Chapter 6

THE DEFINITE INTEGRAL

§ 6.1. Statement of the Problem. The Lower and Upper Integral Sums

Let a function $f(x)$ be defined in the closed interval $[a, b]$. The following is called the *integral sum:*

$$I_n = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i,$$

where $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$,

$$\Delta x_i = x_{i+1} - x_i; \quad \xi_i \in [x_i, x_{i+1}] \quad (i = 0, 1, \ldots, n-1).$$

The sum $S_n = \sum\limits_{i=0}^{n-1} M_i \Delta x_i$ is called the *upper* (integral) *sum*, and $s_n = \sum\limits_{i=0}^{n-1} m_i \Delta x_i$ is called the *lower* (integral) *sum*, where $M_i = \sup f(x)$ $[m_i = \inf f(x)]$ for $x \in [x_i, x_{i+1}]$.

The *definite integral* of the function $f(x)$ on the interval $[a, b]$ is the limit of the integral sums

$$\int_a^b f(x)\,dx = \lim \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i \text{ when } \max|\Delta x_i| \to 0.$$

If this limit exists, the function is called *integrable* on the interval $[a, b]$. Any continuous function is integrable.

6.1.1. For the integral

$$\int_0^{\pi} \sin x\,dx$$

find the upper and lower integral sums corresponding to the division of the closed interval $[0, \pi]$ into 3 and 6 equal subintervals.

Solution. Divide the closed interval $[0, \pi]$ into 3 equal parts by the points:

$$x_0=0, \quad x_1=\frac{\pi}{3}, \quad x_2=\frac{2\pi}{3}, \quad x_3=\pi.$$

The function $\sin x$ increases monotonically on the interval $\left[0, \frac{\pi}{3}\right]$, and therefore for this interval we have $m_0=\sin 0=0$, $M_0=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$. The least value of the function on the interval $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ is $m_1=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, and the greatest value is $M_1=\sin\frac{\pi}{2}=1$. On the interval $\left[\frac{2\pi}{3}, \pi\right]$ the function $\sin x$ decreases monotonically and therefore

$$m_2=\sin\pi=0, \quad M_2=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}.$$

Since all Δx_k are equal to $\frac{\pi}{3}$,

$$s_3=\sum_{k=0}^{2} m_k\,\Delta x_k=\frac{\pi}{3}\left(0+\frac{\sqrt{3}}{2}+0\right)=\frac{\pi\sqrt{3}}{6}\approx 0.907,$$

$$S_3=\sum_{k=0}^{2} M_k\,\Delta x_k=\frac{\pi}{3}\left(\frac{\sqrt{3}}{2}+1+\frac{\sqrt{3}}{2}\right)=\frac{\pi(\sqrt{3}+1)}{3}\approx 2.86.$$

When subdividing the closed interval $[0, \pi]$ into 6 equal intervals by the points $x_0=0$, $x_1=\frac{\pi}{6}$, $x_2=\frac{\pi}{3}$, $x_3=\frac{\pi}{2}$, $x_4=\frac{2\pi}{3}$, $x_5=\frac{5\pi}{6}$, $x_6=\pi$, we find by analogy:

$$m_0=0, \qquad M_0=\sin\frac{\pi}{6}=\frac{1}{2},$$

$$m_1=\sin\frac{\pi}{6}=\frac{1}{2}, \qquad M_1=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2},$$

$$m_2=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}, \qquad M_2=\sin\frac{\pi}{2}=1,$$

$$m_3=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}, \qquad M_3=\sin\frac{\pi}{2}=1,$$

$$m_4=\sin\frac{5\pi}{6}=\frac{1}{2}, \qquad M_4=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2},$$

$$m_5=\sin\pi=0, \qquad M_5=\sin\frac{5\pi}{6}=\frac{1}{2}.$$

For this division we obtain

$$s_6=\frac{\pi}{6}(m_0+m_1+\ldots+m_5)=\frac{\pi}{6}(1+\sqrt{3})\approx 1.43,$$

$$S_6=\frac{\pi}{6}(M_0+M_1+\ldots+M_5)=\frac{\pi}{6}(3+\sqrt{3})\approx 2.48.$$

As would be expected, the inequalities

$$s_3\leqslant s_6\leqslant\int_0^{\pi}\sin x\,dx\leqslant S_6\leqslant S_3$$

hold true (the exact value of the integral is equal to 2).

6.1.2. At what $\delta>0$ does the relation

$$\left|\int_0^{\pi}\sin x\,dx-\sum_{i=0}^{n-1}\sin\xi_k\,\Delta x_k\right|<0.001$$

follow from the inequality $\max\Delta x_i<\delta$.

Solution. Since $s_n<I_n<S_n$, then for the required inequality to hold true it is sufficient that the upper and the lower integral sums differ by less than 0.001:

$$0<S_n-s_n<0.001.$$

But

$$S_n-s_n=\sum_{i=0}^{n-1}(M_i-m_i)\,\Delta x_i<\delta\sum_{i=0}^{n-1}(M_i-m_i),$$

where M_i and m_i are the greatest and the least values of the function $\sin x$ on the interval $[x_i,\ x_{i+1}]$ $(i=0,\ 1,\ \ldots,\ n-1)$. Assuming for simplicity that the point $\frac{\pi}{2}$ is chosen as one of the points of division and taking advantage of monotonicity of the function $\sin x$ on the intervals $\left[0,\ \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2},\ \pi\right]$, we obtain

$$\sum_{i=0}^{n-1}(M_i-m_i)=2\left(\sin\frac{\pi}{2}-\sin 0\right)=2.$$

Consequently, the required inequality is satisfied if $2\delta<0.001$, i.e. $\delta<0.0005$.

6.1.3. Show that the Dirichlet function [see Problem **1.14.4** (b)] is not integrable in the interval [0, 1].

Solution. In dividing the closed interval [0, 1] into a fixed number of parts we must take into consideration, in particular, two possible cases: (1) all points ξ_i are rational; (2) all points ξ_i are irrational. In the first case the integral sum is equal to unity, in the second to zero. Hence, no matter how we reduce the maximum length of subintervals, we always get integral sums equal to unity and integral sums equal to zero. Therefore, the limit of integral sums is non-existent, which means that the Dirichlet function is not integrable on the interval [0, 1].

6.1.4. Find the distance covered by a body in a free fall within the time interval from $t=a$ sec to $t=b$ sec.

Solution. A body moves in a free fall with constant acceleration g and initial velocity $v_0=0$. Consequently, the velocity at the instant t is equal to the velocity increment within the time interval from 0 to t, i. e. $v(t)=\Delta v$. For a short time period Δt the velocity increment is approximately equal to the acceleration at the instant t multiplied by Δt. But in our case acceleration is constant, therefore $\Delta v=g\Delta t$, and hence, $v(t)=gt$, since $\Delta t=t-0=t$.

Let us subdivide the time interval from $t=a$ to $t=b$ into n equal parts; then the duration Δt of each subinterval will be equal to $\Delta t=\frac{b-a}{n}$. We assume that during each subinterval of time the body moves uniformly with a velocity equal to its velocity at the beginning of this interval, i.e.

$$\begin{aligned}
&v_0=ga,\\
&v_1=g\left(a+1\,\frac{b-a}{n}\right),\\
&v_2=g\left(a+2\,\frac{b-a}{n}\right),\\
&\dots\dots\dots\dots\dots\\
&v_{n-1}=g\left[a+(n-1)\,\frac{b-a}{n}\right].
\end{aligned}$$

Whence we find the distance covered by the body during the ith subinterval: $\frac{v_i(b-a)}{n}$. The entire distance covered by the body is approximately equal to

$$\begin{aligned}
s\approx s_n&=\frac{b-a}{n}(v_0+v_1+\dots+v_{n-1})=\\
&=\frac{b-a}{n}g\left[na+1\,\frac{b-a}{n}+2\,\frac{b-a}{n}+\dots+(n-1)\,\frac{b-a}{n}\right]=\\
&=(b-a)\,g\left[a+\frac{b-a}{n^2}\frac{n(n-1)}{2}\right].
\end{aligned}$$

With n increasing the distance covered can be evaluated more accurately. The exact value of s is found as the limit s_n as $n \to \infty$:

$$s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} g(b-a)\left[a + \frac{1}{2}(b-a)\left(1 - \frac{1}{n}\right)\right] =$$
$$= g(b-a)\left[a + \frac{1}{2}(b-a)\right] = \frac{g}{2}(b^2 - a^2).$$

Since s_n is an integral sum

$$s_n = \sum_{i=0}^{n-1} v_i \Delta t_i \quad \left(\Delta t_i = \Delta t = \frac{b-a}{n}\right),$$

the distance s is an integral:

$$s = \int_a^b v\,dt = \int_a^b gt\,dt = \frac{g}{2}(b^2 - a^2).$$

6.1.5. Proceeding from the definition, compute the integral

$$\int_0^1 x\,dx.$$

Solution. By definition,

$$\int_0^1 x\,dx = \lim \sum_{i=0}^{n-1} \xi_i \Delta x_i \quad \text{as} \quad \max \Delta x_i \to 0,$$

where

$$0 = x_0 < x_1 < \ldots < x_n = 1, \quad \xi_i \in [x_i, x_{i+1}],$$
$$\Delta x_i = x_{i+1} - x_i.$$

1. Subdivide the closed interval $[0, 1]$ into n equal parts by the points $x_i = \frac{i}{n}$ $(i = 0, 1, 2, \ldots, n)$.

The length of each subinterval is equal to $\Delta x_i = \frac{1}{n}$, and $\frac{1}{n} \to 0$ as $n \to \infty$.

Let us take the right-hand end-points of the subintervals as the points ξ_i: $\xi_i = x_{i+1} = \frac{i+1}{n}$ $(i = 0, 1, \ldots, n-1)$.

Form an integral sum:

$$I_n = S_n = \sum_{i=0}^{n-1} \frac{i+1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \quad (1 + 2 + \ldots + n) = \frac{n(n+1)}{2n^2}.$$

As $n \to \infty$ the limit of this sum is equal to

$$\lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Hence,

$$\int_0^1 x\,dx = \frac{1}{2}.$$

2. Using this example, we will show that for any other choice of points ξ_i the limit of the integral sum will be the same.

Take, for instance, the mid-points of the subintervals as ξ_i: $\xi_i = \dfrac{i+\frac{1}{2}}{n}$ $(i=0, 1, \dots, n-1)$.

Form an integral sum

$$I_n = \sum_{i=0}^{n-1} \frac{2i+1}{2n} \cdot \frac{1}{n} = \frac{1}{2n^2}\left[1+3+5+\dots+(2n-1)\right] = \frac{2n^2}{4n^2} = \frac{1}{2}.$$

Hence

$$\lim_{n\to\infty} I_n = \frac{1}{2}.$$

6.1.6. Proceeding from the definition, compute the integral:

$$\int_a^b x^m\,dx \quad (m \neq -1,\ 0 < a < b).$$

Solution. In this example the following points can be conveniently chosen as points of division:

$$x_0 = a;\ x_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n}}, \dots, x_i = a\left(\frac{b}{a}\right)^{\frac{i}{n}}, \dots, x_n = a\left(\frac{b}{a}\right)^{\frac{n}{n}} = b.$$

They form a geometric progression with the common ratio

$$q = \left(\frac{b}{a}\right)^{\frac{1}{n}} > 1.$$

The length of the ith subinterval is equal to

$$\Delta x_i = aq^{i+1} - aq^i = aq^i(q-1).$$

Therefore the maximum length of the subintervals equals $\max \Delta x_i = aq^{n-1}(q-1) = a\left(\frac{b}{a}\right)^{\frac{n-1}{n}}\left[\left(\frac{b}{a}\right)^{\frac{1}{n}} - 1\right]$ and tends to zero with increasing n, since $\lim\limits_{n\to\infty} q = 1$.

Now let us choose the right-hand end-points of the subintervals as ξ_i: $\xi_i = x_{i+1} = aq^{i+1}$ $(i=0, 1, 2, \dots, n-1)$.

Form an integral sum:

$$I_n=\sum_{i=0}^{n-1}\xi_i^m\Delta x_i=\sum_{i=0}^{n-1}a^m q^{(i+1)m}aq^i(q-1)=$$
$$=a^{m+1}(q-1)q^m[1+q^{m+1}+\ldots+q^{(n-1)(m+1)}]=$$
$$=a^{m+1}(q-1)q^m\frac{q^{(m+1)n}-1}{q^{m+1}-1}=(b^{m+1}-a^{m+1})q^m\frac{q-1}{q^{m+1}-1}.$$

Let us calculate the limit of the integral sum as $\max\Delta x_i\to 0$, i.e. as $q\to 1$:

$$\lim I_n=(b^{m+1}-a^{m+1})\lim_{q\to 1}q^m\frac{q-1}{q^{m+1}-1}=(b^{m+1}-a^{m+1})\frac{1}{m+1}.$$

Thus,

$$\int_a^b x^m\,dx=\frac{1}{m+1}(b^{m+1}-a^{m+1}).$$

6.1.7. Proceeding from the definition, compute the integral:

$$\int_1^2\frac{dx}{x}.$$

Solution. Subdivide the interval [1, 2] into n parts so that the points of division x_i ($i=0, 1, 2, \ldots, n$) form the geometric progression:

$$x_0=1;\ x_1=q;\ x_2=q^2;\ x_3=q^3;\ \ldots;\ x_n=q^n=2,$$

whence $q=\sqrt[n]{2}$.

The length of the ith subinterval is equal to

$$\Delta x_i=q^{i+1}-q^i=q^i(q-1),$$

and so $\max\Delta x_i=q^{n-1}(q-1)\to 0$ as $n\to\infty$, i.e. as $q\to 1$.

Now let us choose the right-hand end-points of the subintervals as the points ξ_i, i.e., $\xi_i=x_{i+1}=q^{i+1}$.

Form an integral sum:

$$I_n=\sum_{i=0}^{n-1}\frac{1}{\xi_i}\Delta x_i=\sum_{i=0}^{n-1}\frac{1}{q^{i+1}}q^i(q-1)=\frac{n}{q}(q-1)=\frac{1}{2^{\frac{1}{n}}}n\left(2^{\frac{1}{n}}-1\right).$$

$$\lim_{n\to\infty}I_n=\lim_{n\to\infty}\frac{n\left(2^{\frac{1}{n}}-1\right)}{2^{\frac{1}{n}}}=\ln 2,$$

since $2^{\frac{1}{n}}-1\sim\frac{1}{n}\ln 2$ as $n\to\infty$.

And so,

$$\int_1^2 \frac{dx}{x} = \ln 2.$$

6.1.8. Evaluate the integral

$$I = \int_0^5 \sqrt{25 - x^2}\, dx,$$

proceeding from its geometric meaning.

Solution. The curve $y = \sqrt{25 - x^2}$ is the upper half of the circle $x^2 + y^2 = 25$. The portion of the curve corresponding to the variation of x from 0 to 5 lies in the first quadrant. Hence, we conclude that the curvilinear trapezoid bounded by the lines $x = 0$; $x = 5$; $y = 0$, and $y = \sqrt{25 - x^2}$ is a quarter of the circle $x^2 + y^2 = 25$; and its area is equal to $\frac{25\pi}{4}$.

Hence,

$$I = \int_0^5 \sqrt{25 - x^2}\, dx = \frac{25\pi}{4}.$$

6.1.9. Evaluate the integral, proceeding from its geometric meaning:

$$I = \int_1^5 (4x - 1)\, dx.$$

6.1.10. Prove that

$$I = \int_0^x \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} \quad (0 < x \leqslant a).$$

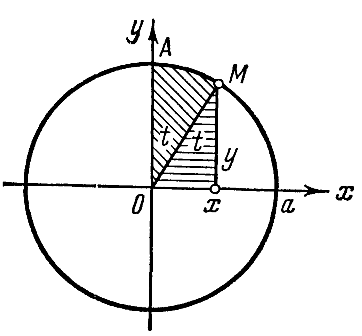

Fig. 59

Solution. The integral

$$I = \int_0^x \sqrt{a^2 - x^2}\, dx$$

expresses the area S_{OAMx} of the portion of a circle of radius a lying in the first quadrant (see Fig. 59). This area equals the sum of the areas of the triangle OMx and the sector OAM.

$$S_{OMx} = \frac{xy}{2} = \frac{x}{2} \sqrt{a^2 - x^2}.$$

The area of the sector

$$S_{OAM}=\frac{1}{2}a^2t,$$

where $\sin t=\frac{x}{a}$.

Hence,

$$S_{OAM}=\frac{a^2}{2}\arcsin\frac{x}{a},$$

and consequently,

$$I=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}.$$

6.1.11. Proceeding from the geometric meaning of the integral, show that

(a) $\int\limits_0^{2\pi}\sin^3 x\,dx=0$; (b) $\int\limits_{-1}^{1}e^{-x^2}\,dx=2\int\limits_0^1 e^{-x^2}\,dx$.

Solution. (a) The graph of the function $y=\sin^3 x$ is shown in Fig. 60. Let us show that the area situated above the x-axis is equal to that lying below this axis. Indeed, let $\pi\leqslant x\leqslant 2\pi$, then $x=\pi+x_1$ where $0\leqslant x_1\leqslant\pi$ and $\sin^3 x=\sin^3(\pi+x_1)=-\sin^3 x_1$. Therefore, the second half of the graph is obtained from the first one by shifting it to the right by π and using the symmetry about the x-axis. Hence,

$$\int\limits_0^{2\pi}\sin^3 x\,dx=0.$$

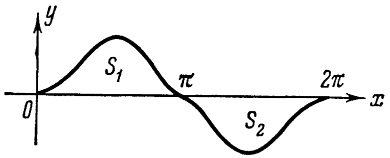

Fig. 60

6.1.12. Given the function $f(x)=x^3$ on the interval $[-2, 3]$, find the lower (s_n) and the upper (S_n) integral sums for the given interval by subdividing it into n equal parts.

6.1.13. Proceeding from the geometric meaning of the definite integral, prove that:

(a) $\int\limits_0^{\pi}\sin 2x\,dx=0$; (b) $\int\limits_0^{2\pi}\cos^3 x\,dx=0$;

(c) $\int\limits_1^{2}(2x+1)\,dx=6$; (d) $\int\limits_{-3}^{3}\sqrt{9-x^2}\,dx=\frac{9\pi}{2}$.

6.1.14. Passing to the limit from the integral sums, compute the integral

$$I=\int_1^4 x^3\,dx,$$

by subdividing the interval [1, 4]:

(a) into equal parts;

(b) by points forming a geometric progression. In both cases choose ξ_i as:

(1) left-hand end-points of the subintervals;

(2) right-hand end-points of the subintervals;

(3) mid-points of the subintervals $[x_i, x_{i+1}]$.

§ 6.2. *Evaluating Definite Integrals by the Newton-Leibniz Formula*

The following is known as the Newton-Leibniz formula:

$$\int_a^b f(x)\,dx=F(x)\Big|_a^b=F(b)-F(a),$$

where $F(x)$ is one of the antiderivatives of the function $f(x)$, i.e.

$$F'(x)\equiv f(x)\quad (a\leqslant x\leqslant b).$$

6.2.1. Evaluate the integral

$$I=\int_1^{\sqrt{3}}\frac{dx}{1+x^2}.$$

Solution. Since the function $F(x)=\operatorname{arc}\tan x$ is one of the antiderivatives of the function $f(x)=\frac{1}{1+x^2}$, using the Newton-Leibniz formula we get

$$I=\int_1^{\sqrt{3}}\frac{dx}{1+x^2}=\operatorname{arc}\tan x\Big|_1^{\sqrt{3}}=\operatorname{arc}\tan\sqrt{3}-\operatorname{arc}\tan 1=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}.$$

6.2.2. Compute the integrals:

(a) $\int_0^{\frac{\pi}{2}}\sin 2x\,dx$; (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos x}{\sin^3 x}\,dx$; (c) $\int_0^2\frac{dx}{\sqrt{16-x^2}}$.

6.2.3. Given the function

$$f(x)=\begin{cases} x^2 & \text{for } 0\leqslant x\leqslant 1, \\ \sqrt{x} & \text{for } 1\leqslant x\leqslant 2. \end{cases}$$

Evaluate $\int\limits_0^2 f(x)\,dx$.

Solution. By the additivity property of the integral

$$\int\limits_0^2 f(x)\,dx=\int\limits_0^1 f(x)\,dx+\int\limits_1^2 f(x)\,dx=\int\limits_0^1 x^2\,dx+\int\limits_1^2 \sqrt{x}\,dx=$$

$$=\frac{x^3}{3}\Big|_0^1+\frac{2}{3}x^{\frac{3}{2}}\Big|_1^2=\frac{1}{3}+\frac{4\sqrt{2}}{3}-\frac{2}{3}=\frac{1}{3}(4\sqrt{2}-1).$$

6.2.4. Evaluate the integral

$$I=\int\limits_0^2 |1-x|\,dx.$$

Solution. Since

$$|1-x|=\begin{cases} 1-x & \text{for } 0\leqslant x\leqslant 1, \\ x-1 & \text{for } 1\leqslant x\leqslant 2, \end{cases}$$

we obtain, taking advantage of the additivity property of the integral,

$$\int\limits_0^2 |1-x|\,dx=\int\limits_0^1 (1-x)\,dx+\int\limits_1^2 (x-1)\,dx=$$

$$=-\frac{(1-x)^2}{2}\Big|_0^1+\frac{(x-1)^2}{2}\Big|_1^2=\frac{1}{2}+\frac{1}{2}=1.$$

6.2.5. Evaluate the integral

$$I=\int\limits_a^b \frac{|x|}{x}\,dx,$$

where $a<b$.

Solution. If $0\leqslant a<b$, then $f(x)=\frac{|x|}{x}=1$, therefore $\int\limits_a^b f(x)\,dx=$ $=b-a$. If $a<b\leqslant 0$, then $f(x)=-1$ and $\int\limits_a^b f(x)dx=-b-(-a)=$ $=a-b$. Finally, if $a<0<b$, then divide the integral $\int\limits_a^b f(x)\,dx$

into two integrals:

$$\int_a^b f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^b f(x)\,dx = b - (-a).$$

The above three cases may be represented by a single formula:

$$\int_a^b \frac{|x|}{x}\,dx = |b| - |a|.$$

Note. When evaluating integrals with the aid of the Newton-Leibniz formula attention should be paid to the conditions of its legitimate use. This formula may be applied to compute the definite integral of a function *continuous* on the interval $[a, b]$ only when the equality $F'(x) = f(x)$ is fulfilled *in the whole interval* $[a, b]$ [$F(x)$ is an antiderivative of the function $f(x)$]. In particular, the antiderivative must be a function continuous on the whole interval $[a, b]$. A discontinuous function used as an antiderivative will lead to the wrong result.

6.2.6. Find a mistake in the following evaluation:

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \frac{1}{2}\arctan\frac{2x}{1-x^2}\Big|_0^{\sqrt{3}} = \frac{1}{2}\left[\arctan(-\sqrt{3}) - \arctan 0\right] = -\frac{\pi}{6},$$

where $\left(\frac{1}{2}\arctan\frac{2x}{1-x^2}\right)' = \frac{1}{1+x^2}\ (x \neq 1)$.

Solution. The result is *a priori* wrong: the integral of a function positive everywhere turns out to be negative. The mistake is due to the fact that the function $\frac{1}{2}\arctan\frac{2x}{1-x^2}$ has a discontinuity of the first kind at the point $x = 1$:

$$\lim_{x \to 1-0} \frac{1}{2}\arctan\frac{2x}{1-x^2} = \frac{\pi}{4}; \quad \lim_{x \to 1+0} \frac{1}{2}\arctan\frac{2x}{1-x^2} = -\frac{\pi}{4}.$$

The correct value of the integral under consideration is equal to

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \arctan x\Big|_0^{\sqrt{3}} = \arctan\sqrt{3} - \arctan 0 = \frac{\pi}{3}.$$

Here the Newton-Leibniz formula is applicable, since the function $F(x) = \arctan x$ is continuous on the interval $\left[0, \frac{\pi}{3}\right]$ and the equality $F'(x) = f(x)$ is fulfilled on the whole interval.

6.2.7. Find a mistake in the following evaluation of the integral:

$$\int_0^{\pi}\frac{dx}{1+2\sin^2 x}=\int_0^{\pi}\frac{dx}{\cos^2 x+3\sin^2 x}=$$

$$=\int_0^{\pi}\frac{\frac{dx}{\cos^2 x}}{1+3\tan^2 x}=\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\tan x)\Big|_0^{\pi}=0.$$

(The integral of a function positive everywhere turns out to be zero!)

Solution. The Newton-Leibniz formula is not applicable here, since the antiderivative $F(x)=\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\tan x)$ has a discontinuity at the point $x=\frac{\pi}{2}$. Indeed,

$$\lim_{x\to\frac{\pi}{2}-0}F(x)=\lim_{x\to\frac{\pi}{2}-0}\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\tan x)=$$

$$=\frac{1}{\sqrt{3}}\arctan(+\infty)=\frac{\pi}{2\sqrt{3}},$$

$$\lim_{x\to\frac{\pi}{2}+0}F(x)=\lim_{x\to\frac{\pi}{2}+0}\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\tan x)=$$

$$=\frac{1}{\sqrt{3}}\arctan(-\infty)=-\frac{\pi}{2\sqrt{3}}.$$

The correct result can be obtained in the following way:

$$\int_0^{\pi}\frac{dx}{\cos^2 x+3\sin^2 x}=\int_0^{\pi}\frac{1}{\cot^2 x+3}\,\frac{dx}{\sin^2 x}=$$

$$=-\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\cot x)\Big|_0^{\pi}=\frac{\pi}{\sqrt{3}}.$$

It can also be found with the aid of the function $F(x)=\frac{1}{\sqrt{3}}\arctan(\sqrt{3}\tan x)$. For this purpose divide the interval of integration $[0, \pi]$ into two subintervals, $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$, and take into consideration the above-indicated limit values of the function $F(x)$ as $x\longrightarrow\frac{\pi}{2}\mp 0$. Then the antiderivative becomes a continuous function on each of the subintervals, and the Newton-Leibniz formula becomes applicable:

$$\int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x} = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{1}{\sqrt{3}} \arctan(\sqrt{3}\tan x)\Big|_0^{\frac{\pi}{2}} + \frac{1}{\sqrt{3}} \arctan(\sqrt{3}\tan x)\Big|_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{1}{\sqrt{3}}\left[\left(\frac{\pi}{2} - 0\right) + \left(0 - \left(-\frac{\pi}{2}\right)\right)\right] = \frac{\pi}{\sqrt{3}}.$$

6.2.8. Compute the integral

$$\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\, dx.$$

Solution. $\sqrt{\frac{1+\cos 2x}{2}} = \sqrt{\frac{2\cos^2 x}{2}} = |\cos x| =$

$$= \begin{cases} \cos x, & 0 \leqslant x \leqslant \frac{\pi}{2}, \\ -\cos x, & \frac{\pi}{2} \leqslant x \leqslant \pi. \end{cases}$$

Therefore

$$\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\, dx = \int_0^{\frac{\pi}{2}} \cos x\, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\, dx =$$

$$= \sin x \Big|_0^{\frac{\pi}{2}} + (-\sin x)\Big|_{\frac{\pi}{2}}^{\pi} = (1-0) + (0-(-1)) = 2.$$

Note. If we ignore the fact that $\cos x$ is negative in $\left[\frac{\pi}{2}, \pi\right]$ and put

$$\sqrt{\frac{1+\cos 2x}{2}} = \cos x,$$

we get the wrong result:

$$\int_0^{\pi} \cos x\, dx = \sin x\Big|_0^{\pi} = 0.$$

6.2.9. Evaluate the integral

$$I = \int_0^{100\pi} \sqrt{1-\cos 2x}\, dx.$$

Solution. We have

$$\sqrt{1-\cos 2x} = \sqrt{2}\,|\sin x|.$$

Since $|\sin x|$ has a period π, then

$$\int_0^{100\pi} \sqrt{1-\cos 2x}\, dx = \sqrt{2} \int_0^{100\pi} |\sin x|\, dx =$$

$$= 100\sqrt{2} \int_0^{\pi} \sin x\, dx = 200\sqrt{2}.$$

6.2.10. Evaluate the integrals:

(a) $I = \int_{-2}^{-1} \frac{dx}{(11+5x)^3}$; (b) $I = \int_{-3}^{-2} \frac{dx}{x^2-1}$;

(c) $I = \int_{-\pi}^{\pi} \sin^2 \frac{x}{2}\, dx$; (d) $I = \int_0^{\frac{\pi}{4}} \frac{x^2}{x^2+1}\, dx$;

(e) $I = \int_e^{e^2} \frac{dx}{x \ln x}$; (f) $I = \int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin \frac{1}{x}}{x^2}\, dx$;

(g) $I = \int_0^1 \frac{e^x}{1+e^{2x}}\, dx$; (h) $I = \int_0^1 \frac{x^3\, dx}{1+x^8}$;

(i) $I = \int_0^3 \frac{x\, dx}{\sqrt{x+1}+\sqrt{5x+1}}$;

(j) $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x}\, dx$;

(k) $I = \int_1^{\sqrt{3}} \frac{dx}{(1+x^2)^{\frac{3}{2}}}$.

§ 6.3. Estimating an Integral. The Definite Integral as a Function of Its Limits

1. If $f(x) \leqslant \varphi(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b \varphi(x)\,dx.$$

In particular,

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx.$$

2.
$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a),$$

where m is the least value, and M the greatest value of the function $f(x)$ on the interval $[a, b]$ *(estimation of an integral)*.

3. If the function $f(x)$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = f(\xi)(b-a), \quad a < \xi < b$$

(mean-value theorem).

4. If the functions $f(x)$ and $\varphi(x)$ are continuous on $[a, b]$, and $\varphi(x)$, in addition, retains its sign on this interval, then

$$\int_a^b f(x)\varphi(x)\,dx = f(\xi)\int_a^b \varphi(x)\,dx, \quad a < \xi < b$$

(generalized mean-value theorem).

5. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$; $\frac{d}{dx}\int_x^a f(t)\,dt = -f(x)$ at each point x of continuity of the function $f(x)$.

6.3.1. Estimate the following integrals:

(a) $I = \int_1^3 \sqrt{3+x^3}\,dx$; (b) $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x}\,dx$;

(c) $I = \int_0^2 \frac{x^2+5}{x^2+2}\,dx$.

Solution. (a) Since the function $f(x) = \sqrt{3+x^2}$ increases monotonically on the interval $[1, 3]$, then $m = 2$, $M = \sqrt{30}$, $b-a = 2$.

Hence, the estimation of the integral has the form

$$2\cdot 2 \leqslant \int_1^3 \sqrt{3+x^3}\,dx \leqslant \sqrt{30}\cdot 2,$$

i. e.

$$4 \leqslant \int_1^3 \sqrt{3+x^3}\,dx \leqslant 2\sqrt{30} \approx 10.95.$$

(b) The integrand $f(x)=\frac{\sin x}{x}$ decreases on the interval $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, since its derivative

$$f'(x)=\frac{x\cos x-\sin x}{x^2}=\frac{(x-\tan x)\cos x}{x^2}<0.$$

Hence, the least value of the function:

$$m=f\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{2\pi},$$

its greatest value being

$$M=f\left(\frac{\pi}{4}\right)=\frac{2\sqrt{2}}{\pi}.$$

Therefore

$$\frac{3\sqrt{3}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \leqslant \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx \leqslant \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right),$$

i. e.

$$0.22 \approx \frac{\sqrt{3}}{8} \leqslant \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\,dx \leqslant \frac{\sqrt{2}}{6} \approx 0.24.$$

6.3.2. Estimate the absolute value of the integral

$$\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx.$$

Solution. Since $|\sin x| \leqslant 1$, for $x \geqslant 10$ the inequality

$$\left|\frac{\sin x}{1+x^8}\right| < 10^{-8} \text{ is fulfilled.}$$

Therefore

$$\left|\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx\right| < (19-10)\,10^{-8} < 10^{-7}$$

(the true value of the integral $\approx -10^{-8}$).

6.3.3. Which of the two integrals

$$\int_0^1 \sqrt{x}\,dx, \quad \int_0^1 x^3\,dx$$

is the greater?

Solution. As is known, $\sqrt{x} > x^3$ for $0 < x < 1$. Therefore

$$\int_0^1 \sqrt{x}\,dx > \int_0^1 x^3\,dx.$$

6.3.4. Prove the inequalities:

(a) $0 < \int_0^1 \frac{x^7\,dx}{\sqrt[3]{1+x^8}} < \frac{1}{8}$; (b) $1 < \int_0^1 e^{x^2}\,dx < e$.

Solution. (a) Since $0 < \frac{x^7}{\sqrt[3]{1+x^8}} < x^7$ for $0 < x \leqslant 1$, then

$$0 < \int_0^1 \frac{x^7\,dx}{\sqrt[3]{1+x^8}} < \int_0^1 x^7\,dx = \left.\frac{x^8}{8}\right|_0^1 = \frac{1}{8}.$$

(b) Since for $0 < x < 1$ there exists the inequality $1 < e^{x^2} < e$, then

$$\int_0^1 dx < \int_0^1 e^{x^2}\,dx < \int_0^1 e\,dx.$$

Hence the inequality under consideration holds true.

6.3.5. Prove the inequality

$$\int_0^{\frac{\pi}{2}} e^{-R\sin x}\,dx < \frac{\pi}{2R}(1-e^{-R}) \quad (R > 0).$$

Solution. Since the function $f(x) = \frac{\sin x}{x}$ decreases on $\left(0, \frac{\pi}{2}\right)$ [see Problem **6.3.1** (b)], then for $0 < x < \frac{\pi}{2}$

$$f(x) = \frac{\sin x}{x} > f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}.$$

Hence, on this interval $\sin x > \frac{2}{\pi}x$, therefore

$$e^{-R\sin x} < e^{-\frac{2R}{\pi}x}$$

and

$$\int_0^{\frac{\pi}{2}} e^{-R\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}x}\,dx = -\frac{\pi}{2R}\left[e^{-\frac{2R}{\pi}x}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2R}(1-e^{-R}).$$

6.3.6. Prove that for any functions $f(x)$ and $g(x)$, integrable on the interval (a, b), the Schwarz-Bunyakovsky inequality takes place:

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leqslant \sqrt{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}.$$

Solution. Consider the function

$$F(x) = [f(x) - \lambda g(x)]^2,$$

where λ is any real number. Since $F(x) \geqslant 0$, then

$$\int_a^b [f(x) - \lambda g(x)]^2\,dx \geqslant 0,$$

or

$$\lambda^2 \int_a^b g^2(x)\,dx - 2\lambda \int_a^b f(x)\,g(x)\,dx + \int_a^b f^2(x)\,dx \geqslant 0.$$

The expression in the left side of the latter inequality is a quadratic trinomial with respect to λ. It follows from the inequality that at any λ this trinomial is non-negative. Hence, its discriminant is non-positive, i. e.

$$\left\{\int_a^b f(x)\,g(x)\,dx\right\}^2 - \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx \leqslant 0.$$

Hence

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leqslant \sqrt{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx},$$

which completes the proof.

6.3.7. Estimate the integral from above

$$I = \int_0^1 \frac{\sin x}{1+x^2}\,dx.$$

Solution. By the generalized mean-value theorem we have

$$\int_0^1 \frac{\sin x}{1+x^2}\,dx = \sin\xi \int_0^1 \frac{dx}{1+x^2} = \sin\xi \arctan x\,\Big|_0^1 = \frac{\pi}{4}\sin\xi\ (0 < \xi < 1).$$

Since the function $\sin x$ increases on the interval $[0, 1]$ then $\sin\xi < \sin 1$. Whence we get an upper estimate of the integral:

$$\int_0^1 \frac{\sin x}{1+x^2}\,dx < \frac{\pi}{4}\sin 1 \approx 0.64.$$

It is possible to get a better estimation if we apply the same theorem in the form

$$\int_0^1 \frac{\sin x}{1+x^2}\,dx = \frac{1}{1+\xi^2}\int_0^1 \sin x\,dx = \frac{1}{1+\xi^2}(1-\cos 1) < 1-\cos 1 \approx 0.46.$$

6.3.8. Proceeding from geometric reasoning, prove that:

(a) if the function $f(x)$ increases and has a concave graph in the interval $[a, b]$, then

$$(b-a)f(a) < \int_a^b f(x)\,dx < (b-a)\frac{f(a)+f(b)}{2};$$

(b) if the function $f(x)$ increases and has a convex graph in the interval $[a, b]$, then

$$(b-a)\frac{f(a)+f(b)}{2} < \int_a^b f(x)\,dx < (b-a)f(b).$$

Solution. (a) Without limitation of generality we may assume $f(x) > 0$. Concavity of the graph of a function means, in particular, that the curve lies below the chord through the points $A(a, f(a))$ and $B(b, f(b))$ (see Fig. 61). Therefore the area of trapezoid $aABb$ is greater than that of the curvilinear trapezoid bounded above by the graph of the function, i. e.

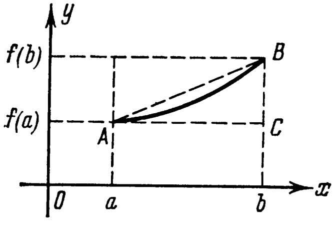

Fig. 61

$$\int_a^b f(x)\,dx < S_{aABb} = (b-a)\cdot\frac{f(a)+f(b)}{2}.$$

The inequality

$$(b-a)f(a) < \int_a^b f(x)\,dx$$

is obvious.

6.3.9. Estimate the integral $\int_0^1 \sqrt{1+x^4}\,dx$ using

(a) the mean-value theorem for a definite integral,
(b) the result of the preceding problem,
(c) the inequality $\sqrt{1+x^4} < 1 + \frac{x^4}{2}$,
(d) the Schwarz-Bunyakovsky inequality (see Problem **6.3.6**).

Solution. (a) By the mean-value theorem

$$I = \int_0^1 \sqrt{1+x^4}\,dx = \sqrt{1+\xi^4}, \text{ where } 0 \leqslant \xi \leqslant 1.$$

But

$$1 < \sqrt{1+\xi^4} < \sqrt{2},$$

whence

$$1 < I < \sqrt{2} \approx 1.414.$$

(b) The function $f(x) = \sqrt{1+x^4}$ is concave on the interval $[0, 1]$, since

$$f''(x) = \frac{2x^2(x^4+3)}{(1+x^4)^{3/2}} > 0, \quad 0 \leqslant x \leqslant 1.$$

On the basis of the preceding problem we get

$$1 < \int_0^1 \sqrt{1+x^4}\,dx < \frac{1+\sqrt{2}}{2} \approx 1.207.$$

(c) $$1 < I = \int_0^1 \sqrt{1+x^4}\,dx < \int_0^1 \left(1+\frac{x^4}{2}\right)dx = 1 + \frac{1}{10} = 1.1.$$

(d) Put $f(x) = \sqrt{1+x^4}$, $g(x) = 1$ and take advantage of the Schwarz-Bunyakovsky inequality

$$\left|\int_0^1 \sqrt{1+x^4}\,dx\right| = \int_0^1 \sqrt{1+x^4}\,dx = I < \sqrt{\int_0^1 (1+x^4)\,dx \cdot \int_0^1 1^2\,dx} = \sqrt{1.2} \approx 1.095.$$

6.3.10. Find the derivative with respect to x of the following functions:

(a) $F(x) = \int_{x^2}^{x^3} \ln t\,dt \quad (x > 0)$,

(b) $F(x) = \int_{\frac{1}{x}}^{\sqrt{x}} \cos(t^2)\,dt \quad (x > 0)$.

Solution. (a) Write the given integral in the following way:

$$F(x)=\int_{x^2}^{c}\ln t\,dt+\int_{c}^{x^3}\ln t\,dt=\int_{c}^{x^3}\ln t\,dt-\int_{c}^{x^2}\ln t\,dt,$$

where $c>0$ is an arbitrary constant.

Now let us find the derivative $F'(x)$ using the rule for differentiating a composite function and the theorem on the derivative of an integral with respect to the upper limit:

$$F'_x(x)=\left[\int_{c}^{x^3}\ln t\,dt\right]'_{x^3}(x^3)'_x-\left[\int_{c}^{x^2}\ln t\,dt\right]'_{x^2}(x^2)'_x=\ln x^3\,3x^2-\ln x^2\,2x=$$

$$=(9x^2-4x)\ln x.$$

(b) $$F(x)=\int_{\frac{1}{x}}^{c}\cos(t^2)\,dt+\int_{c}^{\sqrt{x}}\cos(t^2)\,dt=$$

$$=-\int_{c}^{\frac{1}{x}}\cos(t^2)\,dt+\int_{c}^{\sqrt{x}}\cos(t^2)\,dt;$$

$$F'(x)=-\left[\int_{c}^{\frac{1}{x}}\cos(t^2)\,dt\right]'_{\frac{1}{x}}\left(\frac{1}{x}\right)'_x+\left[\int_{c}^{\sqrt{x}}\cos(t^2)\,dt\right]'_{\sqrt{x}}(\sqrt{x})'_x=$$

$$=-\cos\frac{1}{x^2}\left(-\frac{1}{x^2}\right)+\cos x\cdot\frac{1}{2\sqrt{x}}=\frac{1}{x^2}\cos\frac{1}{x^2}+\frac{1}{2\sqrt{x}}\cos x.$$

6.3.11. Find the derivative with respect to x of the following functions:

(a) $F(x)=\int_{0}^{2x}\frac{\sin t}{t}\,dt$; (b) $F(x)=\int_{x}^{0}\sqrt{1+t^4}\,dt$.

6.3.12. Find the points of extremum of the function $F(x)=\int_{0}^{x}\frac{\sin t}{t}\,dt$ in the domain $x>0$.

Solution. Find the derivative

$$F'(x)=\left[\int_{0}^{x}\frac{\sin t}{t}\,dt\right]'_x=\frac{\sin x}{x}.$$

The critical points are:

$$x=n\pi\quad(n=1,\ 2,\ \ldots),\qquad \text{where } \sin x=0.$$

Find the second derivative at these points:

$$F''(x) = \frac{x\cos x - \sin x}{x^2};$$

$$F''(n\pi) = \frac{1}{n\pi}\cos(n\pi) = \frac{1}{n\pi}(-1)^n \neq 0.$$

Since the second derivative is non-zero at the points $x = n\pi$ ($n = 1, 2, \ldots$), these points are points of extremum of the function, namely: maxima if n is odd, and minima if n is even.

6.3.13. Find the derivative of y, with respect to x, of the function represented parametrically:

$$x = \int_1^{t^3} \sqrt[3]{z}\ln z\,dz; \quad y = \int_{\sqrt{t}}^{3} z^2 \ln z\,dz.$$

Solution. As is known, $y'_x = \dfrac{y'_t}{x'_t}$.

Find x'_t and y'_t:

$$x'_t = \left(\int_1^{t^3} \sqrt[3]{z}\ln z\,dz\right)'_{t^3} (t^3)'_t = t\ln t^3 \cdot 3t^2 = 9t^3 \ln t;$$

$$y'_t = \left(\int_{\sqrt{t}}^{3} z^2 \ln z\,dz\right)'_{\sqrt{t}} (\sqrt{t})'_t = -t\ln\sqrt{t}\,\frac{1}{2\sqrt{t}} = -\frac{1}{4}\sqrt{t}\ln t;$$

whence

$$y'_x = \frac{9t^3 \ln t}{-\frac{1}{4}\sqrt{t}\ln t} = -36t^2\sqrt{t} \quad (t > 0).$$

6.3.14. Find the limits:

(a) $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \sin\sqrt{x}\,dx}{x^3}$; (b) $\lim\limits_{x\to +\infty} \dfrac{\int_0^{x} (\arctan x)^2\,dx}{\sqrt{x^2+1}}$;

(c) $\lim\limits_{x\to +\infty} \dfrac{\left(\int_0^{x} e^{x^2}\,dx\right)^2}{\int_0^{x} e^{2x^2}\,dx}$.

Solution. (a) At $x = 0$ the integral $\int_0^{x^2} \sin\sqrt{x}\,dx$ equals zero; it is easy to check the fulfilment of the remaining conditions that ensure the legitimacy of using the L'Hospital rule. Therefore

$$\lim_{x\to 0}\frac{\int\limits_0^{x^2}\sin\sqrt{x}\,dx}{x^3}=\lim_{x\to 0}\frac{\left[\int\limits_0^{x^2}\sin\sqrt{x}\,dx\right]'_{x^2}(x^2)'_x}{3x^2}=\lim_{x\to 0}\frac{2x\sin x}{3x^2}=\frac{2}{3}.$$

(c) We have an indeterminate form of the type $\frac{\infty}{\infty}$. Use the L'Hospital rule:

$$\lim_{x\to+\infty}\frac{\left(\int\limits_0^{x}e^{x^2}\,dx\right)^2}{\int\limits_0^{x}e^{2x^2}\,dx}=\lim_{x\to+\infty}\frac{2\int\limits_0^{x}e^{x^2}\,dx\cdot e^{x^2}}{e^{2x^2}}=$$

$$=\lim_{x\to+\infty}\frac{2\int\limits_0^{x}e^{x^2}\,dx}{e^{x^2}}=\lim_{x\to+\infty}2\frac{e^{x^2}}{e^{x^2}\cdot 2x}=0.$$

6.3.15. Find the derivative $\frac{dy}{dx}$ of the following implicit functions:

(a) $\int\limits_0^{y}e^{-t^2}\,dt+\int\limits_0^{x^2}\sin^2 t\,dt=0;$

(b) $\int\limits_0^{y}e^{t}\,dt+\int\limits_0^{x}\sin t\,dt=0;$

(c) $\int\limits_{\frac{\pi}{2}}^{x}\sqrt{3-2\sin^2 z}\,dz+\int\limits_0^{y}\cos t\,dt=0.$

Solution. (a) Differentiate the left side of the equation with respect to x, putting $y=y(x)$:

$$\left[\int\limits_0^{y}e^{-t^2}\,dt\right]'_y\cdot\frac{dy}{dx}+\left[\int\limits_0^{x^2}\sin^2 t\,dt\right]'_{x^2}(x^2)'_x=0;$$

$$e^{-y^2}\frac{dy}{dx}+\sin^2 x^2\cdot 2x=0.$$

Hence, solving the equation with respect to $\frac{dy}{dx}$, we get

$$\frac{dy}{dx}=-2xe^{+y^2}\sin^2 x^2.$$

(c) Differentiate the left side of the equation with respect to x, putting $y=y(x)$:

$$\left[\int_{\frac{\pi}{2}}^{x} \sqrt{3-2\sin^2 z}\, dz\right]_x' + \left[\int_0^y \cos t\, dt\right]_y' \frac{dy}{dx} = 0.$$

Whence

$$\sqrt{3-2\sin^2 x} + \cos y \frac{dy}{dx} = 0; \quad \frac{dy}{dx} = -\frac{\sqrt{3-2\sin^2 x}}{\cos y}.$$

6.3.16. Find: (a) the points of extremum and the points of inflection on the graph of the function

$$I = \int_0^x (t-1)(t-2)^2\, dt;$$

(b) curvature of the line defined by the parametric equations:

$$\begin{cases} x = a\sqrt{\pi} \int_0^t \cos \frac{\pi t^2}{2}\, dt, \\ y = a\sqrt{\pi} \int_0^t \sin \frac{\pi t^2}{2}\, dt \end{cases}$$

(the Cornu spiral).

Solution. (a) The function is defined and continuously differentiable throughout the entire number scale. Its derivative

$$I_x' = (x-1)(x-2)^2$$

equals zero at the points $x_1 = 1$, $x_2 = 2$, and when passing through the point x_1 it changes sign from minus to plus, whereas in the neighbourhood of the point x_2 the sign remains unchanged. Consequently, there is a minimum at the point $x_1 = 1$, and there is no extremum at the point $x_2 = 2$.

The second derivative

$$I_x'' = 3x^2 - 10x + 8$$

vanishes at the points $x_1 = \frac{4}{3}$, $x_2 = 2$ and changes sign when passing through these points. Hence, these points are the abscissas of the points of inflection.

(b) We have

$$x_t' = a\sqrt{\pi} \cos \frac{\pi t^2}{2}, \quad y_t' = a\sqrt{\pi} \sin \frac{\pi t^2}{2},$$

hence,

$$y'_x = \frac{y'_t}{x'_t} = \tan\frac{\pi t^2}{2}, \quad y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{\sqrt{\pi}\, t}{a\cos^3\frac{\pi t^2}{2}};$$

whence the curvature

$$K = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} = \frac{\sqrt{\pi}\, t}{a}.$$

6.3.17. Prove that the function $L(x)$, defined in the interval $(0, \infty)$ by the integral

$$L(x) = \int_1^x \frac{dt}{t},$$

is an inverse of the function e^x.

Solution. Let us take the derivative

$$L'(x) = \frac{1}{x} \quad (x > 0).$$

Since the derivative is positive, the function $y = L(x)$ increases and, hence, has an inverse function

$$x = L^{-1}(y).$$

The derivative of this inverse function is equal to

$$\frac{dx}{dy} = \frac{1}{L'(x)} = x,$$

whence it follows (see Problem **3.1.10**) that

$$x = Ce^y.$$

To find C, substitute $x = 1$. Since

$$L(1) = 0, \text{ i.e. } y|_{x=1} = 0,$$

then

$$1 = Ce^0 = C,$$

which proves our assertion:

$$x = L^{-1}(y) = e^y.$$

6.3.18. Given the graph of the function $y = f(x)$ (Fig. 62), find the shape of the graph of the antiderivative $I = \int_0^x f(t)\,dt$.

Solution. On the interval $[0, a]$ the given function is positive; consequently, the antiderivative increases. On the interval

$\left[0, \frac{a}{2}\right]$ the derivative of the given function is positive; hence, the curve $I=I(x)$ is concave. On the interval $\left[\frac{a}{2}, a\right]$ the derivative of the given function is negative; consequently, the curve $I=I(x)$ is convex, the point $x=\frac{a}{2}$ being a point of inflection. The interval $[a, 2a]$ is considered in a similar way. The point $x_1=0$ is a point of minimum, since the derivative $I'(x)=f(x)$ changes its sign from minus to plus; the point $x_2=a$ is a point of maximum, since the sign of the derivative changes from plus to minus.

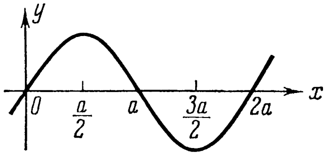

Fig. 62

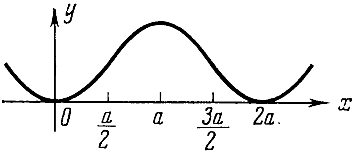

Fig. 63

The antiderivative $I(x)$ is a periodic function with period $2a$, since the areas lying above and below the x-axis are mutually cancelled over intervals of length $2a$. Taking all this into account, we can sketch the graph of the antiderivative (see Fig. 63).

6.3.19. Find the polynomial $P(x)$ of the least degree that has a maximum equal to 6 at $x=1$, and a minimum equal to 2 at $x=3$.

Solution. The polynomial is an everywhere-differentiable function. Therefore, the points of extremum can only be roots of the derivative. Furthermore, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x_1=1$ and $x_2=3$ has the form $a(x-1)(x-3)$. Hence,

$$P'(x)=a(x-1)(x-3)=a(x^2-4x+3).$$

Since at the point $x=1$ there must be $P(1)=6$, we have

$$P(x)=\int_1^x P'(x)\,dx+6=a\int_1^x (x^2-4x+3)\,dx+6=$$

$$=a\left(\frac{x^3}{3}-2x^2+3x-1\frac{1}{3}\right)+6.$$

The coefficient a is determined from the condition $P(3)=2$, whence $a=3$. Hence,

$$P(x)=x^3-6x^2+9x+2.$$

6.3.20. Find the polynomial $P(x)$ of the least degree whose graph has three points of inflection: $(-1, -1)$, $(1, 1)$ and a point with abscissa 0 at which the curve is inclined to the axis of abscissas at an angle of 60°.

Solution. Since the required function is a polynomial, the abscissas of the points of inflection can only be among the roots of the second derivative. The polynomial of the least degree with roots -1, 0, 1 has the form $ax(x^2-1)$. Consequently,

$$P''(x) = a(x^3 - x).$$

Since at the point $x=0$ the derivative $P'(0) = \tan 60° = \sqrt{3}$, we have

$$P'(x) = \int_0^x P''(x)\,dx + \sqrt{3} = a\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + \sqrt{3}.$$

Then, since $P(1) = 1$, we get

$$P(x) = \int_1^x P'(x)\,dx + 1 = a\left(\frac{x^5}{20} - \frac{x^3}{6} + \frac{7}{60}\right) + \sqrt{3}(x-1) + 1.$$

The coefficient a is determined from the last remaining condition $P(-1) = -1$, whence $a = \frac{60(\sqrt{3}-1)}{7}$. Hence,

$$P(x) = \frac{\sqrt{3}-1}{7}(3x^5 - 10x^3) + x\sqrt{3}.$$

6.3.21. Taking advantage of the mean-value theorem for the definite integral, prove that

(a) $3 < \int_0^1 \sqrt{q + x^2}\,dx < 10,$

(b) $\frac{\pi}{2} < \int_0^{\pi/2} \sqrt{1 + \frac{1}{2}\sin^2 x}\,dx < \frac{\pi}{2}\sqrt{\frac{3}{2}},$

(c) $\frac{2\pi}{13} < \int_0^{2\pi} \frac{dx}{10 + 3\cos x} < \frac{2\pi}{7}.$

6.3.22. Using the Schwarz-Bunyakovsky inequality, prove that $\int_0^1 \sqrt{1 + x^3}\,dx < \frac{\sqrt{5}}{2}$. Make sure that the application of the mean-value theorem yields a rougher estimate.

6.3.23. Find the derivatives of the following functions:

(a) $F(x)=\int\limits_1^x \ln t\,dt\ (x>0)$; (b) $F(x)=\int\limits_{\frac{2}{x}}^{x^2} \frac{dt}{t}$.

6.3.24. Find the derivative $\frac{dy}{dx}$ of functions represented parametrically:

(a) $x=\int\limits_2^t \frac{\ln z}{z}\,dz,\quad y=\int\limits_5^{\ln t} e^z\,dz$;

(b) $x=\int\limits_{c^2}^{\sin t} \arcsin z\,dz,\quad y=\int\limits_{n}^{\sqrt{t}} \frac{\sin z^2}{z}\,dz$.

6.3.25. Find the points of extremum of the following functions:

(a) $F(x)=\int\limits_1^x e^{-\frac{t^2}{2}}(1-t^2)\,dt$;

(b) $F(x)=\int\limits_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt$.

§ 6.4. Changing the Variable in a Definite Integral

If a function $x=\varphi(t)$ satisfies the following conditions:

(1) $\varphi(t)$ is a continuous single-valued function defined in $[\alpha, \beta]$ and has in this interval a continuous derivative $\varphi'(t)$;

(2) with t varying on $[\alpha, \beta]$ the values of the function $x=\varphi(t)$ do not leave the limits of $[a, b]$;

(3) $\varphi(\alpha)=a$ and $\varphi(\beta)=b$,

then the formula for *changing the variable* (or *substitution*) *in the definite integral* is valid for any function $f(x)$ which is continuous on the interval $[a, b]$:

$$\int\limits_a^b f(x)\,dx=\int\limits_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt.$$

Instead of the substitution $x=\varphi(t)$ the inverse substitution $t=\psi(x)$ is frequently used. In this case the limits of integration α and β are determined directly from the equalities $\alpha=\psi(a)$ and $\beta=\psi(b)$. In practice, the substitution is usually performed with the aid of monotonic, continuously differentiable functions. The change in the limits of integration is conveniently expressed in the tabular form:

$$\begin{array}{|c|c|}x & t \\ \hline a & \alpha \\ b & \beta \end{array}\,.$$

6.4.1. Compute the integral $\int\limits_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\,dx$.

Solution. Make the substitution $x=2\sin t$, assuming that $-\frac{\pi}{3}\leqslant t\leqslant\frac{\pi}{3}$. The function $x=\varphi(t)=2\sin t$ on the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ satisfies all the conditions of the theorem on changing the variable in a definite integral, since it is continuously differentiable, monotonic and

$$\varphi\left(-\frac{\pi}{3}\right)=-\sqrt{3}, \quad \varphi\left(\frac{\pi}{3}\right)=\sqrt{3}.$$

And so,

$$x=2\sin t; \quad dx=2\cos t\,dt; \quad \sqrt{4-x^2}=2\,|\cos t\,|=2\cos t,$$

since $\cos t>0$ on the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

Thus,

$$\int\limits_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\,dx=4\int\limits_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2 t\,dt=2\int\limits_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (1+\cos 2t)\,dt=$$

$$=2\left[t+\frac{1}{2}\sin 2t\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}=\frac{4\pi}{3}+\sqrt{3}.$$

6.4.2. Compute the integral $\int\limits_{2}^{4} \frac{\sqrt{x^2-4}}{x^4}\,dx$.

Solution. Make the substitution

$$x=2\sec t; \quad dx=2\frac{\sin t}{\cos^2 t}\,dt; \qquad \begin{array}{|c|c|}x & t \\ \hline 2 & 0 \\ 4 & \frac{\pi}{3}\end{array}\,.$$

On the interval $\left[0, \frac{\pi}{3}\right]$ the function $2\sec t$ is monotonic, therefore the substitution is valid.

Hence,

$$\int_2^4 \frac{\sqrt{x^2-4}}{x^4}\,dx = \int_0^{\frac{\pi}{3}} \frac{\sqrt{4\sec^2 t-4}}{16\sec^4 t}\cdot 2\frac{\sin t}{\cos^2 t}\,dt =$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} \sin^2 t\cos t\,dt = \frac{1}{12}\sin^3 t\Big|_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{32}.$$

6.4.3. Compute the integrals:

(a) $\int_0^a x^2\sqrt{a^2-x^2}\,dx$; (b) $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{(1+x^2)^3}}$.

6.4.4. Compute the integrals:

(a) $\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{6-5\sin x+\sin^2 x}$; (b) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Solution. (a) Apply the substitution

$\sin x = t$;
$\cos x\,dx = dt$;

x	t
0	0
$\frac{\pi}{2}$	1

The inverse function $x = \arcsin t$ $\left(0 \leqslant x \leqslant \frac{\pi}{2} \text{ for } 0 \leqslant t \leqslant 1\right)$ satisfies all conditions of the theorem on changing the variable. Hence,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{6-5\sin x+\sin^2 x} = \int_0^1 \frac{dt}{6-5t+t^2} = \ln\frac{t-3}{t-2}\Big|_0^1 = \ln\frac{4}{3}.$$

(b) Make the substitution $t = \tan\frac{x}{2}$

$x = 2\arctan t$, $dx = \frac{2dt}{1+t^2}$,

x	t
0	0
$\frac{\pi}{2}$	1

which is valid due to monotonicity of the function $\tan\frac{x}{2}$ on the interval $\left[0, \frac{\pi}{2}\right]$.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = 2\int_0^1 \frac{dt}{3+t^2} =$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} \Big|_0^1 = \frac{2}{\sqrt{3}} \left(\arctan \frac{1}{\sqrt{3}} - \arctan 0\right) = \frac{\pi}{3\sqrt{3}}.$$

6.4.5. Compute the integral

$$\int_0^{\frac{\pi}{4}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} \qquad (a > 0, \quad b > 0).$$

Solution. Make the substitution

$$\tan x = t, \qquad \frac{dx}{\cos^2 x} = dt,$$

x	t
0	0
$\frac{\pi}{4}$	1

Hence,

$$\int_0^{\frac{\pi}{4}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^1 \frac{dt}{a^2+b^2t^2} = \frac{1}{b^2} \int_0^1 \frac{dt}{\frac{a^2}{b^2}+t^2} =$$

$$= \frac{1}{b^2} \cdot \frac{b}{a} \arctan \frac{bt}{a} \Big|_0^1 = \frac{1}{ab} \arctan \frac{b}{a}.$$

If $a = b = 1$, then $\frac{1}{ab} \arctan \frac{b}{a} = \arctan 1 = \frac{\pi}{4}$, which exactly coincides with the result of the substitution $a = b = 1$ into the initial integral

$$\int_0^{\frac{\pi}{4}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\frac{\pi}{4}} dx = \frac{\pi}{4}.$$

6.4.6. Compute the integrals:

(a) $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2} dx$; (b) $\int_1^{e^2} \frac{dx}{x\sqrt{1+\ln x}}$;

(c) $\int_3^2 \frac{\sqrt[3]{(x-2)^2}}{3+\sqrt[3]{(x-2)^2}} dx$.

6.4.7. Compute the integral $I=\int\limits_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx.$

Solution. Reduce this integral to the sum of two integrals:

$$I=\int\limits_0^{\frac{\pi}{2}}\frac{x\sin x}{1+\cos^2 x}\,dx+\int\limits_{\frac{\pi}{2}}^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx=I_1+I_2.$$

To the integral

$$I_2=\int\limits_{\frac{\pi}{2}}^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx$$

apply the substitution

$x=\pi-t,$
$dx=-dt,$

x	t
$\frac{\pi}{2}$	$\frac{\pi}{2}$
π	0

.

Then

$$I_2=-\int\limits_{\frac{\pi}{2}}^{0}\frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}\,dt=\int\limits_0^{\frac{\pi}{2}}\frac{(\pi-t)\sin t}{1+\cos^2 t}\,dt=$$

$$=\pi\int\limits_0^{\frac{\pi}{2}}\frac{\sin t}{1+\cos^2 t}\,dt-\int\limits_0^{\frac{\pi}{2}}\frac{t\sin t}{1+\cos^2 t}\,dt.$$

Hence

$$I=I_1+I_2=\int\limits_0^{\frac{\pi}{2}}\frac{x\sin x\,dx}{1+\cos^2 x}+\pi\int\limits_0^{\frac{\pi}{2}}\frac{\sin t\,dt}{1+\cos^2 t}-\int\limits_0^{\frac{\pi}{2}}\frac{t\sin t\,dt}{1+\cos^2 t}.$$

Since the first and the third integrals differ only in the notation of the variable of integration, we have

$$I=\pi\int\limits_0^{\frac{\pi}{2}}\frac{\sin t\,dt}{1+\cos^2 t}.$$

To this integral apply the substitution

$$u=\cos t,\qquad du=-\sin t\,dt,$$

t	u
0	1
$\frac{\pi}{2}$	0

,

$$I=-\pi\int_1^0\frac{du}{1+u^2}=\pi\int_0^1\frac{du}{1+u^2}=\frac{\pi^2}{4}.$$

Note. The indefinite integral $\int\frac{x\sin x}{1+\cos^2 x}dx$ is not expressed in elementary functions. But the given definite integral, as we have shown, can be computed with the aid of an artificial method.

6.4.8. Evaluate the integral

$$I=\int_0^1\frac{\ln(1+x)}{1+x^2}\,dx.$$

Solution. Make the substitution

$$x=\tan t,\qquad dx=\frac{dt}{\cos^2 t},$$

x	t
0	0
1	$\frac{\pi}{4}$

.

Hence,

$$I=\int_0^{\frac{\pi}{4}}\frac{\ln(1+\tan t)\sec^2 t}{\sec^2 t}\,dt=\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\,dt.$$

Transform the sum $1+\tan t$:

$$1+\tan t=\tan\frac{\pi}{4}+\tan t=\frac{\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)}{\cos t}.$$

Substituting into the integral, we obtain

$$I=\int_0^{\frac{\pi}{4}}\frac{1}{2}\ln 2\,dt+\int_0^{\frac{\pi}{4}}\ln\sin\left(t+\frac{\pi}{4}\right)dt-\int_0^{\frac{\pi}{4}}\ln\cos t\,dt=$$

$$=\frac{1}{2}t\ln 2\Big|_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\ln\sin\left(t+\frac{\pi}{4}\right)dt-\int_0^{\frac{\pi}{4}}\ln\cos t\,dt=$$

$$=\frac{\pi}{8}\ln 2+\int_0^{\frac{\pi}{4}}\ln\sin\left(t+\frac{\pi}{4}\right)dt-\int_0^{\frac{\pi}{4}}\ln\cos t\,dt=\frac{\pi}{8}\ln 2+I_1-I_2.$$

Now let us show that $I_1 = I_2$. To this end apply the substitution

$$t = \frac{\pi}{4} - z, \qquad dt = -dz,$$

t	z
0	$\frac{\pi}{4}$
$\frac{\pi}{4}$	0

to the integral $I_2 = \int\limits_0^{\frac{\pi}{4}} \ln\cos t\,dt$.

Then

$$I_2 = -\int\limits_{\frac{\pi}{4}}^{0} \ln\cos\left(\frac{\pi}{4} - z\right) dz = \int\limits_0^{\frac{\pi}{4}} \ln\sin\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - z\right)\right] dz =$$

$$= \int\limits_0^{\frac{\pi}{4}} \ln\sin\left(\frac{\pi}{4} + z\right) dz = I_1.$$

Therefore

$$I = \frac{\pi}{8}\ln 2.$$

Note that in this problem, as well as in the preceding one, the indefinite integral $\int \frac{\ln(1+x)}{1+x^2}\,dx$ is not expressed in elementary functions.

6.4.9. Prove that for any given integral with finite limits a and b one can always choose the linear substitution $x = pt + q$ (p, q constants) so as to transform this integral into a new one with limits 0 and 1.

Solution. We notice that the substitution $x = pt + q$ satisfies explicitly the conditions of the theorem on changing the variable. Since t must equal zero at $x = a$ and t must equal unity at $x = b$ we have for p and q the following system of equations

$$a = p \cdot 0 + q,$$
$$b = p \cdot 1 + q,$$

whence $p = b - a$, $q = a$. Hence,

$$\int\limits_a^b f(x)\,dx = (b - a)\int\limits_0^1 f[(b - a)t + a]\,dt.$$

6.4.10. Compute the sum of two integrals

$$\int_{-4}^{-5} e^{(x+5)^2}\,dx+3\int_{\frac{1}{3}}^{\frac{2}{3}} e^{9\left(x-\frac{2}{3}\right)^2}\,dx.$$

Solution. Let us transform each of the given integrals into an integral with limits 0 and 1 (see the preceding problem).

To this end apply the substitution $x=-t-4$ to the first integral. Then $dx=-dt$ and

$$I_1=\int_{-4}^{-5} e^{(x+5)^2}\,dx=-\int_0^1 e^{(-t+1)^2}\,dt=-\int_0^1 e^{(t-1)^2}\,dt.$$

Apply the substitution $x=\frac{t}{3}+\frac{1}{3}$ to the second integral. Then $dx=\frac{dt}{3}$ and

$$I_2=3\int_{\frac{1}{3}}^{\frac{2}{3}} e^{9\left(x-\frac{2}{3}\right)^2}\,dx=\int_0^1 e^{(t-1)^2}\,dt.$$

Hence

$$I_1+I_2=-\int_0^1 e^{(t-1)^2}\,dt+\int_0^1 e^{(t-1)^2}\,dt=0.$$

Note that neither of the integrals $\int e^{(x+5)^2}\,dx$ and $\int e^{9\left(x-\frac{2}{3}\right)^2}\,dx$ is evaluated separately in elementary functions.

6.4.11. Prove that the integral

$$\int_0^{\pi} \frac{\sin 2kx}{\sin x}\,dx$$

equals zero if k is an integer.

Solution. Make the substitution

$$x=\pi-t,\qquad dx=-dt,$$

x	t
0	π
π	0

Then at k an integral number we get:

$$\int_0^{\pi} \frac{\sin 2kx}{\sin x}\,dx=-\int_{\pi}^{0} \frac{\sin 2k(\pi-t)}{\sin(\pi-t)}\,dt=-\int_0^{\pi} \frac{\sin 2kt}{\sin t}\,dt.$$

Since the definite integral does not depend on notation of the variable of integration, we have

$$I=-I, \text{ whence } I=0.$$

6.4.12. Compute the integral

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x\sqrt{1-x^2}}.$$

Solution. Apply the substitution $x=\sin t$ (the given function is not monotonic), $dx=\cos t\,dt$. The new limits of integration t_1 and t_2 are found from the equations $\frac{1}{2}=\sin t$; $\frac{\sqrt{3}}{2}=\sin t$. We may put $t_1=\frac{\pi}{6}$ and $t_2=\frac{\pi}{3}$, but other values may also be chosen, for instance, $t_1=\frac{5\pi}{6}$ and $t_2=\frac{2\pi}{3}$.

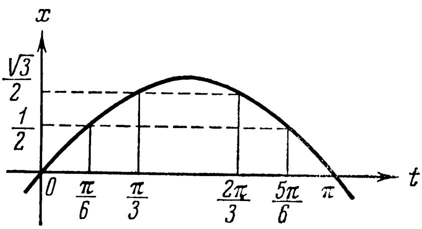

Fig. 64

In both cases the variable $x=\sin t$ runs throughout the entire interval $\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$ (see Fig. 64), the function $\sin t$ being monotonic both on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $\left[\frac{2\pi}{3}, \frac{5\pi}{6}\right]$.

Let us show that the results of the two integrations will coincide. Indeed,

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x\sqrt{1-x^2}}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos t\,dt}{\sin t\cos t}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dt}{\sin t}=\ln\left|\tan\frac{t}{2}\right|\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=$$

$$=\ln\tan\frac{\pi}{6}-\ln\tan\frac{\pi}{12}=\ln\frac{2+\sqrt{3}}{\sqrt{3}}.$$

On the other hand, taking into consideration that $\cos t$ is negative on the interval $\left[\frac{2\pi}{3}; \frac{5\pi}{6}\right]$, we obtain

$$\int\limits_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x\sqrt{1-x^2}} = \int\limits_{\frac{5\pi}{6}}^{\frac{2\pi}{3}} \frac{\cos t\,dt}{\sin t\,(-\cos t)} = \int\limits_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{dt}{\sin t} =$$

$$= \ln\left|\tan\frac{t}{2}\right|\Bigg|_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} = \ln\frac{\tan\frac{5}{12}\pi}{\tan\frac{\pi}{3}} = \ln\frac{2+\sqrt{3}}{\sqrt{3}}.$$

Note. Do not take $t_1 = \frac{5\pi}{6}$, $t_2 = \frac{\pi}{3}$, since, with t varying on the interval $\left[\frac{\pi}{3}, \frac{5\pi}{6}\right]$, the values of the function $x = \sin t$ lie beyond the limits of the interval $\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$.

6.4.13. Prove that the function $L(x)$ defined on the interval $(0, \infty)$ by the integral $L(x) = \int\limits_1^x \frac{dt}{t}$ possesses the following properties:

$$L(x_1x_2) = L(x_1) + L(x_2),$$
$$L\left(\frac{x_1}{x_2}\right) = L(x_1) - L(x_2).$$

Solution. By the additivity property

$$L(x_1x_2) = \int\limits_1^{x_1x_2} \frac{dt}{t} = \int\limits_1^{x_1} \frac{dt}{t} + \int\limits_{x_1}^{x_1x_2} \frac{dt}{t}.$$

Let us change the variable in the second integral

$$t = x_1z, \qquad dt = x_1dz,$$

t	z
x_1	1
x_1x_2	x_2

Then

$$L(x_1x_2) = \int\limits_1^{x_1} \frac{dt}{t} + \int\limits_1^{x_2} \frac{dz}{z} = L(x_1) + L(x_2).$$

Putting here $x_1x_2 = x_3$; $x_2 = \frac{x_3}{x_1}$, we obtain

$$L(x_3) = L(x_1) + L\left(\frac{x_3}{x_1}\right), \text{ i.e. } L\left(\frac{x_3}{x_1}\right) = L(x_3) - L(x_1).$$

It is also easy to obtain the other corollary $L\left(x^{\frac{m}{n}}\right)=\frac{m}{n}L(x)$ for any integral m and n.

Indeed, for positive m and n this follows from the relations

$$L\left(x^{\frac{m}{n}}\right)=mL\left(x^{\frac{1}{n}}\right),\quad L(x)=nL\left(x^{\frac{1}{n}}\right),$$

and for a negative exponent, from

$$L(1)=0,\ L(x^{-1})=L\left(\frac{1}{x}\right)=L(1)-L(x)=-L(x).$$

Now, taking advantage of the continuity of the integral as a function of the upper limit, we get the general property $L(x^a)=aL(x)$.

Note. As is known, $L(x)=\ln x$. Here we have obtained the principal properties of the logarithm proceeding only from its determination with the aid of the integral.

6.4.14. Transform the integral $\int_0^3 (x-2)^2\,dx$ by the substitution $(x-2)^2=t$.

Solution. A formal application of the substitution throughout the interval [0, 3] would lead to the wrong result, since the inverse function $x=\varphi(t)$ is double-valued: $x=2\pm\sqrt{t}$, i.e. the function x has two branches: $x_1=2-\sqrt{t}$; $x_2=2+\sqrt{t}$. The former branch cannot attain values $x>2$, the latter values $x<2$. To obtain a correct result we have to break up the given integral in the following way:

$$\int_0^3 (x-2)^2\,dx=\int_0^2 (x-2)^2\,dx+\int_2^3 (x-2)^2\,dx,$$

and to put $x=2-\sqrt{t}$ in the first integral, and $x=2+\sqrt{t}$ in the second. Then we get

$$I_1=\int_0^2 (x-2)^2\,dx=-\int_4^0 t\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^4 \sqrt{t}\,dt=\frac{8}{3},$$

$$I_2=\int_2^3 (x-2)^2\,dx=\int_0^1 t\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^1 \sqrt{t}\,dt=\frac{1}{3}.$$

Hence, $I=\frac{8}{3}+\frac{1}{3}=3$, which is a correct result. It can be easily verified by directly computing the initial integral:

$$\int_0^3 (x-2)^2\,dx = \left.\frac{(x-2)^3}{3}\right|_0^3 = \frac{1}{3}+\frac{8}{3}=3.$$

6.4.15. Compute the integrals:

(a) $I=\int_0^1 \frac{dx}{1+\sqrt{x}}$; (b) $I=\int_0^5 \frac{dx}{2x+\sqrt{3x+1}}$;

(c) $I=\int_{\pi/4}^{\pi/3} \frac{dx}{1-\sin x}$; (d) $I=\int_0^1 \sqrt{2x-x^2}\,dx$;

(e) $I=\int_0^{\pi/4} \frac{\sin x+\cos x}{3+\sin 2x}\,dx$;

(f) $I=\int_0^a x^2\sqrt{\frac{a-x}{a+x}}\,dx,\quad a>0$;

(g) $I=\int_0^{2a} \sqrt{2ax-x^2}\,dx$; (h) $I=\int_{-1}^1 \frac{dx}{(1+x^2)^2}$.

6.4.16. Applying a suitable change of the variable, find the following definite integrals:

(a) $\int_0^2 \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}$; (b) $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$;

(c) $\int_1^2 \frac{dx}{x(1+x^4)}$; (d) $\int_{\sqrt{(3a^2+b^2)/2}}^{\sqrt{(a^2+b^2)/2}} \frac{x\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$.

6.4.17. Consider the integral $\int_{-2}^{2} \frac{dx}{4+x^2}$. It is easy to conclude that it is equal to $\frac{\pi}{4}$. Indeed,

$$\int_{-2}^{2} \frac{dx}{4+x^2} = \left.\frac{1}{2}\arctan\frac{x}{2}\right|_{-2}^{2} = \frac{1}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\frac{\pi}{4}.$$

On the other hand, making the substitution $x=\frac{1}{t}$, we have

$$dx=-\frac{dt}{t^2},\quad \begin{array}{|c|c|}\hline x & t \\ \hline -2 & -\frac{1}{2} \\ 2 & \frac{1}{2} \\ \hline \end{array},$$

$$\int_{-2}^{2}\frac{dx}{4+x^2}=-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{dt}{t^2\left(4+\frac{1}{t^2}\right)}=-\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{dt}{4t^2+1}=\frac{1}{2}\arctan 2t\Big|_{-\frac{1}{2}}^{\frac{1}{2}}=-\frac{\pi}{4}.$$

This result is obviously wrong, since the integrand $\frac{1}{4+x^2}>0$, and, consequently, the definite integral of this function cannot be equal to a negative number $-\frac{\pi}{4}$. Find the mistake.

6.4.18. Consider the integral $I=\int\limits_0^{2\pi}\frac{dx}{5-2\cos x}$. Making the substitution $\tan\frac{x}{2}=t$ we have

$$\int_0^{2\pi}\frac{dx}{5-2\cos x}=\int_0^0\frac{2\,dt}{(1+t^2)\left(5-2\frac{1-t^2}{1+t^2}\right)}=0.$$

The result is obviously wrong, since the integrand is positive, and, consequently, the integral of this function cannot be equal to zero. Find the mistake.

6.4.19. Make sure that a formal change of the variable $t=x^{\frac{2}{5}}$ leads to the wrong result in the integral $\int\limits_{-2}^{2}\sqrt[5]{x^2}\,dx$. Find the mistake and explain it.

6.4.20. Is it possible to make the substitution $x=\sec t$ in the integral $I=\int\limits_0^1\sqrt{x^2+1}\,dx$?

6.4.21. Given the integral $\int\limits_0^1\sqrt{1-x^2}\,dx$. Make the substitution $x=\sin t$. Is it possible to take the numbers π and $\frac{\pi}{2}$ as the limits for t?

6.4.22. Prove the equality

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) + f(-x)]\,dx$$

for any continuous function $f(x)$.

6.4.23. Transform the definite integral $\int_{0}^{2\pi} f(x)\cos x\,dx$ by the substitution $\sin x = t$.

§ 6.5. Simplification of Integrals Based on the Properties of Symmetry of Integrands

1. If the function $f(x)$ is even on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

2. If the function $f(x)$ is odd on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

3. If the function $f(x)$ is periodic with period T, then

$$\int_{a}^{b} f(x)\,dx = \int_{a+nT}^{b+nT} f(x)\,dx,$$

where n is an integer.

6.5.1. Compute the integral $\int_{-1}^{1} |x|\,dx$.

Solution. Since the integrand $f(x) = |x|$ is an even function, we have

$$\int_{-1}^{1} |x|\,dx = 2\int_{0}^{1} |x|\,dx = 2\int_{0}^{1} x\,dx = x^2\Big|_0^1 = 1.$$

6.5.2. Compute the integral

$$\int_{-7}^{7} \frac{x^4 \sin x}{x^6 + 2}\,dx.$$

Solution. Since the integrand is odd, we conclude at once that the integral equals zero.

6.5.3. Evaluate the integrals

(a) $\int\limits_{-\pi}^{\pi} f(x)\cos nx\,dx$;

(b) $\int\limits_{-\pi}^{\pi} f(x)\sin nx\,dx$,

if: (1) $f(x)$ is an even function; (2) $f(x)$ is an odd function.

6.5.4. Calculate the integral $\int\limits_{-5}^{5} \frac{x^5 \sin^2 x}{x^4+2x^2+1}\,dx$.

6.5.5. Compute the integral $\int\limits_{\pi}^{\frac{5}{4}\pi} \frac{\sin 2x}{\cos^4 x+\sin^4 x}\,dx$.

Solution. The integrand is a periodic function with period π, since

$$f(x+\pi)=\frac{\sin 2(x+\pi)}{\cos^4(x+\pi)+\sin^4(x+\pi)}=\frac{\sin 2x}{\cos^4 x+\sin^4 x}=f(x).$$

Therefore it is possible to subtract the number π from the upper and lower limits:

$$\int\limits_{\pi}^{\frac{5}{4}\pi} \frac{\sin 2x\,dx}{\cos^4 x+\sin^4 x}=\int\limits_{0}^{\frac{\pi}{4}} \frac{\sin 2x\,dx}{\cos^4 x+\sin^4 x}=2\int\limits_{0}^{\frac{\pi}{4}} \frac{\tan x\,dx}{\cos^2 x\,(1+\tan^4 x)}\,.$$

Make the substitution

$$t=\tan x,\qquad dt=\frac{dx}{\cos^2 x},$$

x	t
0	0
$\frac{\pi}{4}$	1

$$2\int\limits_{0}^{\frac{\pi}{4}} \frac{\tan x\,dx}{\cos^2 x(1+\tan^4 x)}=\int\limits_{0}^{1} \frac{2t\,dt}{1+t^4}=\arctan t^2\Big|_0^1=\frac{\pi}{4}\,.$$

6.5.6. Prove the equality

$$\int\limits_{-a}^{a} \cos x f(x^2)\,dx=2\int\limits_{0}^{a} \cos x f(x^2)\,dx.$$

Solution. It is sufficient to show that the integrand is even:

$$\cos(-x)f[(-x)^2]=\cos x f(x^2).$$

6.5.7. Compute the integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\,dx.$$

Solution.

$$\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\,dx =$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7-10x^5-7x^3+x}{x^2+2}\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^2(x^4-4)+1}{x^2+2}\,dx =$$

$$= 0 + 2\int_{0}^{\sqrt{2}} \left[3(x^4-2x^2)+\frac{1}{x^2+1}\right] dx =$$

$$= \frac{6}{5}x^5-4x^3+\frac{2}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\Bigg|_0^{\sqrt{2}} = -\frac{16}{5}\sqrt{2}+\frac{\pi}{2\sqrt{2}}.$$

In calculating we expanded the given integral into the sum of two integrals so as to obtain an odd integrand in the first integral and an even integrand in the second.

6.5.8. Compute the integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \ln\frac{1+x}{1-x}\,dx.$$

Solution. The function $f(x)=\cos x$ is even. Let us prove that the function $\varphi(x)=\ln\frac{1+x}{1-x}$ is odd:

$$\varphi(-x)=\ln\frac{1-x}{1+x}=\ln\left(\frac{1+x}{1-x}\right)^{-1} = -\ln\frac{1+x}{1-x} = -\varphi(x).$$

Thus, the integrand is the product of an even function by an odd one, i. e. an odd function, therefore

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \ln\frac{1+x}{1-x}\,dx = 0.$$

6.5.9. Prove the validity of the following equalities:

(a) $\int\limits_{-\frac{\pi}{8}}^{\frac{\pi}{8}} x^8 \sin^9 x\,dx = 0$; (b) $\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} e^{\cos x} dx = 2\int\limits_{0}^{\frac{1}{2}} e^{\cos x}\,dx$;

(c) $\int\limits_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$ (m and n natural numbers);

(d) $\int\limits_{-a}^{a} \sin x f(\cos x)\,dx = 0$.

6.5.10. Prove the equality

$$\int\limits_a^b f(x)\,dx = \int\limits_a^b f(a+b-x)\,dx.$$

Solution. In the right-hand integral make the substitution

$x = a+b-t, \quad dx = -dt,$

x	t
a	b
b	a

.

Then we obtain

$$\int\limits_a^b f(a+b-x)\,dx = -\int\limits_b^a f(t)\,dt = \int\limits_a^b f(t)\,dt = \int\limits_a^b f(x)\,dx.$$

Note. The relation established between the integrals can be explained geometrically.

The graph of the function $f(x)$, considered on the interval $[a, b]$, is symmetrical to that of the function $f(a+b-x)$, considered on the same interval, about the straight line $x = \frac{a+b}{2}$. Indeed, if the point A lies on the x-axis and has the abscissa x, then the point A', which is symmetrical to it about the indicated straight line, has the abscissa $x' = a+b-x$. Therefore, $f(a+b-x') = f[a+b-(a+b-x)] = f(x)$. But symmetrical figures have equal areas which are expressed by definite integrals. And so, the proved equality is an equality of areas of two symmetrical curvilinear trapezoids.

6.5.11. Prove the equality

$$\int\limits_0^t f(x)\,g(t-x)\,dx = \int\limits_0^t g(x)\,f(t-x)\,dx.$$

Solution. Apply the substitution $t-x=z$ in the right-hand integral; then we have

$$-\int_t^0 g(t-z)\, f(z)\, dz = \int_0^t f(z)\, g(t-z)\, dz.$$

6.5.12. Prove the equality $\int_0^{\frac{\pi}{2}} \sin^m x\, dx = \int_0^{\frac{\pi}{2}} \cos^m x\, dx$ and apply the obtained result in computing the following integrals:

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx \text{ and } \int_0^{\frac{\pi}{2}} \sin^2 x\, dx.$$

Solution. On the basis of Problem **6.5.10** we have

$$\int_0^{\frac{\pi}{2}} \sin^m x\, dx = \int_0^{\frac{\pi}{2}} \sin^m \left(\frac{\pi}{2}-x\right) dx = \int_0^{\frac{\pi}{2}} \cos^m x\, dx.$$

Hence, in particular,

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx;$$

add these integrals:

$$2I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\, dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2};$$

hence, $I = \frac{\pi}{4}$.

6.5.13. Prove the equality

$$\int_0^{\pi} f(\sin x)\, dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\, dx.$$

Solution. Since

$$\int_0^{\pi} f(\sin x)\, dx = \int_0^{\frac{\pi}{2}} f(\sin x)\, dx + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\, dx,$$

it is sufficient to prove that

$$\int_{\pi/2}^{\pi} f(\sin x)\,dx = \int_{0}^{\pi/2} f(\sin x)\,dx.$$

In the left integral make the substitution

$$x = \pi - t, \quad dx = -dt,$$

x	t
$\frac{\pi}{2}$	$\frac{\pi}{2}$
π	0

Then

$$\int_{\pi/2}^{\pi} f(\sin x)\,dx = -\int_{\pi/2}^{0} f[\sin(\pi - t)]\,dt =$$

$$= \int_{0}^{\pi/2} f(\sin t)\,dt = \int_{0}^{\pi/2} f(\sin x)\,dx.$$

6.5.14. Prove the equality

$$\int_{0}^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_{0}^{\pi} f(\sin x)\,dx.$$

Solution. In the left integral make the substitution

$$x = \pi - t, \quad dx = -dt,$$

x	t
0	π
π	0

Then we obtain

$$\int_{0}^{\pi} xf(\sin x)\,dx = -\int_{\pi}^{0} (\pi - t) f[\sin(\pi - t)]\,dt =$$

$$= \int_{0}^{\pi} \pi f(\sin t)\,dt - \int_{0}^{\pi} tf(\sin t)\,dt.$$

Whence

$$2\int_{0}^{\pi} xf(\sin x)\,dx = \pi\int_{0}^{\pi} f(\sin x)\,dx,$$

which is equivalent to the given equality.

6.5.15. Using the equality

$$\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}=\frac{1}{2}+\cos x+\cos 2x+\ldots+\cos nx,$$

prove that

$$\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx=\pi.$$

6.5.16. Prove that if $\varphi(x)=\frac{1}{2}a_0+a_1\cos x+b_1\sin x+a_2\cos 2x+$ $+b_2\sin 2x+\ldots+a_n\cos nx+b_n\sin nx$, then

(a) $\int_0^{2\pi}\varphi(x)\,dx=\pi a_0$; (b) $\int_0^{2\pi}\varphi(x)\cos kx\,dx=\pi a_k$;

(c) $\int_0^{2\pi}\varphi(x)\sin kx\,dx=\pi b_k$ $(k=1, 2, \ldots, n)$.

§ 6.6. *Integration by Parts. Reduction Formulas*

If u and v are functions of x and have continuous derivatives, then

$$\int_a^b u(x)\,v'(x)\,dx=u(x)\,v(x)\Big|_a^b-\int_a^b v(x)\,u'(x)\,dx$$

or, more briefly,

$$\int_a^b u\,dv=uv\Big|_a^b-\int_a^b v\,du.$$

6.6.1. Compute the integral $\int_0^1 xe^x\,dx$.

Solution. Let us put

$$x=u,\qquad e^x\,dx=dv;$$
$$du=dx;\qquad v=e^x,$$

which is quite legitimate, since the functions $u=x$ and $v=e^x$ are continuous and have continuous derivatives on the interval $[0, 1]$.

Using the formula for integration by parts, we obtain

$$\int_0^1 xe^x\,dx=xe^x\Big|_0^1-\int_0^1 e^x\,dx=e-e^x\Big|_0^1=1.$$

6.6.2. Compute the integral $I=\int_0^{\frac{\pi}{b}} e^{ax}\sin bx\,dx$.

Solution. Let us put

$$u=\sin bx, \qquad dv=e^{ax}\,dx;$$
$$du=b\cos bx\,dx, \qquad v=\frac{1}{a}e^{ax}.$$

Since the functions $u=\sin bx$, $v=\frac{1}{a}e^{ax}$ together with their derivatives are continuous on the interval $[0, \pi]$, the formula for integration by parts is applicable:

$$I=\frac{1}{a}e^{ax}\sin bx\,\Big|_0^{\frac{\pi}{b}}-\frac{b}{a}\int_0^{\frac{\pi}{b}} e^{ax}\cos bx\,dx=$$
$$=-\frac{b}{a}\int_0^{\frac{\pi}{b}} e^{ax}\cos bx\,dx=-\frac{b}{a}\,I_1.$$

Now let us integrate by parts the integral I_1. Put

$$u=\cos bx, \qquad dv=e^{ax}\,dx,$$
$$du=-b\sin bx\,dx, \qquad v=\frac{1}{a}e^{ax}.$$

Then

$$I=-\frac{b}{a}\left(\frac{1}{a}e^{ax}\cos bx\,\Big|_0^{\frac{\pi}{b}}+\frac{b}{a}\int_0^{\frac{\pi}{b}} e^{ax}\sin bx\,dx\right)=$$
$$=-\frac{b}{a}\left(-\frac{e^{\frac{a\pi}{b}}}{a}-\frac{1}{a}\right)-\frac{b^2}{a^2}I=\frac{b\left(e^{\frac{a\pi}{b}}+1\right)}{a^2}-\frac{b^2}{a^2}I.$$

Hence

$$\frac{a^2+b^2}{a^2}I=\frac{b\left(e^{\frac{a\pi}{b}}+1\right)}{a^2}, \qquad I=\frac{b\left(e^{\frac{a\pi}{b}}+1\right)}{a^2+b^2}.$$

In particular, at $a=b=1$ we get

$$\int_0^{\pi} e^x\sin x\,dx=\frac{1}{2}(e^{\pi}+1).$$

6.6.3. Compute the integral $\int_1^e \ln^3 x\,dx$.

6.6.4. Compute the integral $\int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\,dx$.

Solution. First make the substitution

$$\sqrt{x}=t,\quad x=t^2,\quad dx=2t\,dt,$$

x	t
0	0
$\frac{\pi^2}{4}$	$\frac{\pi}{2}$

Whence

$$\int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\,dx = 2\int_0^{\frac{\pi}{2}} t\sin t\,dt.$$

Integrate by parts the latter integral.
Put

$$t=u;\qquad \sin t\,dt=dv;$$
$$du=dt;\qquad v=-\cos t.$$

Then

$$2\int_0^{\frac{\pi}{2}} t\sin t\,dt = 2\left[-t\cos t\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t\,dt\right] = 2\sin t\Big|_0^{\frac{\pi}{2}} = 2.$$

6.6.5. Compute the integral $I=\int_0^1 \frac{\arcsin x}{\sqrt{1+x}}\,dx$.

6.6.6. Compute the integral $\int_0^{\frac{\pi}{2}} x^2\sin x\,dx$.

6.6.7. Compute the integral $I_n=\int_0^a (a^2-x^2)^n\,dx$, where n is a natural number.

Solution. The integral can be computed by expanding the integrand $(a^2-x^2)^n$ according to the formula of the Newton binomial, but it involves cumbersome calculations. It is simpler to deduce a formula for reducing the integral I_n to the integral I_{n-1}. To this end let us expand the integral I_n in the following way:

$$I_n=\int_0^a (a^2-x^2)^{n-1}(a^2-x^2)\,dx=a^2I_{n-1}-\int_0^a x(a^2-x^2)^{n-1}x\,dx$$

and integrate the latter integral by parts:

$$u=x; \qquad (a^2-x^2)^{n-1}x\,dx=dv,$$
$$du=dx; \qquad v=-\frac{1}{2n}(a^2-x^2)^n \quad (n\neq 0).$$

We obtain

$$I_n=a^2I_{n-1}+\frac{1}{2n}x(a^2-x^2)^n\Big|_0^a-\frac{1}{2n}\int_0^a (a^2-x^2)^n\,dx=a^2I_{n-1}-\frac{1}{2n}I_n.$$

Whence

$$I_n=a^2\frac{2n}{2n+1}I_{n-1}.$$

This formula is valid at any real n other than 0 and $-\frac{1}{2}$.

In particular, at natural n, taking into account that

$$I_0=\int_0^a dx=a,$$

we get

$$I_n=a^{2n+1}\frac{2n(2n-2)(2n-4)\ldots 6\cdot 4\cdot 2}{(2n+1)(2n-1)(2n-3)\ldots 5\cdot 3}=a^{2n+1}\frac{(2n)!!}{(2n+1)!!},$$

where

$$(2n)!!=2\cdot 4\cdot 6\ldots(2n),$$
$$(2n+1)!!=1\cdot 3\cdot 5\ldots(2n+1).$$

6.6.8. Using the result of the preceding problem obtain the following formula:

$$1-\frac{C_n^1}{3}+\frac{C_n^2}{5}-\frac{C_n^3}{7}+\ldots+(-1)^n\frac{C_n^n}{2n+1}=\frac{(2n)!!}{(2n+1)!!},$$

where C_n^k are binomial coefficients.

Solution. Consider the integral

$$I_n=\int_0^1 (1-x^2)^n\,dx=\frac{(2n)!!}{(2n+1)!!}.$$

Expanding the integrand by the formula of the Newton binomial and integrating within the limits from 0 to 1, we get:

$$I_n = \int_0^1 (1-x^2)^n\, dx =$$

$$= \int_0^1 (1 - C_n^1 x^2 + C_n^2 x^4 - C_n^3 x^6 + \ldots + (-1)^n C_n^n x^{2n})\, dx =$$

$$= \left[x - \frac{C_n^1 x^3}{3} + \frac{C_n^2 x^5}{5} - \frac{C_n^3 x^7}{7} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} \right]_0^1 =$$

$$= 1 - \frac{C_n^1}{3} + \frac{C_n^2}{5} - \frac{C_n^3}{7} + \ldots + \frac{(-1)^n}{2n+1},$$

which completes the proof.

6.6.9. Compute the integral

$$H_m = \int_0^{\frac{\pi}{2}} \sin^m x\, dx = \int_0^{\frac{\pi}{2}} \cos^m x\, dx$$

(m a natural number).

Solution. The substitution

$$\sin x = t, \quad \cos x\, dx = dt,$$

x	t
0	0
$\frac{\pi}{2}$	1

reduces the second integral to the integral

$$H_m = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)^{\frac{m-1}{2}} \cos x\, dx = \int_0^1 (1-t^2)^{\frac{m-1}{2}}\, dt,$$

considered in Problem **6.6.7** with $a=1$ and $n=\frac{m-1}{2}$. Therefore, the reduction formula

$$H_m = \frac{m-1}{m} H_{m-2} \quad (m \neq 0,\ m \neq 1)$$

is valid here, since

$$H_m = I_{\frac{m-1}{2}} = \frac{2\cdot\frac{m-1}{2}}{2\cdot\frac{m-1}{2}+1} I_{\frac{m-1}{2}-1} = \frac{m-1}{m} I_{\frac{m-3}{2}} = \frac{m-1}{m} H_{m-2}.$$

If m is an odd number, the obtained reduction formula reduces H_m to

$$H_1 = \int_0^{\frac{\pi}{2}} \cos x\, dx = 1,$$

therefore

$$H_m = \frac{(m-1)!!}{m!!}.$$

If m is an even number then the reduction formula transforms H_m into

$$H_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2},$$

therefore

$$H_m = \frac{(m-1)!!}{m!!}\,\frac{\pi}{2}.$$

6.6.10. Compute the integral

$$I = \int_0^{\pi} x \sin^m x\, dx$$

(m a natural number).

Solution. Taking advantage of the results of Problems **6.5.14** and **6.5.13**, we get

$$I = \int_0^{\pi} x \sin^m x\, dx = \frac{\pi}{2}\int_0^{\pi} \sin^m x\, dx = \pi \int_0^{\frac{\pi}{2}} \sin^m x\, dx,$$

which, taking into consideration the result of Problem **6.6.9**, gives

$$I = \int_0^{\pi} x \sin^m x\, dx = \begin{cases} \frac{\pi^2}{2} \cdot \frac{(m-1)!!}{m!!} & \text{if } m \text{ is even,} \\ \pi \frac{(m-1)!!}{m!!} & \text{if } m \text{ is odd.} \end{cases}$$

6.6.11. Compute the integral $I_n = \int_0^1 x^m (\ln x)^n\, dx$; $m > 0$, n is a natural number.

Solution. First of all note that, though the integrand $f(x) = x^m (\ln x)^n$ has no meaning at $x = 0$ it can be made continuous on the interval

[0, 1] for any $m>0$ and $n>0$, by putting $f(0)=0$. Indeed,

$$\lim_{x\to+0} x^m (\ln x)^n = \lim_{x\to+0} \left(x^{\frac{m}{n}} \ln x\right)^n = 0$$

by virtue of Problem **3.2.4.**

Hence, in particular, it follows that the integral I_n exists at $m>0$, $n>0$. To compute it we integrate by parts, putting

$$u=(\ln x)^n, \qquad dv = x^m\,dx,$$
$$du = \frac{n(\ln x)^{n-1}}{x}\,dx, \qquad v=\frac{x^{m+1}}{m+1}.$$

Hence,

$$I_n=\int_0^1 x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1}\Bigg|_0^1 - \frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1} dx = -\frac{n}{m+1} I_{n-1}.$$

The formula obtained reduces I_n to I_{n-1}. In particular, with a natural n, taking into account that

$$I_0=\int_0^1 x^m\,dx=\frac{1}{m+1},$$

we get

$$I_n=(-1)^n \frac{n!}{(m+1)^{n+1}}.$$

6.6.12. Compute the integral $I_{m,n}=\int_0^1 x^m(1-x)^n\,dx$,

where m and n are non-negative integers.

Solution. Let us put

$$(1-x)^n=u; \qquad x^m\,dx=dv;$$
$$du=-n(1-x)^{n-1}\,dx; \qquad v=\frac{x^{m+1}}{m+1}.$$

Then

$$I_{m,n}=\left[\frac{x^{m+1}}{m+1}(1-x)^n\right]_0^1+\frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}\,dx=\frac{n}{m+1} I_{m+1,\,n-1}.$$

The obtained formula is valid for all $n>0$, and $m>-1$. If n is a positive integer, then, applying this formula successively n times, we get

$$I_{m,n}=\frac{n}{m+1} I_{m+1,\,n-1}=\frac{n(n-1)}{(m+1)(m+2)} I_{m+2,\,n-2}=\ldots$$
$$\ldots=\frac{n(n-1)\ldots[n-(n-1)]}{(m+1)(m+2)\ldots(m+n)} I_{m+n,\,0}.$$

But

$$I_{m+n,\,0}=\int_0^1 x^{m+n}\,dx=\left.\frac{x^{m+n+1}}{m+n+1}\right|_0^1=\frac{1}{m+n+1}.$$

Hence,

$$I_{m,\,n}=\frac{n\,(n-1)\,(n-2)\ldots 3\cdot 2\cdot 1}{(m+1)\,(m+2)\ldots(m+n)\,(m+n+1)}.$$

The obtained result, with m a non-negative integer, can be written in the form

$$I_{m,\,n}=\frac{m!n!}{(m+n+1)!}.$$

6.6.13. Compute the integrals:

(a) $\int_0^1 \arctan\sqrt{x}\,dx$; (b) $\int_0^1 (x-1)\,e^{-x}\,dx$;

(c) $\int_{\pi/4}^{\pi/3} \frac{x\,dx}{\sin^2 x}$; (d) $\int_0^1 x\arctan x\,dx$;

(e) $\int_0^1 x\ln(1+x^2)\,dx$; (f) $\int_0^{\pi/4} \ln(1+\tan x)\,dx$;

(g) $\int_0^{\pi/2} \sin 2x\arctan(\sin x)\,dx$; (h) $\int_1^{16} \arctan\sqrt{\sqrt{x}-1}\,dx$.

6.6.14. Prove that

$$\int_0^1 (\arccos x)^n\,dx=n\left(\frac{\pi}{2}\right)^{n-1}-n\,(n-1)\int_0^1 (\arccos x)^{n-2}\,dx \quad (n>1).$$

6.6.15. Prove that if $f''(x)$ is continuous on $[a, b]$, then the following formula is valid

$$\int_a^b xf''(x)\,dx=[bf'(b)-f(b)]-[af'(a)-f(a)].$$

§ 6.7. Approximating Definite Integrals

1. **Trapezoidal formula.** Divide the interval $[a, b]$ into n equal parts by points $x_k=a+kh$, where $h=\frac{b-a}{n}$, $k=0,\ 1,\ \ldots,\ n$, and apply the formula

$$\int_a^b f(x)\,dx\approx\frac{b-a}{n}\left[\frac{1}{2}f(x_0)+f(x_1)+\ldots+f(x_{n-1})+\frac{1}{2}f(x_n)\right].$$

The error R in this formula is estimated as follows:

$$|R| \leqslant \frac{M_2 (b-a)^3}{12n^2}, \text{ where } M_2 = \sup_{a \leqslant x \leqslant b} |f''(x)|$$

(assuming that the second derivative is bounded).

2. **Simpson's formula.** Divide the interval $[a, b]$ into $2n$ equal parts by points $x_k = a + kh$, where $h = \frac{b-a}{2n}$, and apply the formula

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6n}\{f(x_0) + f(x_{2n}) + 4[f(x_1) + f(x_3) + \ldots$$
$$\ldots + f(x_{2n-1})] + 2[f(x_2) + f(x_4) + \ldots + f(x_{2n-2})]\}.$$

Assuming that $f^{IV}(x)$ exists and is bounded, the error in this formula is estimated in the following way:

$$|R| \leqslant \frac{M_4 (b-a)^5}{180(2n)^4}, \text{ where } M_4 = \sup_{a \leqslant x \leqslant b} |f^{IV}(x)|.$$

6.7.1. Approximate the integral $I = \int_0^1 \frac{dx}{1+x}$ using the trapezoidal formula at $n = 10$.

Solution. Let us tabulate the values of the integrand, the ordinates $y_i = f(x_i)$ $(i = 0, 1, \ldots, 10)$ being calculated within four decimal places.

x_i	$1+x_i$	$y_i = \frac{1}{1+x_i}$	x_i	$1+x_i$	$y_i = \frac{1}{1+x_i}$
0.0000	1.0000	1.0000	0.6000	1.6000	0.6250
0.1000	1.1000	0.9091	0.7000	1.7000	0.5882
0.2000	1.2000	0.8333	0.8000	1.8000	0.5556
0.3000	1.3000	0.7692	0.9000	1.9000	0.5263
0.4000	1.4000	0.7143	1.0000	2.0000	0.5000
0.5000	1.5000	0.6667			

Using the trapezoidal formula, we obtain

$$I = \int_0^1 \frac{dx}{1+x} \approx \frac{1}{10}\left(\frac{1.0000 + 0.5000}{2} + 0.9091 + 0.8333 + \right.$$
$$+ 0.7692 + 0.7143 + 0.6667 + 0.6250 + 0.5882 + 0.5556 +$$
$$\left. + 0.5263\right) = \frac{1}{10} \cdot 6.9377 = 0.69377 \approx 0.6938.$$

Estimate the error in the result obtained. We have $f''(x)=\frac{2}{(1+x)^3}$. Since $0\leqslant x\leqslant 1$, then $|f''(x)|\leqslant 2$. Consequently, we may take the number 2 as M_2 and estimate the error:

$$|R|\leqslant\frac{2}{12\times 10^2}=\frac{1}{600}<0.0017.$$

We calculated the ordinates accurate to four decimal places, and the round-off error does not exceed $\frac{0.00005}{10}(1+9\times 1)=0.00005$ $\left(\text{more precisely, } \frac{0.00005}{10}\cdot 9=0.000045, \text{ since the ordinates } y_0 \text{ and } y_{10} \text{ are exact numbers}\right)$. Thus, the total error due to using the trapezoidal formula and rounding off the ordinates does not exceed 0.0018.

Note that when computing the given integral by the Newton-Leibniz formula we obtain

$$I=\int_0^1\frac{dx}{1+x}=\ln(1+x)\Big|_0^1=\ln 2\approx 0.69315.$$

Thus, the error in the result obtained does not exceed 0.0007, i. e. we have obtained a result accurate to three decimal places.

6.7.2. Evaluate by Simpson's formula the integral $\int_{0.5}^{1.5}\frac{e^{0.1x}}{x}dx$ accurate to four decimal places.

Solution. To give a value of $2n$ which ensures the required accuracy, we find $f^{IV}(x)$. Successively differentiating $f(x)=\frac{e^{0.1x}}{x}$, we get

$$f^{IV}(x)=\frac{e^{0.1x}}{x^5}(0.0001x^4-0.004x^3+0.12x^2-2.4x+24)=\frac{P(x)}{x^5}e^{0.1x},$$

where $P(x)$ is the polynomial in parentheses. On the interval $[0.5, 1.5]$ the function $\varphi(x)=e^{0.1x}$ increases and therefore reaches its greatest value at $x=1.5$: $\varphi(1.5)=e^{0.15}<1.2$. The upper estimate of the absolute value of the polynomial $P(x)$ divided by x^5 can be obtained as the sum of moduli of its separate terms. The greatest value of each summand is attained at $x=0.5$, therefore

$$\left|\frac{P(x)}{x^5}\right|<\frac{0.0001}{x}+\frac{0.004}{x^2}+\frac{0.12}{x^3}+\frac{2.4}{x^4}+\frac{24}{x^5}\leqslant$$
$$\leqslant 0.0002+0.016+0.96+38.4+768<808.$$

And so, $|f^{IV}(x)|<1.2\times 808<1\,000$. Hence, the number 1 000 may be taken as M_4.

We have to compute the integral accurate to four decimal places. To ensure such accuracy it is necessary that the sum of errors of the method, operations and final rounding off should not exceed 0.0001. For this purpose we choose a value of $2n$ (which will determine the step of integration h) so that the inequality

$$|R| < \frac{1}{2} \cdot 0.0001 = 5 \cdot 10^{-5}$$

is satisfied.

Solving the inequality

$$\frac{1^5 \times 1\,000}{180\,(2n)^4} < 5 \times 10^{-5},$$

we obtain

$$2n > 19.$$

Let us take $2n = 20$; then the step of integration h will be equal to

$$h = \frac{b-a}{2n} = \frac{1}{20} = 0.05.$$

A more accurate calculation shows that at $2n = 20$

$$|R| < 3.5 \times 10^{-5}.$$

If we calculate y_i within five decimal places, i. e. with an error not exceeding 10^{-5}, then the error of the final rounding off will also be not greater than 10^{-5}. Thus, the total error will be less than $4.5 \times 10^{-5} < 0.0001$.

Now compile a table of values of the function $y = \frac{e^{0.1x}}{x}$ for the values of x from 0.5 to 1.5 with the step $h = 0.05$. The calculations are carried out within five decimal places.

i	x_i	$0.1x_i$	$e^{0.1x_i}$	y_i
0	0.50	0.050	1.05127	2.10254
1	0.55	0.055	1.05654	1.92098
2	0.60	0.060	1.06184	1.76973
3	0.65	0.065	1.06716	1.64178
4	0.70	0.070	1.07251	1.53216
5	0.75	0.075	1.07788	1.43717
6	0.80	0.080	1.08329	1.35411
7	0.85	0.085	1.08872	1.28085
8	0.90	0.090	1.09417	1.21574
9	0.95	0.095	1.09966	1.15754
10	1.00	0.100	1.10517	1.10517
11	1.05	0.105	1.11071	1.05782

i	x_i	$0.1x_i$	$e^{0.1x_i}$	y_i
12	1.10	0.110	1.11628	1.01480
13	1.15	0.115	1.12187	0.97554
14	1.20	0.120	1.12750	0.93958
15	1.25	0.125	1.13315	0.90652
16	1.30	0.130	1.13883	0.87602
17	1.35	0.135	1.14454	0.84781
18	1.40	0.140	1.15027	0.82162
19	1.45	0.145	1.15604	0.79727
20	1.50	0.150	1.16183	0.77455

For pictorialness sake we use the tabular data to compile the following calculation chart:

i	x_i	y_i		
		at $i=0$ and $i=20$	at an odd i	at an even i
0	0.50	2.10254		
1	0.55		1.92098	
2	0.60			1.76973
3	0.65		1.64178	
4	0.70			1.53216
5	0.75		1.43717	
6	0.80			1.35411
7	0.85		1.28085	
8	0.90			1.21574
9	0.95		1.15754	
10	1.00			1.10517
11	1.05		1.05782	
12	1.10			1.01480
13	1.15		0.97554	
14	1.20			0.93958
15	1.25		0.90652	
16	1.30			0.87602
17	1.35		0.84781	
18	1.40			0.82162
19	1.45		0.79727	
20	1.50	0.77455		
	Sums	2.87709	12.02328	10.62893

Using Simpson's formula, we get

$$\int_{0.5}^{1.5} \frac{e^{0.1x}}{x}\,dx \approx \frac{1}{60}(2.87709 + 4\times 12.02328 +$$

$$+ 2\times 10.62893) = \frac{1}{60}\cdot 72.22807 = 1.2038.$$

6.7.3. The river is 26 m wide. The table below shows the successive depths of the river measured across its section at steps of 2 m:

x	0	2	4	6	8	10	12	14	16	18	20	22	24	26
y	0.3	0.9	1.7	2.1	2.8	3.4	3.3	3.0	3.5	2.9	1.7	1.2	0.8	0.6

Here x denotes the distance from one bank and y, the corresponding depth (in metres). Knowing that the mean rate of flow is 1.3 m/sec, determine the flowrate per second Q of the water in the river.

Solution. By the trapezoidal formula the area S of the cross-section

$$S=\int_0^{26} y\,dx \approx 2\left[\frac{1}{2}(0.3+0.6)+0.9+1.7+2.1+2.8+3.4+\right.$$
$$\left.+3.3+3.0+3.5+2.9+1.7+1.2+0.8\right]=55.5\ (\text{m}^2).$$

Hence,

$$Q=55.5\times 1.3\approx 72\ (\text{m}^3/\text{sec}).$$

It is impossible to estimate the error accurately in this case. Some indirect methods of estimation enable us to indicate approximately the order of the error. The error in S is about 3 m^2, hence, the error in Q is about 4 m^3/sec.

6.7.4. Compute the following integrals:

(a) $\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx$ accurate to three decimal places, using Simpson's formula;

(b) $\int_0^1 e^{-x^2}\,dx$ accurate to three decimal places, by the trapezoidal formula.

6.7.5. By Simpson's formula, approximate the integral

$$I=\int_{1.05}^{1.36} f(x)\,dx,$$

if the integrand is defined by the following table:

x	1.05	1.10	1.15	1.20	1.25	1.30	1.35
$f(x)$	2.36	2.50	2.74	3.04	3.46	3.98	4.6

§ 6.8. Additional Problems

6.8.1. Given the function

$$f(x)=\begin{cases} 1-x & \text{at } 0\leqslant x\leqslant 1, \\ 0 & \text{at } 1<x\leqslant 2, \\ (2-x)^2 & \text{at } 2<x\leqslant 3. \end{cases}$$

Check directly that the function

$$F(x)=\int_0^x f(t)\,dt$$

is continuous on the interval [0, 3] and that its derivative at each interior point of this interval exists and is equal to $f(x)$.

6.8.2. Show that the function

$$f(x)=\begin{cases} \frac{x\ln x}{1-x} & \text{at } 0<x<1, \\ 0 & \text{at } x=0 \\ -1 & \text{at } x=1 \end{cases}$$

is integrable on the interval [0, 1].

6.8.3. Can one assert that if a function is absolutely integrable on the interval $[a, b]$, then it is integrable on this interval?

6.8.4. A line tangent to the graph of the function $y=f(x)$ at the point $x=a$ forms an angle $\frac{\pi}{3}$ with the axis of abscissas and an angle $\frac{\pi}{4}$ at the point $x=b$.

Evaluate $\int_a^b f''(x)\,dx$, if $f''(x)$ is a continuous function.

6.8.5. Prove that

$$\int_0^x E(x)\,dx=\frac{E(x)(E(x)-1)}{2}+E(x)[x-E(x)].$$

6.8.6. Given the integral $\int_0^{\pi} \frac{dx}{1+\cos^2 x}$. Make sure that the functions

$$F_1(x) = \frac{1}{\sqrt{2}} \arccos \frac{\sqrt{2}\cos x}{\sqrt{1+\cos^2 x}} \text{ and } F_2(x) = \frac{1}{\sqrt{2}} \arctan \frac{\tan x}{\sqrt{2}}$$

are antiderivatives for the integrand. Is it possible to use both antiderivatives for computing the definite integral by the Newton-Leibniz formula? If not, which of the antiderivatives can be used?

6.8.7. For $f(x)$ find such an antiderivative which attains the given magnitude $y = y_0$ at $x = x_0$ (Cauchy's problem).

6.8.8. At what value of ξ is the equality $\int_a^b e^{2x}\,dx = e^{2\xi}(b-a)$ fulfilled? Show that

$$\xi > \frac{a+b}{2}.$$

6.8.9. Investigate the function defined by the definite integral $F(x) = \int_0^x \sqrt{1-t^4}\,dt$.

6.8.10. Show that the inequalities

$$0.692 \leqslant \int_0^1 x^x\,dx \leqslant 1$$

are valid.

6.8.11. With the aid of the inequality $x \geqslant \sin x \geqslant \frac{2}{\pi} x \left(0 \leqslant x \leqslant \frac{\pi}{2}\right)$ show that $1 < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < \frac{\pi}{2}$.

6.8.12. Using the inequality $\sin x \geqslant x - \frac{x^3}{6}$ $(x \geqslant 0)$ and the Schwarz-Bunyakovsky inequality, show that

$$1.096 < \int_0^{\frac{\pi}{2}} \sqrt{x \sin x}\,dx < 1.111.$$

6.8.13. Assume that integrable functions $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ are given on the interval $[a, b]$, the function $p_1(x)$ is non-negative, and the functions $p_2(x)$, $p_3(x)$, $p_4(x)$ satisfy the inequality

$$p_3(x) \leqslant p_2(x) \leqslant p_4(x).$$

Prove that

$$\int_a^b p_3(x)\, p_1(x)\, dx \leqslant \int_a^b p_2(x)\, p_1(x)\, dx \leqslant \int_a^b p_4(x)\, p_1(x)\, dx.$$

6.8.14. Let the function $f(x)$ be positive on the interval $[a, b]$. Prove that the expression

$$\int_a^b f(x)\, dx \cdot \int_a^b \frac{dx}{f(x)}$$

reaches the least value only if $f(x)$ is constant on this interval.

6.8.15. Prove that

$$\int_0^1 \frac{\arctan x}{x}\, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{t}{\sin t}\, dt.$$

6.8.16. Prove that one of the antiderivatives of an even function is an odd function, and any antiderivative of an odd function is an even function.

6.8.17. Prove that if $f(x)$ is a continuous periodic function with period T, then the integral $I = \int_a^{a+T} f(x)\, dx$ does not depend on a.

6.8.18. Prove that if $u = u(x)$, $v = v(x)$ and their derivatives through order n are continuous on the interval $[a, b]$, then

$$\int_a^b uv^{(n)}\, dx = [uv^{(n-1)} - u'v^{(n-2)} + \ldots + (-1)^{n-1} u^{(n-1)} v] \Big|_a^b + (-1)^n \int_a^b u^{(n)} v\, dx.$$