^2}{p^3}\frac{dp}{dr}=\frac{\mu}{r^2} \qquad ...(1).$$

Integrating we have, by Art. 54,

$$v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}+C \qquad ...(2).$$

Now the (p,r) equation of an ellipse and hyperbola, referred to a focus, are respectively

$$\frac{b^2}{p^2}=\frac{2a}{r}-1, \text{ and } \frac{b^2}{p^2}=\frac{2a}{r}+1 \qquad ...(3),$$

where $2a$ and $2b$ are the transverse and conjugate axes.

Hence, when C is negative, (2) is an ellipse; when C is positive, it is a hyperbola.

Also when $C=0$, (2) becomes $\dfrac{p^2}{r}$ = constant, and this is the (p,r) equation of a parabola referred to its focus.

Hence (2) always represents a conic section, whose focus is at the centre of force, and which is an

$$\left.\begin{array}{r}\text{ellipse}\\ \text{parabola}\\ \text{or hyperbola}\end{array}\right\} \text{according as } C \text{ is } \left.\begin{array}{r}\text{negative}\\ \text{zero}\\ \text{or positive}\end{array}\right\},$$

i.e. according as $v^2 \lesseqqgtr \dfrac{2\mu}{r}$, *i.e.* according as the square of the velocity at any point P is $\lesseqqgtr \dfrac{2\mu}{SP}$, where S is the focus.

Again, comparing equations (2) and (3), we have, in the case of the ellipse, $\dfrac{h^2}{b^2}=\dfrac{\mu}{a}=\dfrac{C}{-1}$.

$$\therefore \; h = \sqrt{\mu \frac{b^2}{a}} = \sqrt{\mu \times \text{semi-latus-rectum}}, \text{ and } C = -\frac{\mu}{a}.$$

Hence, in the case of the ellipse, $v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$ ...(4).

So, for the hyperbola, $v^2 = \mu \left(\frac{2}{r} + \frac{1}{a} \right)$, and, for the parabola,

$$v^2 = \frac{2\mu}{r}.$$

It will be noted that in each case the velocity at any point does not depend on the direction of the velocity.

Since h is twice the area described in the unit of time (Art. 54), therefore, if T be the time of describing the ellipse, we have

$$T = \frac{\text{area of the ellipse}}{\frac{1}{2}h} = \frac{\pi ab}{\frac{1}{2}\sqrt{\mu \frac{b^2}{a}}} = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \qquad ...(5),$$

so that the square of the periodic time varies as the cube of the major axis.

COR. 1. If a particle be projected at a distance R with velocity V in any direction the path is an ellipse, parabola or hyperbola, according as $V^2 <=> \frac{2\mu}{R}$.

Now the square of the velocity that would be acquired in falling from infinity to the distance R, by Art. 31.

$$= 2\int_{\infty}^{R} \left(-\frac{\mu}{r^2} \right) dr = \left[\frac{2\mu}{r} \right]_{\infty}^{R} = \frac{2\mu}{R}.$$

Hence the path is an ellipse, parabola or hyperbola according as the velocity at any point is <=> that acquired in falling from infinity to the point.

COR. 2. The velocity V_1 for the description of a circle of radius R is given by

$$\frac{V_1}{R} = \text{ normal acceleration } = \frac{\mu}{R^2}, \text{ so that } V_1^2 = \frac{\mu}{R}$$

$$\therefore \; V_1 = \frac{\text{Velocity from infinity}}{\sqrt{2}}.$$

67. In the previous article the branch of the hyperbola described is the one nearest the centre of force.

If the central acceleration be from the centre and if it vary as the inverse square of the distance, the further branch is described. For in this case the equation of motion is

$$\frac{h^2}{p^3}\frac{dp}{dr} = -\frac{\mu}{r^2}. \qquad \therefore \; \frac{h^2}{p^2} = -\frac{2\mu}{r} + C \qquad ...(1).$$

Now the (p, r) equation of the further branch of a hyperbola is

$$\frac{b^2}{p^2} = 1 - \frac{2a}{r},$$

and this always agrees with (1) provided that $\dfrac{h^2}{b^2} = \dfrac{\mu}{a} = C$,

so that $h = \sqrt{\mu \times \text{ semi-latus-rectum}}$, and $v^2 = \dfrac{h^2}{p^2} = \mu\left(\dfrac{1}{a} - \dfrac{2}{r}\right)$.

68. *Construction of the orbit given the point of projection and the direction and magnitude of the velocity of projection.*

Let S be the centre of attraction, P the point of projection, TPT' the direction of projection, and V the velocity of projection.

Case I. Let $V^2 < \frac{2\mu}{SP}$; then, by Art.66, the path is an ellipse whose major axis $2a$ is given by the equation $V^2 = \mu\left(\frac{2}{R} - \frac{1}{a}\right)$, where $R = SP$, so that $2a = \frac{2R\mu}{2\mu - V^2 R}$.

Draw PS', so that PS' and PS are on the same side of TPT', making $\angle T'PS' = \angle TPS$, and take

$$PS' = 2a - SP = 2a - R = \frac{V^2R^2}{2\mu - V^2R}.$$

Then S' is the second focus and the elliptic path is therefore known.

Case II. Let $V^2 = \frac{2\mu}{SP}$, so that the path is a parabola. Draw the direction PS' as in Case I; in this case this is the direction of the axis of the parabola. Draw SU parallel to PS' to meet TPT' in U; draw SY perpendicular to TPT' and YA perpendicular to SU. Then A is the vertex of the required parabola whose focus is S, and the curve can be constructed.

The semi-latus-rectum $= 2SA = 2\frac{SY^2}{SP} = \frac{2p_0^2}{R}$, where p_0 is the perpendicular from S on the direction of projection.

Case III. Let $V^2 > \frac{2\mu}{SP}$, so that the path is a hyperbola of transverse axis $2a$ given by the equation

$$V^2 = \mu\left(\frac{2}{R} + \frac{1}{a}\right), \text{ and hence } 2a = \frac{2\mu R}{V^2R - 2\mu}.$$

In this case PS' lies on the opposite side of TPT' from PS, such that $\angle TPS = \angle TPS'$, and $S'P - SP = 2a$, so that

$$S'P = R + 2a = \frac{V^2R^2}{V^2R - 2\mu}.$$

The path can then be constructed, since S' is the second focus.

69. **Kepler's Laws.** The astronomer Kepler, after many years of patient labour, discovered three laws connecting the motions of the various planets about the sun. They are:

1. *Each planet describes an ellipse having the sun in one of its foci.*
2. *The areas described by the radii drawn from the planet to the sun are, in the same orbit, proportional to the times of describing them.*
3. *The squares of the periodic times of the various planets are proportional to the cubes of the major axes of their orbits.*

70. From the second law we conclude, by Art. 54, that the acceleration of each planet, and therefore the force on it, is directed towards the Sun.

From the first law it follows, by Art. 55 or Art. 66, that the acceleration of each planet varies inversely as the square of its distance from the Sun.

From the third law it follows, since from Art. 66 we have

$$T^2 = \frac{4\pi^2}{\mu}.a^3,$$

that the absolute acceleration μ (*i.e.* the acceleration at unit distance from the Sun) is the same for all planets.

Laws similar to those of Kepler have been found to hold for the planets and their satellites.

It follows from the foregoing considerations that we may assume Newton's Law of Gravitation to be true throughout the Solar System.

71. Kepler's Laws were obtained by him, by a process of continually trying hypotheses until he found one that was suitable; he started with the observations made and recorded for many years by Tycho Brahé, a Dane, who lived from A.D. 1546 to 1601.

The first and second laws were enunciated by Kepler in 1609 in his book on the motion of the planet Mars. The third law was announced ten years later in a book entitled *On the Harmonies of the World*. The explanation of these laws was given by Newton in his *Principia* published in the year 1687.

72. Kepler's third law, in the form given in Art. 69, is only true on the supposition that the Sun is fixed, or that the mass of the planet is neglected in comparison with that of the Sun.

A more accurate form is obtained in the following manner.

Let S be the mass of the Sun, P that of any of its planets, and γ the constant of gravitation. The force of attraction between the two is thus $\gamma.\frac{S.P}{r^2}$, where r is the distance between the Sun and planet at any instant.

The acceleration of the planet is then $\alpha\left(=\frac{\gamma S}{r^2}\right)$ towards the Sun, and that of the Sun is $\beta\left(=\frac{\gamma P}{r^2}\right)$ towards the planet.

To obtain the acceleration of the planet relative to the Sun we must give to both an acceleration β along the line PS. The acceleration of the Sun is then zero and that of the planet is $\alpha+\beta$ along PS. If, in addition, we give to each a velocity equal and opposite to that of the Sun we have the motion of P relative to the Sun supposed to be at rest.

The relative acceleration of the planet with respect to the Sun then

$$\alpha+\beta=\frac{\gamma(S+P)}{r^2}.$$

Hence the μ of Art. 66 is $\gamma(S+P)$, and, as in that article we then have

$$T=\frac{2\pi}{\sqrt{\gamma(S+P)}}a^{3/2}.$$

If T_1, be the time of revolution and a_1 the semi-major axis of the relative path of another planet P_1, we have similarly

$$T_1=\frac{2\pi}{\sqrt{\gamma(S+P_1)}}a_1^{3/2}, \quad \therefore \quad \frac{S+P}{S+P_1}\cdot\frac{T^2}{T_1^2}=\frac{a^3}{a_1^3}.$$

Since Kepler's Law, that $\frac{T^2}{T_1^2}$ varies as $\frac{a^3}{a_1^3}$, is very approximately true, it follows that $\frac{S+P_1}{S+P}$ is very nearly unity, and hence that P and P_1 are either very nearly equal or very small compared with S. But it is known that the masses of the planets are very different; hence they must be very small compared with that of the Sun.

73. The corrected formula of the last article may be used to give an approximate value to the ratio of the mass of a planet to that of the Sun in the case where the planet has a small satellite, whose periodic time and mean distance from the planet are known.

In the case of the satellite the attraction of the planet is the force which for all practical purposes determines its path.

If P be the mass of the planet and D its mean distance from the Sun, then, as in the previous article,

$$T = \frac{2\pi}{\sqrt{\gamma(S+P)}} D^{3/2}.$$

Similarly, if p be the mass of the satellite, d its mean distance from the planet, and t its periodic time, then

$$t = \frac{2\pi}{\sqrt{\gamma(P+p)}} d^{3/2}.$$

$$\therefore \quad \frac{S+P}{P+p} \cdot \frac{T^2}{t^2} = \frac{D^3}{d^3}.$$

The quantities T, t, D and d being known, this gives a value for $\dfrac{S+P}{P+p}$.

As a numerical example take the case of the Earth E and the Moon m.

Then
$$\frac{S+E}{E+m} = \frac{t^2}{T^2} \cdot \frac{D^3}{d^3}.$$

Now $T = 365\frac{1}{4}$ days, $t = 27\frac{1}{3}$ days, $D = 149,600,000$ km., and $d = 386,000$ km., all the values being approximate.

$$\therefore \quad \frac{S+E}{E+m} = \left(\frac{27\frac{1}{3}}{365\frac{1}{4}}\right)^2 \times \left(\frac{149,600}{386}\right)^3 = 325,900 \text{ approx.}$$

Therefore $S+E = 325,900$ times the sum of the masses of the Earth and Moon. Also $m = E/81$ nearly.

$$\therefore \quad S = 330,000E \text{ nearly.}$$

This is a fairly close approximation to the accurate result.

If the Sun be assumed to be a sphere of 708,000 km. radius and mean density n times that of the Earth, assumed to be a sphere of 6400 km. radius, this gives

$$n \times 708,000^3 = 330,000 \times 6400^3$$

$$\therefore \; n = \frac{330,000}{110^3} \text{ approx.} = \frac{330}{1331} = \text{ about } \frac{1}{4}.$$

Hence the mean density of the Sun $= \frac{1}{4}$ that of the Earth $= \frac{1}{4} \times 5.527 = 1.4g/cm^3$ approx. so that the mean density of the Sun is nearly half as much again as that of water.

74. It is not necessary to know the mean distance and periodic time of the planet P in order to determine its mass, or rather the sum of its mass and that of its satellite.

For if E and m be the masses of the Earth and Moon, R the distance of the Earth from the Sun, r that of the Moon from the Earth, if Y denote a year and y the mean lunar month, then we have

$$Y = \frac{2\pi}{\sqrt{\gamma(S+E)}}.R^{3/2} \qquad ...(1),$$

$$y = \frac{2\pi}{\sqrt{\gamma(E+m)}}.r^{3/2} \qquad ...(2).$$

Also, as in the last article,

$$t = \frac{2\pi}{\sqrt{\gamma(P+p)}}.d^{3/2} \qquad ...(3).$$

From (1) and (3),

$$(P+p)\frac{t^2}{d^3} = (S+E)\frac{Y^2}{R^3} \qquad ...(4).$$

From (2) and (3),

$$(P+p)\frac{t^2}{d^3} = (E+m)\frac{y^2}{r^3} \qquad ...(5).$$

Equation (4) gives the ratio of $P+p$ to $S+E$.

Equation (5) gives the ratio of $P+p$ to $E+m$.

EXAMPLES

1. Show that the velocity of a particle moving in an ellipse about a centre of force in the focus is compounded of two constant velocities, $\frac{\mu}{h}$ perpendicular to the radius and $\frac{\mu e}{h}$ perpendicular to the major axis.
2. A particle describes an ellipse about a centre of force at the focus; show that, at any point of its path, the angular velocity about the other focus varies inversely as the square of the normal at the point.
3. A particle moves with a central acceleration $\left[=\frac{\mu}{(\text{distance})^2}\right]$; it is projected with velocity V at a distance R. show that its path is a rectangular hyperbola if the angle of projection is

$$\sin^{-1}\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}.$$

4. A particle describes an ellipse under a force $\left[=\frac{\mu}{(\text{distance})^2}\right]$ towards the focus; if it was projected with velocity V from a point distant r from the centre of force, show that its periodic time is

$$\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{V^2}{\mu}\right]^{-3/2}$$

5. If the velocity of the Earth at any point of its orbit, assumed to be circular, were increased by about one-half, prove that it would describe a parabola about the Sun as focus.

   Show also that, if a body were projected from the Earth with a velocity exceeding 11.2 km per second, it will not return to the Earth and may even leave the Solar System.
6. A particle is projected from the Earth's surface with velocity v; show that, if the diminution of gravity be taken into account, but the resistance of the air neglected, the path is an ellipse of major axis $\frac{2ga^2}{2ga - v^2}$, where a is the Earth's radius.
7. Show that an unresisted particle falling to the Earth's surface from a great distance would acquire a velocity $\sqrt{2ga}$, where a is the Earth's radius.

   Prove that the velocity acquired by a particle similarly falling into the Sun is to the Earth's velocity in the square root of the ratio of the diameter of the Earth's orbit to the radius of the Sun.
8. If a planet were suddenly stopped in its orbit, supposed circular, show that it would fall into the Sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.
9. The eccentricity of the Earth's orbit round the Sun is $\frac{1}{60}$; show that the Earth's distance from the Sun exceeds the length of the semi-major axis of the orbit during about 2 days more than half the year.
10. The mean distance of Mars from the Sun being 1.524 times that of the Earth, find the time of revolution of Mars about the Sun.
11. The time of revolution of Mars about the Sun is 687 days and his mean distance $227\frac{1}{2} \times 10^6$ km.; the distance of the Satellite

Deimos from Mars is 23,500 km and its time of revolution 30 hrs. 18 mins.; show that the mass of the Sun is a little more than three million times that of Mars.

12. The time of revolution of Jupiter about the Sun is 18.6 years and its mean distance 777×10^6 km; the distance of his first satellite is 420,000 km, and its time of revolution 1 day $18\frac{1}{2}$ hrs.; show that the mass of Jupiter is a little less than one-thousandth of that of the Sun.

13. The outer satellite of Jupiter revolves in $16\frac{2}{3}$ days approximately, and its distance from the planet's centre is $16\frac{1}{2}$ radii of the latter. The last discovered satellite revolves in 12 hours nearly; find its distance from the planet's centre.

    Find also the approximate ratio of Jupiter's mean density to that of the Earth, assuming that the Moon's distance is 60 times the Earth's radius and that her siderial period is $27\frac{1}{3}$ days nearly.

    [Use equations (2) and (3) of Art. 74, and neglect m in comparison with E, and p in comparison with P.]

14. A planet is describing an ellipse about the Sun as focus; show that its velocity away from the Sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is $\dfrac{2\pi ae}{T\sqrt{1-e^2}}$ where $2a$ is the major axis, e the eccentricity, and T the periodic time.

75. *To find the time of description of a given arc of an elliptic orbit starting from the nearer end of the major axis.*

The equation $r^2\dfrac{d\theta}{dt} = h$ of Art. 53 gives

$$ht = \int_0^\theta r^2 d\theta = \int_0^\theta \frac{l^2}{(1+e\cos\theta)^2} d\theta \qquad ...(1).$$

If $e > 1$, then by the well-known result in Integral Calculus

$$\int_0^\theta \frac{d\theta}{1+e\cos\theta} = \frac{2}{\sqrt{1-e^2}} \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2}\right] \qquad ...(2).$$

Differentiating with respect to the constant e we have

$$\int_0^\theta \frac{-\cos\theta}{(1+e\cos\theta)^2} d\theta$$

$$= \frac{2e}{(1-e^2)^{3/2}} \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2}\right] - \frac{1}{1-e^2} \frac{\sin\theta}{1+e\cos\theta} \qquad ...(3).$$

$$\therefore \quad \int_0^\theta \frac{1}{(1+e\cos\theta)^2} d\theta$$

$$= \int_0^\theta \left[\frac{1}{1+e\cos\theta} - \frac{e\cos\theta}{(1+e\cos\theta)^2}\right] d\theta$$

$$= \frac{2}{(1-e^2)^{3/2}} \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2}\right] - \frac{e}{1-e^2} \frac{\sin\theta}{1+e\cos\theta} \qquad ...(4)$$

Hence, since $\dfrac{l^2}{h} = \dfrac{l^2}{\sqrt{\mu l}} = \dfrac{a^{3/2}(1-e^2)^{3/2}}{\sqrt{\mu}}$, we have, by (1),

$$t = \frac{a^{3/2}}{\sqrt{\mu}} \left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan\frac{\theta}{2}\right) - e\sqrt{1-e^2} \frac{\sin\theta}{1+e\cos\theta}\right] \quad ...(5)$$

ALITER. If we change the variable θ into a new variable ϕ given by the relation

$(1+e\cos\theta)(1-e\cos\phi) = 1-e^2$, so that

$$\cos\theta = \frac{\cos\phi - e}{1-e\cos\phi}, \qquad \sin^2\theta = \frac{(1-e^2)\sin^2\phi}{(1-e\cos\phi)^2},$$

and $\sin\theta.d\theta = \dfrac{\sin\phi(1-e^2)}{(1-e\cos\phi)^2}d\phi$, we have $d\theta = \dfrac{\sqrt{1-e^2}}{1-e\cos\phi}d\phi$.

Hence

$$\int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2} = \int_0^\phi \frac{1-e\cos\phi}{(1-e^2)^{3/2}}d\phi = \frac{1}{(1-e^2)^{3/2}}[\phi - e\sin\phi] \text{ ...(6)}$$

Now $\tan^2\dfrac{\phi}{2} = \dfrac{1-\cos\phi}{1+\cos\phi} = \dfrac{1-e}{1+e}\,\dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1-e}{1+e}\tan^2\dfrac{\theta}{2}$, and

$$\sin\phi = \frac{\sqrt{1-e^2}}{1+e\cos\theta}\ \sin\theta$$

Substituting in (6) we have result (4), and proceed as above.

76. *To find the time similarly for a hyperbolic orbit.*

If $e > 1$, then $\displaystyle\int_0^\theta \frac{d\theta}{1+e\cos\theta} = \frac{1}{\sqrt{e^2-1}}\log\frac{\sqrt{e+1}+\sqrt{e-1}\tan\frac{\theta}{2}}{\sqrt{e+1}-\sqrt{e-1}\tan\frac{\theta}{2}}$.

Differentiating with respect to e, we have,

$$\int_0^\theta \frac{-\cos\theta}{(1+e\cos\theta)^2}d\theta$$

$$= \frac{-e}{(e^2-1)^{3/2}}\log\left[\frac{\sqrt{e+1}+\sqrt{e-1}\tan\frac{\theta}{2}}{\sqrt{e+1}-\sqrt{e-1}\tan\frac{\theta}{2}}\right] + \frac{1}{e^2-1}\frac{\sin\theta}{1+e\cos\theta}.$$

$$\therefore \int \frac{d\theta}{(1+e\cos\theta)^2}$$

$$= \int_0^\theta \left[\frac{1}{1+e\cos\theta} - \frac{e\cos\theta}{(1+e\cos\theta)^2}\right] d\theta$$

$$= -\frac{1}{(e^2-1)^{3/2}} \log\left[\frac{\sqrt{e+1}+\sqrt{e-1}\tan\dfrac{\theta}{2}}{\sqrt{e+1}-\sqrt{e-1}\tan\dfrac{\theta}{2}}\right] + \frac{1}{e^2-1}\frac{\sin\theta}{1+e\cos\theta}$$

Hence, since in this case $\dfrac{l^2}{h} = \dfrac{l^{3/2}}{\sqrt{\mu}} = \dfrac{a^{3/2}(e^2-1)^{3/2}}{\sqrt{\mu}}$,

the equation (1) of the last article gives

$$t = \frac{a^{3/2}}{\sqrt{\mu}}\left[e\sqrt{e^2-1}\frac{\sin\theta}{1+e\cos\theta} - \log\frac{\sqrt{e+1}+\sqrt{e-1}\tan\dfrac{\theta}{2}}{\sqrt{e+1}-\sqrt{e-1}\tan\dfrac{\theta}{2}}\right]$$

ALITER. Change the variable θ into a new variable ϕ such that $(1+e\cos\theta)(e\cosh\phi-1) = e^2-1$, so that

$$\cos\theta = \frac{e\cosh\phi}{e\cosh\phi-1}, \sin^2\theta = \frac{(e^2-1)\sinh^2\phi}{(e\cosh\phi-1)^2}, \text{and } d\theta = \frac{\sqrt{e^2-1}}{e\cosh\phi-1}.$$

Then $\displaystyle\int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}$

$$= \frac{1}{(e^2-1)^{3/2}}\int_0^\phi (e\cosh\phi-1)d\phi$$

$$= \frac{1}{(e^2-1)^{3/2}}[e\sinh\phi-\phi].$$

Now $\tanh^2\dfrac{\phi}{2} = \dfrac{\cosh\phi-1}{\cosh\phi+1} = \dfrac{e-1}{e+1}\tanh^2\dfrac{\theta}{2}$, and

$$\sin\phi = \frac{2\tanh\dfrac{\phi}{2}}{1-\tanh^2\dfrac{\phi}{2}} = \sqrt{e^2-1}\frac{\sin\theta}{1+e\cos\theta}.$$

$$\therefore \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}$$

$$= \frac{e}{e^2-1}\frac{\sin\theta}{1+e\cos\theta} - \frac{2}{(e^2-1)}\tanh^{-1}\left[\sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}\right],$$

which is the same as before, since $\tanh^{-1}x = \frac{1}{2}\log\frac{1+x}{1-x}$.

77. *In the case of a parabolic orbit to find the corresponding time.*

The equation to the parabola is $r = \dfrac{d}{1+\cos\theta}$, where $2d$ is the latus-rectum and is measured from the axis. Hence the equation (3) of Art. 53 gives

$$h.t = \int r^2 d\theta = \int \frac{d^2}{(1+\cos\theta)^2}d\theta.$$

$$\therefore \frac{ht}{d^2} = \int_0^\theta \frac{d\theta}{4\cos^4\frac{\theta}{2}}$$

$$= \frac{1}{4}\int_0^\theta \sec^2\frac{\theta}{2}.\sec^2\frac{\theta}{2}d\theta$$

$$= \frac{1}{2}\int_0^\theta \left(1+\tan^2\frac{\theta}{2}\right)d\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\left[\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right].$$

But $\dfrac{h}{d^2} = \dfrac{\sqrt{\mu d}}{d^2} = \dfrac{\sqrt{\mu}}{d^{3/2}}$.

$\therefore \quad t = \dfrac{d^{3/2}}{2\sqrt{\mu}}\left[\tan\dfrac{\theta}{2} + \dfrac{1}{3}\tan^3\dfrac{\theta}{2}\right] = \sqrt{\dfrac{2a^3}{\mu}}\left[\tan\dfrac{\theta}{2} + \dfrac{1}{3}\tan^3\dfrac{\theta}{2}\right]$ if a be the apsidal distance.

78. *Motion of a projectile, vacations of gravity being taken into consideration but the resistance of the air being neglected.*

The attraction of the Earth at a point outside it at a distance r from the centre is $\frac{\mu}{r^2}$. Hence the path of a projectile in vacuo is one of the cases of Art. 66, one of the foci of the path described being at the centre of the Earth.

If R be the radius of the Earth, then $\frac{\mu}{R^2}$ = the value of gravity at the surface of the Earth = g, so that $\mu = gR^2$.

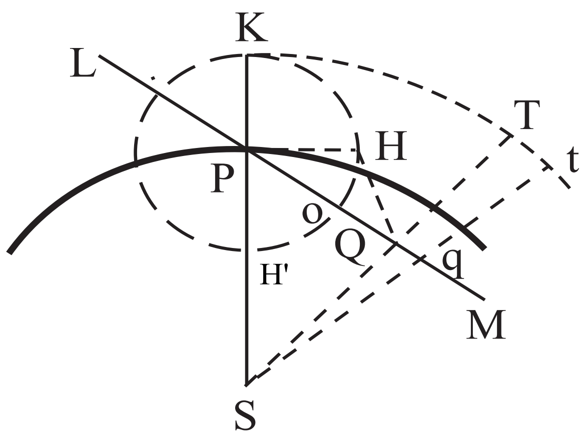
The path of a projectile which starts from a point on the Earth's surface is therefore an ellipse, parabola, or hyperbola according as $V^2 \lesseqqgtr \frac{2\mu}{R}$, *i.e.* according as $V^2 \lesseqqgtr 2gR$.

79. The maximum range of a particle starting from the Earth's surface with a given velocity may be obtained as follows :

Let S be the centre of the Earth and P the point of projection. Let K be the point vertically above P to which the velocity, V, of projection is due, so that, by Art. 31, we have

$$V^2 = 2\mu\left[\frac{1}{SP} - \frac{1}{SK}\right] = 2gR^2\left[\frac{1}{R} - \frac{1}{R+h}\right] \qquad ...(1),$$

where R is the radius of the Earth and PK is h.



If H be the second focus of the path, the semi-major axis is $\frac{1}{2}(R+PH)$.

Hence, by equation (4) of Art. 66,

$$V^2 = \mu\left[\frac{2}{SP} - \frac{2}{R+PH}\right] = 2gR^2\left[\frac{1}{R} - \frac{1}{R+PH}\right].$$

By comparing this with equation (1) we have $PH = h$, so that the locus of the second focus is, for a constant velocity of projection, a circle whose centre is P and radius h. It follows that the major axis of the path is SP + PH or SK.

The ellipse, whose foci are S and H, meets a plane LPM, passing through the point of projection, in a point Q, such that $SQ+QH = SK$. Hence, if SQ meet in T the circle whose centre is S and radius SK, we have $QT = QH$. Since there is, in general, another point, H', on the circle of foci equidistant with H from Q, we have, in general, two paths for a given range.

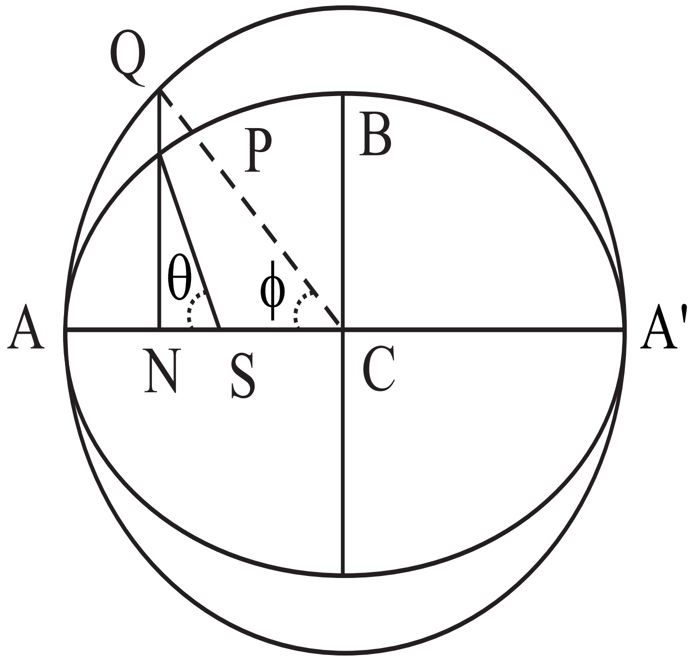
The greatest range on the plane LPM is clearly Pq where qt equals qO. Hence $Sq+qP = Sq+qO+OP = Sq+qt+PK = SK+PK$.

Therefore q lies on an ellipse, whose foci are the centre of the Earth and the point of projection, and which passes through K.

Hence we obtain the **maximum range.**

80. Suppose that the path described by a planet P about the Sun S is the ellipse of the figure. Draw PN perpendicular to the major axis and produce it to meet the auxiliary circle in Q. Let C be the centre.

The points A and A' are called respectively the Perihelion and Aphelion of the path of the planet.



The angle ASP is called the True Anomaly and the angle ACQ the Eccentric Anomaly. In the case of any of the planets the eccentricity of the path is small, being never as large as 0.1 except in the case of Mercury when it is 0.2; the foci of the path are therefore very near C, the ellipse differs little in actual shape from the auxiliary circle, and hence the difference between the, True and Eccentric Anomaly is a small quantity.

If $\dfrac{2\pi}{n}$ be the time of a complete revolution of the planet, so that n is its mean angular velocity, then nt is defined to be the Mean Anomaly and n is the Mean Motion. It is clear therefore that nt would be the Anomaly of an imaginary planet which moved so that its angular velocity was equal to the mean angular velocity of P.

Since $$\frac{2\pi}{n} = \frac{2\pi}{\sqrt{\mu}} a^{3/2} (\text{Art. } 66), \qquad \therefore n = \frac{\sqrt{\pi}}{a^{3/2}}.$$

Let θ be the True Anomaly ASP, and ϕ the Eccentric Anomaly ACQ.

If h be twice the area described in a unit of time, then

$$\begin{aligned}\frac{h}{2}t &= \text{Sectorial area } ASP\\ &= \text{Curvilinear area } ANP + \text{ triangle } SNP\\ &= \frac{b}{a}\times \text{ Curvilinear area } ANQ + \text{ triangle } SNP\\ &= \frac{b}{a}\times \text{ (Sector } ACQ \text{ - triangle } ANQ) + \frac{1}{2}SN.NP\\ &= \frac{b}{a}\left(\frac{1}{2}a^2\phi - \frac{1}{2}a^2\sin\phi\cos\phi\right) + \frac{1}{2}(a\cos\phi - ae).b\sin\phi\\ &= \frac{ab}{2}(\phi - e\sin\phi).\end{aligned}$$

By the polar equation to a Conic Section, we have

$$SP = \frac{l}{1+e\cos\theta} = \frac{a(1-e^2)}{1+e\cos\theta}, \text{ and } SP = a - e\,CN = a(1-e\cos\phi).$$

$$\therefore\ (1-e\cos\phi)(1+e\cos\theta) = 1-e^2,$$

$$\text{and}\quad \therefore\ \cos\theta = \frac{\cos\phi - e}{l - e\cos\phi} \qquad ...(2).$$

81. If e be small, a first approximation from (1) to the value of ϕ is nt, and a second approximation is $nt + e\sin nt$.

From (2), a first approximation to the value of θ is ϕ, and a second approximation is $\phi + \lambda$ where

$$\cos\phi - \lambda\sin\phi = \frac{\cos\phi - e}{1 - e\cos\phi}, \text{ and}$$

$$\therefore\ \lambda = \frac{e\sin\phi}{1-e\cos\phi} = e\sin\phi \quad \text{approx.}$$

Hence, as far as the first power of e,

$$\theta = \phi + e\sin\phi = nt + e\sin nt + e\sin(nt + e\sin nt) = nt + 2e\sin nt.$$

Also $SP = \dfrac{a(1-e^2)}{1+e\cos\theta} = a(1-e\cos\theta)$, to the same approximation,

$$= a - ae\cos(nt + 2e\sin nt) = a - ae\cos nt.$$

If an approximation be made as far as squares of e, the results are found to be

$\phi = nt + e\sin nt + \dfrac{e^2}{2}\sin 2nt$, $\theta = nt + 2e\sin nt + \dfrac{5e^2}{4}\sin 2nt$, and

$r = a\left\{1 - e\cos nt + \dfrac{e^2}{2}(1-\cos 2nt)\right\}.$

82. From equation (2) of Art. 80, we have

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{(1+e)(1-\cos\phi)}{(1-e)(1+\cos\phi)} = \frac{1+e}{1-e}\tan^2\frac{\phi}{2}$$

so that

$\phi = 2\tan^{-1}\left[\sqrt{\dfrac{1-e}{1+e}}\tan\dfrac{\theta}{2}\right]$, and

$$\sin\phi = \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{2\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}}{1+\frac{1-e}{1+e}\tan\frac{\theta}{2}} = \sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}.$$

Hence, from equation (1) of the same article, remembering that $n = \sqrt{\mu}/a^{3/2}$, we have

$$t = \frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right\} - e\sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}\right].$$

This is the result of Art. 75 and gives the time of describing any arc of the ellipse, starting from perihelion.

83. When a particle is describing an elliptic orbit, it may happen that at some point of the path it receives an impulse so that it describes another path; or the strength of the centre of force may be altered so that the path is altered. To obtain the new orbit we shall want to know how the major axis has been altered in magnitude and position, what is the new eccentricity, etc.

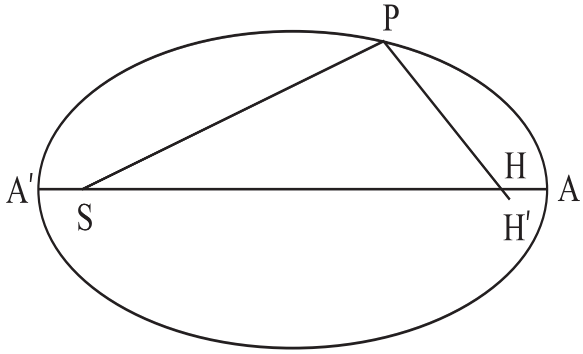
The new orbit will not necessarily be an ellipse and the student will find it a useful exercise to examine the various cases for himself. Similar consideration apply when the initial orbit is a circle, hyperbola, or parabola.

84. **Tangential disturbing force.**

Let APA' be the path of a particle moving about a centre of force at S, and let H be the other focus.

When the particle has arrived at P let its velocity be changed to $v+u$, the direction being unaltered; let $2a'$ be the new major axis. Then we have

$$v^2 = \mu\left[\frac{2}{SP} - \frac{1}{a}\right];\ (v+u)^2 = \left[\frac{2}{SP} - \frac{1}{a'}\right] \qquad ...(1).$$



Hence, by subtraction, we have $\frac{1}{a'}$ Since the direction of motion is unaltered at P, the new focus lies on PH; and, if H' be its position, we have $HH' = (H'P+SP)-(HP+SP) = 2a'-2a$.

If the change of velocity u be small and equal to δv, say, then by differentiating the first of equations (1) we have $2v\delta v = \dfrac{\mu}{a^2}\delta a$.

[For SP is constant as far as these instantaneous changes are concerned.]

Hence δa, the increase in the semi-major axis $= \dfrac{2v.\delta v.a^2}{\mu}$ ...(2).

Again, since HH' is now small, we have

$$\tan HSH' = \frac{HH' \sin H}{2ae + HH' \cos H} = \frac{2\delta a.\sin H}{2ae}.$$

Hence $\delta\psi$, the angle through which the major axis moves

$$= HSH' = \frac{\delta a \sin H}{ae} = \frac{2va}{e\mu}.\sin H.\delta v \qquad ...(3).$$

Since the direction of motion at P is unaltered by the blow, the value of h is altered in the ratio $\dfrac{v+\delta v}{v}$ so that $\delta h = \dfrac{\delta v}{v}h$.

But $h^2 = \mu a(1-e^2)$.

$\therefore \ 2hdh = \mu\delta a(a-e^2) - \mu a.2e\delta e.$

$\therefore \ \mu a.2e\delta e = 2v\delta v.a^2(1-e^2) - 2\dfrac{\delta v}{v}h^2,$

$$\text{so that} \quad \delta e = \frac{\delta v}{v}.\frac{(1-e^2)}{e}.\frac{av^2-\mu}{\mu} \qquad ...(4).$$

This gives the increase in the value of the eccentricity.

Since the periodic time $T = \dfrac{2\pi}{\sqrt{\mu}}a^{3/2}$.

$$\therefore \ \frac{\delta T}{T} = \frac{3}{2}\frac{\delta a}{a} = \frac{3va\delta v}{\mu} \qquad ...(5).$$

85. If the disturbing force is not tangential, the velocity it produces must be compounded with the velocity in the orbit to give the new velocity and tangent at the point P. The equations (1) or (2) of the last article now give the magnitude, $2a'$, of the new major axis.

Also since the moment of the velocity of the point P about the focus S is equal to

$$\sqrt{\mu \times \text{ semi-latus-rectum}},\ \textit{i.e.}\ \text{ to }\ \mu\sqrt{a'(1-e'^2)}$$

we obtain the new eccentricity.

Finally by drawing a line making with the new tangent at P an angle equal to that made by SP, and taking on it a point H', such that $SP+H'P$ is equal to the new major axis, we obtain the new second focus and hence the new position of the major axis of the orbit.

86. *Effect on the orbit of an instantaneous change in the value of the absolute acceleration* μ.

When the particle is at a distance r from the centre of force, let the value of μ be instantaneously changed to μ', and let the new values of the major axis and eccentricity be $2a'$ and e'.

Since the velocity is instantaneously unaltered in magnitude, we have

$$\mu\left(\frac{2}{r}-\frac{1}{a}\right)=v^2=\mu'\left(\frac{2}{r}-\frac{1}{a'}\right) \qquad ...(1),$$

an equation to give a'.

The moment of the velocity about S being unaltered, h remains the same, so that

$$\sqrt{\mu a(1-e^2)}=h=\sqrt{\mu' a'(1-e'^2)} \qquad ...(2),$$

giving e'.

The direction of the velocity at distance r being unaltered, we obtain the new positions of the second focus and of the new major axis as in the previous article.

If the change $\delta\mu$ in μ, be very small the change δa in a is obtained by differentiating the first equation in (1), where v and r are treated as constants, and we have $\dfrac{\delta a}{a^2} = -v^2.\dfrac{\delta\mu}{\mu^2}$.

So, from (2), we have, on taking logarithmic differentials,

$$\frac{\delta\mu}{\mu}+\frac{\delta a}{a}-\frac{2e\delta e}{1-e^2}=0. \quad \therefore \quad \frac{2e\delta e}{1-e^2}=\frac{\delta\mu}{\mu}-\frac{v^2a}{\mu^2}\delta\mu=\frac{\delta\mu}{\mu}\left(1-\frac{v^2a}{\mu}\right).$$

Again, since the periodic time $T=\dfrac{2\pi}{\sqrt{\mu}}a^{3/2}$,

$$\therefore \quad \frac{dT}{T}=\frac{3}{2}\frac{\delta a}{a}-\frac{1}{2}\frac{\delta\mu}{\mu}=-\frac{1}{2}\frac{\delta\mu}{\mu}\left(1+\frac{3av^2}{\mu}\right).$$

EXAMPLES

1. If the period of a planet be 365 days and the eccentricity e is $\dfrac{1}{60}$, show that the times of describing the two halves of the orbit, bounded by the latus rectum passing through the centre of force, are $\dfrac{365}{2}\left[1+\dfrac{1}{15\pi}\right]$ very nearly.
2. The perihelion distance of a comet describing a parabolic path is $\dfrac{1}{n}$ of the radius of the Earth's path supposed circular; show that the time that the comet will remain within the Earth's orbit is

$$\frac{2}{3\pi}.\frac{n+2}{n}.\sqrt{\frac{n-1}{2n}} \text{ of a year.}$$

[If S be the Sun, a the radius of the Earth's path, A the perihelion of the comet's path, and P the intersection of the paths of the earth and comet, then $a = SP = \dfrac{\frac{2a}{n}}{1+\cos\theta}$, so that $\cos\theta = \dfrac{2}{n} - 1$, and therefore $\tan\dfrac{\theta}{2} = \sqrt{n-1}$.

Now use the formula of Art. 77, remembering that $\dfrac{2\pi}{\sqrt{\mu}}a^{3/2} =$ one year.]

3. The Earth's path about the Sun being assumed to be a circle, show that the longest time that a comet, which describes a parabolic path, can remain within the Earth's orbit is $\dfrac{2}{3\pi}$ of a year.
4. A planet, of mass M and periodic time T, when at its greatest distance from the Sun comes into collision with a meteor of mass m, moving in the same orbit in the opposite direction with velocity v; if $\dfrac{m}{M}$ be small, show that the major axis of the planet's path is reduced by
$$\frac{4m}{M}.\frac{vT}{\pi}\sqrt{\frac{1-e}{1+e}}.$$
5. When a periodic comet is at its greatest distance from the Sun its velocity v is increased by a small quantity δv. Show that the comet's least distance from the Sun is increased by the quantity
$$4\delta v.\left\{\frac{a^3(1-e)}{\mu(1+e)}\right\}^{1/2}.$$
6. A small meteor, of mass m, falls into the Sun when the Earth is at the end of the minor axis of its orbit; if M be the mass of the Sun, show that the major axis of the Earth's orbit is lessened by $2a\dfrac{m}{M}$, that the periodic time is lessened by $\dfrac{2m}{M}$ of a year, and that the major axis of its orbit is turned through an angle $\dfrac{b}{ae}.\dfrac{m}{M}$.

7. The Earth's present orbit being taken to be circular, find what its path would be if the Sun's mass were suddenly reduced to $\frac{1}{n}$ of what it is now.
8. A comet is moving in a parabola about the Sun as focus; when at the end of its latus-rectum its velocity suddenly becomes altered in the ratio $n : 1$, where $n < 1$; show that the comet will describe an ellipse whose eccentricity is $\sqrt{1-2n^2+2n^4}$, and whose major axis is $\frac{l}{1-n^2}$, where $2l$ was the latus-rectum of the parabolic path.
9. A body is moving in an ellipse about a centre of force in the focus; when it arrives at P the direction of motion is turned through a right angle, the speed being unaltered; show that the body will describe an ellipse whose eccentricity varies as the distance of P from the centre.
10. Two particles, of masses m_1 and m_2, moving in co-planar parabolas round the Sun, collide at right angles and coalesce when their common distance from the Sun is R. Show that the subsequent path of the combined particles is an ellipse of major axis
$$\frac{(m_1+m_2)^2}{2m_1m_2}R.$$
11. A particle is describing an ellipse under the action of a force to one of its foci. When the particle is at one extremity of the minor axis a blow is given to it and the subsequent orbit is a circle; find the magnitude and direction of the blow.
12. A particle m is describing an ellipse about the focus with angular momentum mh, and when at the end of the minor axis receives a small impulse mu along the radius vector to the focus. Show that the major axis of the path is altered by $\frac{4abeu}{h}$, that the eccentric-

ity is altered by $\frac{ua}{h}(1-e^2)^{3/2}$, and that the major axis is turned through the angle $\frac{au(1-e^2)}{h}$, where a, b are the semi-axes and e the eccentricity of the ellipse.

13. A particle is describing a parabolic orbit (latus-rectum $4a$) about a centre of force (μ) in the focus, and on its arriving at a distance r from the focus moving towards the vertex the centre of force ceases to act for a certain time τ. When the force begins again to operate prove that the new orbit will be an ellipse, parabola or hyperbola according as

$$\tau <=> 2r\sqrt{\frac{r-a}{2\mu}}.$$

14. Show that the maximum range of a projectile on a horizontal plane through the point of projection is $2h.\frac{R+h}{R+2h'}$, where R is the radius of the Earth, and h is the greatest height to which the projectile can be fired. [Use the the result of Art. 79.]

15. When variations of gravity and the spherical shape of the Earth are taken into account, show that the maximum range attainable by a gun placed at the sea level is $2R\sin^{-1}\left(\frac{h}{R}\right)$, and that the necessary angle of elevation is $\frac{1}{2}\cos^{-1}\left(\frac{h}{R}\right)$, where R is the Earth's radius and h is the greatest height above the surface to which the gun can send the ball.

16. Show that the least velocity with which a body must be projected from the Equator of the Earth so as to hit the surface again at the North Pole is about 7.2 km per second, and that the corresponding direction of projection makes an angle of $67\frac{1}{2}^o$ with the vertical at the point of projection.

ANSWERS WITH HINTS

Art. 74 EXAMPLES.

10. 686.7 days **13.** 2.56; .23

Art. 86 EXAMPLES.

7. Hyperbola, $n > 2$