CBSE Board
Class XII Mathematics
Sample Paper 2 – Solution

Section A

1. Correct option: A
Explanation:-

$$P(A \cap B) = \frac{7}{10},\ P(B) = \frac{17}{20}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A/B) = \frac{\frac{7}{10}}{\frac{17}{20}} = \frac{14}{17}$$

2. Correct option: D
Explanation:-

Edges of parallelepiped are $5-2,\ 9-3,\ 7-5 \Rightarrow 3,\ 6,\ 2$

Length of the diagonal $= \sqrt{9+36+4}$

Length of the diagonal $= 7$

3. Correct option: D
Explanation:-

$$\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) = \tan^{-1}\left(-\sin\frac{\pi}{2}\right) = \tan^{-1}(-1)$$

As tan(−x) = −tan x

$$\therefore \tan^{-1}(-1) = \tan^{-1}\left(-\tan\frac{\pi}{4}\right) = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] = -\frac{\pi}{4}$$

Hence, $\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{4}$

4. Correct option: A
Explanation:-

Coordinate of a point on x − axis $(a,0,0)$

The distance of the point $P(a,b,c)$ from x − axis

$$= \sqrt{(a-a)^2 + b^2 + c^2}$$

$$= \sqrt{b^2 + c^2}$$

5. **Correct option: B**

Explanation:-

$f(x) = 2x^2 - kx + 5$

$f'(x) = 4x - k$

$f(x)$ is increasing

$4x - k > 0$ on $[1,2]$

$k < 4x$

Minimum value of k is 4.

$k < 4$

$k \in (-\infty, 4)$

6. **Correct option: D**

Explanation:-

Matrix of order 3 × 3 has 9 elements.

Now the entries have to be either 0 or 1 so that each of the 9 places can be filled with 2 choices 0 or 1.

So $2^9 = 512$ matrices are possible.

7. **Correct option: C**

Explanation:-

Given that xy = 1

Consider,

$\tan^{-1}x + \tan^{-1}y$

$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$= \tan^{-1}(-\infty)$ .....$(\because x < 0, y < 0)$

$= -\frac{\pi}{2}$

8. **Correct option: A**

Explanation:-

$\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k};$

$\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$

$\Rightarrow \vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

$\frac{\vec{a} + \vec{b}}{2} = 3\hat{i} + \hat{j} - 4\hat{k}$

9. Correct option: C

Explanation:-

Given differential equation is $\cos^2 x \frac{dy}{dx} + y = \tan x$

$\Rightarrow \frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$

It is a linear differential equation with $P(x) = \frac{1}{\cos^2 x}$ and $Q(x) = \frac{\tan x}{\cos^2 x}$

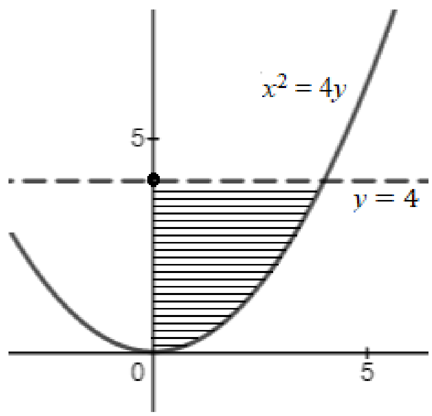
Integrating factor is $e^{\int P(x)dx} = e^{\int \frac{1}{\cos^2 x} dx} = e^{\int \sec^2 x dx} = e^{\tan x}$

Hence, the integrating factor is $e^{\tan x}$.

10. Correct option: A

Explanation:-

Graph of the curve $x^2 = 4y$ and the line $y = 4$ is given by



Area is bounded between the lines y = 0 and y = 4

So, the required area is

$$A = \int_0^4 2\sqrt{y}\, dy = \left[2 \times \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^4 = \frac{4}{3}\left[y^{\frac{3}{2}} \right]_0^4 = \frac{4}{3}[8] = \frac{32}{3}$$

Hence, the area is $\frac{32}{3}$ sq. units.

11. Correct option: A

Explanation:-

Given: $y = 10^{10^x}$

Taking log on both the sides, we get

$\log_{10} y = 10^x (\log 10)$

Differentiating w.r.t x, we get

$\frac{1}{y}\frac{dy}{dx} = (\log 10)\frac{d}{dx}(10^x)$ ... (i)

Let $m = 10^x$

Therefore, $\log_{10} m = x \log 10$

$$\frac{1}{m}\frac{dm}{dx} = \log 10$$

$$\Rightarrow \frac{dm}{dx} = m \log 10$$

$$\Rightarrow \frac{d}{dx}\left(10^{x}\right) = 10^{x}\log 10$$

From (i), we get

$$\frac{dy}{dx} = 10^{10^{x}}\left(\log 10\right)10^{x}\left(\log 10\right)$$

$$\Rightarrow \frac{dy}{dx} = 10^{10^{x}} \cdot 10^{x}\left(\log 10\right)^{2}$$

12. Correct option: B

Explanation:-

Vectors $\vec{a}$ and $\vec{b}$ have the same magnitude

$$\Rightarrow \left|\vec{a}\right| = \left|\vec{b}\right| \text{..........(i)}$$

Let θ be the angle between the two vectors $\Rightarrow \theta = 60° \text{........(ii)}$

Also, $\vec{a} \cdot \vec{b} = \frac{9}{2} \text{.........(iii)}$

$$\Rightarrow \left|\vec{a}\right| \left|\vec{b}\right| \cos 60° = \frac{9}{2}$$

$$\Rightarrow \left|\vec{a}\right| \left|\vec{a}\right| \cos 60° = \frac{9}{2} \text{..... From (i)}$$

$$\Rightarrow \left|\vec{a}\right|^{2} \times \frac{1}{2} = \frac{9}{2}$$

$$\Rightarrow \left|\vec{a}\right|^{2} = 9$$

$$\Rightarrow \left|\vec{a}\right| = 3$$

$$\Rightarrow \left|\vec{a}\right| = \left|\vec{b}\right| = 3 \text{ .... From (i)}$$

13. Correct option: C

Explanation:-

$$f(x) = \frac{-x}{2} + \sin x$$

$$\Rightarrow f'(x) = \frac{-1}{2} + \cos x$$

$\because \cos x > \frac{1}{2}$ for $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

$\Rightarrow \frac{-1}{2} + \cos x > 0$

$\Rightarrow f'(x) > 0$

Hence, f is increasing on $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

14. Correct option: D

Explanation:-

Total number of binary operations on a set containing n elements is $(n)^{n^2}$

So, for n = 2

The no. of binary operations defined on a set of 2 elements = $(2)^{2^2} = 2^4 = 16$

15. Correct option: C

Explanation:-

$f(0) = \lim_{x \to 0} f(x)$

$= \lim_{x \to 0} (x + 1)^{\cot x}$

$= \lim_{x \to 0} \left[(1 + x)^{\frac{1}{x}}\right]^{x \cot x}$

$= \lim_{x \to 0} [e]^{x \cot x}$

$= \lim_{x \to 0} [e]^{\frac{x}{\tan x}}$

$= [e]^{\lim_{x \to 0} \frac{1}{\frac{\tan x}{x}}}$

$= e$

16. Correct option: B

Explanation:-

Given differential equation is $\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$

Integrating on both sides,

$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$

$\Rightarrow \log|y| = \log|x| + \log k$

$\Rightarrow \log\left(\frac{y}{x}\right) = \log k$

$\Rightarrow y = kx$

17. Correct option: C

Explanation:-

$$\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix}$$

$= x(6x - 6x) - 2(6x^2 - 6x) + x(x^3 - x^2)$

$= 0 - 12x^2 + 12x + x^4 - x^3$

$= x^4 - x^3 - 12x^2 + 12x$

Comparing with RHS $ax^4 + bx^3 + cx^2 + dx + e$, we have

$a = 1, b = -1, c = -12, d = 12, e = 0$

$\Rightarrow 5a + 4b + 3c + 2d + e = 5 - 4 - 36 + 24 = -11$

18. Correct option: B

Explanation:-

Let I = $\int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx$

$I = \int \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx$

$= \int \left(\frac{2\sin^2 x}{2\cos^2 x} \right) dx$

$= \int \tan^2 x dx = \int (\sec^2 x - 1) dx$

$= \int \sec^2 x dx - \int dx$

$= \tan x - x + c$

19. Correct option: B

Explanation:-

R = {(2, 8), (3, 27)}

$\therefore$ The range set of R is {8, 27}.

20. Correct option: D

Explanation:-

Let $I = \int_1^{\sqrt{3}} \frac{1}{1 + x^2} dx$

$\therefore I = \left[\tan^{-1} x \right]_1^{\sqrt{3}}$

$I = \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$

$I = \frac{\pi}{3} - \frac{\pi}{4}$

$I = \frac{\pi}{12}$

Section B

21. We know that the slope of the tangent is given by $\frac{dy}{dx}$

According to the question, $\frac{dy}{dx} = y + e^x$

Or, $\frac{dy}{dx} - y = e^x$ ...... (i)

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

where, P = -1 and Q = e^x

Therefore,

$\text{I.F.} = e^{\int P dx} = e^{-\int 1 dx} = e^{-x}$

Solution of (i) is given by

$ye^{-x} = \int e^x e^{-x} dx + c$

$\Rightarrow ye^x = x + c$

This is the required family of curves.

OR

Given: y = A cos 2x + B sin 2x

Differentiating w.r.t. x , we get

$\frac{dy}{dx} = A(-\sin 2x) \times 2 + B(\cos 2x) \times 2$

$\frac{dy}{dx} = -2A \sin 2x + 2B \cos 2x$

Again differentiating w. r. t. x, we get

$\frac{d^2y}{dx^2} = -2A \cos 2x \cdot 2 + 2B(-\sin 2x) \cdot 2$

$\frac{d^2y}{dx^2} = -4(A \cos 2x + B \sin 2x)$

$\frac{d^2y}{dx^2} = -4y \Rightarrow \frac{d^2y}{dx^2} + 4y = 0.$

22. Let I = $\int \frac{\sin x}{(1-\cos x)(2-\cos x)} dx$

Here substitute -cosx = t ⇒ sinx dx = dt

$\int \frac{\sin x}{(1-\cos x)(2-\cos x)} dx = \int \frac{dt}{(1+t)(2+t)}$

Let $\frac{1}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)}$

1 = A(2 + t) + B(1 + t)

Solving the equation we get

B = -1

A = 1

$$\int \frac{dt}{(1+t)(2+t)} = \int \frac{dt}{1+t} - \int \frac{dt}{2+t}$$

$$= \log|1+t| - \log|2+t| + C$$

$$= \log\left|\frac{1+t}{2+t}\right| + C$$

And so

$$\int \frac{\sin x}{(1-\cos x)(2-\cos x)} dx = \log\left|\frac{1-\cos x}{2-\cos x}\right| + C$$

OR

Let $I = \int \frac{2x^2 - x + 4}{x^3 + 4x} dx$

Now, $\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{(x^2+4)x}$

Let $\frac{2x^2 - x + 4}{(x^2+4)x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$ ... (By partial fractions)

$$\Rightarrow 2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$\Rightarrow 2x^2 - x + 4 = (A + B)x^2 + Cx + 4A$$

Equating the corresponding coefficients, we get

A = 1, B = 1 and C = –1

Substituting the values of A, B and C we have

$$I = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4} \right) dx$$

$$= \int \frac{dx}{x} + \int \left(\frac{x-1}{x^2-4} \right) dx$$

$$= \int \frac{dx}{x} + \frac{1}{2}\int \frac{2x}{x^2-4} dx - \int \frac{1}{x^2 - 2^2} dx$$

$$= \log x + \frac{1}{2}\log(x^2 - 4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

23. To prove the continuity of f(x) at x = 0, we need to prove that

$$\lim_{x \to 0} f(x) = f(0)$$

Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{\sin x}{x} + \cos x \right)$$

$$= \lim_{x \to 0} \frac{\sin x}{x} + \lim_{x \to 0} \cos x$$

$$= 1 + \cos 0 \ \ldots \left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \right)$$

$$= 1 + 1$$

$$= 2$$

Therefore, $\lim_{x \to 0} f(x) = 2$

Also, f(0) = 2

$\therefore \lim_{x \to 0} f(x) = f(0)$

Hence, f(x) is continuous at x = 0.

OR

Given: sin y = x sin(a + y) … (i)

To Prove: $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

$$\frac{\sin y}{\sin(a+y)} = x \ \ldots \text{From (i)}$$

$$\frac{\sin(a+y-a)}{\sin(a+y)} = x$$

$$\frac{\sin(a+y)\cos a - \cos(a+y)\sin a}{\sin(a+y)} = x$$

$$\cos a - \cot(a+y)\sin a = x$$

$$\operatorname{cosec}^2(a+y)\sin a . \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin a \operatorname{cosec}^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

24. f : N → N is defined as

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

Let $f(n_1) = f(n_2)$

Case 1 : n_1, n_2 are odd

Let $f(n_1) = f(n_2)$

$\Rightarrow \frac{n_1+1}{2} = \frac{n_2+1}{2}$

$\Rightarrow n_1 = n_2$

Case 2: n_1, n_2 are even

$f(n_1) = f(n_2) \Rightarrow \frac{n_1}{2} = \frac{n_2}{2} \Rightarrow n_1 = n_2$

Case3: n_1 is odd and n_2 is even

$f(n_1) = f(n_2) \Rightarrow \frac{n_1+1}{2} = \frac{n_2}{2}$

$\Rightarrow n_1 + 1 = n_2$

$\Rightarrow n_1 \neq n_2$

Hence,

$f(n_1) = f(n_2)$ does not imply $n_1 = n_2 \ \forall \ n_1, n_2 \in N$

$\therefore$ f is not one – one

Function f is onto and hence, f is surjective.

Hence, f is not bijective.

25. Let the shooter fire n times. Then n fires are Bernoulli's trials

Let p = probability of hitting the target = $\frac{3}{4}$

q = probability of not hitting the target = $\frac{1}{4}$

$\Rightarrow P(X = r) = {}^nC_r\, q^{n-r} p^r$

$$= {}^nC_r \left(\frac{1}{4}\right)^{n-r} \left(\frac{3}{4}\right)^r = {}^nC_r \frac{3^r}{4^n}$$

$\Rightarrow$ P(hitting the target at least once) > 0.99

$P(X \geq 1) > 0.99$

$1 - P(X = 0) > 0.99$

$1 - {}^nC_0 \frac{1}{4^n} > 0.99$

${}^nC_0 \frac{1}{4^n} < 0.01$

$\frac{1}{4^n} < 0.01$

$4^n > \frac{1}{0.01} = 100$

The minimum value of n is 4

Thus the shooter must fire at least 4 times.

26. Given: $\tan^{-1}\left(\frac{2x-4}{2x-3}\right)+\tan^{-1}\left(\frac{2x+4}{2x+3}\right)=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x-4}{2x-3}+\frac{2x+4}{2x+3}}{1-\left(\frac{2x-4}{2x-3}\right)\left(\frac{2x+4}{2x+3}\right)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{(2x-4)(2x+3)+(2x+4)(2x-3)}{(2x-3)(2x+3)-(2x-4)(2x+4)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{4x^2-2x-12+4x^2+2x-12}{4x^2-9-4x^2+16}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{8x^2-24}{-7}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{24-8x^2}{7}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{24-8x^2}{7}=\tan\frac{\pi}{4}$$

$$\Rightarrow 24-8x^2=7$$

$$\Rightarrow 8x^2=17$$

$$\Rightarrow x=\pm\sqrt{\frac{17}{8}}$$

27. To prove that $\begin{vmatrix}3x+y & 2x & x\\ 4x+3y & 3x & 3x\\ 5x+6y & 4x & 6x\end{vmatrix}=x^3$

Consider, LHS

$$=\begin{vmatrix}3x+y & 2x & x\\ 4x+3y & 3x & 3x\\ 5x+6y & 4x & 6x\end{vmatrix}=\begin{vmatrix}3x & 2x & x\\ 4x & 3x & 3x\\ 5x & 4x & 6x\end{vmatrix}+\begin{vmatrix}y & 2x & x\\ 3y & 3x & 3x\\ 6y & 4x & 6x\end{vmatrix}$$

$$=x^3\begin{vmatrix}3 & 2 & 1\\ 4 & 3 & 3\\ 5 & 4 & 6\end{vmatrix}+x^2y\begin{vmatrix}1 & 2 & 1\\ 3 & 3 & 3\\ 6 & 4 & 6\end{vmatrix}$$

$$=x^3\begin{vmatrix}3 & 2 & 1\\ 4 & 3 & 3\\ 5 & 4 & 6\end{vmatrix}+x^2y\times 0 \quad \ldots\left[\because C_1 \text{ and } C_3 \text{ are identical}\right]$$

$$=x^3\begin{vmatrix}3 & 2 & 1\\ 4 & 3 & 3\\ 5 & 4 & 6\end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$

$$= x^3 \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$$= x^3 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$= x^3 \times (3 - 2)$ … [Expanding along C_1]

$= x^3$

= R.H.S.

Hence, $\begin{vmatrix} 3x + y & 2x & x \\ 4x + 3y & 3x & 3x \\ 5x + 6y & 4x & 6x \end{vmatrix} = x^3.$

28. Given: $x = \dfrac{a}{1 + t^3}$ and $y = \dfrac{at}{1 + t^3}$

Differentiating x w.r.t t, we get

$$\frac{dx}{dt} = -a\left(1 + t^3\right)^{-2} \frac{d}{dt}\left(1 + t^3\right)$$

$$\therefore \frac{dx}{dt} = -\frac{3t^2 a}{\left(1 + t^3\right)^2} \text{ … (i)}$$

Differentiating y w.r.t t, we get

$$\frac{dy}{dt} = \frac{\left(1 + t^3\right)a - at\left(3t^2\right)}{\left(1 + t^3\right)^2}$$

$$\frac{dy}{dt} = \frac{\left(1 + t^3 - 3t^3\right)a}{\left(1 + t^3\right)^2}$$

$$\therefore \frac{dy}{dt} = \frac{\left(1 - 2t^3\right)a}{\left(1 + t^3\right)^2} \text{ … (ii)}$$

From (i) and (ii), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{3t^2 a}{\left(1 + t^3\right)^2}}{\frac{\left(1 - 2t^3\right)a}{\left(1 + t^3\right)^2}} = \frac{3t^2 a}{\left(1 - 2t^3\right)a}$$

$$\therefore \frac{dy}{dx} = \frac{3t^2}{1 - 2t^3}$$

29. Let $I = \int_0^4 \left|x^2 - 4x + 3\right| dx$

Now, $x^2 - 4x + 3 = (x - 3)(x - 1)$

$|x^2 - 4x + 3| = x^2 - 4x + 3$ if $x^2 - 4x + 3 > 0$

i.e. $(x - 3)(x - 1) > 0$

i.e. $x - 3 > 0$ & $x - 1 > 0$ OR $x - 3 < 0$ & $x - 1 < 0$

i.e. $x > 3$ & $x > 1$ OR $x < 3$ & $x < 1$

i.e. $x > 3$ OR $x < 1$

Therefore, $|x^2 - 4x + 3| = x^2 - 4x + 3$ for $x \in (-\infty, 1) \cup (3, \infty)$ ... (i)

$|x^2 - 4x + 3| = -(x^2 - 4x + 3)$ if $x^2 - 4x + 3 < 0$

i.e. $(x - 3)(x - 1) < 0$

i.e. $x - 3 > 0$ & $x - 1 < 0$ OR $x - 3 < 0$ & $x - 1 > 0$

i.e. $x > 3$ & $x < 1$ OR $x < 3$ & $x > 1$

i.e. $1 < x < 3$ i.e. $x \in (1, 3)$

Therefore, $|x^2 - 4x + 3| = -(x^2 - 4x + 3)$ for $x \in (1, 3)$ ... (ii)

$$I = \int_0^1 \left(x^2 - 4x + 3\right) dx - \int_1^3 \left(x^2 - 4x + 3\right) dx + \int_3^4 \left(x^2 - 4x + 3\right) dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$

$$= \left[\frac{1}{3} - 2 + 3\right] - \left[9 - 18 + 9 - \frac{1}{3} + 2 - 3\right] + \left[\frac{64}{3} - 32 + 12 - 9 + 18 - 9\right]$$

$$= \left[\frac{1}{3} + 1\right] - \left[-\frac{1}{3} - 1\right] + \left[\frac{64}{3} - 20\right]$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{64}{3} + 1 + 1 - 20$$

$$= \frac{66}{3} - 18$$

$$= 22 - 18$$

$$= 4$$

30. $\vec{a} \neq 0, \vec{b}$ and $\vec{c}$ are three vectors such that $\left|\vec{a}\right| = \left|\vec{b}\right| = 1$ and $\left|\vec{c}\right| = 4$

As $\left|\vec{b} \times \vec{c}\right| = \sqrt{15}$

$\Rightarrow \left|\vec{b}\right|\left|\vec{c}\right| \sin\theta = \sqrt{15}$ ... (θ is the angle between $\vec{b}$ and $\vec{c}$)

$\Rightarrow 1 \times 4 \sin\theta = \sqrt{15}$

$\Rightarrow \sin\theta = \frac{\sqrt{15}}{4}$

$\Rightarrow \cos\theta = \sqrt{1 - \sin^2\theta} = \frac{1}{4}$ ... (i)

Also, $\vec{c} - 2\vec{b} = \lambda\vec{a}$

$$\Rightarrow \left|\vec{c} - 2\vec{b}\right|^2 = \left|\lambda\vec{a}\right|^2$$

$$\Rightarrow \left|\vec{c}\right|^2 + 4\left|\vec{b}\right|^2 - 4\vec{b}\cdot\vec{c} = \lambda^2\left|\vec{a}\right|^2$$

$$\Rightarrow \left|\vec{c}\right|^2 + 4\left|\vec{b}\right|^2 - 4\left|\vec{c}\right|\left|\vec{b}\right|\cos\theta = \lambda^2\left|\vec{a}\right|^2$$

$$\Rightarrow 4^2 + 4(1)^2 - 4(4)(1)\times\frac{1}{4} = \lambda^2(1)^2 \text{ ... From (i)}$$

$$\Rightarrow \lambda^2 = 16$$

$$\Rightarrow \lambda = \pm 4$$

Hence, the values of λ are -4 and 4.

31. Let l, m and n be the direction ratios of the given line.

Since the line passes through the point (–1, 3, –4), so the equation will be of the form

$$\frac{x-(-1)}{l} = \frac{y-3}{m} = \frac{z-(-4)}{n}$$

i.e. $\frac{x+1}{l} = \frac{y-3}{m} = \frac{z+4}{n}$ ... (i)

As this line is perpendicular to the plane $x + 2y - 5z + 9 = 0$

So, the direction ratios of the line will be proportional to the direction ratios of the given line.

$$\therefore \frac{l}{1} = \frac{m}{2} = \frac{n}{-5} = \lambda \text{ ... (As 1, 2 \& } -5 \text{ are direction ratios)}$$

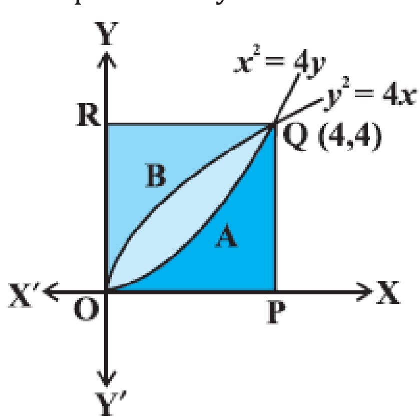
$$\therefore l = \lambda,\ m = 2\lambda \text{ and } n = -5\lambda$$

Putting these values in (i), we get

$\frac{x+1}{1} = \frac{y-3}{2} = \frac{z+4}{-5}$ which is the equation of the line

Section C

32. The point of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ are (0, 0) and (4, 4)



Now, the area of the region OAQBO bounded by curves $y^2 = 4x$ and $x^2 = 4y$

$$\int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \left[2\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{sq.units} \text{ .... (i)}$$

Again, the area of the region OPQAO bounded by the curves $x^2 = 4y$, $x = 0$, $x = 4$ and the x-axis.

$$\int_0^4 \frac{x^2}{4}dx = \left[\frac{x^3}{12}\right]_0^4 = \left(\frac{64}{12}\right) = \frac{16}{3} \text{sq.units} \text{ .... (ii)}$$

Similarly, the area of the region OBQRO bounded by the curve $y^2 = 4x$ and the y-axis, $y = 0$ and $y = 4$

$$\int_0^4 \frac{y^2}{4}dy = \left[\frac{y^3}{12}\right]_0^4 = \frac{16}{3} \text{sq.units} \text{ .... (iii)}$$

From (i) (ii), and (iii), it is concluded that the area of the region OAQBO = area of the region OPQAO = area of the region OBQRO.

i.e., the parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ in three equal parts.

OR

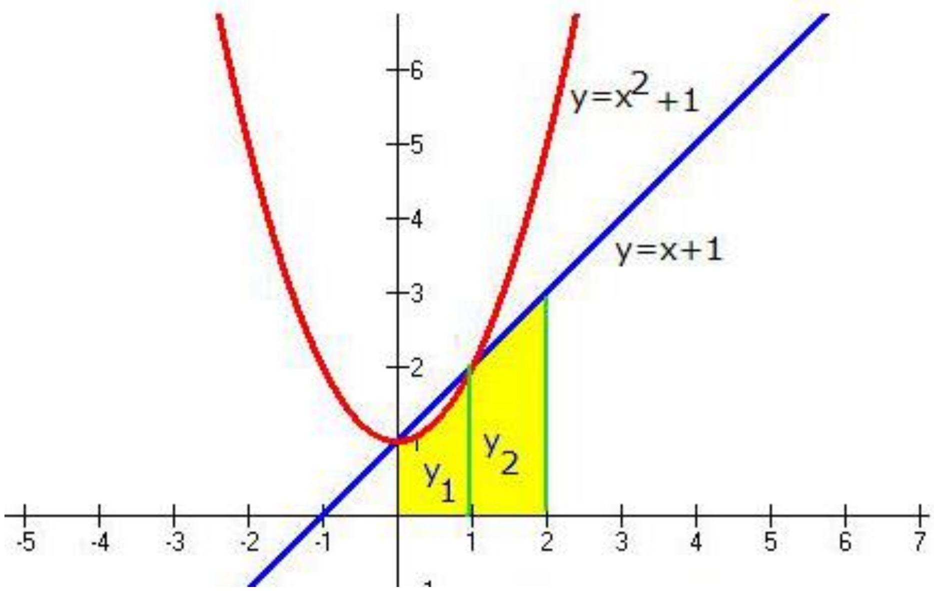
Points of intersection of $y = x^2 + 1$, $y = x + 1$

$x^2 + 1 = x + 1$

$\Rightarrow x(x - 1) = 0$

$\Rightarrow x = 0, 1$

So points of intersection are P(0, 1) and Q(1, 2). The graph is represented as



Required area is given by

$$A = \int_0^1 y_1\, dx + \int_1^2 y_2\, dx,$$

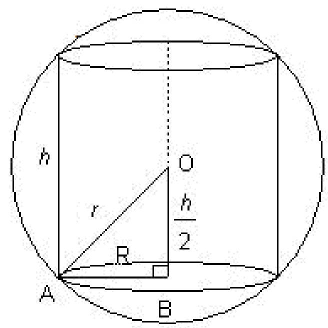
where y_1 and y_2 represent the y co-ordinate of the parabola and straight line respectively.

$$\therefore A = \int_0^1 \left(x^2 + 1\right) dx + \int_1^2 (x + 1)\, dx$$

$$= \left[\left(\frac{x^3}{3} + x\right)\right]_0^1 + \left(\frac{x^2}{2} + x\right)_1^2$$

$$= \left[\left(\frac{1}{3} + 1\right) - 0\right] + \left[(2 + 2) - \left(\frac{1}{2} + 1\right)\right] = \frac{23}{6} \text{ sq. units}$$

33. Radius of the sphere is r. Let *h* and *R* be the height and radius of the cylinder inscribed in the sphere.



Volume of cylinder $(V) = \pi R^2 h$ ..... (1)

In right ΔOBA,

$$AB^2 + OB^2 = OA^2$$

$$R^2 + \frac{h^2}{4} = r^2$$

$$\text{So}, R^2 = r^2 - \frac{h^2}{4}$$

Putting the value of R^2 in equation (1), we get

$$V = \pi\left(r^2 - \frac{h^2}{4}\right) \cdot h$$

$$V = \pi\left(r^2 h - \frac{h^3}{4}\right) \text{ .... (3)}$$

$$\therefore \frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right) \text{ .... (4)}$$

For stationary point, $\frac{dV}{dh} = 0$

$$\pi\left(r^2 - \frac{3h^2}{4}\right) = 0$$

$$r^2 = \frac{3h^2}{4} \Rightarrow h^2 = \frac{4r^2}{3} \Rightarrow h = \frac{2r}{\sqrt{3}}$$

Now, $\dfrac{d^2V}{dh^2} = \pi\left(-\dfrac{6}{4}h\right)$

$\therefore \left[\dfrac{d^2V}{dh^2}\right]_{\left(at\ h=\frac{2r}{\sqrt{3}}\right)} = \pi\left(-\dfrac{3}{2}\cdot\dfrac{2r}{\sqrt{3}}\right) < 0$

$\therefore$ Volume is maximum at $h = \dfrac{2r}{\sqrt{3}}$

Maximum volume is

$= \pi\left(r^2.\dfrac{2r}{\sqrt{3}} - \dfrac{1}{4}.\dfrac{8r^3}{3\sqrt{3}}\right)$

$= \pi\left(\dfrac{2r^3}{\sqrt{3}} - \dfrac{2r^3}{3\sqrt{3}}\right)$

$= \pi\left(\dfrac{6r^3 - 2r^3}{3\sqrt{3}}\right)$

$= \dfrac{4\pi r^3}{3\sqrt{3}}$ cu. unit

OR

The given line is x + 3y = 4 i.e. $y = -\dfrac{1}{3}x + \dfrac{4}{3}$

$\therefore$ Slope of the given line is $-\dfrac{1}{3}$

$\Rightarrow$ Slope of the required normal = $-\dfrac{1}{3}$ … (i) (As required normal is parallel to the given line)

Let the point of contact be (x_1, y_1).

Now, the given curve is $3x^2 - y^2 = 8$

$\Rightarrow 6x - 2y\dfrac{dy}{dx} = 0$ … (Diff w.r.t. x)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3x}{y}$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{3x_1}{y_1}$

$\therefore$ Slope of the normal = $\dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}} = \dfrac{-y_1}{3x_1}$ …. (ii)

Thus from (i) and (ii), we have

$\dfrac{-y_1}{3x_1} = -\dfrac{1}{3}$

$\Rightarrow x_1 = y_1$ …. (iii)

Also, since (x_1, y_1) lies on the given curve, we have

$3x_1^2 - y_1^2 = 8$

$\Rightarrow 3x_1^2 - x_1^2 = 8$ ... From (iii)

$\Rightarrow 2x_1^2 = 8$ or $\Rightarrow x_1^2 = 4$ or $\Rightarrow x_1 = \pm 2$

$\Rightarrow y_1 = \pm 2$ ... From (iii)

Thus, the points of contact are (2, 2) and (−2, −2).

The equation of the required normal at (2, 2) is $\frac{y-2}{x-2} = \frac{-1}{3}$ i.e. $x + 3y - 8 = 0$.

The equation of the required normal at (−2, −2) is $\frac{y+2}{x+2} = \frac{-1}{3}$ i.e. $x + 3y + 8 = 0$.

34. Given:

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$A = AI$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{2}{3} R_1$$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & \frac{2}{3} \\ 0 & 4 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{3}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 3 & \frac{2}{3} \\ 0 & 4 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{4}{3} R_2$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 3 & \frac{2}{3} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = A \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{8}{9} & -\frac{4}{3} & 1 \end{bmatrix}$$

$R_3 \to 9R_3$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 3 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 8 & -12 & 9 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 8 & -12 & 9 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} 3 & -4 & 3 \\ 0 & 1 & 0 \\ 8 & -12 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} 3 & -4 & 3 \\ -6 & 9 & -6 \\ 8 & -12 & 9 \end{bmatrix}$$

$R_2 \to \frac{R_2}{3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$$

OR

Given: $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$$BA = \begin{bmatrix} 2\times 1 + 2\times 2 - 4\times 0 & 2\times(-1) + 2\times 3 - 4\times 1 & 2\times 0 + 2\times 4 - 4\times 2 \\ -4+4 & 4+6-4 & 8-8 \\ 2-2 & -2-3+5 & -4+10 \end{bmatrix}$$

$$BA = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I$$

System of equations x – y = 3, 2x + 3y + 4z = 17, y + 2z = 7, can be written as

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, C = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$AX = C$$

BA = 6I ⇒ B = 6I A^{-1} ⇒ $A^{-1} = \frac{1}{6}B$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

$$X = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$X = \frac{1}{6}\begin{bmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{12}{6} \\ \frac{-6}{6} \\ \frac{24}{6} \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

x = 2, y = −1, z = 4

35. Let the equation of the variable plane be

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ….. (i)

This plane cuts the coordinate axes at A, B, C i.e. x-axis, y-axis and z-axis at the points A(a, 0, 0), B(0, b, 0) and C(0, 0, c) respectively.

Let (p, q, r) be the coordinates of the centroid of ΔABC.

Then, $p = \frac{a + 0 + 0}{3}, q = \frac{0 + b + 0}{3}, r = \frac{0 + 0 + c}{3}$

$\Rightarrow p = \frac{a}{3}, \ q = \frac{b}{3}, \ r = \frac{c}{3}$

$\Rightarrow a = 3p, \ b = 3q, \ c = 3r$ ….. (ii)

Therefore, 3k = length of the perpendicular from (0, 0, 0) to the plane (i)

$$\Rightarrow 3k = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = \frac{1}{3k}$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{9k^2}$$

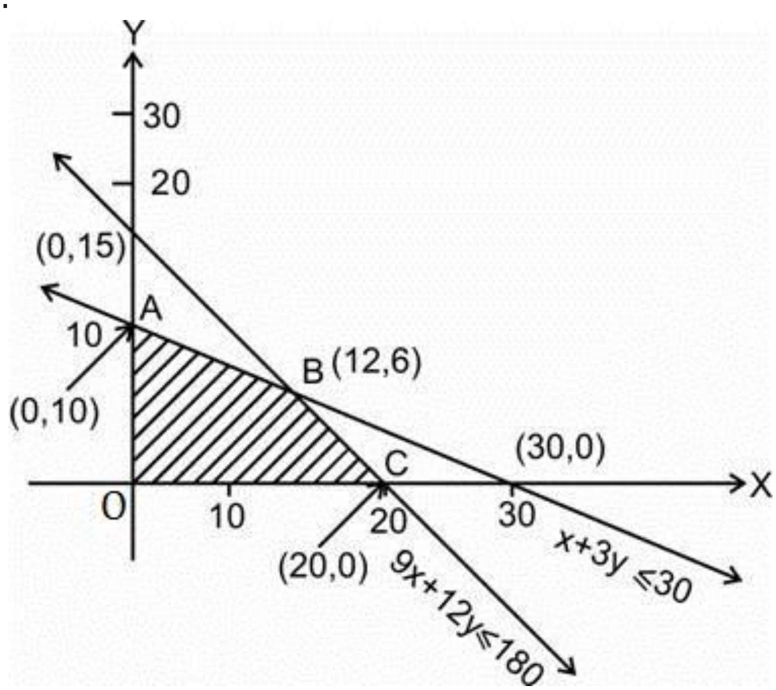
$$\Rightarrow \frac{1}{9p^2}+\frac{1}{9q^2}+\frac{1}{9r^2} = \frac{1}{9k^2}$$

$$\Rightarrow \frac{1}{p^2}+\frac{1}{q^2}+\frac{1}{r^2} = \frac{1}{k^2}$$

$$\Rightarrow p^{-2}+q^{-2}+r^{-2} = k^{-2}$$

Hence, the required locus is $x^{-2}+y^{-2}+z^{-2}=k^{-2}$.

36. Suppose *x* is the number of pieces of Model A and *y* is the number of pieces of Model B.
Then, total profit (in Rs.) = $8000x + 12000y$
Let Z = $8000x + 12000y$
Mathematical statement for the given problem is as follows:
Maximise Z = $8000\,x + 12000\,y$ ... (1)
subject to the constraints,
$9x + 12y \le 180$ (Fabrication constraint) i.e. $3x + 4y \le 60$ ……… (2)
$x + 3y \le 30$ (Finishing constraint) ……… (3)
$x \ge 0, y \ge 0$ ……… (4)
The feasible region (shaded) OABC determined by the linear inequalities (2) to (4) is shown below.



Corner Point	$Z = 8000x + 12000y$
A(0, 10)	120000
B(12, 6)	168000 ⟵ Maximum
C(20, 0)	160000

The company should produce 12 pieces of Model A and 6 pieces of Model B to realise maximum profit and the maximum profit will be Rs. 1, 68, 000.

37. Let Success: Getting a purple ball on a draw

Let E_1 be the event that a red ball is transferred from bag A to bag B

Let E_2 be the event that a black ball is transferred from bag A to bag B

$\therefore E_1$ and E_2 are mutually exclusive and exhaustive.

$P(E_1) = 3/7$; $P(E_2) = 4/7$

Let E be the event that a red ball is drawn from bag

$$P(E|E_1) = \frac{4+1}{(4+1)+5} = \frac{5}{10} = \frac{1}{2}$$

$$P(E|E_2) = \frac{3+1}{(5+1)+4} = \frac{4}{10} = \frac{2}{5}$$

(a)

$$\therefore \text{ Required probability } = P(E_2|E) = \frac{P(E|E_2)P(E_2)}{P(E|E_1)P(E_1) + P(E|E_2)P(E_2)}$$

$$= \frac{\frac{4}{10} \times \frac{4}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{4}{10} \times \frac{4}{7}} = \frac{\frac{16}{70}}{\frac{3}{14} + \frac{16}{70}} = \frac{\frac{16}{70}}{\frac{31}{70}} = \frac{16}{31}$$

(b)

$$\therefore \text{ Required probability } = P(E_1|E) = \frac{P(E|E_1)P(E_1)}{P(E|E_1)P(E_1) + P(E|E_2)P(E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{4}{10} \times \frac{4}{7}} = \frac{\frac{3}{14}}{\frac{3}{14} + \frac{16}{70}} = \frac{\frac{3}{14}}{\frac{31}{70}} = \frac{15}{31}$$

OR

The events A, E_1, E_2, E_3, and E_4 are given by

A = event when doctor visits patients late

E_1 = doctor comes by train

E_2 = doctor comes by bus

E_3 = doctor comes by scooter

E_4 = doctor comes by other means of transport

So, $P(E_1) = \frac{3}{10}$, $P(E_2) = \frac{1}{5}$, $P(E_3) = \frac{1}{10}$, $P(E_4) = \frac{2}{5}$

$P(A/E_1)$ = Probability that the doctor arrives late, given that he is comes by train.

$= \frac{1}{4}$

Similarly $P(A/E_2) = \frac{1}{3}$, $P(A/E_3) = \frac{1}{12}$, $P(A/E_4) = 0$

Required probability of the doctor arriving late by train by using Baye's theorem, $P(E_1/A)$

$$= \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3) + P(E_4)P(A/E_4)}$$

$$= \frac{\frac{3}{10} \times \frac{1}{4}}{\frac{3}{10} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{10} \times \frac{1}{12} + \frac{2}{5} \times 0}$$

$$= \frac{3}{40} \times \frac{120}{18} = \frac{1}{2}$$

Hence the required probability is $\frac{1}{2}$.