1. Limits

Limits (An Introduction)

Limit of a function

Let y = f(x) be a function of x. If at x = a, f(x) takes indeterminate form, then we consider the values of the function which are very near to 'a'. If these values tend to a definite unique number as x tends to 'a', then the unique number so obtained is called the limit of f(x) at x = a and we write it as

$\lim_{x \to a} f(x)$.

Left hand and right hand limit

Consider the values of the functions at the points which are very near to a on the left of a. If these values tend to a definite unique number as x tends to a, then the unique number so obtained is called left-hand limit of f(x) at x = a and symbolically we write it as

$$f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h).$$

Similarly we can define right-hand limit of f(x) at x = a which is expressed as

$$f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h).$$

Method for finding L.H.L. and R.H.L.

(i) For finding right hand limit (R.H.L.) of the function, we write x + h in place of x, while for left hand limit (L.H.L.) we write x – h in place of x.
(ii) Then we replace x by 'a' in the function so obtained.
(iii) Lastly we find limit h → 0.

Existence of limit

$\lim_{x \to a} f(x)$ exists when,

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.* L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.* L.H.L. = R.H.L.

Fundamental theorems on limits

The following theorems are very useful for evaluation of limits if $\lim_{x \to 0} f(x) = l$ and $\lim_{x \to 0} g(x) = m$ (l and m are real numbers) then

(1) $\lim_{x \to a}(f(x) + g(x)) = l + m$ (Sum rule)

(2) $\lim_{x \to a}(f(x) - g(x)) = l - m$ (Difference rule)

(3) $\lim_{x \to a}(f(x).g(x)) = l.m$ (Product rule)

(4) $\lim_{x \to a} k \; f(x) = k.l$ (Constant multiple rule)

(5) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}, m \neq 0$ (Quotient rule)

(6) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$

(7) $\lim_{x \to a} \log\{f(x)\} = \log\{\lim_{x \to a} f(x)\}$

(8) If $f(x) \leq g(x)$ for all x, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

(9) $\lim_{x \to a}[f(x)]^{g(x)} = \{\lim_{x \to a} f(x)\}^{\lim_{x \to a} g(x)}$

(10) If p and q are integers, then $\lim_{x \to a}(f(x))^{p/q} = l^{p/q}$, provided $(l)^{p/q}$ is a real number.

(11) If $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(m)$ provided 'f' is continuous at $g(x) = m$. e.g. $\lim_{x \to a} \ln[f(x)] = \ln(l)$, only if $l > 0$.

Limits Problems with Solutions

1.

If $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $\lim_{x \to 0} f(x) =$

(a) 1 (b) 0

(c) −1 (d) None of these

Solution:

(b) Here $f(0)=0$

Since $-1 \le \sin\frac{1}{x} \le 1 \Rightarrow -|x| \le x\sin\frac{1}{x} \le |x|$

We know that $\lim_{x\to 0}|x|=0$ and $\lim_{x\to 0}-|x|=0$

In this way $\lim_{x\to 0} f(x)=0$.

2.

$$\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}=$$

(a) 0 (b) 1

(c) 2 (d) –2

Solution:

(c) $$\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}=\lim_{x\to 0}\left(\frac{x^3\cot x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right)$$

$$=\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3\times\lim_{x\to 0}\cos x\times\lim_{x\to 0}(1+\cos x)=2$$

3.

$$\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x}=$$

(a) 0 (b) ∞

(c) –2 (d) 2

Solution:

(d) $$\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x}=\lim_{x\to 0}\frac{2x(e^x-1)}{4.\sin^2\frac{x}{2}}$$

$$=2\lim_{x\to 0}\left[\frac{(x/2)^2}{\sin^2\frac{x}{2}}\right]\left(\frac{e^x-1}{x}\right)=2.$$

4.

$$\lim_{x\to 1}\frac{1}{|1-x|}=$$

(a) 0 (b) 1

(c) 2 (d) ∞

Solution:

(d) $\lim_{x \to 1-} \frac{1}{|1-x|} = \lim_{h \to 0} \frac{1}{1-(1-h)} = \infty$

and $\lim_{x \to 1+} \frac{1}{|1-x|} = \lim_{h \to 0} \frac{1}{1+h-1} = \infty$

Hence $\lim_{x \to 1} \frac{1}{|1-x|} = \infty$.

5.

$$\lim_{n \to \infty} \frac{n(2n+1)^2}{(n+2)(n^2+3n-1)} =$$

(a) 0 (b) 2

(c) 4 (d) ∞

Solution:

(c) $\lim_{n \to \infty} \frac{n(2n+1)^2}{(n+2)(n^2+3n-1)} = \lim_{n \to \infty} \frac{4n^3+4n^2+n}{n^3+5n^2+5n-2}$

$$= \lim_{n \to \infty} \frac{n^3\left(4+\frac{4}{n}+\frac{1}{n^2}\right)}{n^3\left(1+\frac{5}{n}+\frac{5}{n^2}-\frac{2}{n^3}\right)} = 4$$

Evaluating Limits

Methods of evaluation of limits

We shall divide the problems of evaluation of limits in five categories.

(1) Algebraic limits:

Let f(x) be an algebraic function and 'a' be a real number. Then $\lim_{x \to a} f(x)$ is known as an algebraic limit.

1. **Direct substitution method:** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.
2. **Factorisation method:** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.
3. **Rationalisation method:** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.
4. **Based on the form when x → ∞:** In this case expression should be expressed as a function 1/x and then after removing indeterminate form, (if it is there) replace 1/x by 0.

(2) Trigonometric limits:

To evaluate trigonometric limit the following results are very important.

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x}$

(ii) $\lim_{x \to 0} \frac{\tan x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan x}$

(iii) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin^{-1} x}$

(iv) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan^{-1} x}$

(v) $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$

(vi) $\lim_{x \to 0} \cos x = 1$

(vii) $\lim_{x \to a} \frac{\sin(x-a)}{x-a} = 1$

(viii) $\lim_{x \to a} \frac{\tan(x-a)}{x-a} = 1$

(ix) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a, |a| \leq 1$

(x) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a; |a| \leq 1$

(xi) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a, -\infty < a < \infty$

(xii) $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x \to \infty} \frac{\sin(1/x)}{(1/x)} = 1$

(3) Logarithmic limits:

To evaluate the logarithmic limits we use following formulae:

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots\text{to } \infty$ where $-1 < x \le 1$ and expansion is true only if base is e.

(ii) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(iii) $\lim_{x \to e} \log_e x = 1$

(iv) $\lim_{x \to 0} \frac{\log(1-x)}{x} = -1$

(v) $\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, \ne 1$

(4) Exponential limits:

(i) Based on series expansion:

We use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\infty$

To evaluate the exponential limits we use the following results:

(a) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(b) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(c) $\lim_{x \to 0} \frac{e^{\lambda x} - 1}{x} = \lambda \ (\lambda \ne 0)$

(ii) Based on the form 1^∞: To evaluate the exponential form $\mathbf{1^\infty}$ we use the following results.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$ or when $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$.

Then $\lim_{x\to a} \{f(x)\}^{g(x)} = \lim_{x\to a} [1 + f(x) - 1]^{g(x)} = e^{\lim_{x\to a} (f(x)-1)g(x)}$

(b) $\lim_{x\to 0} (1 + x)^{1/x} = e$

(c) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

(d) $\lim_{x\to 0} (1 + \lambda x)^{1/x} = e^\lambda$

(e) $\lim_{x\to\infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda$

(f) $\lim_{x\to\infty} a^x = \begin{cases} \infty, \text{if } a > 1 \\ 0\ , \text{if } a < 1 \end{cases}$ *i.e.*, $a^\infty = \infty$, if $a > 1$ and $a^\infty = 0$ if $a < 1$.

(5) L-Hospital's rule:

If f(x) and g(x) be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) Both are continuous at $x = a$

(iii) Both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$.

The above rule is also applicable if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L' Hospital rule, we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get, $\lim_{x\to a} \frac{f'(x)}{g'(x)} =$

$\lim_{x\to a} \frac{f''(x)}{g''(x)}$.

Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

Evaluating Limits Problems with Solutions

1.

$$\lim_{x \to a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{x - a} =$$

(a) $\sqrt{2a}$ (b) $1/\sqrt{2a}$

(c) $2a$ (d) $1/2a$

Solution:

(b) $$\lim_{x \to a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{(x - a)} \times \frac{\sqrt{3x - a} + \sqrt{x + a}}{\sqrt{3x - a} + \sqrt{x + a}}$$

$$= \frac{2}{2\sqrt{2a}} = \frac{1}{\sqrt{2a}}$$

Aliter : Apply L-Hospital's rule

$$\lim_{x \to a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{x - a} = \lim_{x \to a} \frac{3}{2\sqrt{3x - a}} - \frac{1}{2\sqrt{x + a}}$$

$$= \frac{3}{2\sqrt{2a}} - \frac{1}{2\sqrt{2a}} = \frac{1}{\sqrt{2a}}.$$

2.

If $f(x) = \begin{cases} x, \text{ when } 0 \le x \le 1 \\ 2 - x, \text{ when } 1 < x \le 2 \end{cases}$, then $\lim_{x \to 1} f(x) =$

(a) 1 (b) 2

(c) 0 (d) Does not exist

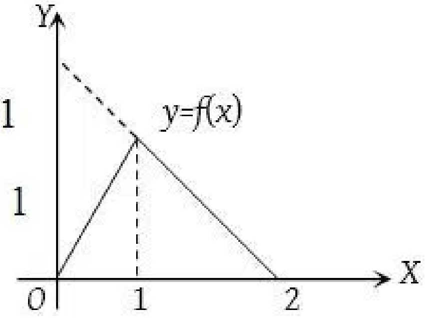
Solution:

(a) Hence $\lim_{x \to 1} f(x) = 1$

Aliter : $\lim_{x \to 1-} f(x) = \lim_{h \to 0} (1 - h) = 1$

and $\lim_{x \to 1+} f(x) = \lim_{h \to 0} 2 - (1 + h) = 1$

Hence limit of function is 1.



3.

$$\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} =$$

(a) 0 (b) 1

(c) −1 (d) Does not exist

Solution:

(d) $f(x) = \left(\frac{e^{1/x} - 1}{e^{1/x} + 1} \right)$, then

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} \left(\frac{e^{1/h} - 1}{e^{1/h} + 1} \right) = \lim_{h \to 0} \frac{e^{1/h}\left(1 - \frac{1}{e^{1/h}}\right)}{e^{1/h}\left(1 + \frac{1}{e^{1/h}}\right)} = 1$$

Similarly $\lim_{x \to 0-} f(x) = -1$. Hence limit does not exist.

4.

$$\lim_{x \to 0} \frac{\log \cos x}{x} =$$

(a) 0 (b) 1

(c) ∞ (d) None of these

Solution:

(a) $$\lim_{x \to 0} \frac{\log \cos x}{x} = \lim_{x \to 0} \frac{\log\left[1 - 2\sin^2 \frac{x}{2}\right]}{x}$$

$$= \lim_{x \to 0} \frac{-\left[2\sin^2 \frac{x}{2} + \left(\frac{2\sin^2 \frac{x}{2}}{2}\right)^2 + \ldots\ldots\right]}{x} = 0$$

Aliter : Apply L-Hospital's rule,

$$\lim_{x \to 0} \frac{\log \cos x}{x} = \lim_{x \to 0} \frac{-\tan x}{1} = 0.$$

5.

$$\lim_{x \to 0} \frac{|x|}{x} =$$

(a) 1 (b) −1

(c) 0 (d) Does not exist

Solution:

(d) Since $\lim_{x\to 0-}\frac{|x|}{x}=-1$ and $\lim_{x\to 0+}\frac{|x|}{x}=1$, hence limit does not exist.

6.

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=$$

(a) m/n (b) n/m

(c) $\frac{m^2}{n^2}$ (d) $\frac{n^2}{m^2}$

Solution:

(c) $$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\lim_{x\to 0}\left\{\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}}\right\}$$

$$=\lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2\frac{m^2x^2}{4}\cdot\frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}\cdot\frac{4}{n^2x^2}\right]$$

$$=\frac{m^2}{n^2}\times 1=\frac{m^2}{n^2}.$$

Aliter : Apply L-Hospital's rule,

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\lim_{x\to 0}\frac{m\sin mx}{n\sin nx}=\lim_{x\to 0}\frac{m^2\cos mx}{n^2\cos nx}=\frac{m^2}{n^2}.$$

7.

$$\lim_{x\to 0}\frac{e^{\sin x}-1}{x}=$$

(a) 1 (b) e

(c) $1/e$ (d) None of these

Solution:

(a) $$\lim_{x\to 0}\frac{e^{\sin x}-1}{x}=\lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\times\frac{\sin x}{x}$$

$$=\lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\times\lim_{x\to 0}\frac{\sin x}{x}=1\times 1=1.$$

Aliter : Apply L-Hospital's rule,

$$\lim_{x\to 0}\frac{e^{\sin x}-1}{x}=\lim_{x\to 0}\frac{\cos x\, e^{\sin x}}{1}=1.e^0=1.$$

8.

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2} =$$

(a) $\frac{a^2 - b^2}{2}$ (b) $\frac{b^2 - a^2}{2}$

(c) $a^2 - b^2$ (d) $b^2 - a^2$

Solution:

(b) $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$

$$= \lim_{x \to 0} \frac{2 \sin\left(\frac{a+b}{2}\right)x \,.\, \sin\left(\frac{b-a}{2}\right)x}{\left(\frac{a+b}{2}\right)x \,.\, \frac{2}{a+b} \,.\, \frac{2}{b-a} \,.\left(\frac{b-a}{2}\right)x} = \frac{b^2 - a^2}{2}$$

Aliter : Apply L-Hospital's rule,

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2} = \lim_{x \to 0} \frac{-a \sin ax + b \sin bx}{2x}$$

$$= \lim_{x \to 0} \frac{-a^2 \cos ax + b^2 \cos bx}{2} = \frac{b^2 - a^2}{2}.$$

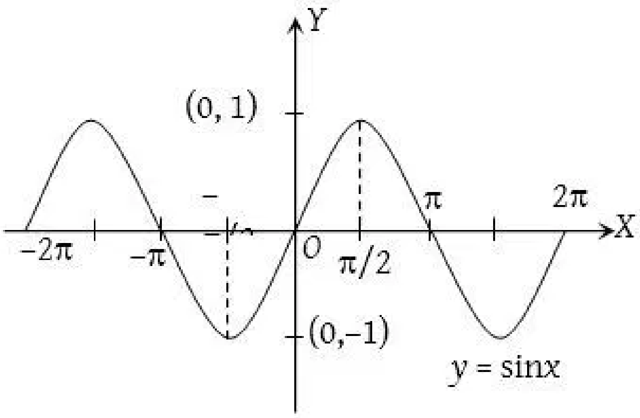
Continuous Function

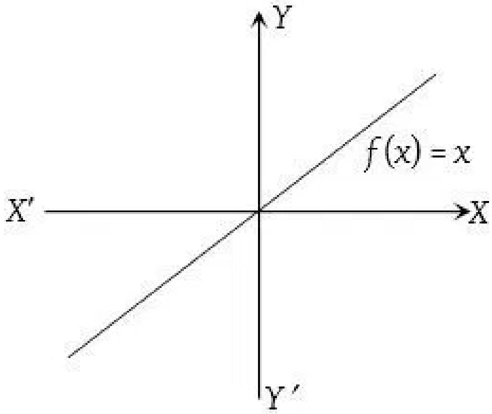
Continuity

The word ‘continuous’ means without any break or gap. If the graph of a function has no break or gap or jump, then it is said to be continuous.

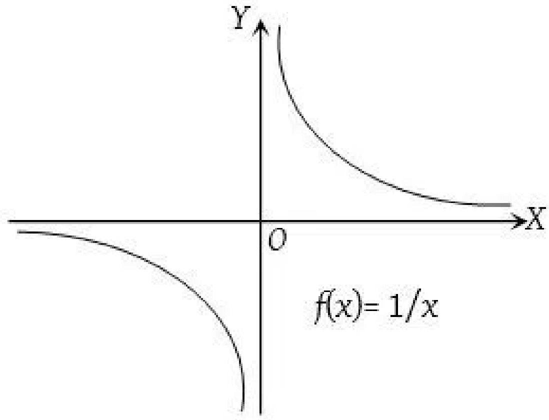
A function which is not continuous is called a discontinuous function. While studying graphs of functions, we see that graphs of functions sin x, x, cos x, e^x etc. are continuous but greatest integer function [x] has break at every integral point, so it is not continuous. Similarly tan x, cot x, sec x, 1/x etc. are also discontinuous function.

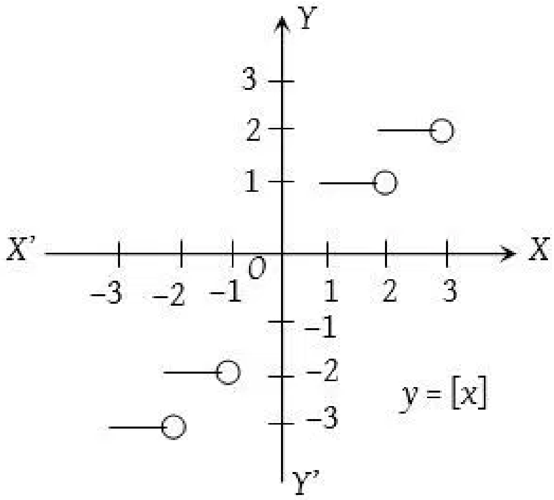
Continuous function





Discontinuous function





Continuity of a function at a point

A function f(x) is said to be continuous at a point x = a of its domain if and only if it satisfies the following three conditions :

(1) $f(a)$ exists. ('a' lies in the domain of f)

(2) $\lim_{x \to a} f(x)$ exist *i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim_{x \to a} f(x) = f(a)$ (limit equals the value of function).

Cauchy's definition of continuity:
A function f is said to be continuous at a point a of its domain D if for every ε > 0 there exists δ > 0 (dependent on ε) such that |x − a| < δ ⇒ |f(x) − f(a)| < ε.

Comparing this definition with the definition of limit we find that $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$

i.e., if $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a+} f(x)$.

Continuity from left and right

Function f(x) is said to be

(1) Left continuous at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Thus a function f(x) is continuous at a point x = a if it is left continuous as well as right continuous at x = a.

Properties of continuous functions

Let f(x) and g(x) be two continuous functions at x = a Then

1. A function f(x) is said to be everywhere continuous if it is continuous on the entire real line R i.e.(-∞, ∞). e.g., polynomial function, e^x, sin x, x, cos x, constant, x^n, |x − a| etc.
2. Integral function of a continuous function is a continuous function.
3. If g(x) is continuous at x = a and f(x) is continuous at x = g(a) then (fog) (x) is continuous at x = a.
4. If f(x) is continuous in a closed interval [a,b] then it is bounded on this interval.
5. If f(x) is a continuous function defined on [a, b] such that f(a) and f(b) are of opposite signs, then there is atleast one value of x for which f(x) vanishes. i.e. if f(a) > 0, f(b) < 0 ⇒ ∃c ∈ (a, b) such that f(c) = 0.

Discontinuous function

Discontinuous function: A function 'f' which is not continuous at a point in its domain is said to be discontinuous there at x = a. The point 'a' is called a point of discontinuity of the function.
The discontinuity may arise due to any of the following situations.

(i) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both may not exist.

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.

Continuous Function Problems with Solutions

1.

If $f(x) = |x - 2|$, then

(a) $\lim_{x \to 2+} f(x) \neq 0$

(b) $\lim_{x \to 2-} f(x) \neq 0$

(c) $\lim_{x \to 2+} f(x) \neq \lim_{x \to 2-} f(x)$

(d) $f(x)$ is continuous at $x = 2$

Solution:

(d) Here $f(2)=0$

$$\lim_{x \to 2-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} |\ 2-h-2| = 0$$

$$\lim_{x \to 2+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} |\ 2+h-2|\ = 0$$

Hence it is continuous at $x = 2$.

2.

If the function $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, \text{when } x \neq \dfrac{\pi}{2} \\ 3, \quad \text{when } x = \dfrac{\pi}{2} \end{cases}$ be continuous at $x = \dfrac{\pi}{2}$, then k =

(a) 3 (b) 6

(c) 12 (d) None of these

Solution:

(b) $f(\pi/2) = 3$. Since $f(x)$ is continuous at $x = \pi/2$

$$\Rightarrow \lim_{x \to \pi/2} \left(\frac{k \cos x}{\pi - 2x} \right) = f\left(\frac{\pi}{2} \right) \Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6.$$

3.

Let $f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, \text{if } x \neq 2 \\ k, \text{if } x = 2 \end{cases}$.

If $f(x)$ be continuous for all x, then k =

(a) 7 (b) –7

(c) ±7 (d) None of these

Solution:

(a) For continuous $\lim_{x \to 2} f(x) = f(2) = k$

$$\Rightarrow k = \lim_{x \to 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4x + 4)\ (x+5)}{(x-2)^2} = 7 \ .$$

4.

If $f(x)=\begin{cases}\dfrac{x^2-4x+3}{x^2-1}, \text{ for } x\neq 1\\ 2, \text{ for } x=1\end{cases}$, then

(a) $\lim\limits_{x\to 1+} f(x)=2$

(b) $\lim\limits_{x\to 1-} f(x)=3$

(c) $f(x)$ is discontinuous at $x=1$

(d) None of these

Solution:

(c) $f(x)=\left\{\dfrac{x^2-4x+3}{x^2-1}\right\}$, for $x\neq 1$

$= 2$, for $x=1$

$$f(1)=2,\ f(1+)=\lim_{x\to 1+}\frac{x^2-4x+3}{x^2-1}=\lim_{x\to 1+}\frac{(x-3)}{(x+1)}=-1$$

$$f(1-)=\lim_{x\to 1-}\frac{x^2-4x+3}{x^2-1}=-1\Rightarrow f(1)\neq f(1-)$$

Hence the function is discontinuous at $x=1$.

5.

If $f(x)=\begin{cases}-x^2, \text{ when } x\leq 0\\ 5x-4, \text{ when } 0<x\leq 1\\ 4x^2-3x, \text{ when } 1<x<2\\ 3x+4, \text{ when } x\geq 2\end{cases}$, then

(a) $f(x)$ is continuous at $x=0$

(b) $f(x)$ is continuous $x=2$

(c) $f(x)$ is discontinuous at $x=1$

(d) None of these

Solution:

(b) $\lim_{x \to 0-} f(x) = 0$

$f(0) = 0, \lim_{x \to 0+} f(x) = -4$

$f(x)$ discontinuous at $x = 0$.

and $\lim_{x \to 1-} f(x) = 1$ and $\lim_{x \to 1+} f(x) = 1, f(1) = 1$

Hence $f(x)$ is continuous at $x = 1$.

Also $\lim_{x \to 2-} f(x) = 4(2)^2 - 3 . 2 = 10$

$f(2) = 10$ and $\lim_{x \to 2+} f(x) = 3(2) + 4 = 10$

Hence $f(x)$ is continuous at $x = 2$.