7 Applications of Definite Integration

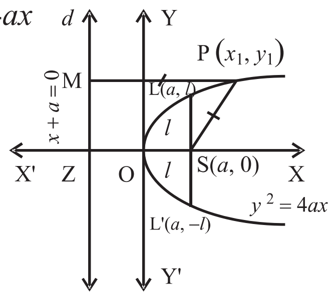
Introduction

The theory of integration has a large variety of applications in Science and Engineering. In this chapter we shall use integration for finding the area of a bounded region. For this, we first draw the sketch (if possible) of the curve which encloses the region. For evaluation of area bounded by the certain curves, we need to know the nature of the curves and their graphs.

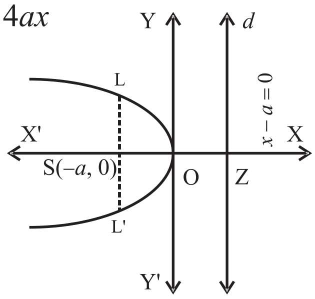
The shapes of different types of curves are discussed below.

7.1 Standard forms of parabola & their shapes

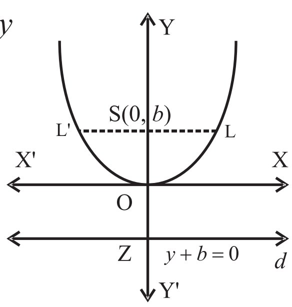
1. $y^2 = 4ax$



2. $y^2 = -4ax$



3. $x^2 = 4by$



4. $x^2 = -4by$

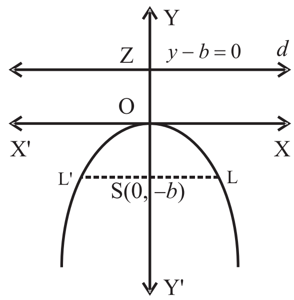


Fig. 7.1

7.2 Standard forms of ellipse

1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad (a > b)$

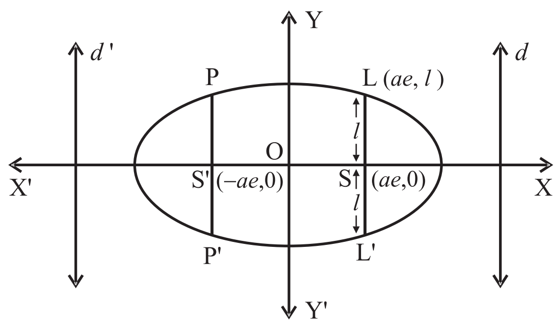


Fig. 7.2

2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad (a < b)$

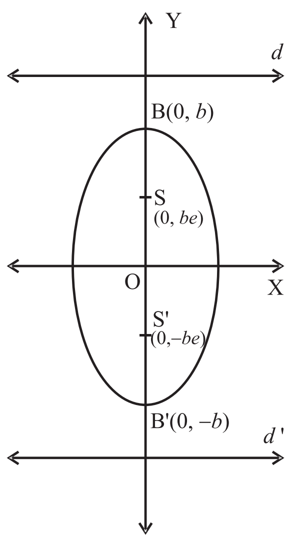


Fig.7.3

7.3 Area under the curve

To find the area under the curve, we state only formulae without proof.

(1) The area "A" bounded by the curve $y = f(x)$, X-axis and bounded between the lines $x = a$ and $x = b$ (fig 7.4) is given by

A = Area of the region PRSQ

$$= \int_{a}^{b} y\,dx = \int_{x=a}^{x=b} f(x)\,dx$$

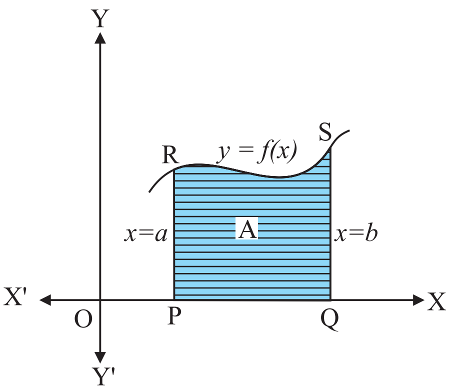


Fig. : 7.4

(2) The area A bounded by the curve $x = g(y)$, Y-axis and bounded between the lines $y = c$ and $y = d$ (Fig. 7.5) is given by

$$A = \int_{c}^{d} x.dy = \int_{y=c}^{y=d} g(y)\,dy$$

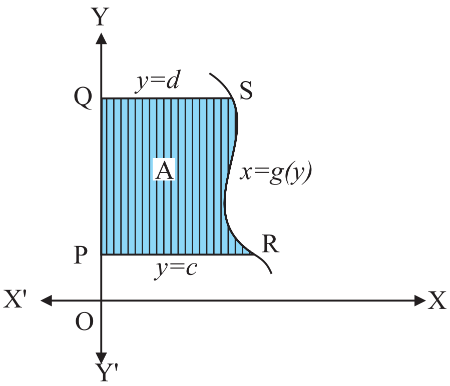


Fig. : 7.5

(3) The area of the shaded region bounded by two curves $y = f(x)$, $y = g(x)$ as shown in fig. 7.6 is obtained by

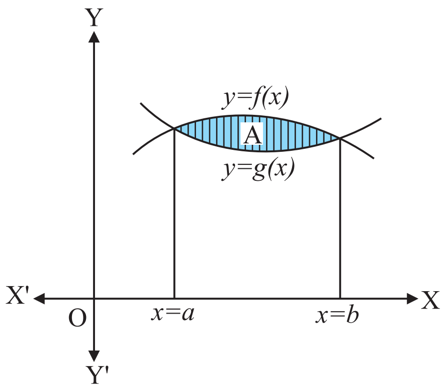


Fig. : 7.6

$$A = \left| \int_{x=a}^{x=b} f(x)\,dx - \int_{x=a}^{x=b} g(x)\,dx \right|$$

where the curve $y = f(x)$ and $y = g(x)$ intersect at points $(a, f(a))$ and $(b, f(b))$.

Remarks:

(i) If the curve under consideration is below the X-axis, then the area bounded by the curve, X-axis and lines $x = a$, $x = b$ is negative (fig. 7.7).

We consider the absolute value in this case.

Thus, required area $= \left| \int_{x=a}^{x=b} f(x)\,dx \right|$

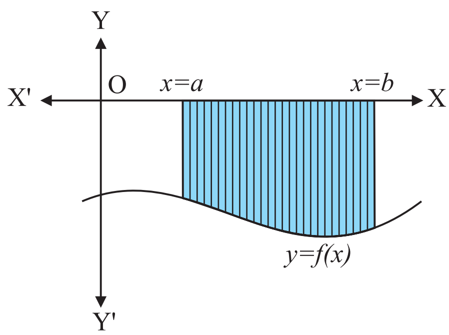


Fig. : 7.7

(ii) The area of the portion lying above the X-axis is positive.

(iii) If the curve under consideraion lies above as well as below the X-axis, say A_1 lies below X-axis and A_2 lies above X-axis (as in Fig. 7.8), then A, the area of the region is given by,

$A = A_1 + A_2$

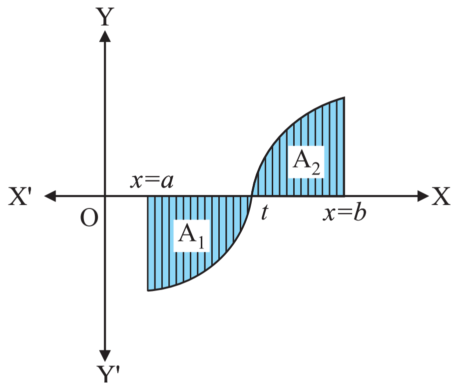


Fig. : 7.8

$$A_1 = \left|\int_a^t f(x)\,dx\right| \quad and \quad A_2 = \int_t^b f(x)\,dx$$

Area A bounded by the curve $y = 2x$, X-axis and lines $x = -2$ and x = 4 is $A_1 + A_2$.

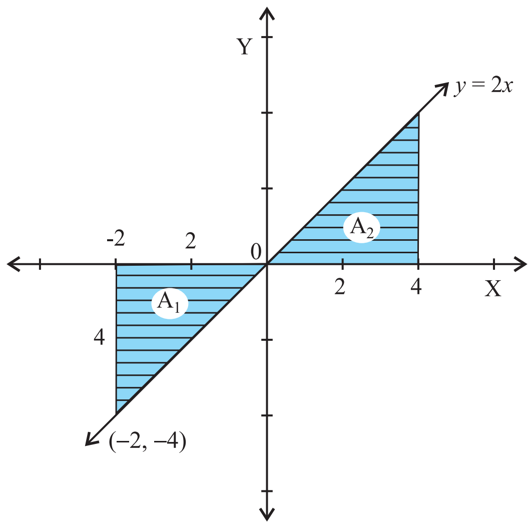


Fig. : 7.9

$$|A_1| = \int_{x=-2}^{0} y\,dx = \left|\int_{-2}^{0} (2x)\,dx\right|$$

$$= \left|2.\int_{-2}^{0} x\,dx\right|$$

$$= \left|\left[2.\frac{x^2}{2}\right]_{-2}^{0}\right|$$

$$= |0-4| = 4 \text{ sq. units}$$

$$A_2 = \int_0^4 2x\,dx = 2\left[\frac{x^2}{2}\right]_0^4 = (4^2-0^2) = 16-0 = 16$$

$A = A_1 + A_2 = 4 + 16 = 20$ sq. units

SOLVED EXAMPLES

1. Find the area of the regions bounded by the following curves, the X-axis and the given lines.

(a) $y = x^2, \ x = 1, \ x = 3$

(b) $y^2 = 4x, \ x = 1, \ x = 4$

(c) $y = -2x, \ x = -1, \ x = 2$

Solution: Let A denote the required area in each case.

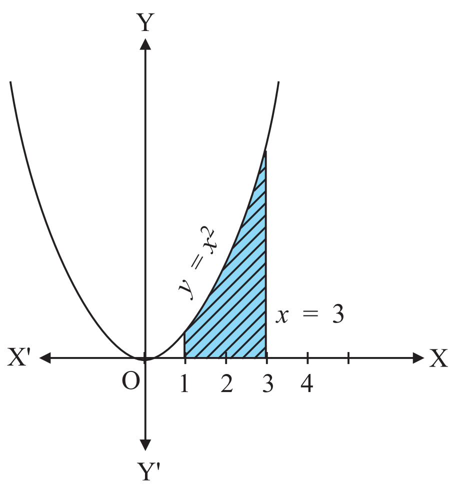


Fig. : 7.10

(a) $A = \int_1^3 y\,dx$

$= \int_1^3 x^2\,dx$

$= \frac{1}{3}[x^3]_1^3 = \frac{1}{3}(3^3 - 1^3) = \frac{1}{3}(27-1)$

$= \frac{26}{3}$ sq. units

(b) $A = \int_1^4 y\,dx$

$= \int_1^4 2\sqrt{x}\,dx$

$= 2.\frac{2}{3}[x^{3/2}]_1^4 = \frac{4}{3}(4^{3/2} - 1^{3/2})$

$= \frac{4}{3}(8-1) = \frac{28}{3}$ sq. units

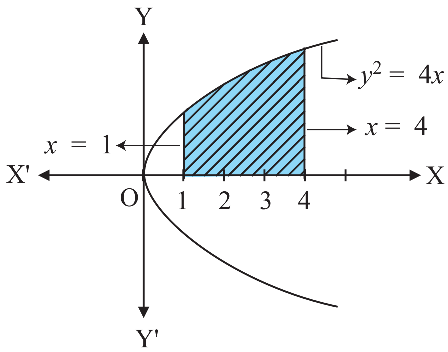


Fig. : 7.11

(c) A = (Area below X-axis) + (Area above X-axis)

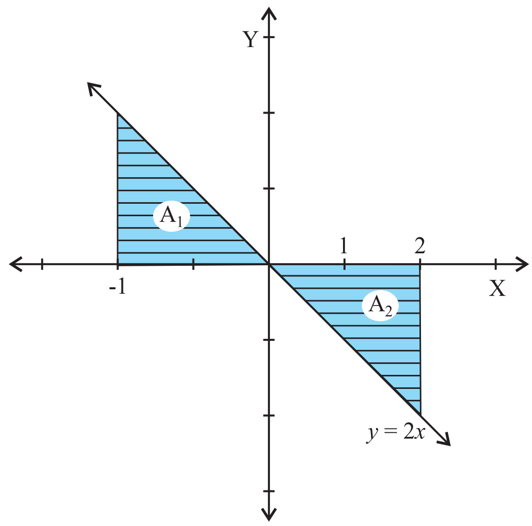


Fig. : 7.12

Required area $A = A_1 + |A_2|$

$A_2 = \int_{-1}^{0}(-2x)\,dx + \left|\int_0^2(-2x)\,dx\right|$

$= \left[-2\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{2x^2}{2}\right]_0^2$

$= \left[-x^2\right]_{-1}^{0} + \left[x^2\right]_0^2$

$= (0+1) + (4-0)$

A = 5 sq. units

2. Find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.

Solution: $y^2 = 16x$

$\therefore \quad y = \pm 4\sqrt{x}$

$\therefore \quad$ A = Area POCP + Area QOCQ

= 2(Area POCP) (why?)

$= 2\int_0^4 y\,dx$

$= 2\int_0^4 4.\sqrt{x}\,dx$

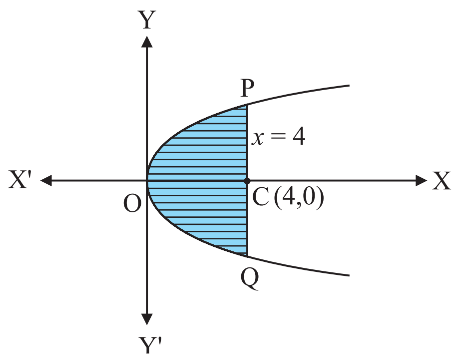


Fig. : 7.13

$\because$ y lies above X-axis

$= 8.\frac{2}{3}.[x^{3/2}]_0^4$

$= \frac{16}{3}.[8] = \frac{128}{3}$ sq.units

3. Find the area of the region bounded by the curve $x^2 = 16y$, $y = 1$, $y = 4$, and the Y - axis lying in the first quadrant.

Solution: Required area = $\int_1^4 x.dy$

$$\therefore \quad A = \int_1^4 \sqrt{16y}\; dy = 4\int_1^4 y^{1/2}.dy$$

$$= \left[4.\frac{2}{3}y^{3/2}\right]_1^4 = \frac{8}{3}\times 7$$

$$= \frac{56}{3} \text{ sq. units.}$$

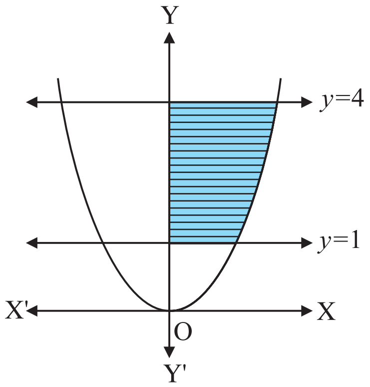


Fig. : 7.14

4. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Given

$$\left(\begin{array}{l}\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\\ \sin^{-1}(1) = \frac{\pi}{2}\ ,\ \sin^{-1}(0) = 0\end{array}\right)$$

Solution: From the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\therefore \quad \frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

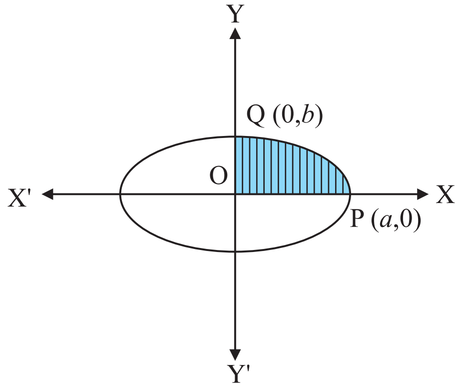


Fig. : 7.15

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$\because$ In first quadrant, $y > 0$

$$\therefore \quad A = 4.\int_0^a y\,dx$$

$$= 4.\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}.\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left\{\frac{a^2}{2}\sin^{-1}(1)-\frac{a^2}{2}\sin^{-1}(0)\right\}$$

$$= \frac{4b}{a}.\frac{a^2}{2}.\frac{\pi}{2}-0$$

$$= \pi ab \text{ sq.units.}$$

5. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution:

Equation of curve is $y = x^2$ .... (i)

and equation of line is $y = 4$ .... (ii)

Because of symmentry,

Required area = 2 [Area in first quadrant]

$$A = 2.\int_0^4 x.dy$$

$$= 2.\int_0^4 \sqrt{y}\, dy$$

$$= 2\times\frac{2}{3}[y^{3/2}]_0^4 = \frac{4}{3}(4^{3/2} - 0^{3/2})$$

$$= \frac{4}{3}(8-0) = \frac{32}{3} \text{ sq.units.}$$

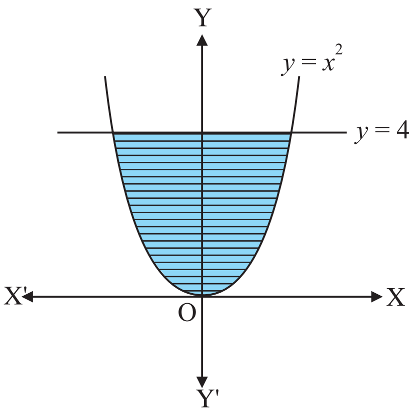


Fig. : 7.16

EXERCISE 7.1

1. Find the area of the region bounded by the following curves, the X-axis and the given lines:
   i) $y = x^4, x = 1, x = 5$
   ii) $y = \sqrt{6x+4}\ , x = 0, x = 2$
   iii) $y = \sqrt{16-x^2}\ , x = 0\ , x = 4$
   iv) $2y = 5x + 7, x = 2, x = 8$
   v) $2y + x = 8, x = 2, x = 4$
   vi) $y = x^2 + 1, x = 0, x = 3$
   vii) $y = 2 - x^2, x = -1, x = 1$
2. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $x = 3$.
3. Find the area of circle $x^2 + y^2 = 25$
4. Find the area of ellipse $\frac{x^2}{4}+\frac{y^2}{25}=1$

MISCELLANEOUS EXERCISE - 7

I) Choose the correct alternative.

1) Area of the region bounded by the curve $x^2 = y$, the X-axis and the lines $x = 1$ and $x = 3$ is ____________
   a) $\frac{26}{3}$ sq. units b) $\frac{3}{26}$ sq.units
   c) 26 sq. units d) 3 sq. units

2) The area of the region bounded by $y^2 = 4x$, the X-axis and the lines $x = 1$ & $x = 4$ is ____________
   a) 28 sq. units b) 3 sq. units
   c) $\frac{28}{3}$ sq. units d) $\frac{3}{28}$ sq. units

3) Area of the region bounded by $x^2 = 16y$, $y = 1$ & $y = 4$ and the Y=axis. lying in the first quadrant is ____________
   a) 63 sq. units b) $\frac{3}{56}$ sq. units
   c) $\frac{56}{3}$ sq.units d) $\frac{63}{7}$ sq.units

4) Area of the region bounded by $y = x^4$, $x = 1$, $x = 5$ and the X-axis is ____________
   a) $\frac{3142}{5}$ sq.units b) $\frac{3124}{5}$ sq. units
   c) $\frac{3142}{3}$ sq. units d) $\frac{3124}{3}$ sq. units

5) Using definite integration area of circle $x^2 + y^2 = 25$ is ____________
   a) 5π sq. units b) 4π sq. units
   c) 25πsq. units d) 25 sq. units

II. Fill in the blanks.

1) Area of the region bounded by $y = x^4$, $x = 1$, $x = 5$ and the X-axis is ____________

2) Using definite integration area of the circle $x^2 + y^2 = 49$ is __________

3) Area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and the Y-axis lying in the first quadrant is ______________

4) The area of the region bounded by the curve $x^2 = y$, the X-axis and the lines $x = 3$ and $x = 9$ is ____________

5) The area of the region bounded by $y^2 = 4x$, the X-axis and the lines $x = 1$ & $x = 4$ is ____________

III) State whether each of the following is True or False.

1) The area bounded by the curve $x = g(y)$, Y-axis and bounded between the lines $y = c$ and $y = d$ is given by $\int_c^d x\,dy = \int_{y=c}^{y=d} g(y)\,dy$

2) The area bounded by two curves $y = f(x)$, $y = g(x)$ and X-axis is $\left|\int_a^b f(x)\,dx - \int_b^a g(x)\,dx\right|$

3) The area bounded by the curve $y = f(x)$, X-axis and lines $x = a$ and $x = b$ is $\left|\int_a^b f(x)\,dx\right|$

4) If the curve, under consideration, is below the X-axis, then the area bounded by curve, X-axis and lines $x = a$, $x = b$ is positive.

5) The area of the portion lying above the X-axis is positive.

IV) Solve the following.

1) Find the area of the region bounded by the curve $xy = c^2$, the X-axis, and the lines $x = c$, $x = 2c$.

2) Find the area between the parabolas $y^2 = 7x$ and $x^2 = 7y$.

3) Find the area of the region bounded by the curve $y = x^2$ and the line $y = 10$.

4) Find the area the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

5) Find the area of the region bounded by $y = x^2$, the X-axis and $x = 1$, $x = 4$.

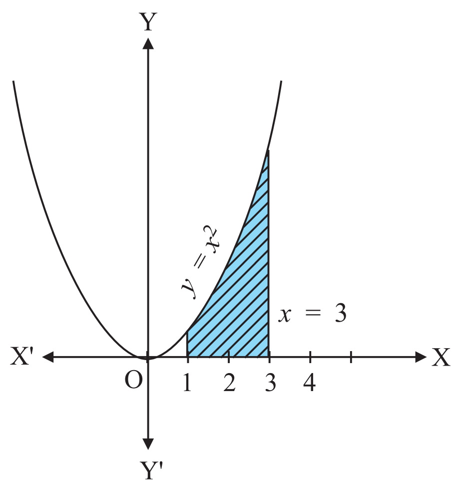
6) Find the area of the region bounded by the curve $x^2 = 25y$, $y = 1$, $y = 4$ and the Y-axis.

7) Find the area of the region bounded by the parabola $y^2 = 25x$ and the line $x = 5$.

Activities

From the following information find the area of the shaded regions.

1)



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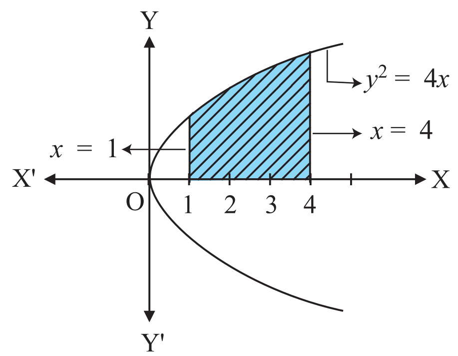

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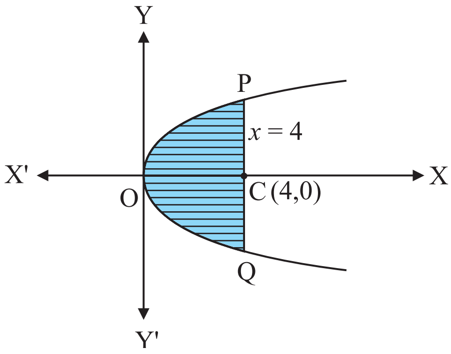
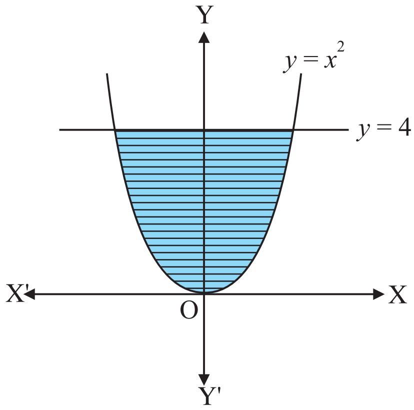
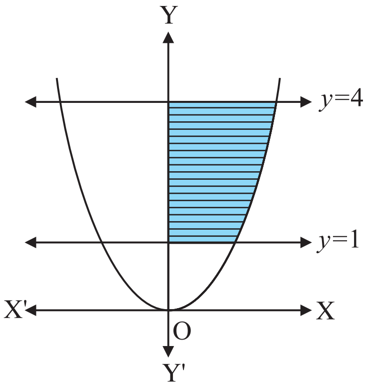


5)



4)



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