Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. Prove that, for any three vectors $\vec{a}, \vec{b}, \vec{c}$

$$[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}] = 2\,[\vec{a} \;\; \vec{b} \;\; \vec{c}]$$

Ans.

$$\text{LHS} = [\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}] = (\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$$

$$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\}$$

$$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\}$$

$$[\because \; \vec{c}\times\vec{c}=\vec{0}]$$

$$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}$$

$$= [\vec{a} \;\; \vec{b} \;\; \vec{c}]+[\vec{a} \;\; \vec{b} \;\; \vec{a}]+[\vec{a} \;\; \vec{c} \;\; \vec{a}]+[\vec{b} \;\; \vec{b} \;\; \vec{c}]+[\vec{b} \;\; \vec{b} \;\; \vec{a}]+[\vec{b} \;\; \vec{c} \;\; \vec{a}]$$

$$= [\vec{a} \;\; \vec{b} \;\; \vec{c}]+\vec{0}+\vec{0}+\vec{0}+\vec{0}+[\vec{b} \;\; \vec{c} \;\; \vec{a}]$$ [By property of scalar triple product]

$$= [\vec{a} \;\; \vec{b} \;\; \vec{c}]+[\vec{b} \;\; \vec{c} \;\; \vec{a}]$$

$$= [\vec{a} \;\; \vec{b} \;\; \vec{c}]+[\vec{a} \;\; \vec{b} \;\; \vec{c}]$$ [By property of circularly rotation]

$$= 2[\vec{a} \;\; \vec{b} \;\; \vec{c}] = \text{RHS}$$

Q.2. Find the value of *x* such that the point *A*(3, 2, 1), *B*(4, *x*, 5), *C*(4, 2, –2) and *D*(6, 5, –1) are coplanar.

Ans.

We have $A(3, 2, 1)$, $B(4, x, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$

$\overrightarrow{AB} = \hat{i} + (x-2)\hat{j} + 4\hat{k}$; $\overrightarrow{AC} = \hat{i} + 0\hat{j} - 3\hat{k}$; $\overrightarrow{AD} = 3\hat{i} + 3\hat{j} - 2\hat{k}$

$\because$ Points A, B, C and D are coplanar $\Rightarrow \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar

$\Rightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$\Rightarrow 1\ (0+9) - (x-2)(-2+9) + 4(3-0) = 0$

$\Rightarrow 9 - 7x + 14 + 12 = 0$

$\Rightarrow 7x = 35 \qquad \Rightarrow x = 5$

Q.3. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ **are coplanar, if** $\vec{a} + \vec{b}$, and $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ **are coplanar.**

Ans.

If part: Let $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Rightarrow$Scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ is zero

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow \qquad \vec{a}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} \times \vec{a}) = \vec{c}.(\vec{a} \times \vec{b}) = 0$

Now, $[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}] = (\vec{a} + \vec{b}).\{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\}$

$= (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\}$

$= (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\}$ $\quad [\because \vec{c} \times \vec{c} = 0]$

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{a}) + \vec{a}.(\vec{c} \times \vec{a}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{a}) + \vec{b}.(\vec{c} \times \vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}] + \vec{0} + \vec{0} + \vec{0} + \vec{0} + [\vec{b}\ \vec{c}\ \vec{a}]$ [By property of scalar triple product]

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}] = 2[\vec{a}\ \vec{b}\ \vec{c}] = 2$

$= 2 \times 0 = 0$ $\quad [\because [\vec{a}\ \vec{b}\ \vec{c}] = 0]$

Hence, $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar

Only if part: Let $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are coplanar.

$\Rightarrow [\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}] = 0$

$\Rightarrow (\vec{a} + \vec{b}).\{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\} = 0$

$\Rightarrow (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\} = 0$

$\Rightarrow (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\} = 0$ $\quad [\because \vec{c} \times \vec{c} = 0]$

$\Rightarrow \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{a}) + \vec{a}.(\vec{c} \times \vec{a}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{a}) + \vec{b}.(\vec{c} \times \vec{a}) = 0$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] + \vec{0} + \vec{0} + \vec{0} + \vec{0} + [\vec{b}\ \vec{c}\ \vec{a}] = 0$

$\Rightarrow 2[\vec{a}\ \vec{b}\ \vec{c}] = 0$ $\quad [\because [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]]$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

Hence, $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Q.4.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

(*a*) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

(*b*) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

Ans.

Given $\vec{a} = \hat{i} + \hat{j} + \hat{k}$; $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

(*a*) Since $\vec{a}$ $\vec{b}$ and $\vec{c}$ vectors are coplanar

$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0 \qquad [\text{Given that } c_1 = 1 \text{ and } c_2 = 2]$$

$$\Rightarrow 1(0-0) - 1(c_3 - 0) + 1(2-0) = 0$$

$$\Rightarrow -c_3 + 2 = 0 \Rightarrow c_3 = 2$$

(*b*) To make $\vec{a}$ $\vec{b}$ and $\vec{c}$ coplanar.

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = 0 \qquad [\text{Given that } c_2 = -1 \text{ and } c_3 = 1]$$

$$\Rightarrow 1(0-0) - 1(1-0) + 1(-1-0) = 0$$

$\Rightarrow -1 - 1 = 0$

$\Rightarrow -2 = 0$ which is never possible.

Hence, if $c_2 = -1$ and $c_3 = 1$, there is no value of c_1 which can make $\vec{a}$ $\vec{b}$ and $\vec{c}$ coplanar.

Q.5.

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$

Ans.

Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = x$ (say)

Since $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors. Therefore,

$\vec{a} . \vec{b} = \vec{b} . \vec{c} = \vec{c} . \vec{a} = 0 = \vec{b} . \vec{a} = \vec{c} . \vec{b} = \vec{a} . \vec{c}$

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c})$

$= \vec{a} . \vec{a} + \vec{a} . \vec{b} + \vec{a} . \vec{c} + \vec{b} . \vec{a} + \vec{b} . \vec{b} + \vec{b} . \vec{c} + \vec{c} . \vec{a} + \vec{c} . \vec{b} + \vec{c} . \vec{c}$

$= x^2 + 0 + 0 + 0 + x^2 + 0 + 0 + 0 + x^2 = 3x^2$

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}x$

Let θ_1 and θ_2 and θ_3 be the angles made by $(\vec{a} + \vec{b} + \vec{c})$ with $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\therefore \cos\theta_1 = \frac{a.(\vec{a} + \vec{b} + \vec{c})}{|\vec{a}|.|\vec{a} + \vec{b} + \vec{c}|} = \frac{\vec{a} . \vec{a} + \vec{a} . \vec{b} + \vec{a} . \vec{c}}{x.\sqrt{3}x}$

$= \frac{x^2+0+0}{\sqrt{3}x^2} = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Similarly, we have $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

i.e.,

$(\vec{a} + \vec{b} + \vec{c})$ is equally inclined with $\vec{a}, \vec{b}$ and $\vec{c}$

⇒

$42 + 14\lambda = 0 \quad \Rightarrow \quad 14\lambda = -42 \quad \Rightarrow \quad \lambda = -3$

Q.6. Find a vector of magnitude 6, perpendicular to each of the vectors

$\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Ans.

$$\vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{a} - \vec{b} = -\hat{j} - 2\hat{k}$$

Now vector perpendicular to $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = (-6+4)\hat{i} - (-4-0)\hat{j} + (-2-0)\hat{k} = -2\hat{i} + 4\hat{j} - 2\hat{k}$$

$$\therefore \quad \text{Required vector} = \pm 6 \frac{(-2\hat{i} + 4\hat{j} - 2\hat{k})}{\sqrt{(-2)^2 + 4^2 + (-2)^2}} = \pm \frac{6}{\sqrt{24}} (-2\hat{i} + 4\hat{j} - 2\hat{k})$$

$$= \pm \frac{6}{2\sqrt{6}} (-2\hat{i} + 4\hat{j} - 2\hat{k}) = \pm\sqrt{6}\,(-\hat{i} + 2\hat{j} - \hat{k})$$

Q.7. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}| = 5, |\vec{b}| = 12$ and $|\vec{c}| = 13$, and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

Ans.

$\because\ \vec{a} + \vec{b} + \vec{c} = \vec{0}$ ...(i)

$\Rightarrow\ \vec{a} \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \cdot \vec{0} \quad \Rightarrow \quad \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0$

$\Rightarrow \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = -\left|\vec{a}\right|^2 \quad \left[\because \vec{a} \cdot \vec{a} = \left|\vec{a}\right|^2\right]$

$\Rightarrow \vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{a} = -25$ ...(ii) $\left[\because \vec{a} \cdot \vec{c} = \vec{c} \cdot \vec{a}\right]$

Similarly taking dot product of both sides of (i) by $\vec{b}$ and $\vec{c}$ respectively, we get

$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} = -|\vec{b}|^2 = -144$ ...(iii)

and $\vec{c} \cdot \vec{a} + \vec{b} \cdot \vec{c} = -|\vec{c}|^2 = -169$ ...(iv)

Adding (ii), (iii) and (iv), we get

$= \vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{b} \cdot \vec{c} = 25 - 144 - 169$

$2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -338$

$= \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{338}{2} - 169$

Q.8. If $\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$ **then express** $\vec{\beta}$ **in the form ,** $\vec{\beta} = \vec{\beta_1} + \vec{\beta}_2$ **where** $\vec{\beta}_1$ **is parallel to** $\vec{\alpha}$ **and** $\vec{\beta}_2$ **is perpendicular to** $\vec{\alpha}$ **.**

Ans.

$\because \vec{\beta}_1$ is parallel to $\vec{\alpha}$, $\Rightarrow$ $\vec{\beta}_1 = \lambda\vec{\alpha}$ where λ is any scalar quantity

$\Rightarrow \vec{\beta}_1 = 3\lambda\hat{i} + 4\lambda\hat{j} + 5\lambda\hat{k}$

Also if, $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$

$\Rightarrow 2\hat{i} + \hat{j} - 4\hat{k} = (3\lambda\hat{i} + 4\lambda\hat{j} + 5\lambda\hat{k}) + \vec{\beta}_2$

$\Rightarrow \vec{\beta}_2 = (2- 3\lambda)\hat{i} + (1- 4\lambda)\hat{j} - (4+5\lambda)\hat{k}$

It is given $\vec{\beta}_2 \perp \vec{\alpha}$

$\Rightarrow (2- 3\lambda). 3 + (1- 4\lambda). 4- (4+5\lambda).5 = 0$

$\Rightarrow 6 - 9\lambda + 4 - 16\lambda - 20 - 25\lambda = 0$

$\Rightarrow - 10 - 50\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{-1}{5}$

Therefore, $\vec{\beta}_1 = -\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} - \hat{k}$

$\vec{\beta}_2 = \left(2 + \frac{3}{5}\right)\hat{i} + \left(1 + \frac{4}{5}\right)\hat{j} - (4 - 1)\hat{k} = \frac{13}{5}\hat{i} + \frac{9}{5}\hat{j} - 3\hat{k}$

The required expression is

$(2\hat{i} + \hat{j} - 4\hat{k}) = \left(-\frac{3}{5}\hat{i} - \frac{4}{5}j - \hat{k}\right) + \left(\frac{13}{5}\hat{i} + \frac{9}{5}\hat{j} - 3\hat{k}\right)$

Q.9. Show that the four points *A, B, C* and *D* with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, - \hat{j} - \hat{k}, 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(- \hat{i} + \hat{j} + \hat{k})$ respectively are coplanar.

Ans.

Position vector of $A \equiv 4\hat{i} + 5\hat{j} + \hat{k}$; Position vector of $B \equiv - \hat{j} - \hat{k}$

Position vector of $C \equiv 3\hat{i} + 9\hat{j} + 4\hat{k}$; Position vector of $D \equiv -4\hat{i} + 4\hat{j} + 4\hat{k}$

$\therefore \overrightarrow{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}, \quad \overrightarrow{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}, \quad \overrightarrow{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$

Now, $\overrightarrow{AB} . (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$

$= -4(12 + 3) + 6(-3 + 24) - 2(1 + 32) = -60 + 126 - 66 = 0$

i.e., $\overrightarrow{AB} . (\overrightarrow{AC} \times \overrightarrow{AD}) = 0$

Hence, $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar *i.e.,* points *A, B, C, D* are coplanar.

[**Note:** Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, if the scalar triple product of these three vectors is zero.]

Q.10. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, **then find a vector** $\vec{c}$ **such that** $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} . \vec{c} = 3$.

Ans.

Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then,

$$(\vec{a} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = (c_3 - c_2)\hat{i} + (c_1 - c_3)\hat{j} + (c_2 - c_1)\hat{k}$$

$\therefore (\vec{a} \times \vec{c}) = \vec{b}$

$\Rightarrow (c_3 - c_2)\hat{i} + (c_1 - c_3)\hat{j} + (c_2 - c_1)\hat{k} = \hat{j} - \hat{k}$

$\Rightarrow c_3 - c_2 = 0, c_1 - c_3 = 1$ and $c_2 - c_1 = -1$ ...(*i*)

Also, $\vec{a} . \vec{c} = (\hat{i} + \hat{j} + \hat{k}) . (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$\Rightarrow \vec{a}\,.\,\vec{c} = c_1 + c_2 + c_3$

$\Rightarrow c_1 + c_2 + c_3 = 3$ $\quad [\because \vec{a}\,.\,\vec{c} = 3]$ $\quad ...(ii)$

$\Rightarrow c_1 + c_2 + c_1 - 1 = 3$ $\quad [\because c_1 - c_3 = 1]$ $\quad ...(iii)$

$\Rightarrow 2c_1 + c_2 = 4$

On solving $c_1 - c_2 = 1$ and $2c_1 + c_2 = 4$ we get

$3c_1 = 5 \quad \Rightarrow \quad c_1 = \frac{5}{3}$

$\therefore c_2 = (c_1 - 1) = \left(\frac{5}{3} - 1\right) = \frac{2}{3}$ and $c_3 = c_2 = \frac{2}{3}$

Hence, $\vec{c} = \left(\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$.

Q.11.

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$ then show that the angle between $\vec{a}$ and $\vec{b}$ is 60°.

Ans.

$\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow (\vec{a} + \vec{b})^2 = (-\vec{c})^2$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = \vec{c}\,.\,\vec{c}$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\,.\,\vec{b} = |\vec{c}|^2$ $\quad \Rightarrow 9 + 25 + 2\vec{a}\,.\,\vec{b} = 49$

$\Rightarrow 2\vec{a}\,.\,\vec{b} = 49 - 25 - 9$

$\Rightarrow 2|\vec{a}|\,|\vec{b}|\cos\theta = 15$ $\quad \Rightarrow 30\cos\theta = 15$

$\Rightarrow \cos\theta = \frac{1}{2} = \cos 60°$ $\quad \theta = 60°$

Q.12.

If $\hat{i}+\hat{j}+\hat{k}$. $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of the points *A, B, C* and *D*, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Ans.

Given, $\overrightarrow{OA}=\hat{i}+\hat{j}+\hat{k}$, $\overrightarrow{OB}=2\hat{i}+5\hat{j}$

$\overrightarrow{OC}=3\hat{i}+2\hat{j}-3\hat{k}$, $\overrightarrow{OD}=\hat{i}-6\hat{j}-\hat{k}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\hat{i}+4\hat{j}-\hat{k}$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=-2\hat{i}-8\hat{j}+2\hat{k}$

$\overrightarrow{CD}=-2(\hat{i}+4\hat{j}-\hat{k}) \Rightarrow \overrightarrow{CD}=-2\overrightarrow{AB}$

Therefore, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel vector so $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear and angle between them is zero.

Q.13.

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}.\vec{b}=\vec{a}.\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\neq 0$, then show that $\vec{b}=\vec{c}$.

Ans.

Given, $\vec{a}.\vec{b}=\vec{a}.\vec{c}$

$\Rightarrow \vec{a}.\vec{b}-\vec{a}.\vec{c}=0 \Rightarrow \vec{a}.(\vec{b}-\vec{c})=0$

$\Rightarrow$ either $\vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

Also given $\vec{a}\times\vec{b}=\vec{a}\times\vec{c} \Rightarrow \vec{a}\times\vec{b}-\vec{a}\times\vec{c}=0$

$\Rightarrow \vec{a}\times(\vec{b}-\vec{c})=0 \Rightarrow \vec{a}\,||\,\vec{b}-\vec{c}$ or $\vec{b}=\vec{c}$

But $\vec{a}$ cannot be both parallel and perpendicular to $(\vec{b}-\vec{c})$. Hence, $\vec{b}=\vec{c}$

Q.14. Find the position vector of a point *R* which divides the line joining two points *P* and *Q*, whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ respectively, externally in the ratio 1 : 2. Also, show that *P* is the mid point of the line segment *RQ*.

Ans.

The position vector of the point R dividing the join of P and Q externally in the ratio 1 : 2 is

Position vector of R

$$(\overrightarrow{OR}) = \frac{1(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{1 - 2}$$
$$= \frac{\vec{a} - 3\vec{b} - 4\vec{a} - 2\vec{b}}{-1} = \frac{-3\vec{a} - 5\vec{b}}{-1} = 3\vec{a} + 5\vec{b}$$

Mid-point of the line segment RQ is $\frac{(3\vec{a} + 5\vec{b}) + (\vec{a} - 3\vec{b})}{2} = 2\vec{a} + \vec{b}$

As it is same as position vector of point P, so P is the mid-point of the line segment RQ.

Q.15. If $\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ then find a vector of magnitude 6 units which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Ans.

Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k},\ \vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

Consider, $\vec{r} = 2\vec{a} - \vec{b} + 3\vec{c}$

$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 4\hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k} = \hat{i} - 2\hat{j} + 2\hat{k}$

Since the required vector has magnitude 6 units and parallel to $\vec{r}$.

Required vector = $\frac{6\vec{r}}{|\vec{r}|}$, where $|\vec{r}| = \sqrt{(1)^2 + (-2)^2 + (2)^2}$

$$= 6\left[\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}}\right] = 6\left[\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}}\right] = 2\hat{i} - 4\hat{j} + 4\hat{k}$$

Q.16.

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}.\vec{c} = 18$.

Ans.

Given, $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$, $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

Vector $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ *i.e.,* $\vec{p}$ is parallel to vector $\vec{a} \times \vec{b}$.

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 4 & 2 \\ -2 & 7 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = 32\hat{i} - \hat{j} - 14\hat{k}$$

Since $\vec{p}$ is parallel to $\vec{a} \times \vec{b}$

$$\therefore \vec{p} = \mu(32\hat{i} - \hat{j} - 14\hat{k})$$

Also, $\vec{p}.\vec{c} = 18$

$$\Rightarrow \mu\ (32\hat{i} - \hat{j} - 14\hat{k}).\ (2\hat{i} - \hat{j} + 4\hat{k}) = 18$$

$$\Rightarrow \mu\ (64 + 1 - 56) = 18 \quad \Rightarrow \quad 9\mu = 18 \quad \text{or} \quad \mu = 2$$

$$\therefore \vec{p} = 2\ (32\hat{i} - \hat{j} - 14\hat{k}) = 64\hat{i} - 2\hat{j} - 28\hat{k}$$

Q.17. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Ans.

Given, two vectors are $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

If $\vec{c}$ is the resultant vector of $\vec{a}$ and $\vec{b}$ then

$\vec{c} = \vec{a} + \vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k}) = 3\hat{i} + \hat{j} + 0.\hat{k}$

Now, a vector having magnitude 5 and parallel to $\vec{c}$ is given by

$\frac{5\vec{c}}{|\vec{c}|} = \frac{5(3\hat{i} + \hat{j} + 0\hat{k})}{\sqrt{3^2+1^2+0^2}} = \frac{15}{\sqrt{10}}\hat{i} + \frac{5}{\sqrt{10}}\hat{j}$

It is required vector.

[**Note:** A vector having magnitude l and parallel to $\vec{a}$ is given by $l.\frac{\vec{a}}{|\vec{a}|}$.]

Q.18. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$, then prove that vector $2\vec{a} + \vec{b}$ is perpendicular to vector $\vec{b}$.

Ans.

$\because \left|\vec{a}+\vec{b}\right| = \left|\vec{a}\right|$

$\Rightarrow \left|\vec{a}+\vec{b}\right|^2 = \left|\vec{a}\right|^2$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = \left|\vec{a}\right|^2$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = \left|\vec{a}\right|^2$

$\Rightarrow \left|\vec{a}\right|^2 + 2\vec{a}.\vec{b} + \vec{b}.\vec{b} = \left|\vec{a}\right|^2$ $\quad [\because \vec{a}.\vec{b} = \vec{b}.\vec{a}]$

$\Rightarrow 2\vec{a}.\vec{b} + \vec{b}.\vec{b} = 0$

$\Rightarrow (2\vec{a}+\vec{b}).\vec{b} = 0$

$\Rightarrow (2\vec{a}+\vec{b})$ is perpendicular to $\vec{b}$.

Q.19. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$ **then find the value of λ, so that** $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ **are perpendicular vectors. 1**

Ans.

Here $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$; $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

$\therefore \vec{a}+\vec{b} = 6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}$; $\vec{a}-\vec{b} = -4\hat{i} + (7-\lambda)\hat{k}$

$\because (\vec{a}+\vec{b})$ is perpendicular to $(\vec{a}-\vec{b})$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 0$ $\quad \Rightarrow -24 + (7+\lambda).(7-\lambda) = 0$

$\Rightarrow -24 + 49 - \lambda^2 = 0$ $\quad \Rightarrow \lambda^2 = 25$

$\Rightarrow \lambda = \pm 5.$

Q.20. The magnitude of the vector product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

Ans.

Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$; $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$; $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

From question

$$\left|\vec{a}\times\frac{\vec{b}+\vec{c}}{\left|\vec{b}+\vec{c}\right|}\right|=\sqrt{2}\Rightarrow\left|\frac{\vec{a}\times(\vec{b}+\vec{c})}{\left|\vec{b}+\vec{c}\right|}\right|=\sqrt{2}\quad\ldots(i)$$

$$\vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

$$\therefore\left|\vec{b}+\vec{c}\right|=\sqrt{(2+\lambda)^2+6^2+(-2)^2}$$

$$=\sqrt{4+\lambda^2+4\lambda+36+4}=\sqrt{\lambda^2+4\lambda+44}$$

$$\Rightarrow\vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&1&1\\2+\lambda&6&-2\end{vmatrix}$$

$$=(-2-6)\hat{i}-(-2-2-\lambda)\hat{j}+(6-2-\lambda)\hat{k}$$

$$=-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}$$

Putting it in (i), we get

$$\left|\frac{-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right|=\sqrt{2}$$

$$\Rightarrow\quad\frac{\sqrt{(-8)^2+(4+\lambda)^2+(4-\lambda)^2}}{\sqrt{\lambda^2+4\lambda+44}}=\sqrt{2}$$

Squaring both sides, we get

$$\frac{64+16+\lambda^2+8\lambda+16+\lambda^2-8\lambda}{\lambda^2+4\lambda+44}=2$$

$$\Rightarrow \quad \frac{96+2\lambda^2}{\lambda^2+4\lambda+44}=2$$

$$\Rightarrow 8\lambda = 8 \quad \Rightarrow \quad \lambda = 1$$

Q.21. Show that the points *A, B, C* with position vectors

$$2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k} \text{ and } 3\hat{i}-4\hat{j}-4\hat{k}$$

respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle.

Ans.

Given, Position vector of $A = 2\hat{i}-\hat{j}+\hat{k}$

Position vector of $B = \hat{i}-3\hat{j}-5\hat{k}$

Position vector of $C = 3\hat{i}-4\hat{j}-4\hat{k}$

$$\Rightarrow \overrightarrow{AB} = -\hat{i}-2\hat{j}-6\hat{k};\quad \overrightarrow{AC}=\hat{i}-3\hat{j}-5\hat{k} \quad \text{and} \quad \overrightarrow{BC}=2\hat{i}-\hat{j}+\hat{k}$$

$$\text{Now, } \left|\overrightarrow{AB}\right|^2 = \overrightarrow{AB}\cdot\overrightarrow{AB} = 1+4+36 = 41$$

$$\left|\overrightarrow{AC}\right|^2 = 1 + 9 + 25 = 35$$

$$\left|\overrightarrow{BC}\right|^2 = 4 + 1 + 1 = 6$$

$$\therefore \left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{BC}\right|^2$$

$\Rightarrow$ *A, B, C* are the vertices of right triangle.

$$\text{Now, } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -6 \\ 1 & -3 & -5 \end{vmatrix}$$

$$= \hat{i}(10 - 18) - \hat{j}(5 + 6) + \hat{k}(3 + 2) = -8\hat{i} - 11\hat{j} + 5\hat{k}$$

$$\therefore \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-8)^2 + (-11)^2 + 5^2} = \sqrt{64 + 121 + 25} = \sqrt{210}$$

$$\therefore \quad \text{Area } (\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{\sqrt{210}}{2} \text{ sq. units}$$

Alternate method to find area:

$$\text{Area of } \Delta ABC = \frac{1}{2} \times \left|\overrightarrow{BC}\right| \times \left|\overrightarrow{AC}\right| = \frac{1}{2} \times \sqrt{35} \times \sqrt{6} = \frac{\sqrt{210}}{2} \text{ sq. units}$$

Q.22. Find a unit vector perpendicular to each of the vectors $\vec{a} + 2\vec{b}$ **and** $2\vec{a} + \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Ans.

Given, $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$$\vec{a} + 2\vec{b} = (3\hat{i} + 2\hat{j} + 2\hat{k}) + (2\hat{i} + 4\hat{j} - 4\hat{k}) = 5\hat{i} + 6\hat{j} - 2\hat{k}$$

$$2\vec{a} + \vec{b} = (6\hat{i} + 4\hat{j} + 4\hat{k}) + (\hat{i} + 2\hat{j} - 2\hat{k}) = 7\hat{i} + 6\hat{j} + 2\hat{k}$$

Now, perpendicular vector of $(\vec{a} + 2\vec{b})$ and $(2\vec{a} + \vec{b})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} = (12+12)\hat{i} - (10+14)\hat{j} + (30-42)\hat{k} = 24\hat{i} - 24\hat{j} - 12\hat{k} = 12(2\hat{i} - 2\hat{j} - \hat{k})$$

Required unit vector

$$= \pm \frac{12(2\hat{i} - 2\hat{j} - \hat{k})}{12\sqrt{2^2+(-2)^2+(-1)^2}}$$

$$= \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} = \pm\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$$

Q.23. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find $(\vec{r} \times \hat{i}).(\vec{r} \times \hat{j}) + \text{xy}$.

Ans.

Here, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

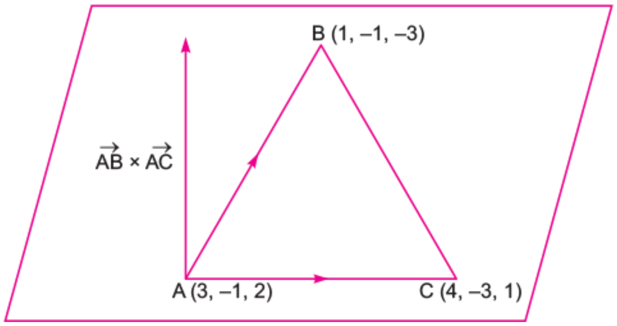
Now, $(\vec{r} \times \hat{i}).(\vec{r} \times \hat{j}) + \text{xy} = \{(x\hat{i} + y\hat{j} + z\hat{k})\times\hat{i}\}.\{(x\hat{i} + y\hat{j} + z\hat{k})\times\hat{j}\} + \text{xy}$

$$= (-y\hat{k} + z\hat{j}).(x\hat{k} - z\hat{i}) + \text{xy}$$
$$= (0\hat{i} + z\hat{j} - y\hat{k}).(-z\hat{i} + 0\hat{j} + x\hat{k}) + \text{xy}$$

$$= 0 + 0 - xy + xy = 0$$

Q.24. Find a unit vector perpendicular to the plane of triangle *ABC*, where the coordinates of its vertices are *A*(3, –1, 2), *B*(1, –1, –3) and *C*(4, –3, 1).

Ans.



Here, $\overrightarrow{AB} = (1-3)\hat{i} + (-1+1)\hat{j} + (-3-2)\hat{k}$

$= -2\hat{i} + 0.\ \hat{j} - 5\hat{k}$

And $\overrightarrow{AC} = (4-3)\hat{i} + (-3+1)\hat{j} + (1-2)\hat{k}$

$= \hat{i} - 2\hat{j} - \hat{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$= (0-10)\ \hat{i} - (2+5)\hat{j} + (4-0)\hat{k} = -10\hat{i} - 7\hat{j} + 4\hat{k}$

Since, $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane of triangle *ABC*.

$\therefore$ Required vector $= \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}$

$= \dfrac{-10\hat{i} - 7\hat{j} + 4\hat{k}}{\sqrt{(-10)^2 + (-7)^2 + 4^2}} = \dfrac{1}{\sqrt{165}}(-10\hat{i} - 7\hat{j} + 4\hat{k})$

$= \dfrac{-10}{\sqrt{165}}\hat{i} - \dfrac{7}{\sqrt{165}}\hat{j} + \dfrac{4}{\sqrt{165}}\hat{k}$

Q.25. Find the area of a parallelogram *ABCD* whose side *AB* and the diagonal *AC* are given by the vectors $3\hat{i} + \hat{j} + 4\hat{k}$ and $4\hat{i} + 5\hat{k}$ **respectively.**

Ans.

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$$

$$= -3\hat{i} - \hat{j} - 4\hat{k} + 4\hat{i} + 5\hat{k} = \hat{i} - \hat{j} + \hat{k}$$

$$\therefore \quad \overrightarrow{AD} = \overrightarrow{BC} = \hat{i} - \hat{j} + \hat{k}$$

$$\therefore \text{ Area of parallelogram } = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \left|(1+4)\hat{i} - (3-4)\hat{j} + (-3-1)\hat{k}\right| = \left|5\hat{i} + \hat{j} - 4\hat{k}\right|$$

$$= \sqrt{5^2 + 1^2 + (-4)^2} = \sqrt{25 + 1 + 16} = \sqrt{42} \text{ sq. units.}$$

Q.26.

If $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$ then express $\vec{b}$ in the from of $\vec{b} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1$ is parallel to $\vec{a}$ and $\vec{b}_2$ is perpendicular to $\vec{a}$.

Ans.

Since $\vec{b}_1 || \vec{a}$

$$\Rightarrow \quad \vec{b}_1 = \lambda\vec{a} = \lambda(2\hat{i} - \hat{j} - 2\hat{k}) = 2\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k}$$

$$\because \quad \vec{b}_1 + \vec{b}_2 = \vec{b} \quad \Rightarrow \quad \vec{b}_2 = \vec{b} - \vec{b}_1$$

$$= (7\hat{i} + 2\hat{j} - 3\hat{k}) - (2\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k})$$

$$= (7 - 2\lambda)\hat{i} + (2 + \lambda)\hat{j} - (3 - 2\lambda)\hat{k}$$

It is given that $\vec{b}_2$ is perpendicular to $\vec{a}$.

$$\Rightarrow \vec{b}_2 . \vec{a} = 0 \qquad \Rightarrow (7 - 2\lambda).2 - (2 + \lambda).1 + (3 - 2\lambda).2 = 0$$

$$\Rightarrow 14 - 4\lambda - 2 - \lambda + 6 - 4\lambda = 0 \qquad \Rightarrow -9\lambda + 18 = 0$$

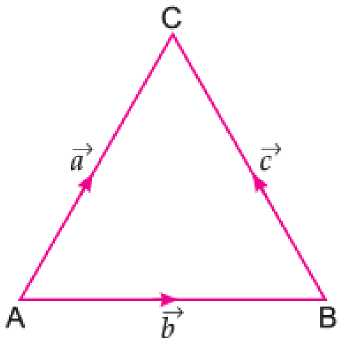
$$\Rightarrow \lambda = \frac{18}{9} = 2$$

Hence, $\vec{b}_1 = 4\hat{i} - 2\hat{j} - 4\hat{k}; \qquad \vec{b}_2 = 3\hat{i} + 4\hat{k} + \hat{k}$

Now, $7\hat{i} + 2\hat{j} - 3\hat{k} = (4\hat{i} - 2\hat{j} - 4\hat{k}) + (3\hat{i} + 4\hat{j} + \hat{k})$, *i.e.*, $\vec{b} = \vec{b}_1 + \vec{b}_2$

Q.27. Given that vectors $\vec{a}, \vec{b}, \vec{c}$ **form a triangle such that** $\vec{a} = \vec{b} + \vec{c}$. **Find *p, q, r, s* such that area of triangle is** $5\sqrt{6}$ **where** $\vec{a} = p\hat{i} + q\hat{j} + r\hat{k}$, $\vec{b} = s\hat{i} + 3\hat{j} + 4\hat{k}$ **and** $\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$.

Ans.



Given, $\vec{a} = \vec{b} + \vec{c}$

$$\Rightarrow p\hat{i} + q\hat{j} + r\hat{k} = (s\hat{i} + 3\hat{j} + 4\hat{k}) + (3\hat{i} + \hat{j} - 2\hat{k})$$

$$\Rightarrow p\hat{i} + q\hat{j} + r\hat{k} = (s + 3)\hat{i} + 4\hat{j} + 2\hat{k}$$

Equating the co-efficient of $\hat{i}, \hat{j}, \hat{k}$ from both sides, we get

$\Rightarrow s + 3 = p \qquad \Rightarrow q = 4$ and $r = 2 \qquad ...(i)$

Now, area of triangle $= \frac{1}{2}\left|\vec{b} \times \vec{c}\right|$

$$\Rightarrow 5\sqrt{6} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ s & 3 & 4 \\ 3 & 1 & -2 \end{vmatrix} = \frac{1}{2}\left|(-6 - 4)\hat{i} - (-2s - 12)\hat{j} + (s - 9)\hat{k}\right|$$

$$\Rightarrow 5\sqrt{6} = \frac{1}{2}\sqrt{10^2 + (2s + 12)^2 + (s - 9)^2}$$

$$\Rightarrow 5\sqrt{6} = \frac{1}{2}\sqrt{100 + 4s^2 + 144 + 48s + s^2 + 81 - 18s}$$

$$\Rightarrow 5\sqrt{6} = \frac{1}{2}\sqrt{325 + 5s^2 + 30s}$$

Squaring both sides

$\Rightarrow 150 = \frac{1}{4}(325 + 5s^2 + 30s)$

$\Rightarrow 600 - 325 = 5s^2 + 30s \quad \Rightarrow \quad 5s^2 + 30s - 275 = 0$

$\Rightarrow s = \frac{-30 \pm \sqrt{900 + 4 \times 5 \times 275}}{10} = \frac{-30 \pm \sqrt{6400}}{10} = \frac{-30 \pm 80}{10}$

$\Rightarrow s = -11, 5 \quad \ldots(ii)$

From (i) and (ii)

$s = -11, 5; \quad p = -8, 8$

$q = 4$ and $r = 2$

Q.28. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ for $\vec{a} - \sqrt{2}\,\vec{b}$ to be a unit vector?

Ans.

Given, $\vec{a} - \sqrt{2}\,\vec{b}$ is an unit vector

$\Rightarrow \left|\vec{a} - \sqrt{2}\,\vec{b}\right| = 1 \quad \Rightarrow \left|\vec{a} - \sqrt{2}\,\vec{b}\right|^2 = 1$

$\Rightarrow (\vec{a} - \sqrt{2}\,\vec{b}).(\vec{a} - \sqrt{2}\,\vec{b}) = 1 \Rightarrow$

$\vec{a}\,.\,\vec{a} - \sqrt{2}\,\vec{a}\,.\,\vec{b} - \sqrt{2}\,\vec{b}\,.\,\vec{a} + 2\,\vec{b}\,.\,\vec{b} = 1$

$$\Rightarrow |a|^2 - 2\sqrt{2}\ \vec{a}.\ \vec{b} + 2|b|^2 = 1 \qquad \Rightarrow [\ \vec{a}\ .\ \vec{b}\ =\ \vec{b}\ .\ \vec{a}]$$

$$\Rightarrow 1 - 2\sqrt{2}\ \vec{a}\ .\ \vec{b}\ +\ 2\ = 1 \qquad [\because\ |\vec{a}| = |\vec{b}| = 1]$$

$$\Rightarrow -2\sqrt{2}\ \vec{a}.\ \vec{b} = -2 \qquad \Rightarrow \quad \vec{a}.\ \vec{b}\ =\ \frac{-2}{-2\sqrt{2}}$$

$$\Rightarrow \vec{a}\ .\ \vec{b} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow |\vec{a}|.|\vec{b}|\cos\theta = \frac{1}{\sqrt{2}} \qquad [\ \vec{a}.\ \vec{b} = |\vec{a}|.|\vec{b}|\cos\theta]$$

$$\Rightarrow 1\ .\ 1\ .\ \cos\theta = \frac{1}{\sqrt{2}} \qquad \Rightarrow \cos\theta\ = \cos\frac{\pi}{4} \qquad \Rightarrow \theta = \frac{\pi}{4}$$

Q.29. Using vectors, find the area of the triangle with vertices *A* (1, 1, 2), *B* (2, 3, 5) and *C* (1, 5, 5).

Ans.

Given, $A \equiv (1, 1, 2); B \equiv (2, 3, 5); C \equiv (1, 5, 5)$

$$\therefore\ \overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k}$$

$$\overrightarrow{AC} = (1-1)\hat{i} + (5-1)\hat{j} + (5-2)\hat{k} = 0.\hat{i} + 4\hat{j} + 3\hat{k}$$

$\therefore$ The area of required triangle $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = \{(6-12)\hat{i} - (3-0)\hat{j} + (4-0)\hat{k}\} = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore\ \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-6)^2 + (-3)^2 + (4)^2} = \sqrt{61}$$

$\therefore$ Required area $= \frac{1}{2}\sqrt{61} = \frac{\sqrt{61}}{2}$ sq units.

Q.30. Show that four points *A, B, C* and *D* whose position vectors are $4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} - \hat{k}, 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ respectively are coplanar.

Ans.

Position vector of $A = 4\hat{i} + 5\hat{j} + \hat{k}$ and Position vector of $B = -\hat{j} - \hat{k}$

Position vector of $C = 3\hat{i} + 9\hat{j} + 4\hat{k}$ and Position vector of $D = 4(-\hat{i} + \hat{j} + \hat{k})$

$\therefore \ \overrightarrow{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k}; \quad \overrightarrow{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}$ and $\overrightarrow{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$

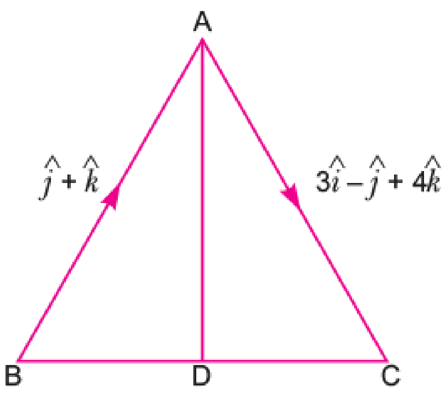
Now, $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12+3) + 6(-3+24) - 2(1+32) = -60 + 126 - 66 = 0$

$\Rightarrow \quad \overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

$\Rightarrow \quad A, B, C, D$ are coplanar.

Q.31. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two side vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively of triangle *ABC*. Find the length of the median through *A*.

Ans.



Here $\overrightarrow{AB} = \hat{j} + \hat{k}$ and $\overrightarrow{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$

$\therefore \quad \overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= \quad -\overrightarrow{AB} + \overrightarrow{AC} = -\hat{j} - \hat{k} + 3\hat{i} - \hat{j} + 4\hat{k} = 3\hat{i} - 2\hat{j} + 3\hat{k}$

$\because \ \overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$

$\Rightarrow \ \overrightarrow{BD} = \frac{1}{2}(3\hat{i} - 2\hat{j} + 3\hat{k})$

$\Rightarrow \ \overrightarrow{BD} = \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}$

Now, $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$= \quad (\hat{j} + \hat{k}) + \left(\frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}\right) \quad \Rightarrow \quad \frac{3}{2}\hat{i} + \frac{5}{2}\hat{k}$

Length of $AD = \left|\overrightarrow{AD}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}$ units .

Q.32. Show that the four points *A*(4, 5, 1), *B*(0, –1, –1), *C*(3, 9, 4) and *D*(– 4, 4, 4) are coplanar.

Ans.

Given four points are $A(4, 5, 1)$, $B(0, -1, -1)$, $C(3, 9, 4)$ and $D(-4, 4, 4)$

Now, $\overrightarrow{AB} = (0 - 4)\hat{i} + (-1 - 5)\hat{j} + (-1 - 1)\hat{k} = -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\overrightarrow{AC} = (3 - 4)\hat{i} + (9 - 5)\hat{j} + (4 - 1)\hat{k} = -\hat{i} + 4\hat{j} + 3\hat{k}$

$\overrightarrow{AD} = (-4 - 4)\hat{i} + (4 - 5)\hat{j} + (4 - 1)\hat{k} = -8\hat{i} - \hat{j} + 3\hat{k}$

$$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$= -4(12 + 3) + 6(-3 + 24) \ -2(1 + 32)$

$= -60 + 126 - 66 = 0$

$\Rightarrow \overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar vectors

$\Rightarrow$ A, B, C and D are coplanar points.

Q.33.

If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then show that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$, it is being given that $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

Ans.

Given, $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{c} \times \vec{d} - \vec{b} \times \vec{d}$ $\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} + \vec{b} \times \vec{d} - \vec{c} \times \vec{d} = \vec{0}$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) + (\vec{b} - \vec{c}) \times \vec{d} = \vec{0}$ [By left and right distributive law]

$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c}) = \vec{0}$ $[\because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}]$

$\Rightarrow (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$ [By right distributive law]

$\Rightarrow (\vec{a} - \vec{d}) \| (\vec{b} - \vec{c})$

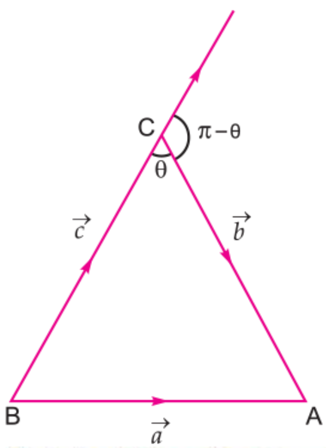
Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. (Cosine formula) If *a, b, c* are the lengths of the opposite sides respectively to the angles *A, B, C* of a triangle *ABC* then show that:

$$\cos\theta = \frac{a^2+b^2-c^2}{2\,\text{ab}}$$

Ans.



By triangle law of vector addition, we have

$$\Rightarrow \quad \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA} \qquad \Rightarrow \qquad \overrightarrow{BC} + \overrightarrow{CA} = -\overrightarrow{AB}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = (-\vec{c}).(-\vec{c})$$

$$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = \vec{c}.\vec{c}$$

$$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = \vec{c}.\vec{c}$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$$

$$i.e., |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos(\pi - \theta) = |\vec{c}|^2$$

$$\Rightarrow a^2 + b^2 - 2ab \cos q = c^2$$

$$\Rightarrow 2ab\cos\theta = a^2 + b^2 - c^2 \qquad \Rightarrow \quad \cos\theta = \frac{a^2+b^2-c^2}{2\,\text{ab}}$$

Q.2. If the vectors $a\hat{i}+a\hat{j}+c\hat{k},\ \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ **are coplanar, then show that** $c^2 = ab$**.**

Ans.

Let $\vec{P} = a\hat{i}+a\hat{j}+c\hat{k}, \vec{Q} = \hat{i}+\hat{k}$ and $\vec{R} = c\hat{i}+c\hat{j}+b\hat{k}$

Since $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are coplanar vectors, therefore,

$$[\vec{P}\ \vec{Q}\ \vec{R}] = 0 \Rightarrow \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

$$\Rightarrow a(0-c) - a(b-c) + c(c-0) = 0$$

$$\Rightarrow -ac - ab + ac + c^2 = 0 \quad \Rightarrow \quad c^2 = ab$$

Q.3.

If $\vec{a}+\vec{b}+\vec{c}=0$ **, then prove that** $(\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c}) = (\vec{c}\times\vec{a})$

Ans.

We have, $\vec{a}+\vec{b}+\vec{c}=0$

$$\Rightarrow \vec{a}+\vec{b} = -\vec{c} \quad \Rightarrow \quad (\vec{a}+\vec{b})\times\vec{b} = (-\vec{c})\times\vec{b}$$

$$\Rightarrow (\vec{a}\times\vec{b})+(\vec{b}\times\vec{b}) = (-\vec{c})\times\vec{b} \quad \text{[By the distributive law]}$$

$$\Rightarrow (\vec{a}\times\vec{b})+0 = (\vec{b}\times\vec{c}) \quad [\because \vec{b}\times\vec{b}=0 \text{ and } (-\vec{c})\times\vec{b} = \vec{b}\times\vec{c}]$$

$$\Rightarrow (\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c}) \quad \ldots(i)$$

Also, $\vec{a}+\vec{b}+\vec{c}=0 \quad \Rightarrow \quad \vec{b}+\vec{c} = -\vec{a}$

$$\Rightarrow (\vec{b}+\vec{c})\times\vec{c} = (-\vec{a})\times\vec{c}$$

$$\Rightarrow (\vec{b}\times\vec{c})+(\vec{c}\times\vec{c}) = (-\vec{a})\times\vec{c} \quad \text{[By the distributive law]}$$

$\Rightarrow (\vec{b} \times \vec{c}) + 0 = (\vec{c} \times \vec{a}) \qquad [\because \vec{c} \times \vec{c} = 0 \text{ and } (-\vec{a}) \times \vec{c} = \vec{c} \times \vec{a}]$

$\Rightarrow (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a}) \qquad ...(ii)$

From (i) and (ii), we get $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})$.

Q.4. Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i} + \hat{k}$ and the other is perpendicular to $\vec{b}$.

Ans.

Let $\vec{a} = \vec{c} + \vec{d}$ such that $\vec{c}$ is parallel to $\vec{b}$ and $\vec{d}$ is perpendicular to $\vec{b}$. $\qquad ...(i)$

Now, $\vec{c} = \lambda\vec{b} = 3\lambda\hat{i} + \lambda\hat{k}$

Also $\vec{d} = \vec{a} - \vec{c}$

$\Rightarrow \vec{d} = (5\hat{i} - 2\hat{j} + 5\hat{k}) - (3\lambda\hat{i} + \lambda\hat{k})$

$\Rightarrow \vec{d} = (5 - 3\lambda)\hat{i} - 2\hat{j} + (5 - \lambda)\hat{k}$

Again, $\because \vec{d}$ is perpendicular to $\vec{b}$.

$\Rightarrow \vec{d} \cdot \vec{b} = 0$

$\Rightarrow (5 - 3\lambda).\,3 + (5 - \lambda).1 = 0 \quad \Rightarrow \quad 15 - 9\lambda + 5 - \lambda = 0$

$\Rightarrow -10\lambda + 20 = 0 \qquad \Rightarrow \qquad \lambda = 2$

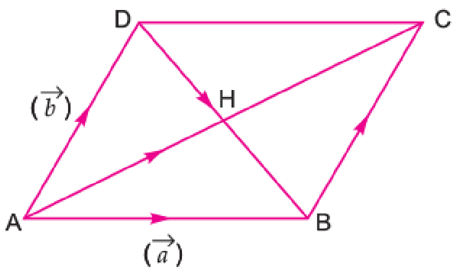
Hence, $\vec{c} = 6\hat{i} + 2\hat{k}$ and

$\vec{d} = -\hat{i} - 2\hat{j} + 3\hat{k} \qquad ...(ii)$

$\therefore \quad \vec{a} = (6\hat{i} + 2\hat{k}) + (-\hat{i} - 2\hat{j} + 3\hat{k}) \qquad$ [From (i) and (ii)]

Q.5. Prove by vector method that the diagonals of a parallelogram bisect each other.

Ans.



Let A be at origin and $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$

Again, let AC and BD intersect each other at H.

We have to prove that H is middle point of AC and BD.

Let $\overrightarrow{AH} = x\overrightarrow{AC}$ and $\overrightarrow{HB} = y\overrightarrow{DB}$ ... (i)

Now, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + \vec{b}$

Also, $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{AB} - \overrightarrow{AD} = \vec{a} - \vec{b}$

From (i) $\overrightarrow{AH} = x(\vec{a} + \vec{b})$ and $\overrightarrow{HB} = y(\vec{a} - \vec{b})$

Now, $\overrightarrow{AB} = \overrightarrow{AH} + \overrightarrow{HB}$

$\Rightarrow \vec{a} = x(\vec{a} + \vec{b}) + y(\vec{a} - \vec{b})$ $\Rightarrow \vec{a} = x\vec{a} + x\vec{b} + y\vec{a} - y\vec{b}$

$\Rightarrow \vec{a} = (x+y)\vec{a} + (x-y)\vec{b}$

Equating the co-efficient of $\vec{a}$ and $\vec{b}$, we get

$x + y = 1$ and $x - y = 0$

$2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$

$(i) \Rightarrow \overrightarrow{AH} = \frac{1}{2}\overrightarrow{AC}$ and $\overrightarrow{HB} = \frac{1}{2}\overrightarrow{DB}$

Hence, H is middle point of AC and BD or diagonals of parallelogram bisect each other.

Q.6. Prove that : $\left|\vec{a} \times \vec{b}\right|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$.

Ans.

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

LHS $\left|\vec{a} \times \vec{b}\right|^2 = (\vec{a} \times \vec{b}).(\vec{a} \times \vec{b})$

$= (\text{ab} \sin\theta)\hat{n}.(\text{ab} \sin\theta)\hat{n} = (a^2b^2 \sin^2\theta)(\hat{n}.\hat{n}) = a^2b^2 \sin^2\theta$

$= a^2b^2(1-\cos^2\theta) = a^2b^2 - (\text{ab} \cos\theta)^2$

$= (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - \left(\vec{a}.\vec{b}\right)^2$ ...(*i*)

Also, RHS = $\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix} = (\vec{a}.\vec{a}).(\vec{b}.\vec{b}) - (\vec{a}.\vec{b}).(\vec{a}.\vec{b})$

$= (\vec{a}.\vec{a}).(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2$ ...(*ii*)

(*i*) and (*ii*) RHS = LHS [Hence proved]

Q.7. If $\vec{a}, \vec{b}$ are unit vectors such that the vector $\vec{a} + 3\vec{b}$ is perpendicular to $7\vec{a} - 5\vec{b}$ and $\vec{a} - 4\vec{b}$ is perpendicular to $7\vec{a} - 2\vec{b}$, then find the angle between $\vec{a}$ and $\vec{b}$.

Ans.

Let angle between $\vec{a}$ and $\vec{b}$ be θ

Given, $(\vec{a}+3\vec{b}) \perp (7\vec{a}-5\vec{b})$ $\Rightarrow (\vec{a}+3\vec{b}).(7\vec{a}-5\vec{b})=0$

$\Rightarrow 7|\vec{a}|^2+16(\vec{a}.\vec{b})-15|\vec{b}|^2=0$

$\Rightarrow 7+16\cos\theta-15=0$ $[\because |\vec{a}|^2=|\vec{b}|^2=1]$

$\Rightarrow \cos\theta=\frac{8}{16}=\frac{1}{2}$ $\Rightarrow \theta=\frac{\pi}{3}$

Also, given that $(\vec{a}-4\vec{b}) \perp (7\vec{a}-2\vec{b})$

$\Rightarrow (\vec{a}-4\vec{b}).(7\vec{a}-2\vec{b})=0$ $\Rightarrow 7|\vec{a}|^2+8|\vec{b}|^2-30(\vec{a}.\vec{b})=0$

$\Rightarrow 15-30\cos\theta=0$

$\Rightarrow \cos\theta=\frac{1}{2}$ $\Rightarrow \theta=\frac{\pi}{3}$

Q.8. If the vector $-\hat{i}+\hat{j}-\hat{k}$ bisects the angle between the vector $\vec{c}$ and the vector $3\hat{i}+4\hat{j}$, then find the unit vector in the direction of $\vec{c}$.

Ans.

Let $x\hat{i}+y\hat{j}+z\hat{k}$ be the unit vector along $\vec{c}$. Since $-\hat{i}+\hat{j}-\hat{k}$ bisects the angle between $\vec{c}$ and $3\hat{i}+4\hat{j}$.

Therefore,

$\lambda(-\hat{i}+\hat{j}-\hat{k})=(x\hat{i}+y\hat{j}+z\hat{k})+\frac{3\hat{i}+4\hat{j}}{5}$

$\Rightarrow x+\frac{3}{5}=-\lambda,\ y+\frac{4}{5}=\lambda$ and $z=-\lambda$

Now, $x^2+y^2+z^2=1$ $[\because x\hat{i}+y\hat{j}+z\hat{k}$ is a unit vector$]$

$\Rightarrow \left(-\lambda-\frac{3}{5}\right)^2+\left(\lambda-\frac{4}{5}\right)^2+\lambda^2=1$ $\Rightarrow 3\lambda^2-\frac{2}{5}\lambda=0$

$\Rightarrow \lambda=0$ or $\lambda=\frac{2}{15}$

But $\lambda \neq 0$, because $\lambda=0$ implies that the given vectors are parallel.

$\therefore \lambda=\frac{2}{15}$ $\Rightarrow x=-\frac{11}{15},\ y=\frac{-10}{15}$ and $z=\frac{-2}{15}$

Hence, $x\hat{i}+y\hat{j}+z\hat{k}=-\frac{1}{15}(11\hat{i}+10\hat{j}+2\hat{k})$