CHAPTER 5

Solutions and Properties of Triangle

- Standard Symbols
- Sine Rule
- Cosine Rule
- Projection Formula
- Half-Angle Formulae
- Area of Triangle
- Escribed Circles of A Triangle And Their Radii
- Miscellaneous Topics
- Solution of Triangles (Ambiguous Cases)

STANDARD SYMBOLS

The following symbols in relation to ΔABC are universally adopted.

- $m\angle BAC = A$
  $m\angle ABC = B$
  $m\angle BCA = C$
  $A + B + C = \pi$
  $AB = c, BC = a, CA = b$
- Semi-perimeter of the triangle, $s = \dfrac{a+b+c}{2}$

  So, $a + b + c = 2s$

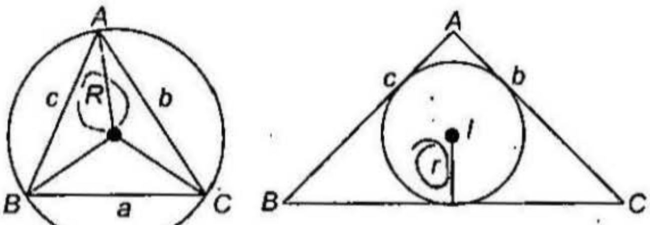


Fig. 5.1

- The radius of the circumcircle of the triangle, i.e., circumradius = R
- The radius of the incircle of the triangle, i.e., inradius = r
- Area of the triangle = Δ

SINE RULE

The sine rule is $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

We shall prove here that $\dfrac{a}{\sin A} = 2R$.

Case I:

$0 < A < \dfrac{\pi}{2}$

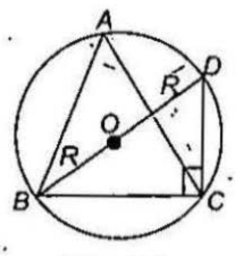


Fig. 5.2

Suppose O is the circumcentre of ΔABC.

$\overline{BO}$ intersects the circumcircle at D.

Here, $BD = 2OB = 2R$ and $\angle D = \angle BDC \cong \angle A$ [angles in the same segment] (i)

Now in ΔBCD, $m\angle BCD = \dfrac{\pi}{2}$ [angle in a semicircle]

$\Rightarrow \sin D = \dfrac{BC}{BD} = \dfrac{a}{2R}$

$\Rightarrow \quad \sin A = \dfrac{a}{2R}$ [using Eq. (i)]

$\Rightarrow \quad \dfrac{a}{\sin A} = 2R$

Case II:

ΔABC is right angled and $A = \dfrac{\pi}{2}$

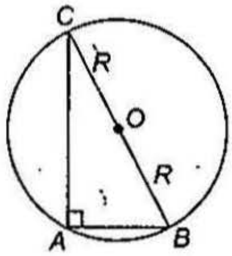


Fig. 5.3

$\overline{BC}$ is a diameter of the circumcircle.

$\therefore \quad BC = 2R$

Now, $a = BC = 2R = 2R \sin \dfrac{\pi}{2} = 2R \sin A$

$\therefore \quad \dfrac{a}{\sin A} = 2R$

Case III:

$\dfrac{\pi}{2} < A < \pi$

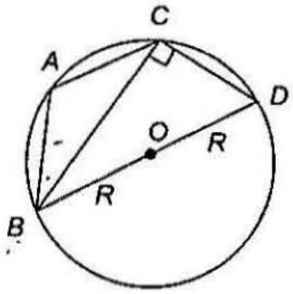


Fig. 5.4

As $\angle A$ is obtuse, so A is on the minor arc BC.
Now, take any point D on the major arc BC.

Here, $m\angle BDC = \pi - A < \dfrac{\pi}{2} \qquad \left[\dfrac{\pi}{2} < A < \pi\right]$ (ii)

Now in ΔBCD, $\sin(\angle BDC) = \dfrac{BC}{BD}$

$\Rightarrow \quad \sin(\pi - A) = \dfrac{a}{2R}$ [using Eq. (ii)]

$\Rightarrow \quad \sin A = \dfrac{a}{2R}$

$\Rightarrow \quad a = 2R \sin A$

Thus, in each case, $\frac{a}{\sin A} = 2R$.

Similarly, we can prove that $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$.

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Nepier's Formula

i. $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot \frac{A}{2}$

ii. $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a} \cot \frac{B}{2}$

iii. $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot \frac{C}{2}$

Proof:

i. From the sine rule, we have $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \frac{\sin B}{\sin C} = \frac{b}{c}$$

$$\Rightarrow \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b-c}{b+c}$$

$$\Rightarrow \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)} = \frac{b-c}{b+c}$$

$$\Rightarrow \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}$$

$$\Rightarrow \tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}$$

$$\Rightarrow \frac{\tan\left(\frac{B-C}{2}\right)}{\cot\frac{A}{2}} = \frac{b-c}{b+c}$$

$$\Rightarrow \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$$

Similarly, we can prove the other formulae.

Note:

These formulae are also known as tangent rules. These are useful in calculating the remaining parts of a triangle when two sides and included angle are given.

Example 5.1 **The perimeter of a triangle ABC is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then find angle A.**

Sol.

Given that $a+b+c=6\times\dfrac{\sin A+\sin B+\sin C}{3}$

$\Rightarrow \quad 2R(\sin A+\sin B+\sin C)=2(\sin A+\sin B+\sin C) \qquad \Rightarrow R=1$

Now, $\dfrac{a}{\sin A}=2R$

$\Rightarrow \quad \sin A=\dfrac{1}{2}\ (\because a=1) \qquad \Rightarrow A=30°$

Example 5.2 **If $\dfrac{\cos A}{a}=\dfrac{\cos B}{b}=\dfrac{\cos C}{c}$ and the side $a=2$, then find the area of the triangle.**

Sol. $\dfrac{\cos A}{a}=\dfrac{\cos B}{b}=\dfrac{\cos C}{c}$

$\Rightarrow \quad \dfrac{\cos A}{2R\sin A}=\dfrac{\cos B}{2R\sin B}=\dfrac{\cos C}{2R\sin C}$

$\Rightarrow \quad \tan A=\tan B=\tan C$

$\Rightarrow \quad \Delta$ is equilateral

$\Rightarrow \quad$ Area $\Delta=\dfrac{\sqrt{3}}{4}a^2=\sqrt{3}$ (as $a=2$)

Example 5.3 **If $A=75°, B=45°$, then prove that $b+c\sqrt{2}=2a$.**

Sol. $A=75°, B=45° \Rightarrow C=60°$

$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$

$\Rightarrow \quad \dfrac{a}{\sin 75°}=\dfrac{b}{\sin 45°}=\dfrac{c}{\sin 60°}=2R$

$\Rightarrow b+c\sqrt{2}=\dfrac{\sin 45°}{\sin 75°}a+\sqrt{2}\dfrac{\sin 60°}{\sin 75°}a$

$$=\frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a+\sqrt{2}\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a=\frac{2}{\sqrt{3}+1}a+\frac{2\sqrt{3}a}{\sqrt{3}+1}=2a$$

Example 5.4 **If the base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, then prove that the altitude of the triangle is equal to $\frac{1}{2}$ of its base.**

Sol.

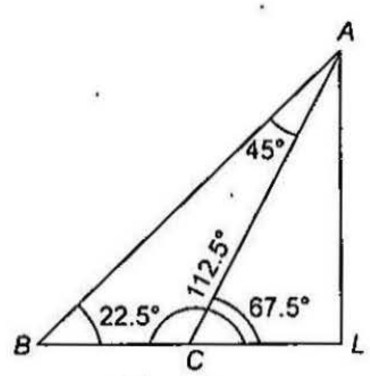


Fig. 5.5

In ΔABC, $\dfrac{BC}{\sin 45^\circ} = \dfrac{AC}{\sin 22\frac{1}{2}^\circ}$ (i)

In ΔALC, $\dfrac{AL}{AC} = \sin 67\frac{1}{2}^\circ$

$\therefore \quad AL = AC \cos 22\frac{1}{2}^\circ$

$\Rightarrow \quad AL = \dfrac{BC}{\sin 45^\circ} \sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ$ [using Eq. (i)]

$= \dfrac{1}{2} BC \dfrac{\sin 45^\circ}{\sin 45^\circ}$

$\Rightarrow \quad AL = \dfrac{1}{2} BC = \dfrac{1}{2}$ of base.

Example 5.5 **Prove that** $\dfrac{\mathbf{1+\cos(A-B)\cos C}}{\mathbf{1+\cos(A-C)\cos B}} = \dfrac{\mathbf{a^2+b^2}}{\mathbf{a^2+c^2}}$.

Sol.

$$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}$$

$$= \frac{1-(\cos^2 A - \sin^2 B)}{1-(\cos^2 A - \sin^2 C)} = \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C}$$

$$= \frac{a^2+b^2}{a^2+c^2} \text{ [using } a = 2R\sin A\text{, etc.]}$$

Example 5.6 **Prove that** $\dfrac{\mathbf{a^2\sin(B-C)}}{\mathbf{\sin B+\sin C}} + \dfrac{\mathbf{b^2\sin(C-A)}}{\mathbf{\sin C+\sin A}} + \dfrac{\mathbf{c^2\sin(A-B)}}{\mathbf{\sin A+\sin B}} = \mathbf{0}$.

Sol.

$$\frac{a^2\sin(B-C)}{\sin B+\sin C} = \frac{4R^2\sin^2 A\sin(B-C)}{\sin B+\sin C}$$

$$= \frac{4R^2\sin A\sin(B+C)\sin(B-C)}{\sin B+\sin C}$$

$$= \frac{4R^2\sin A(\sin^2 B-\sin^2 C)}{\sin B+\sin C}$$

$$= 4R^2\sin A(\sin B-\sin C)$$

Similarly,

$$\frac{b^2\sin(C-A)}{\sin C+\sin A} = 4\text{R}^2\sin B(\sin C-\sin A)$$

and $\dfrac{c^2 \sin(A-B)}{\sin A + \sin B} = 4R^2 \sin C(\sin A - \sin B)$

Adding, we get

$$\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$$

Example 5.7 **In any triangle, if $\dfrac{a^2-b^2}{a^2+b^2}=\dfrac{\sin(A-B)}{\sin(A+B)}$, then prove that the triangle is either right angled or isosceles.**

Sol. $\dfrac{a^2-b^2}{a^2+b^2}=\dfrac{\sin(A-B)}{\sin(A+B)}$

$\Rightarrow \dfrac{4R^2\sin^2 A-4R^2\sin^2 B}{4R^2\sin^2 A+4R^2\sin^2 B}=\dfrac{\sin(A-B)}{\sin(A+B)}$

$\Rightarrow \dfrac{\sin(A+B)\sin(A-B)}{\sin^2 A+\sin^2 B}=\dfrac{\sin(A-B)}{\sin(A+B)}$.

$\Rightarrow \sin(A-B)=0$ or $\dfrac{\sin(\pi-C)}{\sin^2 A+\sin^2 B}=\dfrac{1}{\sin(\pi-C)}$

$\Rightarrow A=B$ or $\sin^2 C=\sin^2 A+\sin^2 B$

$\Rightarrow A=B$ or $c^2=a^2+b^2$ [from the sine rule]

Therefore, the triangle is isosceles or right angled.

Example 5.8 **$ABCD$ is a trapezium such that $AB \parallel CD$ and CB is perpendicular to them. If $\angle ADB=\theta$, $BC=p$ and $CD=q$, show that $AB=\dfrac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$.**

Sol.

Let $\angle ABD=\angle BDC=\alpha$

$\therefore \quad \angle BAD=180°-(\theta+\alpha)$

By the sine formula, in ΔABD, we have

$$\frac{AB}{\sin\theta}=\frac{BD}{\sin(180°-(\theta+\alpha))}$$

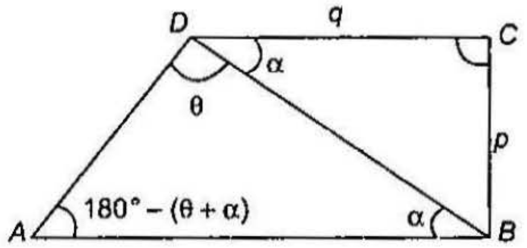


Fig. 5.6

$\therefore \quad AB=\dfrac{BD\sin\theta}{\sin(\theta+\alpha)}=\dfrac{BD\sin\theta}{\sin\theta\cos\alpha+\cos\theta\sin\alpha}$ ...(i)

In ΔBCD, $\sin\alpha = p/BD$
and $\cos\alpha = q/BD$. Also $BD^2 = p^2 + q^2$
Therefore, from Eq. (i), we have

$$AB = \frac{BD\sin\theta}{\sin\theta\,(q/BD) + \cos\theta\,(p/BD)} = \frac{BD^2\sin\theta}{q\sin\theta + p\cos\theta} = \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Example 5.9 **In a triangle ABC, $\angle A = 60°$ and $b : c = \sqrt{3}+1 : 2$, then find the value of $(\angle B - \angle C)$.**

Sol.

$$\frac{b}{c} = \frac{\sqrt{3}+1}{2} \Rightarrow \frac{b-c}{b+c} = \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2} = \frac{\sqrt{3}-1}{(\sqrt{3}+1)}\frac{1}{\sqrt{3}}$$

Now using $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$, we get $\frac{\sqrt{3}-1}{(\sqrt{3}+1)}\frac{\sqrt{3}}{\sqrt{3}} = 2-\sqrt{3} \Rightarrow \frac{B-C}{2} = 15°$

$\therefore B - C = 30°$

Concept Application Exercise 5.1

1. Prove that $\frac{c}{a+b} = \frac{1-\tan\frac{1}{2}A\tan\frac{1}{2}B}{1+\tan\frac{1}{2}A\tan\frac{1}{2}B}$.
2. If the angles of triangle ABC are in the ratio 3:5:4, then find the value of $a + b + c\sqrt{2}$.
3. Prove that $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$.
4. Find the value of $\frac{a^2+b^2+c^2}{R^2}$ in any right-angled triangle.
5. In triangle ABC, if $\cos^2 A + \cos^2 B - \cos^2 C = 1$, then identify the type of the triangle.
6. If angles A, B and C of a triangle ABC are in A.P. and if $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then find angle A.
7. Prove that $b^2\cos 2A - a^2\cos 2B = b^2 - a^2$.

COSINE RULE

In a ΔABC, we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc},$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca}$$

$$\text{and } \cos C = \frac{a^2+b^2-c^2}{2ab}$$

We shall prove that $\cos A = \dfrac{b^2 + c^2 - a^2}{2ab}$

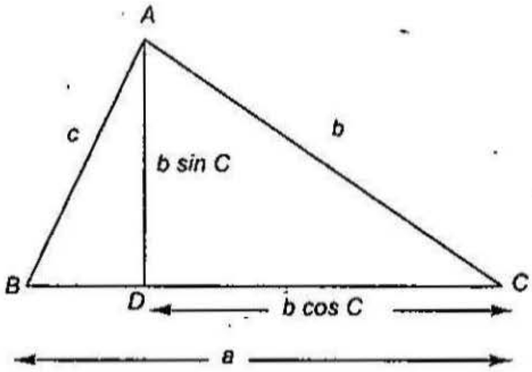


Fig. 5.7

From Fig. 5.7, $BD = a - b\cos C$. Now, in triangle ABD,

$AB^2 = AD^2 + BD^2$

$$\Rightarrow \quad c^2 = (b\sin C)^2 + (a - b\cos C)^2$$

$$= b^2\sin^2 C + a^2 + b^2\cos^2 C - 2ab\cos C$$

$$= (b^2\sin^2 C + b^2\cos^2 C) + a^2 - 2ab\cos C$$

$$= b^2 + a^2 - 2ab\cos C$$

$$\Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Note:

- *The above proof will not change even if $\angle A$ is a right angle or an obtuse angle.*
- *If the lengths of the three sides of a triangle are known, we can find all the angles by using cosine rule because this rule gives us* $\cos A$, $\cos B$ *and* $\cos C$. *We know that A, B and C are in* $(0, \pi)$ *and the* cosine *function is one–one in* $[0, \pi]$, So, *A, B and C are precisely determined. Similarly, if two sides (say b and c) and the included angle A are given, the* cosine *rule* $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ *will give us a and then knowing a, b and c we can find B and C by the cosine rule.*

Example 5.10 **In ΔABC, prove that $(a-b)^2\cos^2\dfrac{C}{2} + (a+b)^2\sin^2\dfrac{C}{2} = c^2$.**

Sol. L.H.S.

$$= (a^2 + b^2 - 2ab)\cos^2\frac{C}{2} + (a^2 + b^2 + 2ab)\sin^2\frac{C}{2}$$

$$= a^2 + b^2 + 2ab\left(\sin^2\frac{C}{2} - \cos^2\frac{C}{2}\right)$$

$$= a^2 + b^2 - 2ab\cos C$$

$$= a^2 + b^2 - (a^2 + b^2 - c^2) = c^2$$

Example 5.11 **In ΔABC, if $(a+b+c)(a-b+c) = 3ac$, then find $\angle B$.**

Sol.

$(a+c)^2 - b^2 = 3ac \Rightarrow a^2 + c^2 - b^2 = ac$

But $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{1}{2} \Rightarrow B = \dfrac{\pi}{3}$

Example 5.12 **If $a = \sqrt{3}$, $b = \dfrac{1}{2}\left(\sqrt{6} + \sqrt{2}\right)$ and $c = \sqrt{2}$, then find ÐA.**

Sol. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{\frac{1}{4}\left(8 + 4\sqrt{3}\right) + 2 - 3}{\sqrt{12} + \sqrt{4}} = \dfrac{1 + \sqrt{3}}{2(1 + \sqrt{3})} = \dfrac{1}{2} \quad \Rightarrow A = \dfrac{\pi}{3}$

Example 5.13 **If the angles A, B, C of a triangle are in A.P. and sides a, b, c are in G.P., then prove that a^2, b^2, c^2 are in A.P.**

Sol.

Given, $2B = A + C \Rightarrow 3B = \pi \Rightarrow B = \pi/3$ (i)

Also a, b, c in G.P. $\Rightarrow b^2 = ac$ (ii)

Now, $\cos B = \cos 60° = \dfrac{1}{2} = \dfrac{c^2 + a^2 - b^2}{2ca}$

$\Rightarrow \quad ca = c^2 + a^2 - b^2$

$\Rightarrow \quad 2b^2 = c^2 + a^2$ [by using Eq. (ii)]

$\Rightarrow \quad a^2, b^2, c^2$ are in A.P.

Example 5.14 **If in a triangle ABC, $\angle C = 60°$, then prove that $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$.**

Sol.

By the cosine formula, we have

$c^2 = a^2 + b^2 - 2ab\cos C \quad \Rightarrow \quad c^2 = a^2 + b^2 - 2ab\cos 60° = a^2 + b^2 - ab$ (i)

Now, $\dfrac{1}{a+c} + \dfrac{1}{b+c} - \dfrac{3}{a+b+c}$

$= \left[\dfrac{(b+c)(a+b+c) + (a+c)(a+b+c) - 3(a+c)(b+c)}{(a+b)(b+c)(a+b+c)}\right]$

$= \dfrac{\left(a^2 + b^2 - ab\right) - c^2}{(a+b)(b+c)(a+b+c)} = 0$ [from Eq. (i)]

$\Rightarrow \quad \dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$

Example 5.15 **In a triangle, if the angles A, B and C are in A.P, show that $2\cos\dfrac{1}{2}(A - C) = \dfrac{a+c}{\sqrt{a^2 - ac + c^2}}$.**

Sol.

Since angles A, B and C are in A.P.

$\therefore \quad A + C = 2B$

But, $A + B + C = 180° \Rightarrow 3B = 180° \Rightarrow B = 60°$

Now, $\cos B = \dfrac{1}{2} = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 + c^2 - b^2 = ac$

$\Rightarrow \quad a^2 - ac + c^2 = b^2$

$\Rightarrow \quad \dfrac{a+c}{\sqrt{a^2 - ac + c^2}} = \dfrac{a+c}{b} = \dfrac{2R(\sin A + \sin C)}{2R\sin B}$

$$= \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin B} = \frac{2\sin 60°}{\sin 60°}\cos\left(\frac{A-C}{2}\right) = 2\cos\left(\frac{A-C}{2}\right)$$

Example 5.16 **The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1, prove that the greatest angle is 120°.**

Sol.

Let $a = x^2+x+1$, $b = 2x+1$ and $c = x^2-1$.

First of all, we have to decide which side is the greatest. We know that in a triangle, the length of each side is greater than zero. Therefore, we have $b = 2x+1 > 0$ and $c = x^2-1 > 0$.

$\Rightarrow \quad x > -1/2$ and $x^2 > 1$

$\Rightarrow \quad x > -1/2$ and $x < -1$ or $x > 1 \Rightarrow x > 1$

$a = x^2+x+1 = (x+1/2)^2 + (3/4)$ is always positive.

Thus, all sides a, b and c are positive when $x > 1$.

Now, $x > 1 \Rightarrow x^2 > x$

$\Rightarrow \quad x^2+x+1 > x+x+1$

$\Rightarrow \quad x^2+x+1 > 2x+1 \Rightarrow a > b$

Also, when $x > 1$,

$x^2+x+1 > x^2-1 \Rightarrow a > c$

Thus, $a = x^2+x+1$ is the greatest side and the angle A opposite to this side is the greatest angle .

$$\therefore \quad \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{-2x^3-x^2+2x+1}{2(2x^3+x^2-2x-1)} = -\frac{1}{2}$$

$$= \cos 120°$$

$$\Rightarrow \quad A = 120°$$

Example 5.17 **Triangle ABC has $BC = 1$ and $AC = 2$. Find the maximum possible value of angle A**

Sol.

Using cosine rule, we have

$$\cos\theta = \frac{x^2+4-1}{4x}$$

$$= \frac{x^2+3}{4x}$$

$$= \frac{1}{4}\left[x + \frac{3}{x}\right]$$

$$= \frac{1}{4}\left[\left(\sqrt{x} - \sqrt{\frac{3}{x}}\right)^2 + 2\sqrt{3}\right]$$

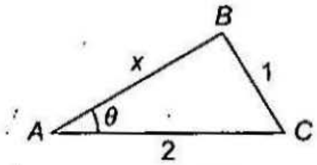


Fig. 5.8

Hence, $\cos\theta$ is minimum if $x = \sqrt{3}$.

Therefore, the minimum value of $\cos\theta = 2 \times \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, and the maximum value of $\theta = \frac{\pi}{6}$

Example 5.18 **Let a, b and c be the three sides of a triangle, then prove that the equation $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$ has imaginary roots.**

Sol.

$b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$
Let $f(x) = b^2x^2 + (2bc\cos A)x + c^2 = 0$
Also in ΔABC, where $A \in (0, \pi)$ in a triangle, we find $\cos A \in (-1, 1)$
$\Rightarrow 2bc\cos A \in (-2bc, 2bc)$
$\Rightarrow D = (2bc\cos A)^2 - 4b^2c^2 = 4b^2c^2(\cos^2 A - 1) < 0$
Hence, the roots are imaginary.

Example 5.19 **Let $a \le b \le c$ be the lengths of the sides of a triangle. If $a^2 + b^2 < c^2$, then prove that Δ is obtuse angled.**

Sol.

$a^2 + b^2 < c^2$
$\Rightarrow a^2 + b^2 < a^2 + b^2 - 2ab\cos C$
$\Rightarrow \cos C < 0$
$\Rightarrow C$ is obtuse

Concept Application Exercise 5.2

1. If the sides of a triangle are a, b and $\sqrt{a^2 + ab + b^2}$, then find the greatest angle.
2. Prove that $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$.
3. If the line segment joining the points $A(a, b)$ and $B(c, d)$ subtends an angle θ at the origin, then prove that $\cos\theta = \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$.
4. Prove that in ΔABC, a^2, b^2, c^2 are in A.P. if and only if $\cot A$, $\cot B$, $\cot C$ are in A.P.
5. If $x, y > 0$, then prove that the triangle whose sides are given by $3x + 4y$, $4x + 3y$ and $5x + 5y$ units is obtuse angled.

6. In ΔABC, angle A is $120°$, $BC + CA = 20$ and $AB + BC = 21$. Find the length of the side BC.
7. In $\Delta ABC, AB = 1, BC = 1$ and $AC = 1/\sqrt{2}$. In $\Delta MNP, MN = 1, NP = 1$ and $\angle MNP = 2\angle ABC$. Find the side MP.

PROJECTION FORMULA

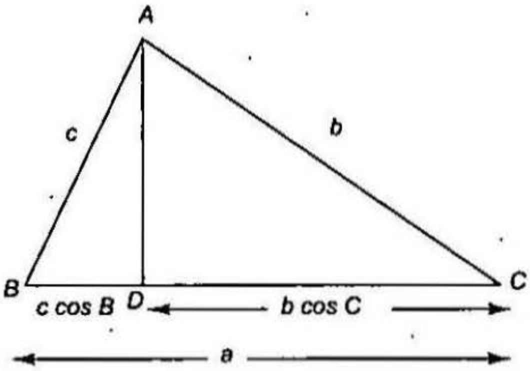


Fig. 5.9

Projection of AB on $BC = BD = c \cos B$

Projection of AC on $BC = CD = b \cos C$

Now, $BC = a = BD + DC = c \cos B + b \cos C$

Similarly, other formulae follow.

Example 5.20 **Prove that $a(b \cos C - c \cos B) = b^2 - c^2$.**

Sol. $a (b \cos C - c \cos B) = (b \cos C + c \cos B)(b \cos C - c \cos B)$

$= b^2 \cos^2 C - c^2 \cos^2 B$

$= b^2 (1 - \sin^2 C) - c^2 (1 - \sin^2 B)$

$= b^2 - c^2 - (b^2 \sin^2 C - c^2 \sin^2 B)$

$= b^2 - c^2$ [as by the sine rule $b \sin C = c \sin B$]

Example 5.21 **If in a triangle $a \cos^2 \frac{C}{2} + c\cos^2 \frac{A}{2} = \frac{3b}{2}$, then find the relation between the sides of the triangle.**

Sol. $a \cos^2 \frac{C}{2} + c\cos^2 \frac{A}{2} = \frac{3b}{2}$

$\Rightarrow$ $a (1 + \cos C) + c (1 + \cos A) = 3b$

$\Rightarrow$ $a + c + (a \cos C + c \cos A) = 3b$

$\Rightarrow$ $a + c + b = 3b$ [by the projection formula]

$\Rightarrow$ $a + c = 2b$

$\Rightarrow$ a, b, c are in A.P.

Example 5.22 **Prove that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = 2s$.**

Sol. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C$

$= (b \cos A + a \cos B) + (c \cos A + a \cos C) + (b \cos C + c \cos B) = c + b + a = 2s$

Concept Application Exercise 5.3

1. In ΔABC, prove that $c\cos(A-\alpha)+a\cos(C+\alpha)=b\cos\alpha$.
2. Prove that $\dfrac{\cos C+\cos A}{c+a}+\dfrac{\cos B}{b}=\dfrac{1}{b}$.
3. Prove that $a(b^2+c^2)\cos A+b(c^2+a^2)\cos B+c(a^2+b^2)\cos C=3abc$.

HALF-ANGLE FORMULAE

1. i. $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

   ii. $\sin\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{ca}}$

   iii. $\sin\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$

Proof:

i. $\sin^2\dfrac{A}{2}=\dfrac{1-\cos A}{2}$

$$=\frac{1}{2}\left[1-\frac{b^2+c^2-a^2}{2bc}\right]$$

$$=\frac{1}{2}\left[\frac{2bc-b^2-c^2+a^2}{2bc}\right]$$

$$=\frac{1}{2}\left[\frac{a^2-(b-c)^2}{2bc}\right]$$

$$=\frac{(a-b+c)(a+b-c)}{4bc}$$

$$=\frac{(a+b+c-2b)(a+b+c-2c)}{4bc}$$

$$=\frac{(2s-2b)(2s-2c)}{4bc} \qquad [\because a+b+c=2s]$$

$$\Rightarrow \sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$$

As $0<\dfrac{A}{2}<\dfrac{\pi}{2}$, so $\sin\dfrac{A}{2}>0$.

Hence, $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

The other formulae can be proved similarly.

2. i. $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$

   ii. $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ca}}$

iii. $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

Proof:

i. $\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$

$= \frac{1}{2}\left[1 + \frac{b^2+c^2-a^2}{2bc}\right]$ [using cosine rule]

$= \frac{1}{2}\left[\frac{2bc+b^2+c^2-a^2}{2bc}\right]$

$= \frac{1}{2}\left[\frac{(b+c)^2-a^2}{2bc}\right]$

$= \frac{1}{2}\left[\frac{(b+c+a)(b+c-a)}{2bc}\right]$

$= \frac{(b+c+a)(a+b+c-2a)}{4bc}$

$= \frac{2s(2s-2a)}{4bc}$ [$\because a+b+c=2s$]

$\therefore \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$

As $0 < \frac{A}{2} < \frac{\pi}{2}$ so $\cos\frac{A}{2} > 0$

Hence, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$.

The other formulae can be proved in the same way.

3. From the above formulae, we can prove

i. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

ii. $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

iii. $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Example 5.23 **If $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}}$, then prove that $a^2+b^2=c^2$.**

Sol. $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}} \Rightarrow \frac{s(s-a)}{bc} = \frac{b+c}{2c}$ [squaring]

$\Rightarrow 2s(2s-2a) = 2b(b+c)$

$\Rightarrow (b+c+a)(b+c-a) = 2b^2+2bc$

$\Rightarrow (b+c)^2 - a^2 = 2b^2+2bc$

$\Rightarrow c^2 = a^2 + b^2$

Example 5.24 **If the cotangents of half the angles of a triangle are in A.P., then prove that the sides are in A.P.**

Sol. $\cot\frac{A}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{s(s-a)}{\Delta}$

Similarly, $\cot\frac{B}{2}=\frac{s(s-b)}{\Delta}$ and $\cot\frac{C}{2}=\frac{s(s-c)}{\Delta}$.

$\cos\frac{A}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ are in A.P.

$\Rightarrow\quad \frac{s(s-a)}{\Delta}, \frac{s(s-b)}{\Delta}$ and $\frac{s(s-c)}{\Delta}$ are in A.P.

$\Rightarrow\quad s-a, s-b$ and $s-c$ are in A.P.

$\Rightarrow\quad a, b$ and c are in A.P.

Concept Application Exercise 5.4

1. If $b+c=3a$, then find the value of cot $\cot\frac{B}{2}\cot\frac{C}{2}$.
2. Prove that $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2$.
3. If in ΔABC, $\tan\frac{A}{2}=\frac{5}{6}$, $\tan\frac{C}{2}=\frac{2}{5}$, then prove that a, b and c are in A.P.

AREA OF TRIANGLE

Different formulae for area of triangle are as follows.

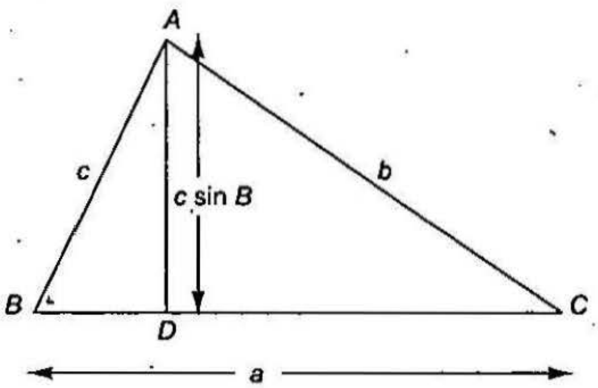


Fig. 5.10

From Fig. 5.10, area of triangle ABC is

$\Delta=\frac{1}{2}AD\times BC=\frac{1}{2}c\sin B\times a=\frac{1}{2}ac\sin B$

Similarly, we can prove that $\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A$

Also, by the sine rule, $\sin A = \dfrac{a}{2R}$

$$\Rightarrow \quad \Delta = \frac{abc}{4R} = \frac{(2R\sin A)(2R\sin B)(2R\sin C)}{4R}$$
$$= 2R^2 \sin A \sin B \sin C$$

Also, $\Delta = \dfrac{1}{2} ac \sin B$

$$= ac \sin\frac{B}{2} \cos\frac{B}{2}$$
$$= ac\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{s(s-b)}{ca}}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}$$

Example 5.25 **If in triangle ABC, $\Delta^2 = a^2 - (b-c)^2$, then find the value of $\tan A$.**

Sol. $\Delta^2 = (a+b-c)(a-b+c)$

$$\Rightarrow \quad \Delta^2 = [2(s-b)\,2(s-c)]^2$$
$$\Rightarrow \quad s(s-a)(s-b)(s-c) = 16(s-b)^2(s-c)^2$$
$$\Rightarrow \quad \frac{(s-b)(s-c)}{s(s-a)} = \frac{1}{16}$$
$$\Rightarrow \quad \tan\frac{A}{2} = \frac{1}{4}$$
$$\Rightarrow \quad \tan^2 A = \frac{2\tan(A/2)}{1-\tan^2(A/2)} = \frac{2.(1/4)}{1-(1/16)} = \frac{8}{15}$$

Example 5.26 **Prove that $a^2 \sin 2B + b^2 \sin 2A = 4\Delta$.**

Sol. $a^2 \sin 2B + b^2 \sin 2A = 4R^2 [\sin^2 A (2\sin B\cos B) + \sin^2 B(2\sin A\cos A)]$ [using sine rule]

$$= 8R^2 \sin A \sin B(\sin A\cos B + \sin B\cos A)$$
$$= 8R^2 \sin A \sin B \sin(A+B)$$
$$= 8R^2 \sin A \sin B \sin C = 4\Delta$$

Example 5.27 **Prove that $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = \sin^2 A$.**

Sol. $$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = \frac{2s\,2(s-a)\;2(s-b)\;2(s-c)}{4b^2c^2}$$
$$= \frac{4\Delta^2}{b^2c^2}$$

$$= \frac{4}{b^2c^2}\left(\frac{1}{2}bc\sin A\right)^2 = \sin^2 A$$

Example 5.28 **If the sides of a triangle are 17, 25 and 28, then find the greatest length of the altitude.**

Sol. We know from geometry that the greatest altitude is perpendicular to the shortest side.
Let $a = 17$, $b = 25$ and $c = 28$.

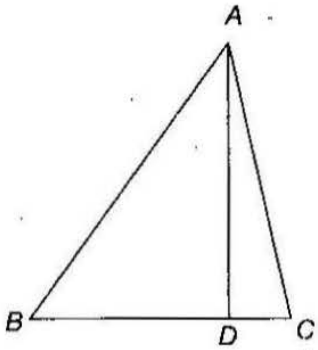


Fig. 5.11

Now, $\Delta = \frac{1}{2} AD \times BC \quad \Rightarrow AD = \frac{2\Delta}{17}$

where $\Delta^2 = s(s-a)(s-b)(s-c) = 210^2$

$\Rightarrow \quad AD = \frac{420}{17}$

Example 5.29 **In equilateral triangle *ABC* with interior point *D*, if the perpendicular distances from *D* to the sides of 4, 5 and 6, respectively, are given, then find the area of ΔABC.**

Sol.

Area of triangle is $\Delta = \dfrac{a\times 4 + a\times 5 + a\times 6}{2}$

$\Rightarrow \dfrac{a(4+5+6)}{2} = \dfrac{3\sqrt{3}}{4}a^2$

$\Rightarrow \dfrac{15}{2} = \dfrac{\sqrt{3}\,a}{4}$

$\Rightarrow a = \dfrac{30}{\sqrt{3}} = 10\sqrt{3}$

$\Rightarrow \Delta = \dfrac{\sqrt{3}}{4} \times 100 \times 3 = 75\sqrt{3}$

Example 5.30 **If area of a triangle is 2 sq. units, then find the value of the product of the arithmetic mean of the lengths of the sides of a triangle and harmonic mean of the lengths of the altitudes of the triangle.**

Sol.

$$a h_1 = b h_2 = c h_3 = 2\Delta$$

$$\Rightarrow \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{a+b+c}{2\Delta}$$

$$\Rightarrow \frac{a+b+c}{3} \times \frac{3}{\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}} = 2\Delta = 4$$

Example 5.31 **A triangle has sides 6, 7 and 8. The line through its incentre parallel to the shortest side is drawn to meet the other two sides at P and Q. Then find the length of the segment PQ**

Sol.

$$\Delta = r \times s$$

$$\therefore \frac{21 \times r}{2} = \frac{6 \times h}{2} = 3h$$

$$\Rightarrow \frac{r}{h} = \frac{2}{7}$$

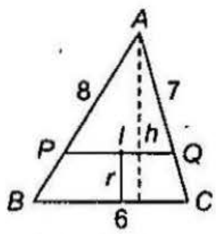


Fig. 5.12

Now APQ and ABC are similiar

$$\Rightarrow \frac{h-r}{h} = \frac{PQ}{6}$$

$$\Rightarrow 1 - \frac{r}{h} = \frac{PQ}{6}$$

$$\Rightarrow 1 - \frac{2}{7} = \frac{PQ}{6}$$

$$\Rightarrow \frac{5}{7} = \frac{PQ}{6} \Rightarrow PQ = \frac{30}{7}$$

Concept Application Exercise 5.5

1. If $c^2 = a^2 + b^2$, then prove that $4s(s-a)(s-b)(s-c) = a^2 b^2$.
2. If the sides of a triangle are in the ratio 3:7:8, then find $R : r$.
3. In triangle ABC, if $a = 2$ and $bc = 9$, then prove that $R = 9/2\Delta$.
4. The area of triangle ABC is equal to $(a^2 + b^2 - c^2)$, where a, b and c are the side of the triangle. Find the value of tan C.
5. Let the lengths of the altitudes drawn from the vertices of ΔABC to the opposite sides are 2, 2 and 3. If the area of ΔABC is Δ, then find the area of triangle.

Different Circles and Centres Connected With Triangle

Circumcircle and Circumcentre(O)

The circle passing through the angular point of ΔABC is called its circumcircle. The centre of this circle is the point of intersection of the perpendicular bisectors of the sides and is called the circumcentre. Its radius is denoted by R.

- Circumcentre of an acute-angled triangle lies inside the triangle.

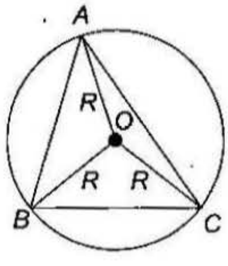


Fig. 5.13

- Circumcentre of an obtuse-angled triangle lies outside the triangle.

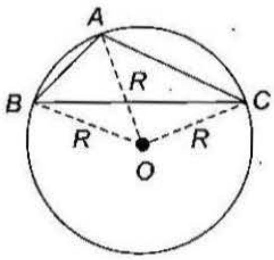


Fig. 5.14

- Circumcentre of a right-angled triangle is the mid-point of the hypotenuse.

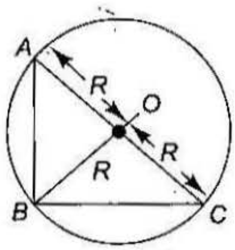


Fig. 5.15

- Distance of the circumcentre from the sides can be calculated as follows.

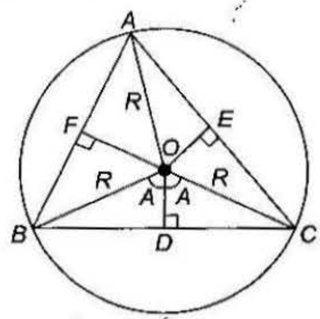


Fig. 5.16

At the circumcentre, the perpendicular bisectors of the sides are concurrent.
Also, $\angle BOC = 2\angle BAC = 2A$. Triangles BOD and COD are congruent.
Hence, $\angle BOD = A$.

Now, in ΔBOD, $\cos A = \dfrac{OD}{OB} = \dfrac{OD}{R}$ $\Rightarrow OD = R\cos A$

Similarly, $OE = R\cos B$ and $OF = R\cos C$.

In-Circle and In-Centre (I)

Point of intersection of the internal bisectors of a triangle is called the in-center of the triangle. Also, it is the centre of the circle touching all the three sides internally. In-centre always lies inside the triangle.

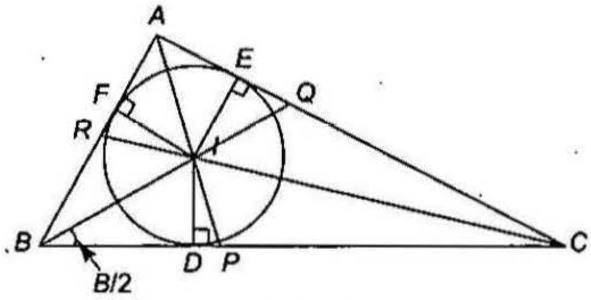


Fig. 5.17

- Points $AFIE$, $BDIF$ and $CEID$ are concyclic
- Internal bisector AP divides side BC in ratio $AB:AC$

or $\dfrac{BP}{PC} = \dfrac{AB}{AC} = \dfrac{c}{b}$ $\Rightarrow BP = ck, CP = bk$

But $BP + CP = a$

$\Rightarrow \quad ck + bk = a$

$\Rightarrow \quad k = \dfrac{a}{b+c}$

$\Rightarrow \quad BP = \dfrac{ac}{b+c}$ and $CP = \dfrac{ab}{b+c}$

Similarly, $AQ = \dfrac{cb}{a+c}, CQ = \dfrac{ab}{a+c}$

and $AR = \dfrac{bc}{a+b}, BR = \dfrac{ac}{a+b}$

- Area of the triangle in terms of r

$$\Delta_{ABC} = \Delta_{IBC} + \Delta_{IAC} + \Delta_{IAB} = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}r(a+b+c) = rs$$

- $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

Proof: $(s-a)\tan\dfrac{A}{2} = (s-a)\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

$$= \frac{\Delta}{s} = r$$

Similarly, the results $r = (s-b)\tan\dfrac{B}{2}$ and $r = (s-c)\tan\dfrac{C}{2}$ can be proved.

- $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

  Proof: $4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 4R \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-c)(s-a)}{ca}} \sqrt{\frac{(s-a)(s-b)}{ab}}$

  $$= 4R \frac{(s-a)(s-b)(s-c)}{abc}$$

  $$= \frac{4R}{abc} \frac{s(s-a)(s-b)(s-c)}{s}$$

  $$= \frac{1}{\Delta} \frac{\Delta^2}{s}$$

  $\left[\text{where } \Delta = \frac{abc}{4R} \text{ and } \Delta = \sqrt{s(s-a)(s-b)(s-c)}\right] = \frac{\Delta}{s} = r$

- **Distance of the in-centre from the vertex**

  In ΔIDB, $\sin \frac{B}{2} = \frac{ID}{BI} = \frac{r}{BI} \quad \Rightarrow BI = \frac{r}{\sin \frac{B}{2}}$

  Similarly, $AI = \frac{r}{\sin \frac{A}{2}}$ and $CI = \frac{r}{\sin \frac{C}{2}}$

- **Length of angle bisector AP**

  Area of ΔABP + Area of ΔACP = Area of ΔABC

  $\Rightarrow (1/2) AB\, AP \sin(A/2) + (1/2) AC\, AP \sin(A/2) = (1/2) AB\, AC \sin A$

  $\Rightarrow (1/2)(c+b) AP \sin(A/2) = (1/2)[cb\, 2\sin(A/2)\cos(A/2)]$

  $\Rightarrow AP = \left(\frac{2bc}{b+c}\right)\cos(A/2)$

  Similarly, length of angle bisector through point B and C is $BQ = \left(\frac{2ac}{a+c}\right)\cos(B/2)$,

  $CR = \left(\frac{2ab}{a+b}\right)\cos(C/2)$

Orthocentre

Orthocentre (H) is the point of intersection of the altitudes of a triangle.

- Orthocentre (H) of an acute-angled triangle lies inside the triangle.
  Here, H is the orthocentre of ΔABC.

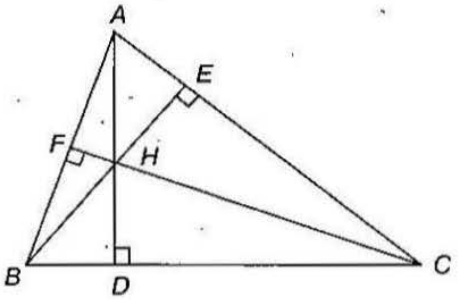


Fig. 5.18

- Orthocentre (H) of an obtuse-angled triangle lies outside the triangle.
  Here, H is orthocenter of ΔABC.

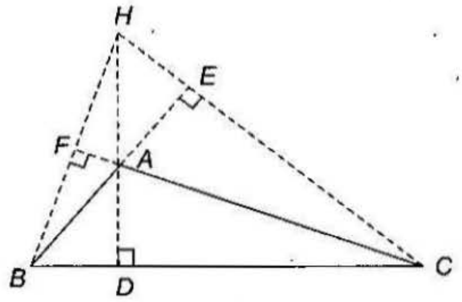


Fig. 5.19

- Orthocentre (H) of a right-angled triangle ABC lies at the right angle itself. In Fig. 5.20, the orthocentre H coincides with the right angle B.

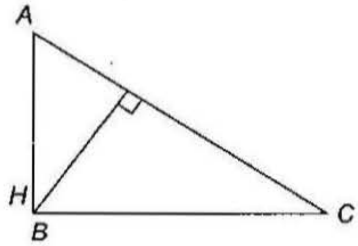


Fig. 5.20

- Image of orthocentre (H) in any side of a triangle lies on the circumcircle.

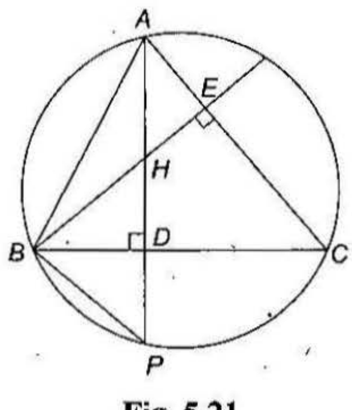


Fig. 5.21

$\angle HBD = \angle EBC = (\pi/2) - C \quad \Rightarrow \angle BHD = C$

Also, $\angle BPD = \angle BPA = \angle BCA = C$

Thus, ΔBPD and ΔBHD are congruent.

This implies $HD = DP \quad \Rightarrow P$ is image in H in BC.

- Distance of the orthocentre from vertices and sides of a triangle can be calculated as follows.

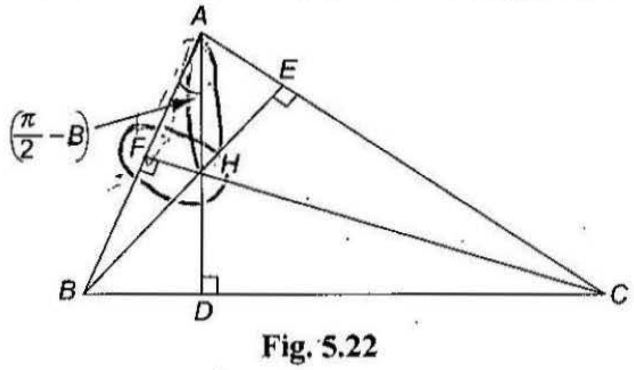


Fig. 5.22

In ΔADB, $\angle BAD = \dfrac{\pi}{2} - B$

In ΔAFC, $AF = b\cos A$ [projection of AC on AB]

In ΔAFH, $\cos\left(\dfrac{\pi}{2} - B\right) = \dfrac{AF}{AH} = \dfrac{b\cos A}{AH}$

$\Rightarrow \quad AH = \dfrac{b\cos A}{\sin B} = 2R\cos A$

Similarly, $BH = 2R\cos B$ and $CH = 2R\cos C$

In ΔAFH, $\tan\left(\dfrac{\pi}{2} - B\right) = \dfrac{FH}{AF} = \dfrac{FH}{b\cos A}$

$\Rightarrow \quad FH = \dfrac{b\cos A\cos B}{\sin B} = 2R\cos A\cos B$

Similarly, $EH = 2R\cos A\cos C$ and $HD = 2R\cos B\cos C$

Pedal Triangle

The triangle formed by the feet of the altitudes on the sides of a triangle is called a pedal triangle.

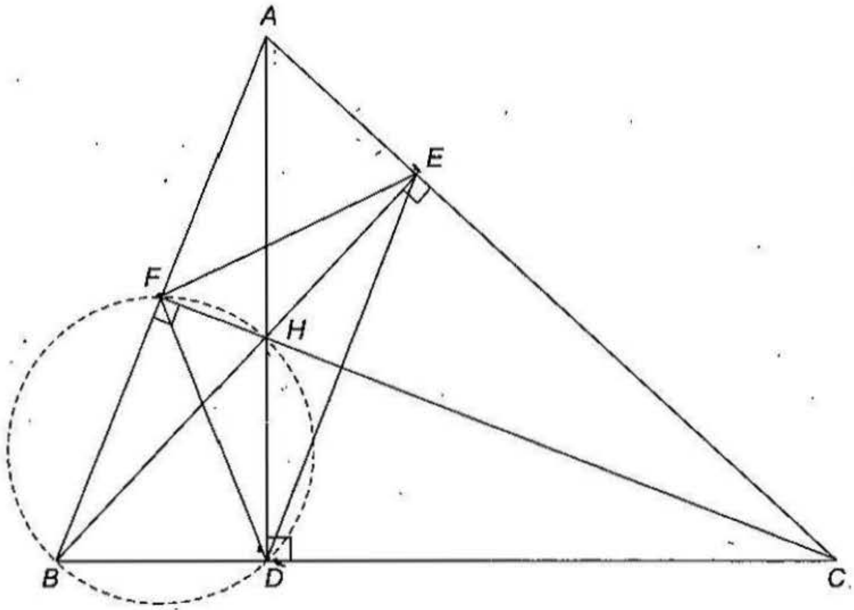


Fig. 5.23

- **In an acute-angled triangle, orthocentre of ΔABC is the in-centre of the pedal triangle DEF.**

Proof:

Points F, H, D and B are concyclic. $\Rightarrow \angle FDH = \angle FBH = \angle ABE = \dfrac{\pi}{2} - A$

Similarly, points D, H, E and C are concyclic $\Rightarrow \angle HDE = \angle HCE = \angle ACF = \dfrac{\pi}{2} - A$.

Thus, $\angle FDH = \angle HDE \Rightarrow AD$ is angle bisector of $\angle FDE$. Hence, altitudes of ΔABC are internal angle bisectors of the pedal triangle. Thus, orthocentre of ΔABC is the in-centre of the pedal triangle DEF.

- **Sides of pedal triangle in acute-angled triangle**

  In ΔAFE, $AF = b\cos A$, $AE = c\cos A$

  By cosine rule, $EF^2 = AE^2 + AF^2 - 2\,AE\,AF\cos(\angle EAF)$

  $\Rightarrow \quad EF^2 = b^2\cos^2 A + c^2\cos^2 A - 2\,bc\cos^3 A$

  $= \cos^2 A\,(b^2 + c^2 - 2\,bc\cos A) = \cos^2 A\,(a^2)$

  $= a\cos A$

- **Circumradius of pedal triangle**

  Let circumradius be R'

  $\Rightarrow \quad 2R' = \dfrac{EF}{\sin(\angle EDF)} = \dfrac{a\cos A}{\sin(\pi - 2A)} = \dfrac{a\cos A}{2\sin A\cos A} = \dfrac{a}{2\sin A} = R$

  $\Rightarrow \quad R' = R/2$

Centroid of Triangle

In $\Delta\,ABC$, the mid-points of the sides BC, CA and AB are D, E and F, respectively. The lines, AD, BE and CF are called medians of the triangle ABC, the points of concurrency of three medians is called the centroid. Generally, it is represented by G.

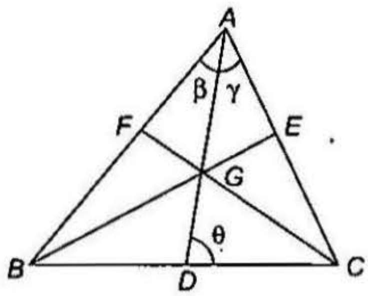


Fig. 5.24

Also, $AG = \dfrac{2}{3}AD$, $BG = \dfrac{2}{3}BE$ and $CG = \dfrac{2}{3}CF$.

- Length of medians and the angles that the medians make with sides

  From Fig. 5.24, we have

  $AD^2 = AC^2 + CD^2 - 2AC \times CD \times \cos C$

  $= b^2 + \dfrac{a^2}{4} - ab\cos C$

  $= b^2 + \dfrac{a^2}{4} - ab\left(\dfrac{b^2 + a^2 - c^2}{2ab}\right)$

  $= \dfrac{2b^2 + 2c^2 - a^2}{4}$

  $\Rightarrow \quad AD = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

  Similarly, $BE = \dfrac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$

  and $CF = \dfrac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

- Apollonius theorem

  $AB^2 + AC^2 = 2(AD^2 + BD^2)$

Proof:

$$2(AD^2+BD^2)=2\left[\frac{1}{4}(2b^2+2c^2-a^2)+\frac{a^2}{4}\right]=b^2+c^2=AB^2+AC^2$$

- Centroid (G) of a triangle is situated on the line joining its circumcentre (O) and orthocenter (H) and divide this line in the ratio 1:2.

Proof:

Let AL be a perpendicular from A on BC, then H lies on AL. If OD is perpendicular from O on BC, then D is mid-point of BC.

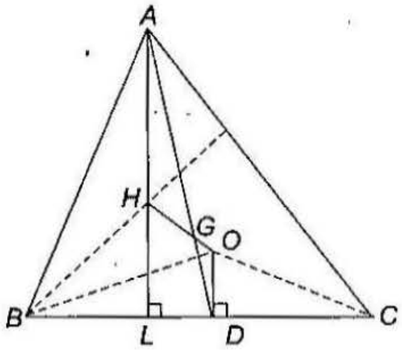


Fig. 5.25

Therefore, AD is a median of ΔABC. Let the line HO meet the median AD at G. Now, we shall prove that G is the centroid of the ΔABC. Obviously, ΔOGD and ΔHGA are similar triangles.

$$\therefore \quad \frac{OG}{HG}=\frac{GD}{GA}=\frac{OD}{HA}=\frac{R\cos A}{2R\cos A}=\frac{1}{2}$$

$$\therefore \quad GD=\frac{1}{2}GA \Rightarrow G \text{ is centroid of } \Delta ABC \text{ and } OG:HG=1:2.$$

Example 5.32 ***ABC* is an acute-angled triangle with circumcentre '*O*' orthocentre *H*. If *AO* = *AH*, then find the angle *A*.**

Sol. $OA=HA$

$R=2R\cos A$

$$\Rightarrow \cos A=\frac{1}{2} \quad \Rightarrow \quad A=\frac{\pi}{3}$$

Example 5.33 **If *x*, *y* and *z* are the distances of incentre from the vertices of the triangle *ABC*, respectively, then prove that** $\dfrac{abc}{xyz}=\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$.

Sol. $x=r\operatorname{cosec}\dfrac{A}{2}$ and $a=r\left(\cot\dfrac{B}{2}+\cot\dfrac{C}{2}\right)$

$$\Rightarrow \frac{a}{x}=\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\sin\frac{A}{2}=\frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Rightarrow \frac{abc}{xyz}=\frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

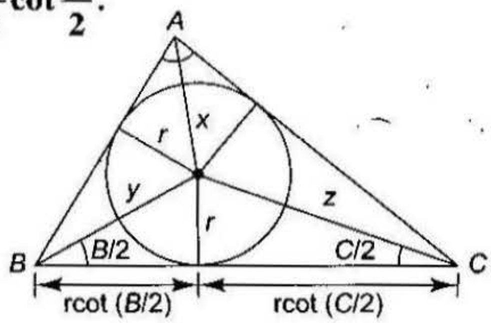


Fig. 5.26

Example 5.34 **Let ABC be a triangle with $\angle BAC = 2\pi/3$ and $AB = x$ such that $(AB)(AC) = 1$. If x varies, then find the longest possible length of the angle bisector AD**

Sol. $AD = y = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{bx}{b+x}$ (as $c = x$)

But $bx = 1 \Rightarrow b = \frac{1}{x}$

$\therefore y = \frac{x}{1+x^2} = \frac{1}{x+\frac{1}{x}}$

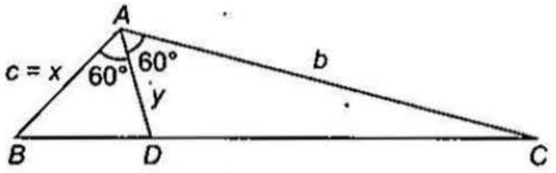


Fig. 5.27

$\Rightarrow y_{max} = \frac{1}{2}$ since the minimum value of the denominator is 2 if $x > 0$.

Example 5.35 **Let ABC be an acute triangle whose orthocentre is at H. If altitude from A is produced to meet the circumcircle of triangle ABC at D, then prove $HD = 4R\cos B\cos C$**

Sol. ΔBHN and ΔBDN are congruent.

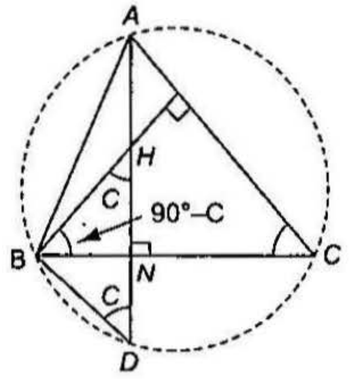


Fig. 5.28

$\therefore HN = ND = 2R\cos B\cos C$

$\therefore HD = 4R\cos B\cos C$

Example 5.36 **In an acute-angled triangle ABC, point D, E and F are the feet of the perpendiculars from A, B and C onto BC, AC and AB, respectively. H is orthocentre. If $\sin A = 3/5$ and $BC = 39$, then find the length of AH.**

Sol. Given $\sin A = 3/5 \Rightarrow \cos A = 4/5$

Also $a = 39$

$\therefore \frac{a}{\sin A} = 2R$

$\Rightarrow \frac{39 \times 5}{3} = 2R$

$\Rightarrow 2R = 65$

$\Rightarrow AH = 2R\cos A = 65 \cdot \frac{4}{5} = 52$

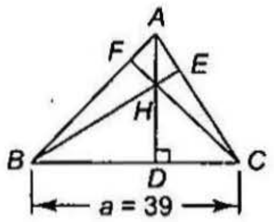


Fig. 5.29

Example 5.37 In triangle ABC, CD is the bisector of the angle C. If $\cos\frac{C}{2} = \frac{1}{3}$ and $CD = 6$, then find the value of $\left(\frac{1}{a}+\frac{1}{b}\right)$.

Sol. $\Delta = \Delta_1 + \Delta_2$

$$\Rightarrow \frac{1}{2}ab\sin C = \frac{1}{2}6b\sin\frac{C}{2} + \frac{1}{2}6a\sin\frac{C}{2}$$

$$\Rightarrow ab\sin\frac{C}{2}\cos\frac{C}{2} = \frac{1}{2}6b\sin\frac{C}{2} + \frac{1}{2}6a\sin\frac{C}{2}$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b} = \frac{1}{9}$$

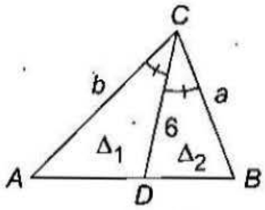


Fig. 5.30

Example 5.38 Let f, g and h be the lengths of the perpendiculars from the circumcentre of ΔABC on the sides a, b and c, respectively, then prove that $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \frac{1}{4}\frac{abc}{fgh}$.

Sol. Distance of circumcentre from to side BC is $R\cos A = f$
Similarly, $g = R\cos B$, $h = R\cos C$

$$\Rightarrow \frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \frac{2R\sin A}{R\cos A}+\frac{2R\sin B}{R\cos B}+\frac{2R\sin C}{R\cos C} = 2(\tan A+\tan B+\tan C)$$

Also, $\frac{a}{f}\frac{b}{g}\frac{c}{h} = 8\tan A\tan B\tan C$

But in triangle, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

$$\Rightarrow \frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \frac{1}{4}\frac{abc}{fgh}$$

Example 5.39 If the incircle of ΔABC touches its sides, respectively, at L, M and N and if x, y, z are the circumradii of the triangles MIN, NIL, and LIM where I is the incenter, then prove that $xyz = \frac{1}{2}Rr^2$.

Sol. In Fig. 5.31, $ANIM$ is a cyclic quadrilateral.
Also, AI is the diameter of circumcircle MNI.
Let $AI = 2x$

$$\Rightarrow \operatorname{cosec}\frac{A}{2} = \frac{2x}{r}$$

$$\Rightarrow x = \frac{r}{2\sin\frac{A}{2}},\ y = \frac{r}{2\sin\frac{B}{2}},\ z = \frac{r}{2\sin\frac{C}{2}}$$

$$\Rightarrow xyz = \frac{r^3}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r^3}{2\frac{r}{R}} = \frac{r^2R}{2}$$

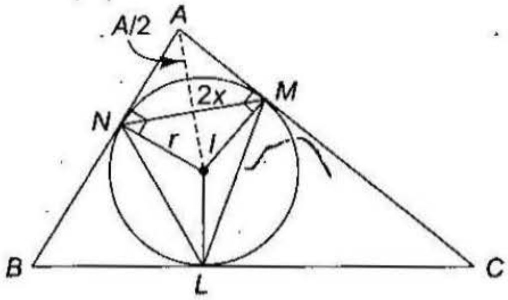


Fig. 5.31

ESCRIBED CIRCLES OF A TRIANGLE AND THEIR RADII

The circle which touches the side BC and two sides AB and AC produced of triangle ABC is called the escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C, respectively. The centres of the escribed circles are called the ex-centres. The centre of the escribed circle opposite to the angle A is the point of intersection of the external bisectors of angles B and C. The internal bisector of angle A also passes through the same point. The centre is generally denoted by I_1.

In any ΔABC, we have

i. $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

ii. $r_1 = s\tan\dfrac{A}{2}$, $r_2 = s\tan\dfrac{B}{2}$, $r_3 = s\tan\dfrac{C}{2}$

iii. $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$r_2 = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$

$r_3 = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

Proof:

Let I_1 be the point of intersection of external bisectors of angles B and C of ΔABC. Suppose the circle touches the side BC at D and sides AB and AC produced at F and E, respectively. Clearly, $I_1D = I_1E = I_1F = r_1$.

i. Area of ΔABC = area of ΔI_1AC + Area of ΔI_1AB − Area of ΔI_1BC

$$\Rightarrow \Delta = \frac{1}{2}r_1 b + \frac{1}{2}r_1 c - \frac{1}{2}r_1 a$$

$$= \frac{1}{2}r_1(b+c-a)$$

$$= \frac{r_1}{2}(2s-2a)$$

$$\Rightarrow r_1 = \frac{\Delta}{s-a}$$

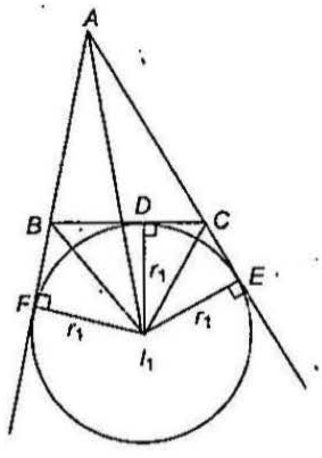


Fig. 5.32

Similarly, it can be shown that $r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$

ii. $s \tan \dfrac{A}{2} = s\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \dfrac{\sqrt{s(s-b)(s-c)(s-a)}}{s-a} = \dfrac{\Delta}{s-a} = r_1$

Similarly, $r_2 = s\tan\dfrac{B}{2}$ and $r_3 = s\tan\dfrac{C}{2}$.

iii. $4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = 4R\sqrt{\dfrac{(s-b)(s-c)}{bc}}\sqrt{\dfrac{s(s-b)}{ac}}\sqrt{\dfrac{s(s-c)}{ab}}$

$$= 4R\,\frac{s(s-a)(s-b)(s-c)}{abc\,(s-a)}$$

$$= \frac{4R}{abc}\frac{\Delta^2}{(s-a)} = \frac{\Delta}{s-a}$$

$$= r_1$$

Similarly, we can prove for r_2 and r_3.

Example 5.40 **Prove that $r_1 + r_2 + r_3 - r = 4R$.**

Sol. $r_1 + r_2 + r_3 - r = \dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c} - \dfrac{\Delta}{s} = \Delta\left[\dfrac{s-b+s-a}{(s-a)(s-b)} + \dfrac{(s-s+c)}{s(s-c)}\right]$

$$= \Delta\left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$$

$$= \Delta c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$$

$$= \frac{\Delta c}{\Delta^2}\,[2s^2 - s(a+b+c) + ab]$$

$$= (c/\Delta)\,[2s^2 - s(2s) + ab] = 4\,(abc/4\Delta) = 4R$$

Example 5.41 **Prove that $\cos A + \cos B + \cos C = 1 + r/R$.**

Sol. $\cos A + \cos B + \cos C = 1 + 4\sin(A/2)\sin(B/2)\sin(C/2)$

$$= 1 + [4R\sin(A/2)\sin(B/2)\sin(C/2)]/R = 1 + r/R$$

Example 5.42 **Prove that $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c} = \dfrac{r}{R}$.**

Sol. We have,

$a\cos A + b\cos B + c\cos C = R(2\sin A\cos A + 2\sin B\cos B + 2\sin A\sin C)$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R\sin A\sin B\sin C$$

and $a+b+c = 2R(\sin A + \sin B + \sin C) = 8R4\cos(A/2)\cos(B/2)\cos(C/2)$

$$\Rightarrow \frac{a\cos A + b\cos B + c\cos C}{a+b+c}$$

$$= \frac{4R\sin A\sin B\sin C}{8R\cos A/2\ \cos B/2\ \cos C/2}$$

$$= [4R\sin(A/2)\sin(B/2)\sin(C/2)]/R = r/R$$

Example 5.43 **If in a triangle $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.**

Sol. We have $r_1 = r_2 + r_3 + r$

$\Rightarrow r_1 - r = r_2 + r_3$

$$\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$\Rightarrow \frac{\Delta a}{s(s-a)} = \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{\Delta a}{(s-b)(s-c)}$$

$\Rightarrow s(s-a) = (s-b)(s-c)$

$\Rightarrow s^2 - sa = s^2 - (b+c)s + bc$

$\Rightarrow 2s(b+c-a) = 2bc$

$\Rightarrow (a+b+c)(b+c-a) = 2bc$

$\Rightarrow (b+c)^2 - a^2 = 2bc$

$\Rightarrow b^2 + c^2 = a^2$

Hence, the triangle is right angled.

Example 5.44 **Prove that $\frac{r_1 + r_2}{1+\cos C} = 2R$.**

Sol. $$r_1 + r_2 = 4R\left(\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2}\right)$$

$$= 4R\left(\cos\frac{C}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{A}{2}\right)\right)$$

$$= 4R\left(\cos^2\frac{C}{2}\right) = 2R(1+\cos C)$$

$$\Rightarrow \frac{r_1 + r_2}{1+\cos C} = 2R$$

Concept Application Exercise 5.6

1. In ΔABC, if $r_1 < r_2 < r_3$, then find the order of lengths of the sides.
2. Find the radius of the in-circle of a triangle where sides are 18, 24 and 30 cm.
3. If in ΔABC, $(a-b)(s-c) = (b-c)(s-a)$, prove that r_1, r_2, r_3 are in A.P.
4. In triangle ABC, $\angle A = \frac{\pi}{2}$, prove that $r + 2R = \frac{1}{2}(b+c+a)$.

5. Prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

6. Prove that $r_1r_2+r_2r_3+r_3r_1=\frac{1}{4}(a+b+c)^2$.

7. In any triangle ABC, find the least value of $\frac{r_1+r_2+r_3}{r}$.

Geometry Relating to Ex-centres

Consider an acute-angled ΔABC.

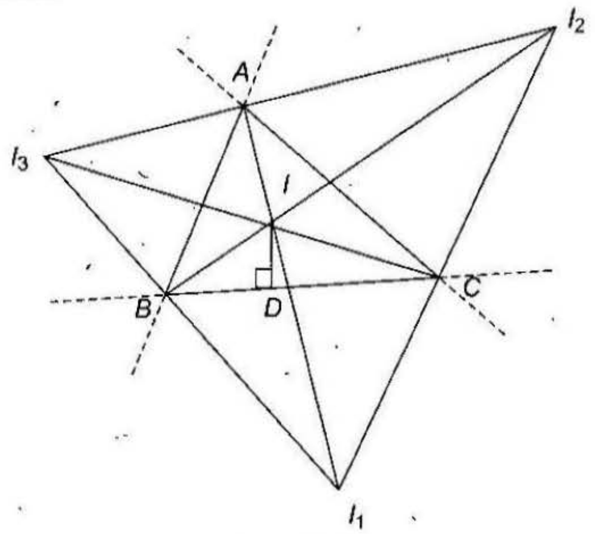


Fig. 5.33

At I_1, external bisectors of $\angle B$ and $\angle C$ and internal bisector of $\angle A$ are concurrent.

Also $\angle IBC = B/2$ and $\angle CBI_1 = \frac{\pi}{2}-\frac{B}{2} \Rightarrow \angle IBI_1 = \pi/2$ or $BI \perp I_1I_3$

Similarly, $CI \perp I_1I_2$ and $AI \perp I_2I_3$.

Thus, the in-centre of triangle ABC is orthocentre of $\Delta\, I_1I_2I_3$ and ABC is the pedal triangle of $\Delta\, I_1I_2I_3$.

Distance Between In-centre and Ex-centre

In $\Delta\, IDB$, $BI = \frac{ID}{\sin\angle IBD} = \frac{r}{\sin(B/2)}$

Also in $\Delta\, IBI_1$, $II_1 = \frac{BI}{\cos\angle BII_1} = \frac{r}{\sin(B/2)\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)} = \frac{r}{\sin(B/2)\sin(C/2)}$

Similarly, $II_2 = \frac{r}{\sin(A/2)\sin(C/2)}$ and $II_3 = \frac{r}{\sin(A/2)\sin(B/2)}$

Distance Between Ex-centres

Let us find I_1I_2. Points B, I, C and I_1 are concyclic.

Hence, $\angle II_1C = \angle IBC = B/2$.

Similarly, points A, I, C and I_2 are concyclic. So, $\angle II_2C = \angle IAC = A/2$.

Then in ΔII_1I_2, $\angle I_1II_2 = \pi - \frac{A+B}{2}$.

Now in ΔI_1I_2 from the sine rule, we get

$$\frac{I_1I_2}{\sin\left(\pi-\frac{A+B}{2}\right)}=\frac{II_1}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad \frac{I_1I_2}{\cos\left(\frac{C}{2}\right)}=\frac{\dfrac{r}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad I_1I_2=\frac{r\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}=4R\cos\left(\frac{C}{2}\right)$$

Similarly, $I_2I_3=4R\cos\left(\frac{A}{2}\right)$ and $I_1I_3=4R\cos\left(\frac{B}{2}\right)$.

MISCELLANEOUS TOPICS

$m-n$ Theorem

Let D be a point on the side BC of a $\Delta\, ABC$ such that $BD{:}DC = m{:}n$ and $\angle ADC=\theta$, $\angle BAD=\alpha$ and $\angle DAC=\beta$. Then

i. $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$
ii. $(m+n)\cot\theta = n\cot B - m\cot C$

Proof:

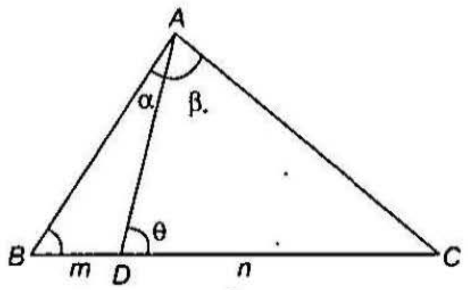


Fig. 5.34

i. Given $\dfrac{BD}{DC}=\dfrac{m}{n}$ and $\angle ADC=\theta$
$\angle ADB=(180°-\theta)$, $\angle BAD=\alpha$ and $\angle DAC=\beta$
$\angle ABD=180°-(\alpha+180°-\theta)=\theta-\alpha=B$

From ΔABD, $\dfrac{BD}{\sin\alpha}=\dfrac{AD}{\sin(\theta-\alpha)}$ (i)

From ΔADC, $\dfrac{DC}{\sin\beta} = \dfrac{AD}{\sin(\theta+\beta)}$ (ii)

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{BD \sin\beta}{DC \sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} \quad \text{(iii)}$$

$$\Rightarrow \quad \frac{m}{n}\frac{\sin\beta}{\sin\alpha} = \frac{\sin\theta\cos\beta + \cos\theta\sin\beta}{\sin\theta\cos\alpha - \cos\theta\sin\alpha}$$

$\Rightarrow \quad m\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha) = n\sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta)$

$\Rightarrow \quad m\cot\alpha - m\cot\theta = n\cot\beta + n\cot\theta$ [dividing both sides by $\sin\alpha\sin\beta\sin\theta$]

$\Rightarrow \quad (m+n)\cot\theta = m\cot\alpha - n\cot\beta$ (iv)

ii. We have $\angle CAD = 180° - (\theta + C)$
$\angle ABC = B, \angle ACD = C, \angle BAD = (\theta - B)$
Putting these values in Eq. (iii), we get
$m\sin(\theta+C)\sin B = n\sin C\sin(\theta - B)$
$\Rightarrow \quad m(\sin\theta\cos C + \cos\theta\sin C)\sin B = n\sin C(\sin\theta\cos B - \cos\theta\sin B)$
Dividing both sides by $\sin\theta\sin B\sin C$, we get
$m(\cot C + \cot\theta) = n(\cot B - \cot\theta)$
$\therefore \quad (m+n)\cot\theta = n\cot B - m\cot C$

Example 5.45 **If the median AD of triangle ABC makes an angle $\pi/4$ with the side BC, then find the value of $|\cot B - \cot C|$.**

Sol.

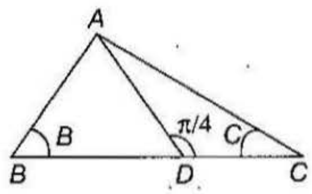


Fig. 5.35

By m–n theorem,
$(BD + DC)\cot(\pi/4) = DC\cot B - BD\cot C \quad \Rightarrow |\cot B - \cot C| = 2$

Inequality

In Chapter 2, we have proved that $\cos A + \cos B + \cos C \le \dfrac{3}{2}$. (i)

Also in ΔABC, $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8} \quad \text{(ii)}$$

In ΔABC, $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} \Rightarrow R \ge 2r$ [using Eq. (ii)]

Example 5.46 **Prove that $a\cos A + b\cos B + c\cos C \le s$.**

Sol. $a\cos A + b\cos B + c\cos C = R(2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C)$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R \sin A \sin B \sin C = \frac{2}{R}\left(2R^2 \sin A \sin B \sin C\right)$$

$$= \frac{2}{R}\Delta = 2\frac{rs}{R} \le s \qquad [\because R \ge 2r]$$

Example 5.47 **In triangle *ABC*, prove that the maximum value of tan $\frac{A}{2}$ tan $\frac{B}{2}$ tan $\frac{C}{2}$ is $\frac{R}{2s}$.**

Sol. For triangle ABC, we have

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s^2}$$

$$= \frac{\Delta}{s^2} = \frac{r}{s} \le \frac{R}{2s}$$

Area of Quadrilateral

$ABCD$ is any quadrilateral where $AB = a$, $BC = b$, $CD = c$, $AD = d$ and $\angle DPA = \alpha$. Let us denote the area of the quadrilateral by S, then ΔDAC = area of ΔAPD + area of ΔDPC.

$$= \frac{1}{2} DP \times AP \times \sin\alpha + \frac{1}{2} DP \times PC \times \sin(\pi - \alpha)$$

$$= \frac{1}{2} DP(AP + PC)\sin\alpha$$

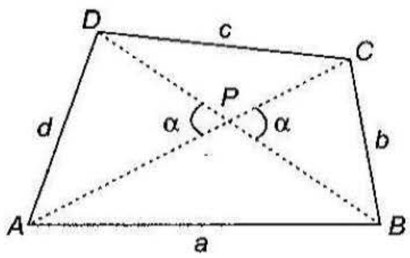


Fig. 5.36

Area of $\Delta DAC = \frac{1}{2} DP \times AC \times \sin\alpha$ (i)

Similarly, area of $\Delta ABC = \frac{1}{2} BP \times AC\sin\alpha$ (ii)

$\therefore$ S = area of ΔDAC + area of ΔABC

$$= \frac{1}{2} DP \times AC\sin\alpha + \frac{1}{2} BP \times AC\sin\alpha \qquad \text{[using (ii) and (iii)]}$$

$$= \frac{1}{2}(DP + BP)AC\sin\alpha \quad \Rightarrow S = \frac{1}{2} BD \times AC\sin\alpha \qquad \text{(iii)}$$

Therefore, area of quadrilateral = $\frac{1}{2}$ (product of the diagonals) × (sine of included angle).

Cyclic Quadrilateral

A cyclic quadrilateral is a quadrilateral which can be circumscribed by a circle.

Note:

- *Sum of the opposite angles of a cyclic quadrilateral is 180°.*
- *In a cyclic quadrilateral, sum of the products of the opposite sides is equal to the product of the diagonals. This is known as Ptolemy's theorem.*
- *If sum of the opposite sides of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.*

Regular Polygon

A regular polygon is a polygon which has equal sides as well as equal angles. In any polygon of n sides, sum of its internal angles is $(n-2)\pi$, then in regular polygon each angle is $\frac{(n-2)\pi}{n}$.

Note:

- *In the regular polygon, the circumcentre and the in-centre are the same.*

Radii of the inscribed and the circumscribed circles and area of a regular polygon of n sides with each side a.

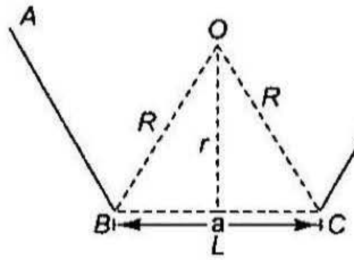


Fig. 5.37

Let AB, BC and CD be three successive sides of the polygon and O be the centre of both the incircle and the circumcircle of the polygon.

$$\angle BOC = \frac{2\pi}{n} \qquad \Rightarrow \angle BOL = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$$

$$a = BC = 2\,BL = 2R\sin\angle BOL = 2R\sin\frac{\pi}{n} \qquad \Rightarrow R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

Again, $a = 2\,BL = 2\,OL\tan\angle BOL = 2r\tan\frac{\pi}{n} \qquad \Rightarrow r = \frac{a}{2}\cot\frac{\pi}{n}$

Now, the area of the regular polygon $= n$ times the area of the $\Delta OBC = n\left(\frac{1}{2}OL \times BC\right)$

$$= n\frac{1}{2}\left(\frac{a}{2}\cot\frac{\pi}{n}\right)a$$

$$= \frac{na^2}{4}\cot\frac{\pi}{n} \quad \text{[in terms of side of polygon]} \qquad \text{(i)}$$

Now, $a = 2r\tan\frac{\pi}{n} \Rightarrow \Delta = nr^2\tan\left(\frac{\pi}{n}\right)$ [from Eq. (i)]

Also, $a = 2R\sin\frac{\pi}{n} \Rightarrow \Delta = \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$ [from Eq. (i)]

Example 5.48 **Find the sum of the radii of the circles, which are, respectively, inscribed and circumscribed about a polygon of n sides, whose side length is a.**

Sol. Radius of the circumscribed circle $= R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$

and radius of the inscribed circle $= r = \frac{1}{2} a \cot(\pi/n)$

$$\Rightarrow \quad R + r = \frac{a}{2\sin(\pi/n)} + \frac{a\cos(\pi/n)}{2\sin(\pi/n)} = \frac{a[1+\cos(\pi/n)]}{2\times 2\sin(\pi/2n)\cos(\pi/2n)} = \frac{1}{2} a \cot\left(\frac{\pi}{2n}\right)$$

Example 5.49 **If the area of the circle is A_1 and the area of the regular pentagon inscribed in the circle is A_2, then find the ratio A_1/A_2.**

Sol.

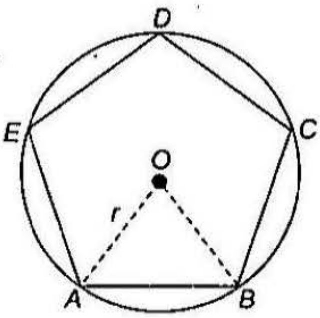


Fig. 5.38

In ΔOAB, $OA = OB = r$ and $\angle AOB = \frac{360°}{5} = 72°$

Therefore, area of $\Delta AOB = \frac{1}{2} r \times r \sin 72° = \frac{1}{2} r^2 \cos 18°$

Area of pentagon $(A_2) = 5$ (area of ΔAOB) $= 5\left(\frac{1}{2} r^2 \cos 18°\right)$ (i)

Also, area of the circle $(A_1) = \pi r^2$ (ii)

Hence, $\frac{A_1}{A_2} = \frac{\pi r^2}{\frac{5}{2} r^2 \cos 18°} = \frac{2\pi}{5} \sec\left(\frac{\pi}{10}\right)$ [from Eqs. (i) and (ii)]

Example 5.50 **Prove that the area of a regular polygon of $2n$ sides inscribed in a circle is the geometric mean of the areas of the inscribed and circumscribed polygons of n sides.**

Sol. Let a be the radius of the circle.
Then,

S_1 = Area of regular polygon of n sides inscribed in the circle $= \frac{1}{2} na^2 \sin(2\pi/n)$

S_2 = Area of regular polygon of n sides circumscribing the circle $= na^2 \tan(\pi/n)$
S_3 = Area of regular polygon of $2n$ sides inscribed in the circle $= na^2 \sin(\pi/n)$
[replacing n by $2n$ is S_1]

Therefore, geometric mean of S_1 and $S_2 = \sqrt{(S_1 S_2)} = na^2 \sin(\pi/n) = S_3$

SOLUTION OF TRIANGLES (AMBIGUOUS CASES)

The three sides a, b, c and the three angles A, B, C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is, the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

- If the three sides a, b and c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ or $\cos A = \frac{b^2+c^2-a^2}{2bc}$. B and C can be obtained in a similar way.
- If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$. Also, $\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated. The third side is given by $a = b\frac{\sin A}{\sin B}$ or $a^2 = b^2 + c^2 - 2bc\cos A$.
- If two sides b and c and the angle B (opposite to side b) are given, then $\sin C = \frac{c}{b}\sin B$, $A = 180° - (B+C)$ and $a = \frac{b\sin A}{\sin B}$ give the remaining elements.

Case I:

$b < c\sin B$

We draw the side c and angle B. Now, it is obvious from Fig. 5.39 that there is no triangle possible.

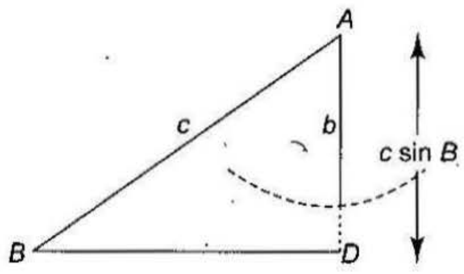


Fig. 5.39

Case II:

$b = c\sin B$ and B is an acute angle, then there is only one triangle possible.

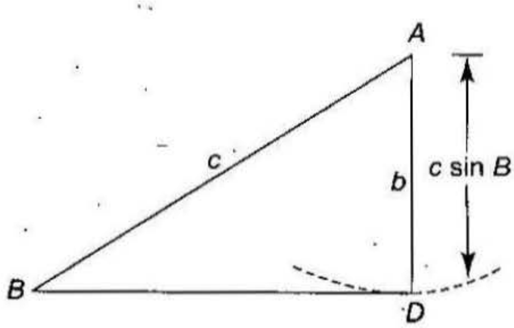


Fig. 5.40

Case III:

$b > c \sin B$, $b < c$ and B is an acute angle, then there are two values of angle C. Hence, two triangles are possible.

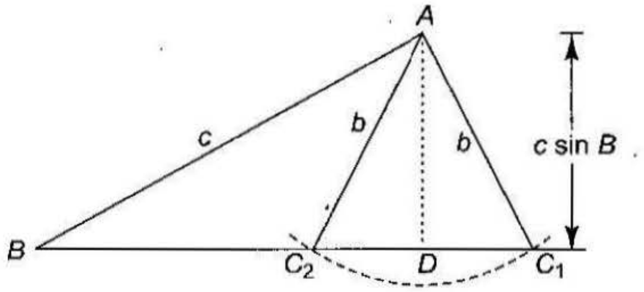


Fig. 5.41

Case IV:

$b > c \sin B$, $c < b$ and B is an acute angle, then there is only one triangle possible.

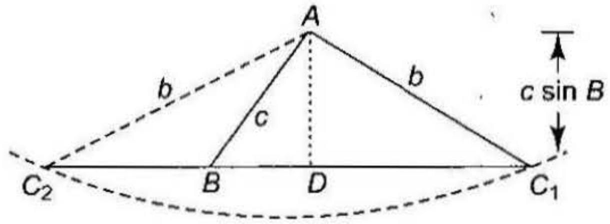


Fig. 5.42

Case V:

$b > c \sin B$, $c > b$ and B is an obtuse angle. For any choice of point C, b will be greater than c which is a contradiction as $c > b$ (given). So, there is no triangle possible.

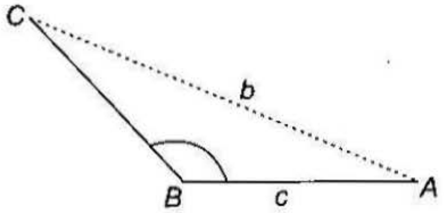


Fig. 5.43

Case VI:

$b > c \sin B$, $c < b$ and B is an obtuse angle. We can see that the circle with A as centre and b as radius will cut the line only in one point. So, only one triangle is possible.

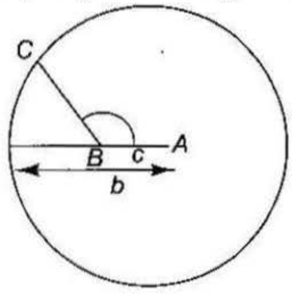


Fig. 5.44

Case VII:

$b > c$ and $B = 90°$

Again the circle with A as centre and b as radius will cut the line only in one point. So, only one triangle is possible.

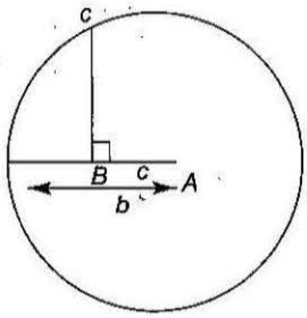


Fig. 5.45

Case VIII:

$b \le c$ and $B = 90°$

The circle with A as centre and b as radius will not cut the line in any point. So, no triangle is possible.

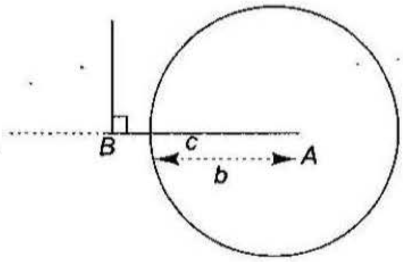


Fig. 5.46

Alternative method:

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$$\Rightarrow \quad a^2 - (2c \cos B)\, a + (c^2 - b^2) = 0$$

$$\Rightarrow \quad a = c \cos B \pm \sqrt{(c \cos B)^2 - (c^2 - b^2)}$$

$$= c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$$

This equation leads to the following cases:

Case I: If $b < c \sin B$, no such triangle is possible.

Case II: Let $b = c \sin B$. There are further following two cases:

a. B is an obtuse angle $\Rightarrow \cos B$ is negative. There exists no such triangle.

b. B is an acute angle $\Rightarrow \cos B$ is positive. There exists only one such triangle.

Case III: Let $b > c \sin B$. There are further following two cases:

a. B is an acute angle $\Rightarrow \cos B$ is positive. In this case, two values of a will exist if and only if $c \cos B > \sqrt{b^2 - (c \sin B)^2}$ or $c > b \Rightarrow$ two such triangles are possible. If $c < b$, only one such triangle is possible.

b. B is an obtuse angle $\Rightarrow \cos B$ is negative. In this case, triangle will exist if and only if $\sqrt{b^2 - (c \sin B)^2} > c|\cos B| \Rightarrow b > c$. So, in this case, only one such triangle is possible. If $b < c$ there exists no such triangle.

Note:

- *If one side a and angles B and C are given, then* $A = 180° - (B + C)$ *and* $b = \frac{a \sin B}{\sin A}$, $c = \frac{a \sin C}{\sin A}$.
- *If the three angles A, B and C are given, we can only find the ratios of the sides a, b and c by using the sine rule (since there are infinite number of similar triangles possible).*

Example 5.51 **If $b = 3$, $c = 4$ and $B = \pi/3$, then find the number of triangles that can be constructed.**

Sol. We have,

$$\frac{\sin B}{b} = \frac{\sin C}{c} \quad \Rightarrow \frac{\sin(\pi/3)}{3} = \frac{\sin C}{4} \quad \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1 \text{ which is not possible.}$$

Hence, no triangle is possible.

Example 5.52 **If $A = 30°$, $a = 7$ and $b = 8$ in ΔABC, then find the number of triangles that can be constructed.**

Sol. We have $\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \sin B = \frac{b \sin A}{a} = \frac{8 \sin 30°}{7} = 4/7$

Thus, we have, $b > a > b \sin A$.

Hence, angle B has two values given by $\sin B = 4/7$.

Example 5.53 **If in triangle ABC, $a = (1 + \sqrt{3})$ cm, $b = 2$ cm and $\angle C = 60°$, then find the other two angles and the third side.**

Sol. From $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, we have

$$\frac{1}{2} = \frac{(1+\sqrt{3})^2 + 4 - c^2}{2(1+\sqrt{3})2}$$

$$\Rightarrow \quad 2 + 2\sqrt{3} = 1 + 3 + 2\sqrt{3} + 4 - c^2$$

$$\Rightarrow \quad c^2 = 6 \Rightarrow c = \sqrt{6} \text{ cm}$$

Also, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \Rightarrow \frac{\sin A}{1+\sqrt{3}} = \frac{\sin B}{2} = \frac{\sqrt{3}/2}{\sqrt{6}} \Rightarrow \sin B = \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad B = 45° \Rightarrow A = 180° - (60° + 45°) = 75°$$

Example 5.54 **In ΔABC, the sides b, c and the angle B are given such that a has two values a_1 and a_2. Then prove that $|a_1 - a_2| = 2\sqrt{b^2 - c^2 \sin^2 B}$.**

Sol. $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

$$\Rightarrow \quad a^2 - 2c \cos B\, a + c^2 - b^2 = 0$$

$$\Rightarrow \quad a_1 + a_2 = 2c \cos B,\ a_1 a_2 = c^2 - b^2$$

$$\Rightarrow \quad (a_1 - a_2)^2 = (a_1 + a_2)^2 - 4a_1 a_2$$

$$= 4c^2 \cos^2 B - 4(c^2 - b^2) = 4b^2 - 4c^2 \sin^2 B = 4(b^2 - c^2 \sin^2 B)$$

$$\Rightarrow \quad |a_1 - a_2| = 2\sqrt{b^2 - c^2 \sin^2 B}$$

Example 5.55 **In ΔABC, a, c and A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then prove that $\sin A = \sqrt{\dfrac{9a^2 - c^2}{8c^2}}$.**

Sol. We have $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ $\Rightarrow b^2 - 2\,bc\cos A + (c^2 - a^2) = 0$

It is given that b_1 and b_2 are the roots of this equation.

Therefore, $b_1 + b_2 = 2c\cos A$ and $b_1 b_2 = c^2 - a^2$

$\Rightarrow$ $3b_1 = 2\,c\cos A,\ 2\,b_1^2 = c^2 - a^2$ ($\because b_2 = 2b_1$ given)

$\Rightarrow$ $2\left(\dfrac{2c}{3}\cos A\right)^2 = c^2 - a^2$

$\Rightarrow$ $8c^2(1 - \sin^2 A) = 9c^2 - 9c^2$

$\Rightarrow$ $\sin A = \sqrt{\dfrac{9a^2 - c^2}{8c^2}}$

EXERCISES

Subjective Type

Solutions on page 5.63

1. O is the circumcentre of ΔABC and R_1, R_2, R_3 are, respectively, the radii of the circumcircles of the triangles OBC, OCA and OAB. Prove that $\dfrac{a}{R_1} + \dfrac{b}{R_2} + \dfrac{c}{R_3} = \dfrac{abc}{R^3}$.
2. In triangle ABC, D is on AC such that $AD = BC$, $BD = DC$, $\angle DBC = 2x$ and $\angle BAD = 3x$, all angles are in degrees, then find the value of x.
3. If in ΔABC, the distances of the vertices from the orthocentre are x, y and z, then prove that $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = \dfrac{abc}{xyz}$.
4. In ΔABC, a semicircle is inscribed, which lies on the side c. If x is the length of the angle bisector through angle C, then prove that the radius of the semicircle is $x\sin(C/2)$.
5. Prove that the distance between the circumcentre and the orthocentre of triangle ABC is $R\sqrt{1 - 8\cos A\cos B\cos C}$.
6. Prove that the distance between the circumcentre and the incentre of triangle ABC is $\sqrt{R^2 - 2Rr}$.
7. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.
8. If p and q are perpendiculars from the angular points A and B of the ΔABC drawn to any line through the vertex C, then prove that $a^2b^2\sin^2 C = a^2p^2 + b^2q^2 - 2\,abpq\cos C$.
9. If I is the incentre of ΔABC and R_1, R_2 and R_3 are, respectively, the radii of the circumcircles of the triangles IBC, ICA and IAB, then prove that $R_1 R_2 R_3 = 2\,rR^2$.
10. In a circle of radius r, chords of lengths a and b cm subtend angles θ and 3θ, respectively at the centre. Show that $r = a\sqrt{\dfrac{a}{3a - b}}$ cm.

11. If in triangle ABC, the median AD and the perpendicular AE from the vertex A to the side BC divide the angle A into three equal parts, show that $\cos\frac{A}{3}\sin^2\frac{A}{3}=\frac{3a^2}{32bc}$.

12. Perpendiculars are drawn from the angles A, B and C of an acute-angled triangle on the opposite sides, and produced to meet the circumscribing circle. If these produced parts are α, β, γ, respectively, then show that $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2(\tan A+\tan B+\tan C)$.

13. Show that the line joining the incentre to the circumcentre of triangle ABC is inclined to the side BC at an angle $\tan^{-1}\left(\frac{\cos B+\cos C-1}{\sin C-\sin B}\right)$.

14. If the tangents of the angles of a triangle are in A.P., prove that the squares of the sides are in the ratio of $x^2(x^2+9):(3+x^2)^2:9(1+x^2)$ where x is the tangent of the least or the greatest angle.

15. In ABC, right angled at C, if $\tan A=\sqrt{\frac{\sqrt{5}-1}{2}}$, show that the sides a, b and c are in G.P.

Objective Type

Solutions on page 5.72

Each question has four choices a, b, c, and d, out of which *only one* answer is correct.

1. If in ΔABC, $\sin A\cos B=\frac{\sqrt{2}-1}{\sqrt{2}}$ and $\sin B\cos A=\frac{1}{\sqrt{2}}$, then the triangle is

   a. equilateral b. isosceles c. right angled d. righ-angled isosceles

2. ABC is an equilateral triangle of side 4 cm. If R, r and h are the circumradius, inradius and altitude, respectively, then $\frac{R+r}{h}$ is equal to

   a. 4 b. 2 c. 1 d. 3

3. In ΔABC, if $\frac{\sin A}{c\sin B}+\frac{\sin B}{c}+\frac{\sin C}{b}=\frac{c}{ab}+\frac{b}{ac}+\frac{a}{bc}$, then the value of angle A is

   a. 120° b. 90° c. 60° d. 30°

4. A piece of paper is in the shape of a square of side 1 m long. It is cut at the four corners to make a regular polygon of eight sides (octagon). The area of the polygon is

   a. $2(\sqrt{2}-1)\,\text{m}^2$ b. $(\sqrt{2}-1)\,\text{m}^2$ c. $\frac{1}{\sqrt{2}}\,\text{m}^2$ d. none of these

5. If A, B and C are angles of a triangle such that angle A is obtuse, then $\tan B\tan C$ will be less than

   a. $\frac{1}{\sqrt{3}}$ b. $\frac{\sqrt{3}}{2}$ c. 1 d. none of these

6. In ΔABC, $\angle B=\pi/3$. The range of values of x, where $x=\sin A\sin C$, is the interval

   a. $\left[-\frac{1}{4},\frac{3}{4}\right]$ b. $\left(0,\frac{3}{4}\right)$ c. $\left(0,\frac{3}{4}\right]$ d. $\left[\frac{1}{4},\frac{3}{4}\right]$

7. If in ΔABC, AC is double of AB, then the value of $\cot\frac{A}{2}\cot\frac{B-C}{2}$ is equal to

a. $\frac{1}{3}$ b. $-\frac{1}{3}$ c. 3 d. $\frac{1}{2}$

8. In a right-angled isosceles triangle, the ratio of the circumradius and inradius is

a. $2(\sqrt{2}+1):1$ b. $(\sqrt{2}+1):1$ c. 2:1 d. $\sqrt{2}:1$

9. In triangle ABC, $a = 5$, $b = 3$ and $c = 7$, the value of $3\cos C + 7\cos B$ is equal to

a. 5 b. 10 c. 7 d. 3

10. If in triangle ABC, $\angle B = 90°$, then $\tan^2(A/2)$ is

a. $\frac{b-c}{b+c}$ b. $\frac{b+c}{b-c}$ c. $\frac{2bc}{b-c}$ d. none of these

11. In ΔABC, if $b^2+c^2=2a^2$, then value of $\frac{\cot A}{\cot B+\cot C}$ is

a. $\frac{1}{2}$ b. $\frac{3}{2}$ c. $\frac{5}{2}$ d. $\frac{5}{3}$

12. If $\sin\theta$ and $-\cos\theta$ are the roots of the equation $ax^2-bx-c=0$, where a, b and c are the sides of a triangle ABC, then $\cos B$ is equal to

a. $1-\frac{c}{2a}$ b. $1-\frac{c}{a}$ c. $1+\frac{c}{2a}$ d. $1+\frac{c}{3a}$

13. In ΔABC, $a^2+b^2+c^2=ac+ab\sqrt{3}$, then the triangle is

a. equilateral b. isosceles c. right angled d. none of these

14. In ΔABC, $(a+b+c)(b+c-a)=kbc$ if

a. $k<0$ b. $k>0$ c. $0<k<4$ d. $k>4$

15. If one side of a triangle is double the other, and the angles on opposite sides differ by 60°, then the triangle is

a. equilateral b. obtuse angled c. right angled d. acute angled

16. In triangle ABC, if $r_1=2r_2=3r_3$, then $a:b$ is equal to

a. $\frac{5}{4}$ b. $\frac{4}{5}$ c. $\frac{7}{4}$ d. $\frac{4}{7}$

17. If in a triangle, $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, then the triangle is

a. right angled b. isosceles c. equilateral d. none of these

18. In an equilateral triangle, the inradius, circumradius and one of the ex-radii are in the ratio

a. 2:3:5 b. 1:2:3 c. 1:3:7 d. 3:7:9

19. In any ΔABC, if $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P., then a, b, c are in

a. A.P. b. G.P. c. H.P. d. none of these

20. In ΔABC, if $A=30°$, $b=2$, $c=\sqrt{3}+1$, then $\frac{C-B}{2}$ is equal to

a. 15° b. 30° c. 45° d. none of these

21. In triangle ABC, if $a:b:c = 7:8:9$, then $\cos A:\cos B$ is equal to

a. $\frac{11}{63}$ b. $\frac{22}{63}$ c. $\frac{2}{9}$ d. none of these

22. In triangle ABC, if $\cos A + \cos B + \cos C = \frac{7}{4}$, then $\frac{R}{r}$ is equal to

a. $\frac{3}{4}$ b. $\frac{4}{3}$ c. $\frac{2}{3}$ d. $\frac{3}{2}$

23. In ΔABC, $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$ is equal to

a. $\frac{\Delta}{r^2}$ b. $\frac{(a+b+c)^2}{abc} 2R$ c. $\frac{\Delta}{r}$ d. $\frac{\Delta}{Rr}$

24. In triangle ABC, $a^2 + c^2 = 2002b^2$, then $\frac{\cot A + \cot C}{\cot B}$ is equal to

a. $\frac{1}{2001}$ b. $\frac{2}{2001}$ c. $\frac{3}{2001}$ d. $\frac{4}{2001}$

25. If the hypotenuse of a right-angled triangle is four times the length of the perpendicular drawn from the opposite vertex to it, then the difference of the two acute angles will be

a. 60° B. 15° c. 75° d. 30°

26. In ΔABC, if $\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$ and $\sin^2 \frac{C}{2}$ are in H.P., then a, b and c will be in

a. A.P. b. G.P. c. H.P. d. none of these

27. Given $b = 2$, $c = \sqrt{3}$, $\angle A = 30°$, then inradius of ΔABC is

a. $\frac{\sqrt{3}-1}{2}$ b. $\frac{\sqrt{3}+1}{2}$ c. $\frac{\sqrt{3}-1}{4}$ d. none of these

28. If P is a point on the altitude AD of the triangle ABC such that $\angle CBP = B/3$, then AP is equal to

a. $2a \sin \frac{C}{3}$ b. $2b \sin \frac{C}{3}$ c. $2c \sin \frac{B}{3}$ d. $2c \sin \frac{C}{3}$

29. In triangle ABC, $\angle A = \pi/2$, then $\tan(C/2)$ is equal to

a. $\frac{a-c}{2b}$ b. $\frac{a-b}{2c}$ c. $\frac{a-c}{b}$ d. $\frac{a-b}{c}$

30. With usual notations, in triangle ABC, $a \cos (B - C) + b \cos (C - A) + c \cos (A - B)$ is equal to

a. $\frac{abc}{R^2}$ b. $\frac{abc}{4R^2}$ c. $\frac{4\,abc}{R^2}$ d. $\frac{abc}{2R^2}$

31. In ΔABC, $\sin A + \sin B + \sin C = 1 + \sqrt{2}$ and $\cos A + \cos + \cos C = \sqrt{2}$ if the triangle is

a. equilateral b. isosceles c. right angled d. right-angled isosceles

32. If $\frac{r}{r_1} = \frac{r_2}{r_3}$, then

a. $A = 90°$ b. $B = 90°$ c. $C = 90°$ d. none of these

33. If in a ΔABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to

a. 90° b. 45° c. 120° d. none of these

34. If $\cos B \cos C + \sin B \sin C \sin^2 A = 1$, then triangle ABC is

a. isosceles and right angled

b. equilateral

c. isosceles whose equal angles are greater than $\pi/4$

d. none

35. In triangle ABC, internal angle bisector $\angle A$ makes an angle θ with side BC. The value of $\sin\theta$ is equal to

a. $\left|\sin\left(\frac{B-C}{2}\right)\right|$ c. $\left|\sin\left(\frac{B}{2}-C\right)\right|$ c. $\cos\left(\frac{B-C}{2}\right)$ d. $\cos\left(\frac{B}{2}-C\right)$

36. In an acute angled triangle ABC, $r+r_1=r_2+r_3$ and $\angle B>\frac{\pi}{3}$, then

a. $b+2c<2a<2b+2c$ b. $b+4c<4a<2b+4c$

c. $b+4c<4a<4b+4c$ d. $b+3c<3a<3b+3c$

37. In triangle ABC, $\angle A=30°$, $BC=2+\sqrt{5}$, then the distance of the vertex A from the orthocentre of the triangle is

a. 1 b. $(2+\sqrt{5})\sqrt{3}$ c. $\frac{\sqrt{3}+1}{2\sqrt{2}}$ d. $\frac{1}{2}$

38. If I is the incentre of a triangle ABC, then the ratio $IA:IB:IC$ is equal to

a. $\operatorname{cosec}\frac{A}{2}:\text{coses}\frac{B}{2}:\operatorname{cosec}\frac{C}{2}$ b. $\sin\frac{A}{2}:\sin\frac{B}{2}:\sin\frac{C}{2}$

c. $\sec\frac{A}{2}:\sec\frac{B}{2}:\sec\frac{C}{2}$ d. none of these

39. In ΔABC, the bisector of the angle A meets the side BC at D and the circumscribed circle at E, then DE equals

a. $\frac{a^2\sec\frac{A}{2}}{2(b+c)}$ b. $\frac{a^2\sin\frac{A}{2}}{2(b+c)}$ c. $\frac{a^2\cos\frac{A}{2}}{2(b+c)}$ d. $\frac{a^2\operatorname{cosec}\frac{A}{2}}{2(b+c)}$

40. If D is the mid-point of the side BC of triangle ABC and AD is perpendicular to AC, then

a. $3b^2=a^2-c^2$ b. $3a^2=b^2-3c^2$ c. $b^2=a^2-c^2$ d. $a^2+b^2=5c^2$

41. Two medians drawn from the acute angles of a right-angled triangle intersect at an angle $\pi/6$. If the length of the hypotenuse of the triangle is 3 units, then the area of the triangle (in sq. units) is

a. $\sqrt{3}$ b. 3 c. $\sqrt{2}$ d. 9

42. For triangle ABC, $R=5/2$ and $r=1$. Let I be the incentre of the triangle and D, E and F be the feet of the perpendiculars from I to BC, CA and AB, respectively. The value of $\frac{ID\times IE\times IF}{IA\times IB\times IC}$ is equal to

a. $\frac{5}{2}$ b. $\frac{5}{4}$ c. $\frac{1}{10}$ d. $\frac{1}{5}$

43. If the median of ΔABC through A is perpendicular to AB, then

a. $\tan A+\tan B=0$ b. $2\tan A+\tan B=0$ c. $\tan A+2\tan B=0$ d. none of these

44. In ΔABC, the median AD divides $\angle BAC$ such that $\angle BAD:\angle CAD=2:1$. Then $\cos(A/3)$ is equal to

a. $\frac{\sin B}{2\sin C}$ b. $\frac{\sin C}{2\sin B}$ c. $\frac{2\sin B}{\sin C}$ d. none of these

45. If in ΔABC, $b=3$ cm, $c=4$ cm and the length of the perpendicular from A to the side BC is 2 cm, then the number of solutions of the triangle is

a. 1 b. 0 c. 3 d. 2

46. In in triangle ABC, $\sum \sin\frac{A}{2}=\frac{6}{5}$ and $\sum II_1=9$ (where I_1, I_2 and I_3 are ex-centres and I is in-centre, then circumradius R is equal to

a. $\frac{15}{8}$ b. $\frac{15}{4}$ c. $\frac{15}{2}$ d. $\frac{4}{12}$

47. In triangle ABC, medians AD and CE are drawn. If $AD=5$, $\angle DAC=\pi/8$ and $\angle ACE=\pi/4$, then the area of the triangle ABC is equal to

a. $\frac{25}{9}$ b. $\frac{25}{3}$ c. $\frac{25}{18}$ d. $\frac{10}{3}$

48. In triangle ABC, if $\tan(A/2)=5/6$ and $\tan(B/2)=20/37$, the sides a, b and c are in

a. A.P. b. GP. c. H.P. d. none of these

49. If H is the orthocentre of a acute-angled triangle ABC whose circumcircle is $x^2+y^2=16$, then circumdiameter of the triangle HBC is

a. 1 b. 2 c. 4 d. 8

50. In triangle ABC, $a=5$, $b=4$ and $c=3$. G is the centroid of the triangle. Circumradius of triangle GAB is equal to

a. $2\sqrt{13}$ b. $\frac{5}{12}\sqrt{13}$ c. $\frac{5}{3}\sqrt{13}$ d. $\frac{3}{2}\sqrt{13}$

51. In triangle ABC, line joining circumcentre and incentre is parallel to side AC, then $\cos A+\cos C$ is equal to

a. -1 b. 1 c. -2 d. 2

52. In triangle ABC, line joining the circumcentre and orthocentre is parallel to side AC, then the value of $\tan A\tan C$ is equal to

a. $\sqrt{3}$ b. 3 c. $3\sqrt{3}$ d. none of these

53. If in ΔABC, $8R^2=a^2+b^2+c^2$, then the triangle ABC is

a. right angled b. isosceles c. equilateral d. none of these

54. In triangle ABC, $\frac{a}{b}=\frac{2}{3}$ and $\sec^2 A=\frac{8}{5}$. Then the number of triangles satisfying these conditions is

a. 0 b. 1 c. 2 d. 3

55. We are given b, c and $\sin B$ such that B is acute and $b < c\sin B$. Then

a. no triangle is possible b. one triangle is possible
c. two triangles are possible d. a right-angled triangle is possible

56. If a, b and A are given in a triangle and c_1, c_2 are the possible values of the third side, then $c_1^2+c_2^2-2c_1c_2\cos A=$ is equal to

a. $4a^2\sin 2A$ b. $4a^2\sin^2 A$ c. $4a^2\cos 2A$ d. $4a^2\cos^2 A$

57. In ΔABC, a, b, A are given and c_1, c_2 are two values of the third side c. The sum of the areas of the two triangles with sides a, b, c_1 and a, b, c_2 is

a. $(1/2)b^2\sin 2A$ b. $(1/2)a^2\sin 2A$ c. $b^2\sin 2A$ d. none of these

58. The area of the circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio of

a. $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ b. $\cos\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ c. $\sin\frac{\pi}{n}:\frac{\pi}{n}$ d. $\cot\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

59. The ratio of the area of a regular polygon of n sides inscribed in a circle to that of the polygon of same number of sides circumscribing the same circle is 3:4. Then the value of n is

a. 6 b. 4 c. 8 d. 12

60. In triangle ABC, if P, Q, R divides sides BC, AC and AB, respectively, in the raito $k : 1$ (in order). If the ratio $\left(\frac{\text{area } PQR}{\text{area } ABC}\right)$ is $\frac{1}{3}$, then k is equal to

a. 1/3 b. 2 c. 3 d. none of these

61. The side of triangle ABC are in A.P. (order being a, b, c) and satisfy $\frac{2!}{1!9!} + \frac{2!}{3!7!} + \frac{1}{5!5!} = \frac{8^a}{(2b)!}$, then the value of $\cos A + \cos B$ is

a. $\frac{12}{7}$ b. $\frac{13}{7}$ c. $\frac{11}{7}$ d. $\frac{10}{7}$

62. Let ABC be a triangle with $\angle B = 90°$. Let AD be the bisector of $\angle A$ with D on BC. Supoose $AC = 6$ cm and the area of the triangle ADC is 10 cm^2. Then the length of BD in cm is equal to

a. $\frac{3}{5}$ b. $\frac{3}{10}$ c. $\frac{5}{3}$ d. $\frac{10}{3}$

63. In any triangle ABC, $\frac{a^2 + b^2 + c^2}{R^2}$ has the maximum value of

a. 3 b. 6 c. 9 d. none of these

64. If a, b and c are the sides of a triangle, then the minimum value of $\frac{2a}{b+c-a} + \frac{2b}{c+a-b} + \frac{2c}{a+b-c}$ is

a. 3 c. 9 c. 6 d. 1

65. In any triangle, the minimum value of $r_1r_2r_3/r^3$ is equal to

a. 1 b. 9 c. 27 d. none of these

66. In a convex quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$ and $DA = d$. This quadrilateral is such that a circle can be inscribed in it and a circle can be also circumscribed about it, then $\tan^2(A/2)$ is equal to

a. $\frac{ad}{bc}$ b. $\frac{ab}{cd}$ c. $\frac{cd}{ab}$ d. $\frac{bc}{ad}$

67. In triangle ABC, $\sin A$, $\sin B$ and $\sin C$ are in A.P, then

a. the altitudes are in H.P. b. the altitudes are in A.P.
c. the altitudes are in G.P. d. none of these

68. In triangle ABC, $\angle C = 2\pi/3$ and CD is the internal angle bisector of $\angle C$, meeting the side AB at D. Length CD is equal to

a. $\frac{ab}{2(a+b)}$ b. $\frac{2ab}{a+b}$ c. $\frac{2ab}{\sqrt{3(a+b)}}$ d. $\frac{ab}{a+b}$

69. In $\Delta\, ABC$, let R = circumradius, r = inradius, if r is the distance between the circumcentre and the incentre, then ratio R/r is equal to

a. $\sqrt{2}-1$ b. $\sqrt{3}-1$ c. $\sqrt{2}+1$ d. $\sqrt{3}+1$

70. In the given figure, AB is the diameter of the circle, centered at 'O'. If $\angle\, COA = 60°$, $AB = 2r$, $AC = d$ and $CD = l$, then l is equal to

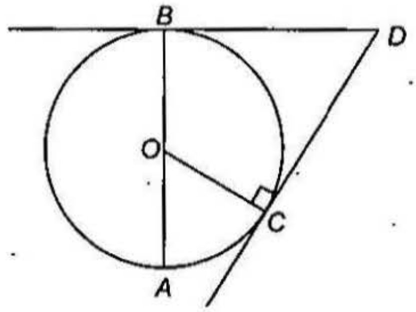


Fig. 5.47

a. $d\sqrt{3}$

b. $d/\sqrt{3}$

c. $3d$

d. $\sqrt{3}d/2$

71. In triangle ABC, base BC and area of triangle Δ are fixed. Locus of the centroid of triangle ABC is a straight line that is

a. parallel to side BC

b. right bisector of side BC

c. right angle of BC

d. inclined at an angle $\sin^{-1}\left(\frac{\sqrt{\Delta}}{BC}\right)$ to side BC

72. Let AD be a median of the ΔABC. If AE and AF are medians of the triangle ABD and ADC, respectively, and $AD = m_1$, $AE = m_2$, $AF = m_3$, then $a^2/8$ is equal to

a. $m_2^2 + m_3^2 - 2m_1^2$

b. $m_1^2 + m_2^2 - 2m_3^2$

c. $m_1^2 + m_3^2 - 2m_2^2$

d. none of these

73. A variable triangle ABC is circumscribed about a fixed circle of unit radius. Side BC always touches the circle at D and has fixed direction. If B and C vary in such a way that $(BD)(CD) = 2$, then locus of vertex A will be a straight line

a. parallel to side BC

b. perpendicular to side BC

c. making an angle $(\pi/6)$ with BC

d. making an angle $\sin^{-1}(2/3)$ with BC

74. In triangle ABC, $\angle A = \pi/3$ and its incircle is of unit radius. If the radius of the circle touching the sides AB, AC internally and incircle externally is x, then the value of x is

a. 1/2 b. 1/4 c. 1/3 d. none of these

75. The radii r_1, r_2, r_3 of the escribed circles of the triangle ABC are in H.P. If the area of the triangle is 24 cm^2 and its perimeter is 24 cm, then the length of its largest side is

a. 10 b. 9 c. 8 d. none of these

76. If 'O' is the circumcentre of ΔABC and R_1, R_2 and R_3 are the radii of the circumcircles of triangles OBC, OCA and OAB, respectively, then $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$ has the value equal to

a. $\frac{abc}{2R^3}$ b. $\frac{R^3}{abc}$ c. $\frac{4\Delta}{R^2}$ d. $\frac{\Delta}{4R^2}$

77. In triangle ABC, If $A - B = 120°$ and $R = 8r$ where R and r have their usual meaning, then $\cos C$ equals

a. 3/4 b. 2/3 c. 5/6 d. 7/8

78. In triangle ABC, $\angle A = 60°$, $\angle B = 40°$ and $\angle C = 80°$. If P is the centre of the circumcircle of triangle ABC with radius unity, then the radius of the circumcircle of triangle BPC is

a. 1 b. $\sqrt{3}$ c. 2 d. $\sqrt{3}/2$

79. Let area of triangle ABC is $(\sqrt{3}-1)/2$, $b = 2$ and $c = (\sqrt{3}-1)$ and $\angle A$ is acute. The measure of the angle C is

a. 15° b. 30° c. 60° d. 75°

80. In triangle ABC, $R(b+c) = a\sqrt{bc}$ where R is the circumradius of the triangle. Then the triangle is

a. isosceles but not right b. right but not isosceles
c. right isosceles d. equilateral

81. In triangle ABC, $\angle ABC = 120°$, $AB = 3$ and $BC = 4$. If perpendicular constructed to the side AB at A and to the side BC at C meets at D, then CD is equal to

a. 3 b. $\frac{8\sqrt{3}}{3}$ c. 5 d. $\frac{10\sqrt{3}}{3}$

Multiple Correct Answers Type

Solutions on page 5.95

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. If the tangents of the angles A and B of triangle ABC satisfy the equation $abx^2 - c^2x + ab = 0$, then

a. $\tan A = a/b$ b. $\tan B = b/a$
c. $\cos C = 0$ d. $\sin^2 A + \sin^2 B + \sin^2 C = 2$

2. In a triangle, the angles are in A.P. and the lengths of the two larger sides are 10 and 9, respectively, then the length of the third side can be

a. $5 + \sqrt{6}$ b. 0.7 c. $5 - \sqrt{6}$ d. none of these

3. In triangle ABC if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$, then angle B is equal to

a. 45° b. 135° c. 120° d. 60°

4. If in triangle ABC, a, b, c and angle A are given and $c \sin A < a < c$, then

a. $b_1 + b_2 = 2c\cos A$ b. $b_1 + b_2 = c\cos A$ c. $b_1b_2 = c^2 - a^2$ d. $b_1b_2 = c^2 + a^2$

5. There exists triangle ABC satisfying

a. $\tan A + \tan B + \tan C = 0$

b. $\frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{7}$

c. $(a+b)^2 = c^2 + ab$ and $\sqrt{2}(\sin A + \cos A) = \sqrt{3}$

d. $\sin A + \sin B = \frac{\sqrt{3}+1}{2}$, $\cos A \cos B = \frac{\sqrt{3}}{4} = \sin A \sin B$

6. CF is the internal bisector of angle C of ΔABC, then CF is equal to

a. $\frac{2ab}{a+b}\cos\frac{C}{2}$ b. $\frac{a+b}{2ab}\cos\frac{C}{2}$ c. $\frac{b\sin A}{\sin\left(B + \frac{C}{2}\right)}$ d. none of these

7. The sides of ΔABC satisfy the equation $2a^2 + 4b^2 + c^2 = 4ab + 2ac$. Then

a. the triangle is isosceles b. the triangle is obtuse
c. $B = \cos^{-1}(7/8)$ d. $A = \cos^{-1}(1/4)$

8. Let ABC be an isosceles triangle with base BC. If 'r' is the radius of the circle inscribed in ΔABC and r_1 is the radius of the circle escribed opposite to the angle A, then the product $r_1 r$ can be equal to

a. $R^2 \sin^2 A$ b. $R^2 \sin^2 2B$ c. $\frac{1}{2}a^2$ d. $\frac{a^2}{4}$

where R is the radius of the circumcircle of the ΔABC.

9. If in a triangle, $\sin^4 A + \sin^4 B + \sin^4 C = \sin^2 B \sin^2 C + 2\sin^2 C \sin^2 A + 2\sin^2 A \sin^2 B$, then its angle A is equal to

a. 30° b. 120° c. 150° d. 60°

10. The area of a regular polygon of n sides is (where r is inradius, R is circumradius and a is side of the triangle)

a. $\frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$ b. $nr^2 \tan\left(\frac{\pi}{n}\right)$ c. $\frac{na^2}{4}\cot\frac{\pi}{n}$ d. $nR^2 \tan\left(\frac{\pi}{n}\right)$

11. In acute-angled triangle ABC, AD is the altitude. Circle drawn with AD as its diameter cuts the AB and AC at P and Q, respectively. Length PQ is equal to

a. $\frac{\Delta}{2R}$ b. $\frac{abc}{4R^2}$ c. $2R \sin A \sin B \sin C$ d. $\frac{\Delta}{R}$

12. If A is the area and $2s$ is the sum of the sides of a triangle, then

a. $A \le \frac{s^2}{4}$ b. $A \le \frac{s^2}{3\sqrt{3}}$ c. $A < \frac{s^2}{\sqrt{3}}$ d. none of these

13. If the angles of a triangle are 30° and 45°, and the included side is $(\sqrt{3}+1)$ cm, then

a. area of the triangle is $\frac{1}{2}(\sqrt{3}+1)$ sq. units

b. area of the triangle is $\frac{1}{2}(\sqrt{3}-1)$ sq. units

c. ratio of greater side to smaller side is $\frac{\sqrt{3}+1}{\sqrt{2}}$

d. ratio of greater side to smaller side is $\frac{1}{4\sqrt{3}}$

14. Sides of ΔABC are in A.P. If $a < \min\{b, c\}$, then $\cos A$ may be equal to

a. $\frac{4b-3c}{2b}$ b. $\frac{3c-4b}{2c}$ c. $\frac{4c-3b}{2b}$ d. $\frac{4c-3b}{2c}$

15. Lengths of the tangents from A, B and C to the incircle are in A.P., then

a. r_1, r_2, r_3 are in H.P. b. r_1, r_2, r_3 are in A.P. c. a, b, c are in A.P. d. $\cos A = \frac{4c-3b}{2b}$

16. If sides of triangle ABC are a, b and c such that $2b = a + c$, then

a. $\frac{b}{c} > \frac{2}{3}$ b. $\frac{b}{c} > \frac{1}{3}$ c. $\frac{b}{c} < 2$ d. $\frac{b}{c} < \frac{3}{2}$

17. In ΔABC, if $\cos\frac{A}{2}=\sqrt{\frac{b+c}{2c}}$, then

a. area of triangle is $\frac{1}{2}ab$

b. circumradius is equal to $\frac{1}{2}c$

c. area of triangle is $\frac{1}{2}bc$

d. circumradius is equal to $\frac{1}{2}a$

18. If the sides of a right-angled triangle are in G.P., then the cosines of the acute angle of the triangle are

a. $\frac{\sqrt{5}-1}{2}$

b. $\frac{\sqrt{5}+1}{2}$

c. $\sqrt{\frac{\sqrt{5}-1}{2}}$

d. $\frac{\sqrt{\sqrt{5}+1}}{2}$

Reasoning Type

Solutions on page 5.102

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1

b Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** If side BC and ratio of r_2 and r_3 of an acute-angled triangle is given, then the locus of A is a hyperbola.
Statement 2: If base of a triangle is given and difference of two variable sides is constant (less than the base), then locus of variable vertex is a hyperbola.
2. **Statement 1:** In any ΔABC, the maximum value of $r_1+r_2+r_3=9R/2$.
Statement 2: In any ΔABC, $R\geq 2r$.
3. In acute-angled ΔABC, $a>b>c$
Statement 1: $r_1>r_2>r_3$.
Statement 2: $\cos A<\cos B<\cos C$.
4. **Statement 1:** The incentre of the triangle formed by the feet of altitudes from the vertices of triangle ABC to the opposite sides is the orthocentre of the triangle ABC.
Statement 2: The incentre of triangle ABC is orthocentre of the triangle $I_1I_2I_3$, where I_1, I_2, I_3 are excentres of triangle ABC.
5. **Statement 1:** If I is incentre of ΔABC and I_1 excentre opposite to A and P is the intersection of II_1 and BC, then $IP\cdot I_1P=BP\cdot PC$.
Statement 2: In ΔABC, I is incentre and I_1 is excentre opposite to A, then IBI_1C must be square.
6. **Statement 1:** If the quadrilateral Q_1 formed by joining mid-points of sides of another quadrilateral Q_2 is cyclic, then Q_1 is necessarily a rectangle.
Statement 2: The quadrilateral Q_1 formed by joining mid-points of sides of another quadrilateral Q_2 is always a parallelogram.
7. Let l_1, l_2, l_3 be the lengths of the internal bisectors of angles A, B, C of ΔABC, respectively.

Statement 1: $$\frac{\cos\frac{A}{2}}{l_1}+\frac{\cos\frac{B}{2}}{l_2}+\frac{\cos\frac{C}{2}}{l_3}=2\left(\frac{l_1}{a}+\frac{l_2}{b}+\frac{l_3}{c}\right)$$

Statement 2: $$l_1^2=bc\left[1-\left(\frac{a}{b+c}\right)^2\right], l_2^2=ca\left[1-\left(\frac{b}{c+a}\right)^2\right], l_3^2=ab\left[1-\left(\frac{c}{a+b}\right)^2\right]$$

8. **Statement 1:** In ΔABC, the centroid (G) divides line joining orthocenter (H) and circumcenter in ratio 2:1.
Statement 2: The centroid (G) divides the median AD in ratio 2:1.
9. **Statement 1:** Circumradius of $\Delta I_1I_2I_3$ is $2R$.
Statement 2: Circumradius of the triangle formed by feet of altitudes of ΔABC is $R/2$.
10. **Statement 1:** If the incircle of the triangle ABC passes through its circumcentre, then

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sqrt{2}.$$

Statement 2: Distance between the circumcentre and incentre is $\sqrt{R^2-2rR}$.
11. **Statement 1:** In triangle ABC, D is a point on the side AB such that $CD^2 = AD \cdot DB$, then the greatest value of $\sin A \sin B$ is $\sin^2(C/2)$.
Statement 2: Greatest value of $\sin A \sin B$ occurs when CD is the angle bisector of angle C.
12. **Statement 1:** If a, b, c are the sides of a triangle, then the minimum value of

$$\frac{2a}{b+c-a}+\frac{2b}{c+a-b}+\frac{2c}{a+b-c} \text{ is } 9.$$

Statement 2: A.M. ≥ G.M. ≥ H.M.
13. **Statement 1:** If $C = 45°$, $B = 60°$, then the line joining A and circumcentre (O) divides BC in ratio $2:\sqrt{3}$.

Statement 2: Line joining A and circumcenter (O) divides BC in ratio $\dfrac{\sin 2C}{\sin 2B}$.
14. **Statement 1:** If $a = 3, b = 7, c = 8$, and internal angle bisector AI meets BC at D (where I is incentre), then $AI/ID = 11/2$.
Statement 2: Internal angle bisector of angle A divides the side BC in ratio AB/AC.

Linked Comprehension Type

Solutions on page 5.106

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1–3

Given that $\Delta = 6, r_1 = 2, r_2 = 3, r_3 = 6$.

1. Circumradius R is equal to
a. 2.5 b. 3.5 c. 1.5 d. none of these
2. Inradius is equal to
a. 2 b. 1 c. 1.5 d. 2.5
3. Difference between the greatest and the least angle is
a. $\cos^{-1}\frac{4}{5}$ b. $\tan^{-1}\frac{3}{4}$ c. $\cos^{-1}\frac{3}{5}$ d. none of these

For Problems 4–6

Let $a = 6, b = 3$ and $\cos(A-B) = \frac{4}{5}$.

4. Area of the triangle is equal to
a. 9 b. 12 c. 11 d. 10
5. Angle C is equal to

a. $\dfrac{3\pi}{4}$ b. $\dfrac{\pi}{4}$ c. $\dfrac{\pi}{2}$ d. none of these

6. Value of sin A is equal to

a. $\dfrac{1}{2\sqrt{5}}$ b. $\dfrac{1}{\sqrt{3}}$ c. $\dfrac{1}{\sqrt{5}}$ d. $\dfrac{2}{\sqrt{5}}$

For Problems 7–9

p_1, p_2, p_3 are altitudes of triangle ABC from the vertices A, B, C and Δ is the area of the triangle.

7. The value of $p_1^{-2} + p_2^{-2} + p_3^{-2}$ is equal to

a. $\dfrac{a+b+c}{\Delta}$ b. $\dfrac{a^2+b^2+c^2}{4\Delta^2}$ c. $\dfrac{a^2+b^2+c^2}{\Delta^2}$ d. none of these

8. The value of $\dfrac{1}{p_1}+\dfrac{1}{p_2}+\dfrac{1}{p_3}$ is equal to

a. $\dfrac{2}{r}$ b. $\dfrac{2s}{\Delta}$ c. $\dfrac{8R}{abc}$ d. none of these

9. The value of $\dfrac{\cos A}{p_1}+\dfrac{\cos B}{p_2}+\dfrac{\cos C}{p_3}$ is equal to

a. $\dfrac{1}{R}$ b. $\dfrac{a^2+b^2+c^2}{2R}$ c. $\dfrac{\Delta}{2R}$ d. none of these

For Problems 10–12

Let O be a point inside a ΔABC such that $\angle OAB = \angle OBC = \angle OCA = \theta$.

10. $\cot A + \cot B + \cot C$ is equal to

a. $\tan^2\theta$ b. $\cot^2\theta$ c. $\tan\theta$ d. $\cot\theta$

11. $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$ is equal to

a. $\cot^2\theta$ b. $\operatorname{cosec}^2\theta$ c. $\tan^2\theta$ d. $\sec^2\theta$

12. Area of ΔABC is equal to

a. $\left(\dfrac{a^2+b^2+c^2}{4}\right)\tan\theta$ b. $\left(\dfrac{a^2+b^2+c^2}{4}\right)\cot\theta$ c. $\left(\dfrac{a^2+b^2+c^2}{2}\right)\tan\theta$ d. $\left(\dfrac{a^2+b^2+c^2}{2}\right)\cot\theta$

For Problems 13–15

Let D, E and F be the feet of altitudes from the vertices of acute-angled triangle ABC to the sides BC, AC and AB, respectively. Triangle DEF is defined as the pedal triangle of triangle ABC. (R and r are circumradius and inradius of triangle ABC, respectively.)

13. Consider the following statements:

i. orthocentre of the triangle ABC is incentre of the triangle DEF

ii. A, B, C are excentres of triangle DEF

a. only (i) is true b. only (ii) is true

c. both (i) and (ii) are true d. both (i) and (ii) are false

14. Circumradius of a pedal triangle of triangle ABC is

a. $R/2$ b. $r/2$ c. $R/4$ d. $r/4$

15. If x, y, z are the sides of a pedal triangle, then $x+y+z$ is equal to

a. $\Delta R/2$ b. $\Delta/2R$ c. ΔR d. none of these

For Problems 16–18

Incircle of ΔABC touches the sides BC, AC and AB at D, E and F, respectively. Then answer the following questions.

16. $\angle DEF$ is equal to

a. $\frac{\pi - B}{2}$ b. $\pi - 2B$ c. $A - C$ d. none of these

17. Area of ΔDEF is

a. $2r^2 \sin(2A) \sin(2B) \sin(2C)$ b. $2r^2 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

c. $2r^2 \sin(A-B) \sin(B-C) \sin(C-A)$ d. none of these

18. The length of side EF is

a. $r \sin\frac{A}{2}$ b. $2r \sin\frac{A}{2}$ c. $r \cos\frac{A}{2}$ d. $2r \cos\frac{A}{2}$

For Problems 19–21

Internal bisectors of ΔABC meet the circumcircle at points D, E and F,

19. The length of side EF is

a. $2R \cos\left(\frac{A}{2}\right)$ b. $2R \sin\left(\frac{A}{2}\right)$ c. $R \cos\left(\frac{A}{2}\right)$ d. $2R \cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

20. Area of ΔDEF is

a. $2R^2 \cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right)$ b. $2R^2 \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

c. $2R^2 \sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right)$ d. $2R^2 \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

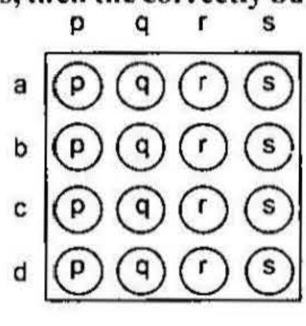
21. Ratio of area of triangle ABC and triangle DEF is

a. ≥ 1 b. ≤ 1 c. $\geq 1/2$ d. $\leq 1/2$

Matrix-Match Type

Solutions on page 5.111

Each question contains statements given in two columns which have to be matched.
Statements (A, B, C, D) in column I have to be matched with statements (p, q, r, s) in column II. If the correct match are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. $b > c \sin B$, $b < c$ and B is an acute angle	p. 0
b. $b > c \sin B$, $c < b$, and B is an acute angle	q. 2
c. $b > c \sin B$, $c < b$ and B is an obtuse angle	r. data insufficient
d. $b > c \sin B$, $c > b$ and B is an obtuse angle	s. 1

2. In acute-angled triangle ABC

Column I	Column II
a. $\cos A$, $\cos B$, $\cos C$ are in A.P.	p. Distances of orthocentre from vertices of triangle are in A.P.
b. $\sin (A/2)$, $\sin (B/2)$, $\sin (C/2)$ are in A.P.	q. Distances of orthocentre from sides of triangle are in H.P.
c. Distances of circumcentre from the vertices of the triangle ABC are in A.P.	r. Distances of incentre from vertices of triangle are in H.P.
d. Circumradii of triangles OBC, OAC and OAB are in H.P. (where O is cicumcentre of triangle ABC)	s. Distances of incentre from excentres of triangle are in A.P.

3.

Column I	Column II
a. If the sines of the angles A and B of a triangle ABC satisfy the equation $c^2x^2 - c(a+b)x + ab = 0$, the triangle can be	p. right angled
b. If one angle of a triangle is 30° and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$, the triangle can be	q. isosceles
c. If two angles of a triangle ABC satisfy the equation $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then the triangle can be ($x \in (0, \pi/2)$)	r. equilateral
d. In triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle can be	s. obtuse angled

4. Let O be the circumcentre, H be the orthocentre, I be the incentre and I_1, I_2, I_3 be the excentres of acute-angled ΔABC

Column I	Column II
a. Angle subtended by OI at vertex A	p. $\vert B-C\vert$
b. Angle subtended by HI at vertex A	q. $\frac{\vert B-C\vert}{2}$
c. Angle subtended by OH at vertex A	r. $\frac{B+C}{2}$
d. Angle subtended by I_2I_3 at I_1	s. $\frac{B}{2}-C$

5.

Column I (Condition)	Column II (Type of ΔABC)
a. $\cot\frac{A}{2}=\frac{b+c}{a}$	p. always right angled
b. $a\tan A+b\tan B=(a+b)\tan\left(\frac{A+B}{2}\right)$	q. always isosceles
c. $a\cos A=b\cos B$	r. may be right angled
d. $\cos A=\frac{\sin B}{2\sin C}$	s. may be right-angled isosceles

Integer Type

Solutions on page 5.115

1. Suppose α, β, γ and δ are the interior angles of regular pentagon, hexagon, decagon and dodecagon, respectively, then the value of $|\cos\alpha\ \sec\beta\ \cos\gamma\ \text{cosec}\ \delta|$ is ________.
2. Let $ABCDEFGHIJKL$ be a regular dodecagon. Then the value of $\frac{AB}{AF}+\frac{AF}{AB}$ is equal to ________.
3. Two equilateral triangles are constructed from a line segment of length L. If M and m are the maximum and minimum value of the sum of the areas of two plane figures, then the value of M/m is ________.
4. In ΔABC, if $r=1$, $R=3$ and $s=5$, then the value of $\frac{a^2+b^2+c^2}{3}$ is ________.
5. Consider a ΔABC in which the sides are $a=(n+1)$, $b=(n+2)$, $c=n$ with $\tan C=4/3$, then the value of $\Delta/12$ is ________.
6. In ΔAEX, T is the midpoint of XE, and P is the midpoint of ET. If ΔAPE is equilateral of side length equal to unity, then the value of $[(AX)^2/2]$ is (where [.] represents greatest integer function) ________.
7. In ΔABC, the incircle touches the sides BC, CA and AB, respectively, at D, E and F. If the radius of the incircle is 4 units and BD, CE and AF are consecutive integers, then the value of $s/3$, where s is a semi-perimeter of triangle, is ________.
8. The altitudes from the angular points A, B and C on the opposite sides BC, CA and AB of ΔABC are 210, 195 and 182, respectively. Then the value of $a/30$ is (where $a=BC$) ________.
9. In ΔABC, if $\angle C=3\angle A$, $BC=27$ and $AB=48$. Then the value of $AC/7$ is ________.
10. The area of a right triangle is 6864 square units. If the ratio of its legs is 143 : 24, then the value of $[r/4]$, where [·] represents the greatest integer function, is ________.
11. In ΔABC, if $\cos A+\sin A-\frac{2}{\cos B+\sin B}=0$, then the value of $\left(\frac{a+b}{c}\right)^4$ is ________.
12. In ΔABC, $\angle C=2\angle A$ and $AC=2\ BC$, then the value of $\frac{a^2+b^2+c^2}{R^2}$ (where R is circum-radius of triangle) is ________.
13. A circle inscribed in a triangle ABC touches the side AB at D such that $AD=5$ and $BD=3$. If $\angle A=60°$, then the value of $[BC/3]$ (where [·] represents greatest integer function) is ________.

14. The sides of triangle ABC satisfy the relations $a + b - c = 2$ and $2ab - c^2 = 4$, then square of the area of triangle is ________.

15. The lengths of the tangents drawn from the vertices A, B and C to the incircle of ΔABC are 5, 3 and 2, respectively. If the lengths of the parts of tangents within the triangle which are drawn parallel to the sides BC, CA and AB of the triangle to the incircle are α, β and γ, respectively, then the value of $[\alpha + \beta + \gamma]$ (where $[\cdot]$ represents greatest integer funciton) is ________.

16. If a, b and c represent the lengths of sides of a triangle, then the possible integral value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ is ________.

17. In triangle ABC, $\sin A \sin B + \sin B \sin C + \sin C \sin A = 9/4$ and $a = 2$, then the value of $\sqrt{3}\Delta$, where Δ is area of triangle, is ________.

18. In ΔABC, $AB = 52$, $BC = 56$, $CA = 60$. Let D be the foot of the altitude from A, and E be the intersection of the internal angle bisector of $\angle BAC$ with BC. Find the length DE is ________.

Archives

Solutions on page 5.121

Subjective

1. ABC is a triangle. D is the middle point of BC. If AD is perpendicular to AC, then prove that $\cos A \cos C = \frac{2(c^2 - a^2)}{3ac}$. **(IIT-JEE, 1980)**

2. ABC is a triangle with $AB = AC$. D is any point on the side BC. E and F are points on the sides AB and AC, respectively, such that DE is parallel to AC and DF is parallel to AB. Prove that $DF + FA + AE + ED = AB + AC$. **(IIT-JEE, 1980)**

3. Let the angles A, B and C of triangle ABC be in A.P. and let $b{:}c$ be $\sqrt{3} : \sqrt{2}$. Find the angle A. **(IIT-JEE, 1981)**

4. The exradii r_1, r_2 and r_3 of ΔABC are in H.P. Show that its sides a, b and c are in A.P. **(IIT-JEE, 1983)**

5. For triangle ABC, it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that the triangle is equilateral. **(IIT-JEE, 1984)**

6. With usual notation, if in triangle ABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$. **(IIT-JEE, 1984)**

7. In triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11 - 6\sqrt{3}}}$ and it divides the angle A into angles 30° and 45°. Find the length of the side BC. **(IIT-JEE, 1985)**

8. If in triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$. Show that $a{:}b{:}c = 1{:}1{:}\sqrt{2}$. **(IIT-JEE, 1986)**

9. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle. **(IIT-JEE, 1997)**

10. In a triangle of base a, the ratio of the other two sides is r (< 1). Show that the altitude of the triangle is less than or equal to $\frac{ar}{1 - r^2}$. **(IIT-JEE, 1997)**

11. Three circles touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contanct is 4. Find the ratio of the product of the radii to the sum of the radii of the circles. **(IIT-JEE, 1992)**
12. Consider the following statements concerning triangle ABC
    i. The sides a, b and c and area (Δ) are rational
    ii. a, $\tan \frac{B}{2}$ and $\tan \frac{C}{2}$ are rational
    iii. a, $\sin A$, $\sin B$ and $\sin C$ are rational
    Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). **(IIT-JEE, 1994)**
13. Let ABC be a triangle with incentre I and inradius r. Let D, E and F be the feet of the perpendiculars from I to the sides BC, CA and AB, respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$, respectively, prove that
    $$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}.$$ **(IIT-JEE, 2000)**
14. If Δ is the area of a triangle with side lengths a, b and c, then show that $\Delta \le \frac{1}{4}\sqrt{(a+b+c)\,abc}$. Also show that the equality occurs in the above inequality if and only if $a = b = c$. **(IIT-JEE, 2001)**
15. If I_n is the area of n-sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that $I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$. **(IIT-JEE, 2003)**
16. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chord subtend angles $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ at the centre, where $k > 0$, then find the value of $[k]$.
    (*Note:*) $[k]$ denotes the largest integer less than or equal to k) **(IIT-JEE, 2010)**
17. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C, respectively. Suppose $a = 6$, $b = 10$ and the area of triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then find the value of r^2. **(IIT-JEE, 2010)**

Objective

Fill in the blanks

1. In ΔABC, $\angle A = 90°$ and AD is an altitude. Complete the relation $\frac{BD}{DA}=\frac{AB}{(\cdots)}$. **(IIT-JEE, 1980)**
2. ABC is a triangle, P is a point on AB and Q is a point on AC such that $\angle AQP = \angle ABC$. Complete the relation $\frac{\text{area of } \Delta APQ}{\text{area of } \Delta ABC}=\frac{(\ )}{AC^2}$. **(IIT-JEE, 1980)**
3. ABC is a triangle with $\angle B$ greater than $\angle C$. D and E are points on BC such that AD is perpendicular to BC and AE is the bisector of angle A. Complete the relation $\angle DAE = \frac{1}{2}[(\cdots) - \angle C]$.
   (IIT-JEE, 1980)

4. The set of all real numbers a such that $a^2+2a, 2a+3$ and a^2+3a+8 are the sides of a triangle is _____.
(IIT-JEE, 1985)

5. In triangle ABC, if $\cot A$, $\cot B$, $\cot C$ are in $A.P.$, then a^2, b^2, c^2 are in _____ progression.
(IIT-JEE, 1985)

6. A polygon of nine sides, each side of length 2, is inscribed in a circle. The radius of the circle is _____.
(IIT-JEE, 1987)

7. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cm, then the area of the triangle is _____.
(IIT-JEE, 1988)

8. If in triangle ABC, $\dfrac{2\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{2\cos C}{c}=\dfrac{a}{c}+\dfrac{b}{ca}$, then the value of the angle A is _____ degrees.
(IIT-JEE, 1993)

9. In triangle ABC, AD is the altitude from A. Given $b>c$, $\angle C=23°$ and $AD=\dfrac{abc}{b^2-c^2}$, then $\angle B=$ _____.
(IIT-JEE, 1994)

10. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is _____.
(IIT-JEE, 1994)

11. In triangle ABC, $a:b:c=4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is _____.
(IIT-JEE, 1996)

Multiple choice questions with one correct answer

1. In triangle ABC, angle A is greater than angle B. If the measures of angles A and B satisfy the equation $3\sin x-4\sin^3 x-k=0$, $0<k<1$, then the measure of angle C is

a. $\dfrac{\pi}{3}$ b. $\dfrac{\pi}{2}$ c. $\dfrac{2\pi}{3}$ d. $\dfrac{5\pi}{6}$

(IIT-JEE, 1990)

2. If the lengths of the sides of triangle are 3, 5 and 7, then the largest angle of the triangle is

a. $\dfrac{\pi}{2}$ b. $\dfrac{5\pi}{6}$ c. $\dfrac{2\pi}{3}$ d. $\dfrac{3\pi}{4}$

(IIT-JEE, 1994)

3. In triangle ABC, $\angle B=\pi/3$ and $\angle C=\pi/4$. Let D divide BC internally in the ratio 1:3. Then $\dfrac{\sin\angle BAD}{\sin\angle CAD}$ equals

a. $\dfrac{1}{\sqrt{6}}$ b. $\dfrac{1}{3}$ c. $\dfrac{1}{\sqrt{3}}$ d. $\sqrt{\dfrac{2}{3}}$

(IIT-JEE, 1995)

4. In triangle ABC, $2ac\sin\left(\dfrac{1}{2}(A-B+C)\right)$ is equal to

a. $a^2+b^2-c^2$ b. $c^2+a^2-b^2$ c. $b^2-c^2-a^2$ d. $c^2-a^2-b^2$

(IIT-JEE, 2000)

5. In triangle ABC, let $\angle C=\pi/2$. If r is the inradius and R is circumradius of the triangle, then $2(r+R)$ is equal to

a. $a+b$ b. $b+c$ c. $c+a$ d. $a+b+c$

(IIT-JEE, 2000)

6. Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)?
   a. a, $\sin A$, $\sin B$ b. a, b, c c. a, $\sin B$, R d. a, $\sin A$, R

   (IIT-JEE, 2002)

7. If the angles of a triangle are in the ratio 4:1:1, then the ratio of the longest side to the perimeter is
   a. $\sqrt{3}:(2+\sqrt{3})$ b. 1:6 c. $1:2+\sqrt{3}$ d. 2:3

   (IIT-JEE, 2003)

8. The side of a triangle are in the ratio $1:\sqrt{3}:2$, then the angles of the triangle are in the ratio
   a. 1:3:5 b. 2:3:4 c. 3:2:1 d. 1:2:3

   (IIT-JEE, 2004)

9. In an equilateral triangle, three coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is

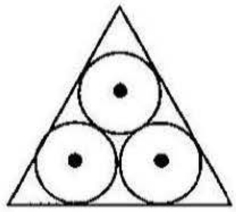


Fig. 5.48

   a. $4+2\sqrt{3}$ b. $6+4\sqrt{3}$ c. $12+\dfrac{7\sqrt{3}}{4}$ d. $3+\dfrac{7\sqrt{3}}{4}$

   (IIT-JEE, 2005)

10. In triangle ABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is given by
    a. $(b-c)\sin\left(\dfrac{B-C}{2}\right)=a\cos\dfrac{A}{2}$ b. $(b-c)\cos\left(\dfrac{A}{2}\right)=a\sin\dfrac{B-C}{2}$
    c. $(b+c)\sin\left(\dfrac{B+C}{2}\right)=a\cos\dfrac{A}{2}$ d. $(b-c)\cos\left(\dfrac{A}{2}\right)=2a\sin\dfrac{B+C}{2}$

    (IIT-JEE, 2005)

11. One angle of an isosceles Δ is 120° and radius of its incircle $=\sqrt{3}$. Then the area of the triangle in sq. units is
    a. $7+12\sqrt{3}$ b. $12-7\sqrt{3}$ c. $12+7\sqrt{3}$ d. 4π

    (IIT-JEE, 2006)

12. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB=2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is
    a. 3 b. 2 c. $\dfrac{3}{2}$ d. 1

    (IIT-JEE, 2007)

13. Let ABC be a triangle such that $\angle ACB = \pi/6$ and let a, b and c denote the lengths of the side opposite to A, B and C, respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

a. $-(2+\sqrt{3})$ b. $1+\sqrt{3}$ c. $2+\sqrt{3}$ d. $4\sqrt{3}$

(IIT-JEE, 2010)

Multiple choice questions with one or more than one correct answers

1. There exists a triangle ABC satisfying the conditions

a. $b \sin A = a, A < \pi/2$
b. $b \sin A > a, A > \pi/2$
c. $b \sin A > a, A < \pi/2$
d. $b \sin A < a, A < \pi/2, b > a$
e. $b \sin A < a, A > \pi/2, b = a$

(IIT-JEE, 1986)

2. In a triangle, the lengths of the two larger sides are 10 and 9, respectively. If the angles are in A.P., then the length of the third side can be

a. $5-\sqrt{6}$ b. $3\sqrt{3}$ c. 5 d. $5+\sqrt{6}$

(IIT-JEE, 1987)

3. If in a triangle PQR, $\sin P$, $\sin Q$, $\sin R$ are in A.P., then

a. the altitudes are in A.P.
b. the altitudes are in H.P.
c. the medians are in G.P.
d. the medians are in A.P.

(IIT-JEE, 1988)

4. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

a. $\frac{3}{4}$ b. $3\sqrt{3}$ c. 3 d. $\frac{3\sqrt{3}}{2}$

(IIT-JEE, 1998)

5. In ΔABC, internal angle bisector of $\angle A$ meets side BC in D. $DE \perp AD$ meets AC in E and AB in F. Then

a. AE is $H.M.$ of b and c b. $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$ c. $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ d. $\Delta\, AEF$ is isosceles

(IIT-JEE, 2006)

6. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

a. $\frac{1}{PS}+\frac{1}{ST}<\frac{2}{\sqrt{QS\times SR}}$
b. $\frac{1}{PS}+\frac{1}{ST}>\frac{2}{\sqrt{QS\times SR}}$
c. $\frac{1}{PS}+\frac{1}{ST}<\frac{4}{QR}$
d. $\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$

(IIT-JEE, 2008)

7. In a triangle ABC with fixed base BC, the vertex A moves such that

$$\cos B + \cos C = 4\sin^2\frac{A}{2}$$

If a, b and c denote the lengths of the sides of the trangle opposite to the angles A, B and C, respectively, then

a. $b+c=4a$
b. $b+c=2a$
c. locus of point A is an ellipse
d. locus of point A is a pair of straight lines

(IIT-JEE, 2009)

ANSWERS AND SOLUTIONS

Subjective Type

1. If O is the circumcentre of ΔABC, then
$OA = OB = OC = R$
Let R_1, R_2 and R_3 be circumradii of ΔOBC, ΔOCA and ΔOAB, respectively.

In ΔOBC, $2R_1 = \dfrac{a}{\sin 2A} \quad \Rightarrow \quad \dfrac{a}{R_1} = 2\sin 2A$

Similarly, $\dfrac{a}{R_2} = 2\sin 2B$ and $\dfrac{a}{R_3} = 2\sin 2C$

$$\Rightarrow \quad \frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3} = 2(\sin 2A + \sin 2B + \sin 2C) = 8\sin A \sin B \sin C = 8\frac{a}{2R}\frac{b}{2R}\frac{c}{2R} = \frac{abc}{R^3}$$

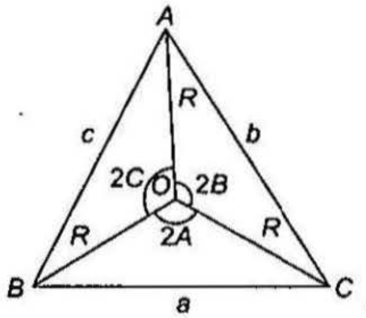


Fig. 5.49

2. In ΔABC,

$\dfrac{AC}{\sin 5x} = \dfrac{BC}{\sin 3x}$

$\Rightarrow \dfrac{a+p}{\sin 5x} = \dfrac{a}{\sin 3x}$ (i)

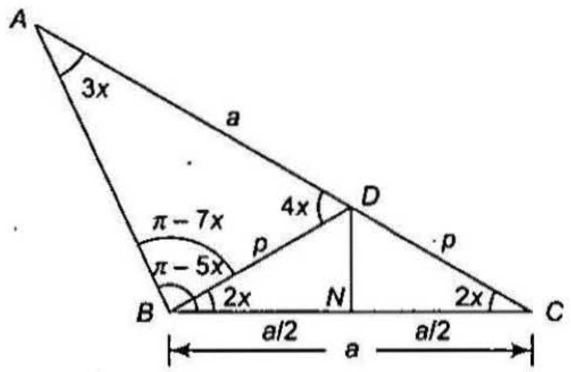


Fig. 5.50

In ΔBDN, $\cos 2x = \dfrac{a}{2p}$

$\Rightarrow a = 2p\cos 2x$

From Eq.(i), $\dfrac{2p\cos 2x + p}{\sin 5x} = \dfrac{2p\cos 2x}{\sin 3x}$

$\Rightarrow 2\sin 3x \cos 2x + \sin 3x = 2\sin 5x \cos 2x$

$\Rightarrow \sin 5x + \sin x + \sin 3x = \sin 7x + \sin 3x$
$\Rightarrow \sin 7x - \sin 5x = \sin x$
$\Rightarrow 2 \cos 6x \sin x = \sin x$

$\Rightarrow \cos 6x = \dfrac{1}{2}$

$\Rightarrow x = 10°$

3. We know that distance of orthocentre (H) from vertex (A) is $2R \cos A$
or $x = 2R \cos A, y = 2R \cos B, z = 2R \cos C$

$$\Rightarrow \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z} = \frac{2R\sin A}{2R\cos A}+\frac{2R\sin B}{2R\cos B}+\frac{2R\sin C}{2R\cos C} = \tan A+\tan B+\tan C = \tan A\tan B\tan C$$

Also, $$\frac{abc}{xyz} = \frac{(2R\sin A)(2R\sin B)(2R\sin C)}{(2R\cos A)(2R\cos B)(2R\cos C)} = \tan A\tan B\tan C$$

4.

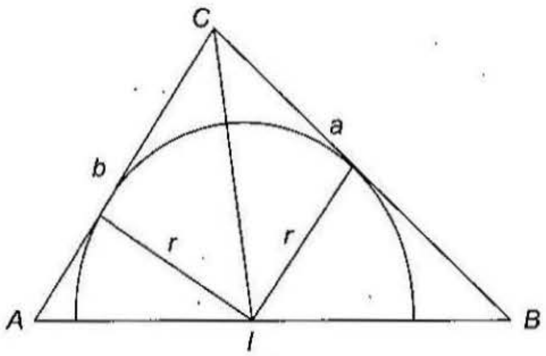


Fig. 5.51

From Fig. 5.51,
$(1/2)\, ra + (1/2)\, rb = (1/2)\, a\, b \sin C$
$\Rightarrow r(a+b) = 2\Delta$

$$\Rightarrow r = \frac{2\Delta}{a+b} = \frac{2\,abc}{4R(2R\sin A+2R\sin B)} = \frac{abc}{4R^2(\sin A+\sin B)} \qquad \text{(i)}$$

Also $x = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$ [length of angle bisector]

From Eq. (i), $$r = \frac{2\times\frac{1}{2}ab\sin C}{a+b}$$

$$= \frac{2ab\sin\frac{C}{2}\cos\frac{C}{2}}{a+b}$$

$$= \frac{2\,ab\cos\frac{C}{2}}{a+b}\cdot\sin\frac{C}{2}$$

$$= x\sin\frac{C}{2}$$

5. Let O and H be the circumcentre and the orthocentre, respectively.
If OF is the perpendicular to AB, we have
$\angle OAF = 90° - \angle AOF = 90° - C$
Also, $\angle HAL = 90° - C$
Hence, $\angle OAH = A - \angle OAF - \angle HAL$
$= A - 2(90° - C)$
$= A + 2C - 180°$
$= A + 2C - (A + B + C) = C - B$
Also, $OA = R$, and $HA = 2R\cos A$

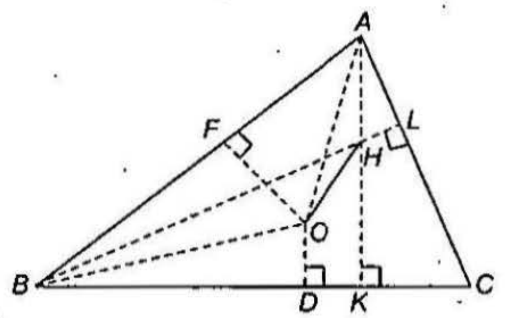


Fig. 5.52

Now in ΔAOH,
$OH^2 = OA^2 + HA^2 - 2OA\ HA\cos(\angle OAH)$
$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$
$= R^2 + 4R^2\cos A\,[\cos A - \cos(C - B)]$
$= R^2 - 4R^2\cos A\,[\cos(B + C) + \cos(C - B)]$
$= R^2 - 8R^2\cos A\cos B\cos C$

Hence, $OH = R\sqrt{1 - 8\cos A\cos B\cos C}$

6. Let O be the circumcentre and OF be the perpendicular to AB.
Let I be the incentre and IE be the perpendicular to AC.
Then $\angle OAF = 90° - C$.

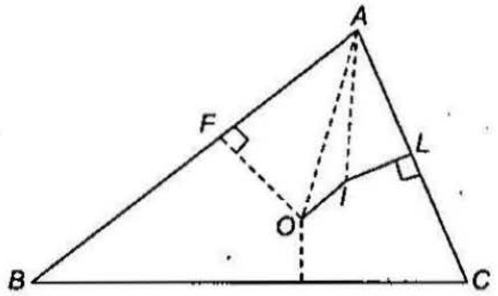


Fig. 5.53

$$\Rightarrow \angle OAI = \angle IAF - \angle OAF = \frac{A}{2} - (90° - C) = \frac{A}{2} + C - \frac{A + B + C}{2} = \frac{C - B}{2}$$

Also, $AI = \dfrac{IE}{\sin\frac{A}{2}} = \dfrac{r}{\sin\frac{A}{2}} = 4R\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

Hence in ΔOAI, $OI^2 = OA^2 + AI^2 - 2OA\,AI\cos\angle OAI$

$$\Rightarrow \quad = R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$$

$$\Rightarrow \quad \frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B+C}{2}$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}$$

Therefore, $OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2 - 2Rr}$

7. Let $ABCD$ be the cyclic quadrilateral in which $AB = 2$ and $BC = 5$, $\angle ABC = 60°$

$\angle ADC = 180° - 60° = 120°$.

Area of cyclic quadrilateral $ABCD$ = Area of ΔABC + Area of ΔACD

$$\Rightarrow \quad 4\sqrt{3} = \frac{1}{2}AB\,BC\sin 60° + \frac{1}{2}CD\,DA\sin 120°$$

$$= \frac{1}{2}2\times5\times(\sqrt{3}/2) + \frac{1}{2}xy(\sqrt{3}/2), \text{ where } CD = x, AD = y$$

$\therefore \quad xy = 6$ (i)

From ΔABC, we get

$AC^2 = AB^2 + BC^2 - 2AB\,BC\cos 60° = 4 + 25 - 20\,(1/2) = 19$ (ii)

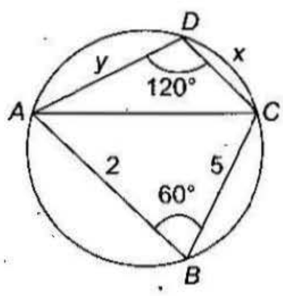


Fig. 5.54

Also from ΔACD,

$AC^2 = CD^2 + DA^2 - 2CD\,DA\cos 120° = x^2 + y^2 + xy = x^2 + y^2 + 6$ [using Eq. (i)] (iii)

Now from Eqs. (ii) and (iii), we have

$x^2 + y^2 + 6 = 19$ or $x^2 + y^2 = 13$ (iv)

Solving Eqs. (ii) and (iv), we get

$x^4 - 13x^2 + 36 = 0 \quad \Rightarrow \quad x^2 = 4, 9 \quad \Rightarrow \quad x = 2, 3 \Rightarrow y = 3, 2$

Hence, the other two sides of the cyclic quadrilateral are 3 and 2.

8. Let $\angle ACE = \alpha$. Clearly, from Fig. 5.55, we get

$$\frac{p}{AC} = \sin\alpha, \frac{q}{BC} = \sin(\alpha + C)$$

$$\Rightarrow \quad \frac{p}{b} = \sin\alpha, \frac{q}{a} = \sin\alpha\cos C + \cos\alpha\sin C$$

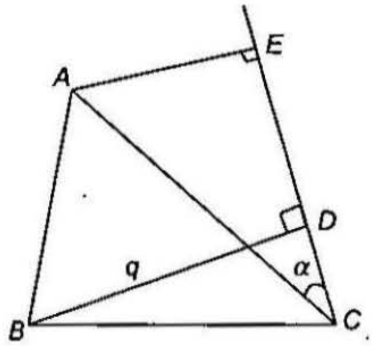


Fig. 5.55

$$\Rightarrow \quad \frac{q}{a} = \frac{p}{b}\cos C + \cos\alpha\sin C$$

$$\Rightarrow \quad \left(\frac{q}{a} - \frac{p}{b}\cos C\right)^2 = \cos^2\alpha\sin^2 C = \left(1 - \frac{p^2}{b^2}\right)\left(1 - \cos^2 C\right)$$

$$\Rightarrow \frac{q^2}{a^2} + \frac{p^2}{b^2}\cos^2 C - \frac{2pq}{ab}\cos C = 1 - \frac{p^2}{b^2} - \left(1 - \frac{p^2}{b^2}\right)\cos^2 C$$

$$\Rightarrow \quad \frac{q^2}{a^2} + \frac{p^2}{b^2} - \frac{2pq}{ab}\cos C = \sin^2 C$$

$$\Rightarrow \quad a^2p^2 + b^2q^2 - 2\,abpq\cos C = a^2b^2\sin^2 C$$

9. Let I be the in-centre of the ΔABC.

In ΔIBC, $\angle BIC = \pi - \dfrac{B+C}{2} = \pi - \dfrac{\pi - A}{2} = \dfrac{\pi + A}{2}$.

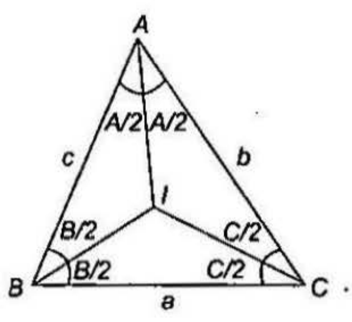


Fig. 5.56

Now, radius of circumcircle of ΔIBC, by sine rule is

$$R_1 = \frac{BC}{2\sin(\angle BIC)} = \frac{a}{2\sin\left(\frac{\pi + A}{2}\right)} = \frac{2R\sin A}{2\cos\frac{A}{2}} = 2R\sin\frac{A}{2}$$

Similarly, radius of circumcircle of ΔICA and ΔIAB are given by

$$R_2 = 2R\sin\frac{B}{2} \text{ and } R_3 = 2R\sin\frac{C}{2} \quad \Rightarrow \quad R_1R_2R_3 = 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2rR^2$$

10.

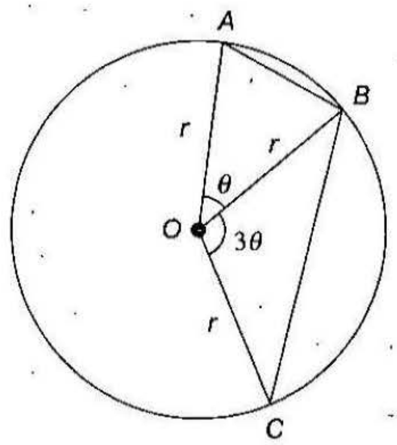


Fig. 5.57

Applying cosine rule in $\Delta\, OAB$, we get

$$\cos\theta = \frac{2r^2 - a^2}{2r^2} \quad \Rightarrow \quad a^2 = 2r^2(1 - \cos\theta) \quad \Rightarrow \quad a = 2r\sin\frac{\theta}{2}$$

Applying cosine rule in $\Delta\, OBC$, we get

$$\cos 3\theta = \frac{2r^2 - b^2}{2r^2}$$

$$\Rightarrow \quad b = 2r\sin\left(\frac{3\theta}{2}\right)$$

$$= 2r\left[3\sin\frac{\theta}{2} - 4\sin^3\frac{\theta}{2}\right]$$

$$= 2r\left[\frac{3a}{2r} - \frac{4a^3}{8r^3}\right]$$

$$= 3a - \frac{a^3}{r^2}$$

$$\Rightarrow \quad r^2 = \frac{a^3}{3a - b}$$

$$\Rightarrow \quad r = a\sqrt{\frac{a}{3a - b}} \text{ cm}$$

11. In ΔAEB, $\dfrac{AE}{\sin\left(90° - \dfrac{A}{3}\right)} = \dfrac{BE}{\sin\dfrac{A}{3}} = c$ (1)

In ΔAED, $\dfrac{AE}{\sin\left(90° - \dfrac{A}{3}\right)} = \dfrac{ED}{\sin\dfrac{A}{3}}$ (2)

Now $BE = ED = a/4$

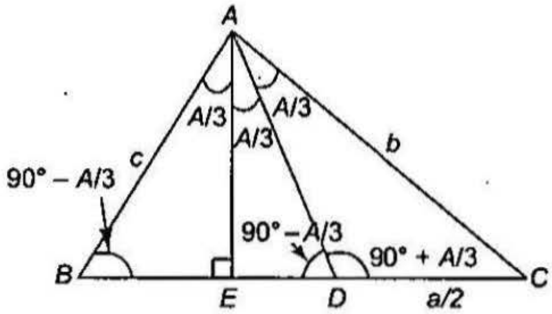


Fig. 5.58

$\Rightarrow \cos\dfrac{A}{3} = \dfrac{AE}{c}$ and $\dfrac{AE}{\cos\dfrac{A}{3}} = \dfrac{a/4}{\sin\dfrac{A}{3}} \Rightarrow \sin\dfrac{A}{3} = \dfrac{a}{4c}$

In ΔAEC, $\sin\dfrac{2A}{3} = \dfrac{EC}{b}$ or $2\sin\dfrac{A}{3}\cos\dfrac{A}{3} = \dfrac{\dfrac{a}{2}+\dfrac{a}{4}}{b} = \dfrac{3a}{4b}$

i.e., $\cos\dfrac{A}{3} = \dfrac{3c}{2b}$

Now, L.H.S. $= \cos\dfrac{A}{2}\sin^2\dfrac{A}{3} = \dfrac{3c}{2b} \times \dfrac{a^2}{16c^2} = \dfrac{3a^2}{32bc}$ = R.H.S.

12. Let AD be the perpendicular from A on BC. When AD is produced, it meets the circumscribing circle at E. From question, $DE = \alpha$.

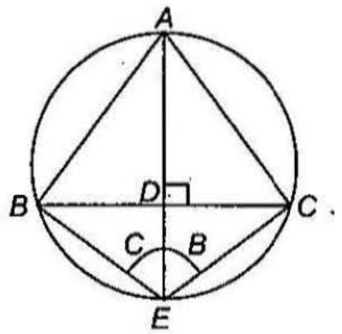


Fig. 5.59

Since angles in the same segment are equal, we have
$\angle AEB = \angle ACB = \angle C$, and $\angle AEC = \angle ABC = \angle B$
From the right-angled triangle BDE,

$\tan C = \dfrac{BD}{DE}$ (i)

From the right-angled triangle CDE,

$$\tan B = \frac{CD}{DE} \quad \text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$\tan B + \tan C = \frac{BD + CD}{DE} = \frac{BC}{DE} = \frac{a}{\alpha} \quad \text{(iii)}$$

Similarly, $\tan C + \tan A = \dfrac{b}{\beta}$ (iv)

and $\tan A + \tan B = \dfrac{c}{\gamma}$ (v)

Adding Eqs. (iii), (iv) and (v), we get

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$$

13.

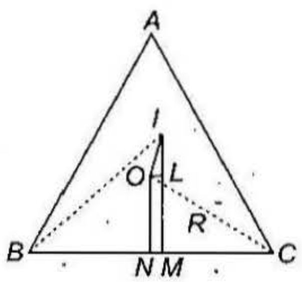


Fig. 5.60

Let I be the incentre and O be the circumcentre of the triangle ABC. Let OL be parallel to BC. Let $\angle IOL = \theta, IM = r, OC = R, \angle NOC = A$

$$\Rightarrow \quad \tan\theta = \frac{IL}{OL} = \frac{IM - LM}{BM - BN}$$

$$= \frac{IM - ON}{BM - NC}$$

$$= \frac{r - R\cos A}{r\cot\frac{B}{2} - R\sin A}$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - R\cos A}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\cot\frac{B}{2} - R\sin A}$$

$$= \frac{\cos A + \cos B + \cos C - 1 - \cos A}{\sin A + \sin C - \sin B - \sin A}$$

$$= \frac{\cos B + \cos C - 1}{\sin C - \sin B}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left[\frac{\cos B + \cos C - 1}{\sin C - \sin B}\right]$$

14. Let $\tan A = x$

$\tan A + \tan C = 2 \tan B \quad \Rightarrow \quad \tan C = 2 \tan B - x$

Also, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ $\quad [\because A + B + C = \pi]$

$\Rightarrow \quad x + \tan B + 2 \tan B - x = x \tan B (2 \tan B - x)$

$\Rightarrow \quad 3 = x (2 \tan B - x)$

$$\Rightarrow \quad \frac{3 + x^2}{2x} = \tan B$$

$$\Rightarrow \quad \tan C = \frac{3 + x^2}{x} - x = \frac{3}{x}$$

Now,

$a^2 : b^2 : c^2 = \sin^2 A : \sin^2 B : \sin^2 C$

$$= \frac{\tan^2 A}{1 + \tan^2 A} : \frac{\tan^2 B}{1 + \tan^2 B} : \frac{\tan^2 C}{1 + \tan^2 C}$$

$$= \frac{x^2}{1 + x^2} : \frac{\left(\frac{3 + x^2}{2x}\right)^2}{1 + \left(\frac{3 + x^2}{2x}\right)^2} : \frac{\left(\frac{3}{x}\right)^2}{1 + \left(\frac{3}{x}\right)^2}$$

$$= \frac{x^2}{1 + x^2} : \frac{(3 + x^2)^2}{(x^2 + 9)(x^2 + 1)} : \frac{9}{x^2 + 9}$$

$$= x^2 (x^2 + 9) : (3 + x^2)^2 : 9 (1 + x^2)$$

15. $A + B = 90°$

$\Rightarrow \quad \tan A = \cot B$

$$\Rightarrow \quad \frac{\sqrt{5} - 1}{2} = \cot^2 B$$

$$\Rightarrow \quad \frac{\cos^2 B}{\sqrt{5} - 1} = \frac{\sin^2 B}{2} = \frac{1}{\sqrt{5} + 1}$$

$$\Rightarrow \quad \frac{b^2}{ac} = \frac{4R^2 \sin^2 B}{4R^2 \sin A \sin C} = \frac{\sin^2 B}{\sin A} = \sin^2 B \sqrt{1 + \cot^2 A} \quad [\because \angle C = 90°]$$

$$= \left(\frac{2}{\sqrt{5} + 1}\right) \sqrt{1 + \frac{2}{\sqrt{5} - 1}} \quad \left[\because \tan A = \sqrt{\frac{\sqrt{5} - 1}{2}}\right]$$

$$= \frac{2}{\sqrt{5} + 1} \sqrt{\frac{\sqrt{5} + 1}{\sqrt{5} - 1}} = \frac{2}{\sqrt{5} + 1} \sqrt{\frac{(\sqrt{5} + 1)^2}{4}} = 1$$

$\Rightarrow \quad b^2 = ac \Rightarrow a, b, c$ are in G.P.

Objective Type

1. c. Adding, $\sin(A+B)=1$

and subtracting, $\sin(A-B)=\dfrac{\sqrt{2}-2}{\sqrt{2}}=1-\sqrt{2}\neq 0$

$\therefore\ A+B=90°, A\neq B$

2. c.

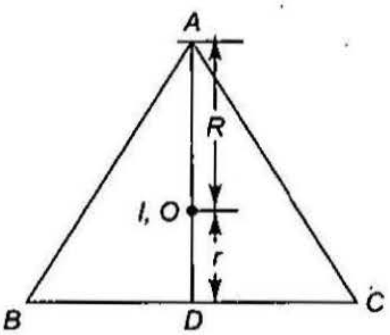


Fig. 5.61

In equilateral triangle, circumcentre (O) and incentre (I) coincide.

Also from the diagram $R+r=h \quad \Rightarrow \quad \dfrac{R+r}{h}=1$

3. b. $\dfrac{\sin A}{c\sin B}+\dfrac{\sin B}{c}+\dfrac{\sin C}{b}=\dfrac{c}{ab}+\dfrac{b}{ac}+\dfrac{a}{bc}$

$\Rightarrow \dfrac{a}{bc}+\dfrac{\sin B}{c}+\dfrac{\sin C}{b}=\dfrac{c}{ab}+\dfrac{b}{ac}+\dfrac{a}{bc}$

$\Rightarrow \dfrac{\sin B}{c}+\dfrac{\sin C}{b}=\dfrac{c}{ab}+\dfrac{b}{ac}$

$\Rightarrow \dfrac{b\sin B+c\sin C}{bc}=\dfrac{c^2+b^2}{abc}$

$\Rightarrow a=\dfrac{b^2+c^2}{b\sin B+c\sin C}=\dfrac{b(2R\sin B)+c(2R\sin C)}{b\sin B+c\sin C}$

$\Rightarrow a=2R$

$\Rightarrow \angle A=\pi/2$

4. a.

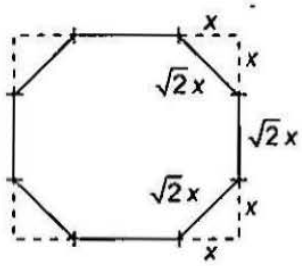


Fig. 5.62

Clearly, $x+\sqrt{2}x+x=1$. So, $x=\dfrac{1}{2+\sqrt{2}}$

The required area $=\left(1^2-4\times\dfrac{1}{2}x^2\right)\text{ m}^2$

$$= \left\{1 - 2 \quad \frac{1}{(2+\sqrt{2})^2}\right\} \text{m}^2$$

$$= \left\{1 - \frac{1}{(\sqrt{2}+1)^2}\right\} \text{m}^2$$

$$= \left[1-(\sqrt{2}-1)^2\right] \text{m}^2 = 2(\sqrt{2}-1)\,\text{m}^2$$

5. c. As A is an obtuse angle,

$90° < A < 180°$

$\Rightarrow \quad 90° < 180 - (B+C) < 180°$

$\Rightarrow \quad 0 < B+C < 90°$

$\Rightarrow \quad B+C < 90° \Rightarrow B < 90° - C$

$\therefore \quad \tan B < \tan(90° - C)$

$\tan B < \cot C \Rightarrow \tan B \tan C < 1$

6. d. $x = \frac{1}{2}\,[\cos(A-C) - \cos(A+C)] = \frac{1}{2}\left[\cos(A-C) + \frac{1}{2}\right]$

But $0 \le \cos(A-C) \le 1 \qquad \Rightarrow \quad \frac{1}{2}\left(0+\frac{1}{2}\right) \le x \le \frac{1}{2}\left(1+\frac{1}{2}\right)$

7. c. Here $b = 2c$.

Now, $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

$$\cot\frac{A}{2}\cot\frac{B-C}{2} = \frac{b+c}{b-c} = \frac{3c}{c} = 3$$

8. b.

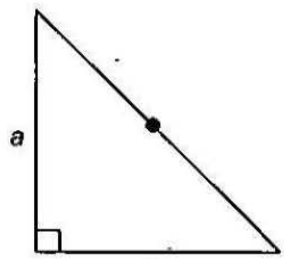


Fig. 5.63

Here, $R = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}a^2}{\frac{1}{2}\left(a+a+\sqrt{2}a\right)} = \frac{a}{2+\sqrt{2}}$$

$$\therefore \quad \frac{R}{r} = \frac{a}{\sqrt{2}} \times \frac{2+\sqrt{2}}{a} = \sqrt{2}+1$$

9. a. $\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{25+9-49}{2\times5\times3} = -\dfrac{1}{2}$

$\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{25+49-9}{2\times5\times7} = \dfrac{13}{14}$

$\Rightarrow \quad 3\cos C + 7\cos B = -\dfrac{3}{2}+\dfrac{13}{2} = 5$

10. a. $\angle B = 90° \Rightarrow \cos A = \dfrac{c}{b}$

$\Rightarrow \quad \dfrac{1-\tan^2\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}} = \dfrac{c}{b}$

$\Rightarrow \quad \dfrac{\left(1+\tan^2\dfrac{A}{2}\right)-\left(1-\tan^2\dfrac{A}{2}\right)}{\left(1+\tan^2\dfrac{A}{2}\right)+\left(1-\tan^2\dfrac{A}{2}\right)} = \dfrac{(b-c)}{(b+c)}$

$\Rightarrow \quad \tan^2\dfrac{A}{2} = \dfrac{b-c}{b+c}$

11. a. $\dfrac{\cot A}{\cot B+\cot C} = \dfrac{\dfrac{R\left(b^2+c^2-a^2\right)}{abc}}{\dfrac{R\left(a^2+c^2-b^2\right)}{abc}+\dfrac{R\left(a^2+b^2-c^2\right)}{abc}}$

$= \dfrac{b^2+c^2-a^2}{2a^2} = \dfrac{2a^2-a^2}{2a^2} = \dfrac{1}{2}$

12. c. Here, $\sin\theta-\cos\theta = \dfrac{b}{a}$ and $\sin\theta\cos\theta = \dfrac{c}{a}$

$\Rightarrow \quad 1-2\sin\theta\cos\theta = \dfrac{b^2}{a^2}$

$\Rightarrow \quad 1-\dfrac{2c}{a} = \dfrac{b^2}{a^2}$

$\Rightarrow \quad a^2-b^2 = 2ac$

Hence, $\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{2ac+c^2}{2ac} = 1+\dfrac{c}{2a}$

13. c. We have $a^2+b^2+c^2-ac-ab\sqrt{3} = 0$

$\Rightarrow \quad \dfrac{a^2}{4}-ac+c^2+\dfrac{3a^2}{4}+b^2-ab\sqrt{3}=0$

$\Rightarrow \quad \left[\dfrac{a}{2}-c\right]^2+\left[\dfrac{\sqrt{3}a}{2}-b\right]^2=0$

$\Rightarrow \quad a = 2c$ and $2b = \sqrt{3}a \qquad \Rightarrow \quad b^2 + c^2 = a^2$

Hence, the triangle is right angled.

14. c. $(a + b + c)(b + c - a) = kbc$

$\Rightarrow \quad (b + c)^2 - a^2 = kbc$

$\Rightarrow \quad b^2 + c^2 - a^2 = (k - 2)\, bc$

$\Rightarrow \quad 2bc \cos A = (k - 2)\, bc$

$\Rightarrow \quad \cos A = \dfrac{k-2}{2}$

Now, A being the angle of a triangle,

$-1 < \cos A < 1 \qquad \Rightarrow \quad -2 < k - 2 < 2$

$\Rightarrow \quad 0 < k < 4$

15. c. $a = 2b$ and $A - B = 60°$

We know that $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$

$\Rightarrow \quad \tan 30° = \dfrac{1}{3} \cot \dfrac{C}{2} \qquad \Rightarrow \quad \tan \dfrac{C}{2} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad C = 60°$

Hence, $A + B = 120° \qquad \Rightarrow \quad 2A = 180° \qquad \Rightarrow \quad A = 90°, B = 30°, C = 60°$

16. a. $r_1 = 2r_2 = 3r_3$

$\Rightarrow \quad \dfrac{\Delta}{s-a} = 2\dfrac{\Delta}{s-b} = 3\dfrac{\Delta}{s-c}$

$\Rightarrow \quad \dfrac{1}{s-a} = \dfrac{2}{s-b} = \dfrac{3}{s-c} = k$ (say)

$\Rightarrow \quad s - a = \dfrac{1}{k},\ s - b = \dfrac{2}{k}$ and $s - c = \dfrac{3}{k}$

Adding, we get $3s - (a + b + c) = \dfrac{6}{k} \qquad \Rightarrow \quad s = \dfrac{6}{k} \qquad \Rightarrow \quad a = \dfrac{5}{k}$ and $b = \dfrac{4}{k} \qquad \Rightarrow \quad \dfrac{a}{b} = \dfrac{5}{4}$

17. a. We have $\left(1 - \dfrac{s-b}{s-a}\right)\left(1 - \dfrac{s-c}{s-a}\right) = 2$

$\Rightarrow \quad 2(b - a)(c - a) = 4(s - a)^2$

$\Rightarrow \quad 2(bc - ac - ab + a^2) = (2s - 2a)^2$

$\Rightarrow \quad 2(bc - ac - ab + a^2) = (b + c - a)^2$

$\Rightarrow \quad a^2 = b^2 + c^2$

Hence, triangle is right angled.

18. b. We have $\Delta = \dfrac{\sqrt{3}}{4}a^2,\ s = \dfrac{3a}{2}$

$\therefore \quad r = \dfrac{\Delta}{s} = \dfrac{a}{2\sqrt{3}},\ R = \dfrac{abc}{4\Delta} = \dfrac{a^3}{\sqrt{3}a^2} = \dfrac{a}{\sqrt{3}}$

and $r_1 = \dfrac{\Delta}{s-a} = \dfrac{\sqrt{3}/4a^2}{a/2} = \dfrac{\sqrt{3}}{2}a$

Hence, $r : R : r_1 = \dfrac{a}{2\sqrt{3}} : \dfrac{a}{\sqrt{3}} : \dfrac{\sqrt{3}}{2}a = 1 : 2 : 3$

19. a. $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

$\Rightarrow \quad 2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2}$

$\Rightarrow \quad 2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$\Rightarrow \quad 2(s-b) = s-a+s-c$

$\Rightarrow \quad 2b = a+c$

$\Rightarrow \quad a, b, c$ are in A.P.

20. b. We have $\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$

$$\therefore \quad \tan\left(\frac{C-B}{2}\right) = \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2}\cot 15^\circ$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+3}\frac{1}{\tan(45-30^\circ)}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+3}\frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}}$$

$$= \tan 30^\circ$$

$\Rightarrow \frac{C-B}{2} = 30^\circ$

21. d. $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{64+81-49}{2\times 8\times 9} = \frac{145-49}{144} = \frac{96}{144}$

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{49+81-64}{2\times 7\times 9} = \frac{66}{126} = \frac{11}{26}$

22. b. $\cos A + \cos B + \cos C = \frac{7}{4}$

$\Rightarrow \quad 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{7}{4}$

$\Rightarrow \quad 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{3}{4}$ $\quad$ [$\because r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$]

$\Rightarrow \quad \frac{r}{R} = \frac{3}{4} \Rightarrow \frac{R}{r} = \frac{4}{3}$

23. a. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$

$$= \frac{s}{\Delta}[3s-(a+b+c)] = \frac{s^2}{\Delta} = \frac{\left(\frac{\Delta}{r}\right)^2}{\Delta} = \frac{\Delta}{r^2}$$

24. b. $\dfrac{\cot A + \cot C}{\cot B} = \dfrac{\sin(A+C)\sin B}{\sin A \sin C \cos B}$

$$= \frac{\sin^2 B}{\sin A \sin C \cos B}$$

$$= \frac{4R^2 b^2}{4R^2 ac \cos B}$$

$$= \frac{2b^2}{2ac\cos B} = \frac{2b^2}{a^2 + c^2 - b^2}$$

$$= \frac{2b^2}{2002b^2 - b^2} = \frac{2}{2001}$$

25. a.

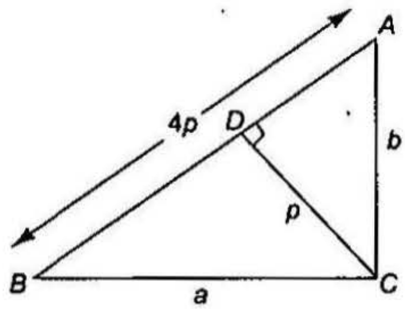


Fig. 5.64

$\Delta = \dfrac{1}{2}ab = \dfrac{1}{2}p\,4p \quad \Rightarrow \quad ab = 4p^2$

Also, $a^2 + b^2 = c^2 = 16p^2$

$\therefore \quad (a-b)^2 = a^2 + b^2 - 2ab = 8p^2$

Also, $(a+b)^2 = a^2 + b^2 + 2ab = 24p^2$

$\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2} = \dfrac{1}{\sqrt{3}}\,1$

$\Rightarrow \quad \dfrac{A-B}{2} = 30° \quad \Rightarrow \quad A - B = 60°$

26. c. $\dfrac{1}{\sin^2\frac{A}{2}}, \dfrac{1}{\sin^2\frac{B}{2}}, \dfrac{1}{\sin^2\frac{C}{2}}$ are in A.P.

$$\Rightarrow \quad \frac{1}{\sin^2\frac{C}{2}} - \frac{1}{\sin^2\frac{B}{2}} = \frac{1}{\sin^2\frac{B}{2}} - \frac{1}{\sin^2\frac{A}{2}}$$

$$\Rightarrow \quad \frac{ab}{(s-a)(s-b)} - \frac{ac}{(s-a)(s-c)} = \frac{ac}{(s-a)(s-c)} - \frac{bc}{(s-b)(s-c)}$$

$$\Rightarrow \left(\frac{a}{s-a}\right)\left(\frac{b(s-c)-c(s-b)}{(s-b)(s-c)}\right)=\left(\frac{c}{s-c}\right)\left(\frac{a(s-b)-b(s-a)}{(s-a)(s-b)}\right)$$

$\Rightarrow \quad ab - ac = ac - bc$

$\Rightarrow \quad ab + bc = 2ac$

$\Rightarrow \quad \frac{1}{c}+\frac{1}{a}=\frac{2}{b}$, i.e., a, b, c are in H.P.

27. a. $b = 2, c = \sqrt{3}, \angle A = 30°$

$$a = \sqrt{b^2 + c^2 - 2bc \cos A} = \sqrt{4 + 3 - 2\times 2\sqrt{3}\ \frac{\sqrt{3}}{2}} = 1$$

$$\Rightarrow \quad r = (s-a)\tan\frac{A}{2} = \frac{b+c-a}{2}\tan\frac{A}{2} = \frac{\sqrt{3}+1}{2}\tan 15° = \frac{\sqrt{3}+1}{2}\ \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{2}$$

28. c.

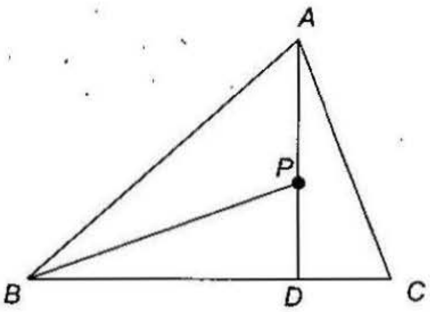


Fig. 5.65

$\angle BPA = 90° + (B/3)$, $\angle ABP = 2B/3$

In ΔABP,

$$\frac{AP}{\sin(2B/3)} = \frac{c}{\sin[90° + (B/3)]} = \frac{c}{\cos(B/3)}$$ [by sine rule]

$$\Rightarrow \quad AP = \frac{c\sin(2B/3)}{\cos(B/3)} = \frac{2c\sin(B/3)\cos(B/3)}{\cos(B/3)}$$

$= 2c\sin(B/3)$

29. d. $\angle A = \frac{\pi}{2} \Rightarrow a^2 = b^2 + c^2$

$$\sin C = \frac{c}{a} = \frac{2\tan\frac{C}{2}}{1+\tan^2\frac{C}{2}}$$

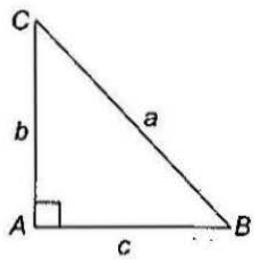


Fig. 5.66

$\Rightarrow \quad c \tan^2 \dfrac{C}{2} - 2a \tan \dfrac{C}{2} + c = 0$

$\Rightarrow \quad \tan \dfrac{C}{2} = \dfrac{2a \pm \sqrt{4a^2 - 4c^2}}{2c} = \dfrac{2a \pm 2b}{2c} = \dfrac{a \pm b}{c}$

$= \dfrac{a - b}{c}$

Because if $\tan \dfrac{C}{2} = \dfrac{a+b}{c}$, then $\tan \dfrac{C}{2} > 1 \quad \Rightarrow \quad \dfrac{C}{2} > \dfrac{\pi}{4} \Rightarrow C > \dfrac{\pi}{2}$ which is not possible.

30. a. $a \cos (B - C) + b \cos (C - A) + c \cos (A - B)$
$= 2R \sin A \cos (B - C) + 2R \sin B \cos (C - A) + 2R \sin C \cos (A - B)$
$= 2R \sin (B + C) \cos (B - C) = R[\sin 2B + \sin 2C]$
$= R[\sin 2B + \sin 2C + \sin 2C + \sin 2A + \sin 2A + \sin 2B]$
$= 2R (\sin 2A + \sin 2B + \sin 2C)$
$= 8R \sin A \sin B \sin C$

$= 8R \dfrac{a}{2R} \dfrac{b}{2R} \dfrac{c}{2R} = \dfrac{abc}{R^2}$

31. d. If the triangle is equilateral

$\sin A + \sin B + \sin C = \dfrac{3\sqrt{3}}{2}$

If the triangle is isosceles, let $A = 30°$, $B = 30°$, $C = 120°$.

Then, $\sin A + \sin B + \sin C = 1 + \dfrac{\sqrt{3}}{2}$

If the triangle is right angled, let $A = 90°$, $B = 30°$, $C = 60°$.

Then, $\sin A + \sin B + \sin C = \dfrac{3 + \sqrt{3}}{2}$

If the triangle is right-angled isosceles, then one of the angles is 90° and the remaining two are 45° each, so that

$\sin A + \sin B + \sin C = 1 + \sqrt{2}$

and $\cos A + \cos B + \cos C = \sqrt{2}$

32. c. $\dfrac{r}{r_1} = \dfrac{r_2}{r_3}$

$\Rightarrow \quad r r_3 = r_1 r_2$

$\Rightarrow \quad \dfrac{\Delta}{s} \dfrac{\Delta}{s - c} = \dfrac{\Delta}{s - a} \dfrac{\Delta}{s - b}$

$\Rightarrow \quad \dfrac{(s - a)(s - b)}{s(s - c)} = 1$

$$\Rightarrow \quad \tan^2\frac{C}{2} = 1 \Rightarrow \tan\frac{C}{2} = 1$$

$$\Rightarrow \quad \frac{C}{2} = 45° \Rightarrow C = 90°$$

33. c. Since $\cos 3A + \cos 3B + \cos 3C = 1$

$$\Rightarrow \quad 4\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2} = 0$$

Either $\frac{3A}{2} = 180°$ or $\frac{3B}{2} = 180°$ or $\frac{3C}{2} = 180°$

Either $A = 120°$ or $B = 120°$ or $C = 120°$

34. a. Given, $\cos B\cos C + \sin B\sin C\sin^2 A = 1$ (i)

Now, we know that $\sin^2 A \le 1$ (ii)

Also, $\sin B$ and $\sin C$ are positive.

$\Rightarrow \quad \sin B\sin C\sin^2 A \le \sin B\sin C$ (iii)

$\Rightarrow \quad 1 - \cos B\cos C \le \sin B\sin C$, [by using Eq. (i)]

$\Rightarrow \quad \cos(B - C) \ge 1 \quad \Rightarrow \quad \cos(B - C) = 1 \quad \Rightarrow \quad B - C = 0 \quad \Rightarrow \quad B = C$

Also, $\sin^2 A = 1$, i.e., $A = \pi/2$. Hence, the triangle is right-angled isosceles.

35. c.

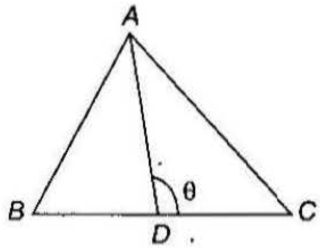


Fig. 5.67

$$\theta = \pi - \left(C + \frac{A}{2}\right) \quad \Rightarrow \quad \sin\theta = \sin\left(C + \frac{A}{2}\right) = \sin\left(C + \frac{\pi}{2} - \frac{B+C}{2}\right) = \cos\left(\frac{B-C}{2}\right)$$

36. d. $r - r_2 = r_3 - r_1$

$$\Rightarrow \quad \frac{\Delta}{s} - \frac{\Delta}{s-b} = \frac{\Delta}{s-c} - \frac{\Delta}{s-a}$$

$$\Rightarrow \quad \frac{-b}{s(s-b)} = \frac{c-a}{(s-a)(s-c)}$$

$$\Rightarrow \quad \frac{(s-a)(s-c)}{s(s-b)} = \frac{a-c}{b}$$

$$\Rightarrow \quad \tan^2\frac{B}{2} = \frac{a-c}{b}$$

But $\frac{B}{2} \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

$$\Rightarrow \quad \tan^2\frac{B}{2} \in \left(\frac{1}{3}, 1\right)$$

$\Rightarrow \quad \frac{1}{3} < \frac{a-c}{b} < 1$

$\Rightarrow \quad b < 3a - 3c < 3b$

$\Rightarrow \quad b + 3c < 3a < 3b + 3c$

37. b.

$$R = \frac{a}{2\sin A} = \frac{2+\sqrt{5}}{2\sin 30°} = \frac{2+\sqrt{5}}{2\times\frac{1}{2}} = \left(2+\sqrt{5}\right)$$

Now, $AH = 2R\cos A = 2(2+\sqrt{5})\cos 30° = (2+\sqrt{5})\sqrt{3}$

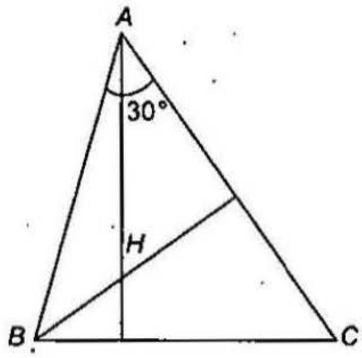


Fig. 5.68

38. a. We know that $IA = \frac{r}{\sin\frac{A}{2}}$

$\Rightarrow \quad IA : IB : IC = \operatorname{cosec}\frac{A}{2} : \operatorname{cosec}\frac{B}{2} : \operatorname{cosec}\frac{C}{2}$

39. a. Using the property of angle bisector, we have $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

$\Rightarrow BD + DC = ck + bk = a$

$\Rightarrow k = \frac{a}{b+c}$

Also $xy = bck^2$ (property of circle)

$$\Rightarrow x = \left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)\frac{bc\,a^2}{(b+c)^2}$$

$$= \frac{a^2\sec\frac{A}{2}}{2(b+c)}$$

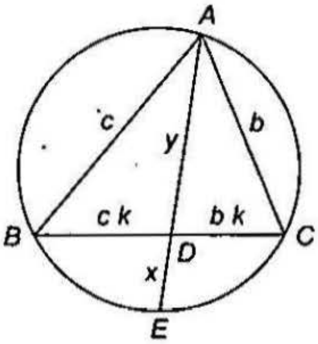


Fig. 5.69

40. a. From the right angled ΔCAD, we have

$\cos C = \frac{b}{a/2} \Rightarrow \frac{2b}{a} = \frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow a^2 + b^2 - c^2 = 4b^2 \Rightarrow a^2 - c^2 = 3b^2$

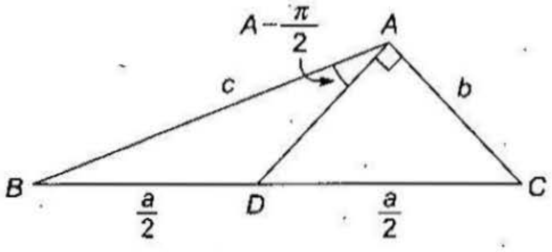


Fig. 5.70

41. a.

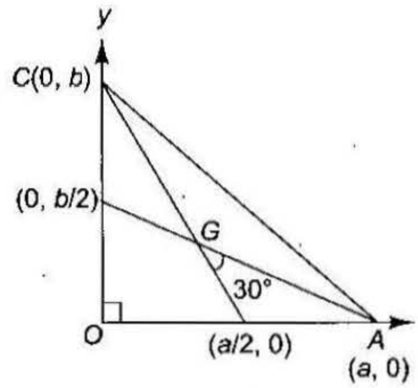


Fig. 5.71

Slope of $GC = m_1 = \frac{-2b}{a}$, slope of $AG = m_2 = \frac{-b}{2a}$

$$\tan 30° = \tan\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{\frac{3b}{2a}}{1 + \frac{b^2}{a^2}} \text{ and } a^2 + b^2 = 9$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{3ba}{2(a^2+b^2)} \qquad \Rightarrow \quad \frac{1}{2}ab = \left(\frac{a^2+b^2}{3\sqrt{3}}\right) = \frac{9}{3\sqrt{3}} = \sqrt{3}$$

42. c.

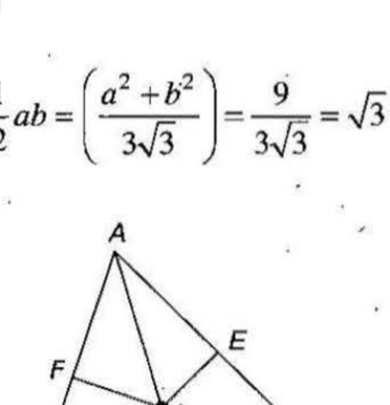


Fig. 5.72

In triangles AIF and AIE,

$$\frac{IF}{\sin(A/2)} = AI = \frac{IE}{\sin(A/2)} \qquad \Rightarrow \quad AI^2 = \frac{IE \cdot IF}{\sin^2(A/2)}$$

$$\Rightarrow \quad \frac{ID \cdot IE \cdot IF}{IA \cdot IB \cdot IC} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R} = \frac{1}{10}$$

43. c. We have $BD = DC$ and $\angle DAB = 90°$. Draw CN perpendicular to BA produced, then in ΔBCN, we have

$DA = \frac{1}{2}CN$ and $AB = AN$

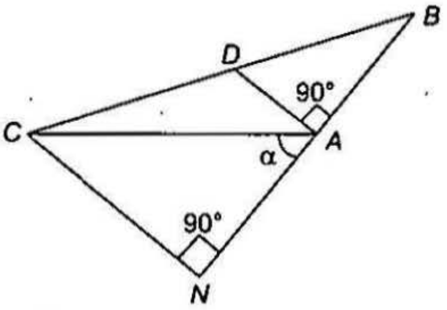


Fig. 5.73

Let $\angle CAN = \alpha$

$\because \tan A = \tan(\pi - \alpha) = -\tan\alpha = -\frac{CN}{NA} = -2\frac{AD}{AB} = -2\tan B \quad \Rightarrow \quad \tan A + 2\tan B = 0$

44. a.

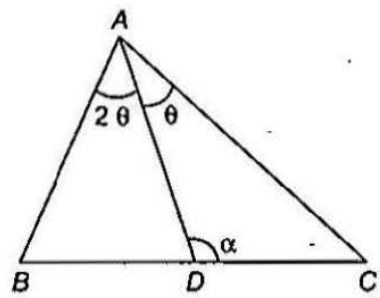


Fig. 5.74

Let $\frac{A}{3} = \angle CAD = \theta$

Now, by *m–n* theorem,

$(1+1)\cot\alpha = 1\cot 2\theta - 1\cot\theta \quad \Rightarrow \quad 2\cot(B+2\theta) = \cot 2\theta - \cot\theta$

$\Rightarrow \cot(B+2\theta) + \cot\theta = \cot 2\theta - \cot(B+2\theta)$

$$\Rightarrow \frac{\sin(B+3\theta)}{\sin(B+2\theta)\sin\theta} = \frac{\sin B}{\sin(B+2\theta)\sin 2\theta}$$

$$\Rightarrow \frac{\sin(B+A)}{\sin\theta} = \frac{\sin B}{\sin 2\theta}$$

$$\Rightarrow \sin C = \frac{\sin B}{2\cos\theta}$$

$$\Rightarrow \cos\frac{A}{3} = \frac{\sin B}{2\sin C}$$