

Relations and Functions

Types of Relations

Equivalence Relation

- A relation R in a set A is called reflexive if $(a, a) \in R$ for every $a \in A$.

- For example: A relation R in set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b; a, b \in A\}$ is a reflexive relation since $\sin a = \sin a \forall a \in A$.

- A relation R in a set A is called symmetric if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

- For example: A relation in the set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b; a, b \in A\}$ is a symmetric relation. Since for $a, b \in A$, $\sin a = \sin b$ implies $\sin b = \sin a$. So, $(a, b) \in R \Rightarrow (b, a) \in R$.

- A relation R in a set A is called transitive if $(a_1, a_2) \in R$, and $(a_2, a_3) \in R$ together imply that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

- For example: A relation in the set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b; a, b \in A\}$ is a transitive relation. Since for $a, b, c \in A$, let $(a, b), (b, c) \in R$.

$$\Rightarrow \sin a = \sin b \text{ and } \sin b = \sin c$$

$$\Rightarrow \sin a = \sin c$$

$$\Rightarrow (a, c) \in R$$

- A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

- For example: Relation R in the set $A = \left[0, \frac{\pi}{2}\right]$ defined by $R = \{\sin a = \sin b; a, b \in A\}$ is an equivalence relation.

Equivalence Classes

- Every arbitrary equivalence relation R in a set X divides X into mutually disjoint subsets (A_i) called partitions or subdivisions of X satisfying the following conditions:

- All elements of A_i are related to each other for all i .
- No element of A_i is related to any element of A_j whenever $i \neq j$.
- $A_i \cup A_j = X$ and $A_i \cap A_j = \Phi, i \neq j$

These subsets (A_i) are called equivalence classes.

- For an equivalence relation in a set X , the equivalence class containing $a \in X$, denoted by $[a]$, is the subset of X containing all elements b related to a .

Trivial Relations

- Trivial relations are of two types:
- Empty relation
- Universal relation
- A relation in a set A is called an empty relation if no element of A is related to any element of A , i.e., $R = \Phi \subset A \times A$.
- For example: Consider a relation R in set $A = \{2, 4, 6\}$ defined by $R = \{(a, b): a + b \text{ is odd, where } a, b \in A\}$. The relation R is an empty relation since for any pair $(a, b) \in A \times A, a + b$ is always even.
- A relation R in a set A is called a universal relation if each element of A is related to every element of A , i.e., $R = A \times A$.
- For example: Let A be the set of all students of class XI. Let R be a relation in set A defined by $R = \{(a, b): \text{the sum of the ages of } a \text{ and } b \text{ is greater than 10 years}\}$. The relation R is a universal relation because it is obvious that the sum of the ages of two students of class XI is always greater than 10 years.

Solved Examples

Example 1

Check whether the relation R in the set of all vowels defined by $R = \{(u, u), (u, a), (a, u)\}$ is reflexive, symmetric or transitive?

Solution:

The relation R is defined in the set $\{a, e, i, o, u\}$ as $R = \{(u, u), (u, a), (a, u)\}$.

The relation R is not reflexive as $(a, a), (e, e), (i, i), (o, o) \notin R$.

Now, (u, a) and $(a, u) \in R$

Hence, R is symmetric.

Now, $(u, u), (u, a) \in R$ implies $(u, a) \in R$

Also, $(u, a), (a, u) \in R$ implies $(u, u) \in R$

Hence, R is transitive.

Thus, the relation R is symmetric and transitive but not reflexive.

Example 2

Show that the relation R defined in the set of real numbers as $R = \{(a, b) : a = b \text{ or } a = -b \text{ for } a, b \in \mathbf{R}\}$ is an equivalence relation. Also, find its equivalence classes.

Solution:

A relation R in \mathbf{R} is defined as $R = \{(a, b) : a = b \text{ or } a = -b, \text{ for } a, b \in \mathbf{R}\}$

Clearly, $(a, a) \in R$ for every $a \in \mathbf{R}$, since $a = a$.

$\therefore R$ is reflexive.

Now, let $(a, b) \in R$ for $a, b \in \mathbf{R}$

$$\Rightarrow a = b \text{ or } a = -b$$

$$\Rightarrow b = a \text{ or } b = -a$$

$$\Rightarrow (b, a) \in R$$

$\therefore R$ is symmetric.

Now, let $(a, b), (b, c) \in R$, for $a, b, c \in \mathbf{R}$

$$\therefore a = b \text{ or } a = -b \text{ and } b = c \text{ or } b = -c$$

Case I

$$a = b, b = c$$

$$\Rightarrow a = c$$

$$\Rightarrow (a, c) \in R$$

Case II

$$a = b, b = -c$$

$$\Rightarrow a = -c$$

$$\Rightarrow (a, c) \in R$$

Case III

$$a = -b, b = c$$

$$\Rightarrow a = -c$$

$$\Rightarrow (a, c) \in R$$

Case IV

$$a = -b, b = -c$$

$$\Rightarrow a = c$$

$$\Rightarrow (a, c) \in R$$

Thus, $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

Equivalence class of $0 = [0] = \{0\}$

Equivalence class of $1 = [1] = \{1, -1\}$

Equivalence class of $2 = [2] = \{2, -2\}$ and so on. . .

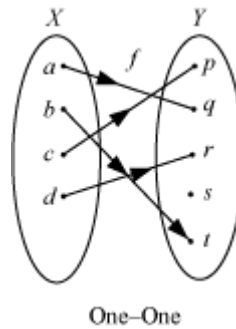
There are an infinite number of equivalence classes.

For every $a \in \mathbf{R}$, $[a] = \{a, -a\}$.

Types of Functions

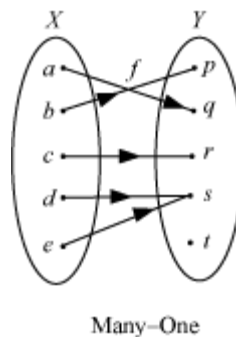
- A function $f: X \rightarrow Y$ is said to be **one-one** (or **injective**) if the images of distinct elements of X under f are distinct. In other words, a function f is one-one if for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

An example of a one-one function from X to Y is shown in the following diagram.



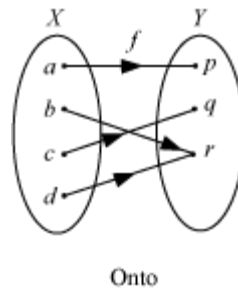
A function $f: X \rightarrow Y$ is said to be **many-one** if the image of distinct elements of X under f are not distinct i.e., a function that is not one-one is called a many-one function.

An example of a many-one function from X to Y is shown in the following diagram.

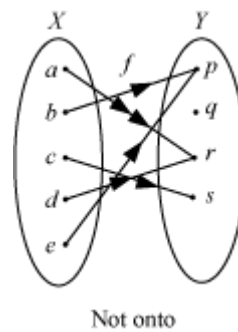


In this case, two elements $f(d) = f(e) = s$.

- A function $f: X \rightarrow Y$ is defined as **onto** (or **surjective**) if every element of Y is the image of some element of x in X under f . In other words, f is onto if and only if, $y \in Y$, there exist $x \in X$ such that $f(x) = y$.
- $f: X \rightarrow Y$ is onto if and only if the range of $f = Y$.
- An example of an onto function from X to Y is shown in the following diagram.

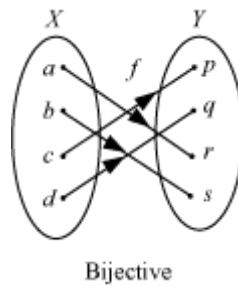


-
- A function from X to Y that is not onto is shown in the following diagram.



- A function $f: X \rightarrow Y$ is said to be **bijective** if it is both one-one and onto.

A bijective function from X to Y is shown in the following diagram.



Solved Examples

Example 1

Check whether the function $h: \mathbf{R} \rightarrow \mathbf{R}$ defined by $h(x) = |2x - 5|$ is an injective function.

Solution:

The given function i.e., $h: \mathbf{R} \rightarrow \mathbf{R}$ is defined by

$$h(x) = |2x - 5|$$

It can be observed that $5, 0 \in \mathbf{R}$ (considering domain). Hence, we have

$$h(5) = |2 \times 5 - 5| = |10 - 5| = 5$$

$$h(0) = |2 \times 0 - 5| = |0 - 5| = 5$$

$$\therefore h(5) = h(0).$$

Hence, the given function i.e., $h(x)$ is not an injective function.

Example 2

Check whether the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = x^5 + 4$ is a bijective function.

Solution:

We know that a function is bijective if it is both one-one and onto.

Now, let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$. Accordingly,

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1^5 + 4 = x_2^5 + 4$$

$$\Rightarrow x_1^5 = x_2^5$$

$$\Rightarrow x_1 = x_2$$

Therefore, the function f is a one-one function.

It is clear that for every $y \in \mathbf{R}$, there exists $(y-4)^{\frac{1}{5}} \in \mathbf{R}$ such

$$\text{that } f\left[(y-4)^{\frac{1}{5}}\right] = \left[(y-4)^{\frac{1}{5}}\right]^5 + 4 = (y-4) + 4 = y$$

Therefore, the function f is an onto function.

Hence, the given function f is a bijective function.

Example 3

Check whether the function $f: \mathbf{N} \rightarrow \mathbf{N}$ defined by $f(x) = 4^x$ is an onto function.

Solution:

The given function $f: \mathbf{N} \rightarrow \mathbf{N}$ is defined by

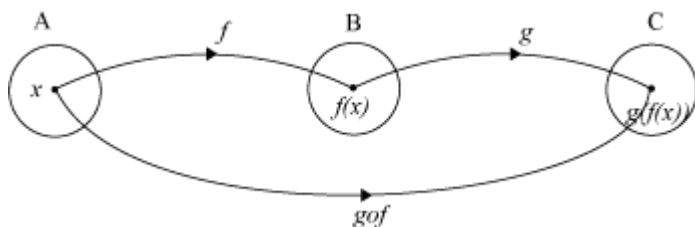
$$f(x) = 4^x$$

We can clearly observe that $2 \in \mathbf{N}$ (co-domain). However, there does not exist any element $y \in \mathbf{N}$ (domain) whose image is 2.

Hence, the given function f is not an onto function.

Composition of Two functions

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions. Accordingly, the composition of f and g is denoted by gof and is defined as the function $gof: A \rightarrow C$ given by $gof(x) = g(f(x))$, for all $x \in A$.



- For example: If $f: \mathbf{N} \rightarrow \mathbf{N}$ is defined by $f(x) = x + 1$ for all $x \in \mathbf{N}$ and $g: \mathbf{N} \rightarrow \mathbf{N}$ is defined by $g(x) = x^2$ for all $x \in \mathbf{N}$, then $gof: \mathbf{N} \rightarrow \mathbf{N}$ is given by

$$gof(x) = g(f(x)) = g(x + 1) = (x + 1)^2, \text{ where } x \in \mathbf{N}.$$

$$\text{Also, } fog(x) = f(g(x)) = f(x^2) = x^2 + 1 \text{ for all } x \in \mathbf{N}$$

- If $f: A \rightarrow B$ and $g: B \rightarrow C$ are one-one, then $gof: A \rightarrow C$ is also one-one.
- If $f: A \rightarrow B$ and $g: B \rightarrow C$ are onto, then $gof: A \rightarrow C$ is also onto.
- If the composite function gof is one-one, then the function f is also one-one. However, the function g may or may not be one-one.
- If the composite function gof is onto, then the function g is also onto. However, the function f may or may not be onto.

Solved Examples

Example 1

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = 12x^2 - x - 11$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ be given by $g(x) = x^2$. Find $fo(gog)$.

Solution:

It is given that

$f: \mathbf{R} \rightarrow \mathbf{R}$ is defined by $f(x) = 12x^2 - x - 11$

$g: \mathbf{R} \rightarrow \mathbf{R}$ is defined by $g(x) = x^2$

Now, $(gog)(x) = g(g(x))$

$$= g(x^2)$$

$$= (x^2)^2$$

$$= x^4$$

$$(fo(gog))(x) = f(gog)(x)$$

$$= f(x^4)$$

$$= 12(x^4)^2 - x^4 - 11$$

$$= 12x^8 - x^4 - 11$$

Example 2

Let $f: \mathbf{R} - \left\{ \frac{11}{3} \right\} \rightarrow \mathbf{R} - \left\{ -\frac{4}{3} \right\}$ be defined by $f(x) = \frac{4x-5}{11-3x}$ and $g: \mathbf{R} - \left\{ -\frac{4}{3} \right\} \rightarrow \mathbf{R} - \left\{ \frac{11}{3} \right\}$ be defined by $g(x) = \frac{11x+5}{4+3x}$. Show that $fog = I_A$ and $gof = I_B$, where I_A and I_B are identity functions on A and B respectively and $A = \mathbf{R} - \left\{ -\frac{4}{3} \right\}$ and $B = \mathbf{R} - \left\{ \frac{11}{3} \right\}$.

Solution:

$$(fog)(x) = f(g(x))$$

$$= f\left(\frac{11x+5}{4+3x}\right)$$

$$\begin{aligned}
&= \frac{4\left(\frac{11x+5}{4+3x}\right) - 5}{11 - 3\left(\frac{11x+5}{4+3x}\right)} \\
&= \frac{44x + 20 - 20 - 15x}{44 + 33x - 33x - 15} \\
&= \frac{29x}{29} \\
&= x
\end{aligned}$$

$$(gof)(x) = g(f(x))$$

$$\begin{aligned}
&= g\left(\frac{4x-5}{11-3x}\right) \\
&= \frac{11\left(\frac{4x-5}{11-3x}\right) + 5}{4 + 3\left(\frac{4x-5}{11-3x}\right)} \\
&= \frac{44x - 55 + 55 - 15x}{44 - 12x + 12x - 15} \\
&= \frac{29x}{29} \\
&= x
\end{aligned}$$

Thus, $(fog)(x) = x$ for all $x \in A \Rightarrow fog = I_A$ and $(gof)(x) = x$ for all $x \in B \Rightarrow gof = I_B$.

Hence proved.

Example 3

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined as $f(x) = 2x^{\frac{1}{3}}$; $g: \mathbf{R} \rightarrow \mathbf{R}$ be defined as $g(x) = x + 2$ and $h: \mathbf{R} \rightarrow \mathbf{R}$ be defined as $h(x) = 4x + 9$. Find $fo(g + h)$ and $(fog) + (foh)$.

Solution:

$(g + h): \mathbf{R} \rightarrow \mathbf{R}$ is given by:

$$(g + h)(x) = g(x) + h(x)$$

$$\begin{aligned}
&= (x + 2) + (4x + 9) \\
&= 5x + 11
\end{aligned}$$

$$\therefore fo(g + h)(x) = f((g + h)(x))$$

$$= f(5x + 11)$$

$$= 2(5x + 11)^{\frac{1}{3}}$$

$$\text{Now, } (fog)(x) = f(g(x))$$

$$= f(x + 2)$$

$$= 2(x + 2)^{\frac{1}{3}}$$

$$(foh)(x) = f(h(x))$$

$$= f(4x + 9)$$

$$= 2(4x + 9)^{\frac{1}{3}}$$

$$\therefore (fog + foh)(x) = (fog)(x) + (foh)(x)$$

$$= 2(x + 2)^{\frac{1}{3}} + 2(4x + 9)^{\frac{1}{3}}$$

$$= 2 \left[(x + 2)^{\frac{1}{3}} + (4x + 9)^{\frac{1}{3}} \right]$$

Invertible Functions

Key Concepts

- A function $f: X \rightarrow Y$ is said to be invertible if there exists a function $g: Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$.
- The function g is called the inverse of f and it is denoted by f^{-1} .
- A function f is invertible if and only if f is one-one and onto.
- If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are invertible functions, then gof is also invertible and $(gof)^{-1} = f^{-1}og^{-1}$

Solved Examples

Example 1

Determine whether the following functions have inverse or not. Find the inverse, if it exists.

(i) $f: \{10, 12, 15\} \rightarrow \{3, 7, 9, 10, 14\}$ is defined as $f = \{(12, 9), (15, 7), (10, 10)\}$.

(ii) $g: \{2, 4, 6, 8\} \rightarrow \{1, 3, 5\}$ is defined as $g = \{(4, 3), (8, 3), (2, 1), (6, 5)\}$.

(iii) $h: \{11, 16\} \rightarrow \{7, 14\}$ is defined as $h = \{(11, 7), (16, 14)\}$.

Solution:

(i) The given function f is one-one. However, f is not onto since the elements $3, 14 \in \{3, 7, 9, 10, 14\}$ are not the image of any element in $\{10, 12, 15\}$ under f .

Hence, function f is not invertible.

(ii) The given function g is onto. However, g is not one-one since, $g(4) = g(8) = 3$.

Hence, the function g is not invertible.

(iii) Clearly, the given function h is both one-one and onto. Hence, h is invertible.

The inverse of h is given by $h^{-1} = \{(7, 11), (14, 16)\}$.

Example 2

Determine whether the functions f and g , defined below, are inverses of each other or not.

$f: \mathbf{R} - \{4\} \rightarrow \mathbf{R} - \{-3\}$ is given as $f(x) = \frac{-3x}{x-4}$, and

$g: \mathbf{R} - \{-3\} \rightarrow \mathbf{R} - \{4\}$ is given as $g(x) = \frac{4x}{x+3}$

Solution:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{4x}{x+3}\right) = \frac{-3\left(\frac{4x}{x+3}\right)}{\frac{4x}{x+3} - 4} = \frac{-12x}{4x - 4x - 12} = \frac{-12x}{-12} = x$$

We have

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{-3x}{x-4}\right) = \frac{4\left(\frac{-3x}{x-4}\right)}{\left(\frac{-3x}{x-4}\right) + 3} = \frac{-12x}{-3x + 3x - 12} = \frac{-12x}{-12} = x$$

Thus, $(f \circ g)(x) = x \forall x \in B$, where $B = \mathbf{R} - \{-3\}$ and $(g \circ f)(x) = x \forall x \in A$, $A = \mathbf{R} - \{4\}$.

$\therefore g \circ f = I_A$ and $f \circ g = I_B$.

Thus, functions f and g are the inverses of each other.

Example 3

Let $f: \mathbf{R}_+ \rightarrow [-3, \infty)$ be defined as $f(x) = 4x^2 - 5x - 3$ where \mathbf{R}_+ is the set of all positive real numbers. Show that f is invertible and find the inverse of f .

Solution:

$f: \mathbf{R}_+ \rightarrow [-3, \infty)$ is defined as $f(x) = 4x^2 - 5x - 3$.

Let y be an arbitrary element of $[-3, \infty)$.

Let $y = 4x^2 - 5x - 3$

$$\Rightarrow y = \left(2x - \frac{5}{4}\right)^2 - 3 - \frac{25}{16} = \left(2x - \frac{5}{4}\right)^2 - \frac{73}{16}$$

$$\Rightarrow \left(2x - \frac{5}{4}\right)^2 = y + \frac{73}{16}$$

$$\Rightarrow 2x - \frac{5}{4} = \sqrt{y + \frac{73}{16}} \quad \left[y \geq -3 \Rightarrow y + \frac{73}{16} \geq 0 \right]$$

$$x = \frac{1}{2} \sqrt{y + \frac{73}{16}} + \frac{5}{8}$$

$\therefore f$ is onto.

Hence, $\text{Range } f = [-3, \infty)$.

Let us define $g: [-3, \infty) \rightarrow \mathbf{R}_+$ as

$$g(y) = \frac{1}{2} \sqrt{y + \frac{73}{16}} + \frac{5}{8}$$

Now, we have

$$(g \circ f)(x) = g(f(x)) = g(4x^2 - 5x - 3) = g\left(\left(2x - \frac{5}{4}\right)^2 - \frac{73}{16}\right) = \frac{1}{2}\sqrt{\left(2x - \frac{5}{4}\right)^2} + \frac{5}{8} = x - \frac{5}{8} + \frac{5}{8} = x$$

$$(f \circ g)(y) = f(g(y)) = f\left(\left(\frac{1}{2}\sqrt{y + \frac{73}{16}}\right) + \frac{5}{8}\right) = \left[2 \times \left(\left(\frac{1}{2}\sqrt{y + \frac{73}{16}}\right) + \frac{5}{8}\right) - \frac{5}{4}\right]^2 - \frac{73}{16} = y + \frac{73}{16} - \frac{73}{16} = y$$

$$\therefore g \circ f = I_{\mathbb{R}_+} \text{ and } f \circ g = I_{[-3, \infty)}$$

Thus, f is invertible and its inverse is given by

$$f^{-1}(y) = g(y) = \frac{1}{2}\left(\sqrt{y + \frac{73}{16}}\right) + \frac{5}{8}$$

Binary Operations

Definition of Binary Operation Properties

- A binary operation $*$ on a set A is a function $*$ from $A \times A \rightarrow A$. We denote $*(a, b)$ by $a * b$.

For example, the operation $*$ defined on \mathbf{N} as $a * b = a^2b$ is a binary operation since $*$ carries each pair (a, b) to a unique element a^2b in \mathbf{N} .

Properties of Binary Operation

- A binary operation $*$ on a set A is called commutative, if $a * b = b * a$, for every $a, b \in A$.

For example, $*, \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ defined by $a * b = 11(a + b + ab)$ is commutative since $a * b = 11(a + b + ab)$ and $b * a = 11(b + a + ba)$. Therefore, $a * b = b * a$.

- A binary operation $*$ on a set A is called associative, if $(a * b) * c = a * (b * c)$, for every $a, b, c \in A$.

For example, $*, \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ defined by $a * b = 5 + a + b$ is associative.

$$a * (b * c) = 10 + a + b + c = 5 + (5 + a + b) + c = ((a * b) * c)$$

- For a binary operation $*, A \times A \rightarrow A$, an element $e \in A$, if it exists, is called its identity element, if $a * e = a = e * a$, for every $a \in A$.

For example: 1 is the identity for multiplication on \mathbf{R} .

- Given a binary operation $*$: $A \times A \rightarrow A$ with the identity element e in A , an element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element $b \in A$, such that $a * b = e = b * a$, and b is called the inverse of a and is denoted by a^{-1} .

For example: $-a$ is the inverse of a for the addition operation on \mathbf{R} , where 0 is the identity element.

Binary Operation Table

- When the number of elements in set A is small, we can express a binary operation $*$ on A through a table called operation table.
- For an operation $*$: $A \times A \rightarrow A$, if $A = \{a_1, a_2, \dots, a_n\}$, then the operation table will have n rows and n columns with $(i, j)^{\text{th}}$ entry being $a_i * a_j$.
- Given any operation with n rows and n columns with each entry being an element of $A = \{a_1, a_2, \dots, a_n\}$, we can define a binary operation $*$ on A given by $a_i * a_j = \text{entry in } i^{\text{th}} \text{ row and } j^{\text{th}} \text{ column of the operation table}$

Example: We can define a binary operation $*$ on $A = \{a, b, c\}$ as follows:

$*$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

Here, $a * b = b = b * a$

$a * c = c = c * a$

$b * c = c = c * b$

\therefore The operation $*$ is commutative.

Solved Examples

Example 1:

A binary operation $A \times A \rightarrow A$, where $A = \{a, b, c\}$, is defined as follows:

*	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

Determine whether the operation $*$ is commutative and associative. Also, find the identity for the operation $*$, if it exists.

Solution:

From the table, it can be observed that

$$a * b = a = b * a$$

$$a * c = a = c * a$$

$$b * c = c = c * b$$

The given binary operation $*$ is commutative since for all $x, y, \in A = \{a, b, c\}$.

$$x * y = y * x$$

$$\text{Now, consider } a * (b * c) = a * c = a$$

$$(a * b) * c = a * c = a$$

$$\text{Thus, } a * (b * c) = (a * b) * c$$

Similarly, we can prove that $(x * y) * z = x * (y * z)$ for all $x, y, z \in A$.

Thus, the given binary operation $*$ is associative.

Also, we can observe that for any element $x \in A$, we have $x * a = x = a * x$.

Thus, a is the identity element for the given binary operation $*$.

Example 2:

Determine whether the binary operation on the set \mathbf{R} , defined by $a * b = \frac{2a+5b}{4}$, $a, b \in \mathbf{R}$, is commutative or not.

Solution:

We have $*$: $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ defined by $a * b = \frac{2a+5b}{4}$, $a, b \in \mathbf{R}$.

We know that a binary operation $*$ defined on set A is commutative, if $a * b = b * a$ $\forall a, b \in A$.

Now, $a * b = \frac{2a+5b}{4}$ and $b * a = \frac{2b+5a}{4}$

$\therefore a * b \neq b * a$

Hence, the given binary operation $*$ is not commutative.

Example 3:

A binary operation $*$ on the set $\{5, 6, 9\}$ is defined by the following table:

$*$	5	6	9
5	5	6	9
6	6	9	5
9	9	5	6

Compute $(5 * 9) * 6$ and $5 * (9 * 6)$. Are they equal?

Solution:

From the given binary operation table, we have $(5 * 9) = 9$

$$\therefore (5 * 9) * 6 = 9 * 6 = 5$$

Then, $(9 * 6) = 5$

$$\therefore 5 * (9 * 6) = 5 * 5 = 5$$

Thus, $(5 * 9) * 6 = 5 * (9 * 6)$