Relations and Functions

Cartesian Product of Sets

- Let P and Q be two non-empty sets. The **Cartesian product** of sets P and Q is denoted by $P \times Q$ and it is defined as the set of all ordered pairs of elements from P and Q i.e., $P \times Q = \{(p, q): p \in P, q \in Q\}$

- For e.g., The Cartesian product of sets $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ is $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

- Two ordered pairs are **equal** if and only if their corresponding first elements are equal and the second elements are also equal.

- If $n(A) = p$ and $n(B) = q$,then $n(A \times B) = pq$

- If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.

- $A \times A \times A = \{(a, b, c): a, b, c \in A\}$. Here, (a, b, c) is called an **ordered triplet**.

Example 1: Let $A = \{1, 9\}$, $B = \{2, 4, 10, 11\}$ and $C = \{2, 4, 6, 10\}$.

Find $A \times (B \cap C)$ and show that it equals $(A \times B) \cap (A \times C)$.

Solution:

$B \cap C = \{2, 4, 10\}$

$A \times (B \cap C) = \{1, 9\} \times \{2, 4, 10\}$

$A \times (B \cap C) = \{(1, 2), (1, 4), (1, 10), (9, 2), (9, 4), (9, 10)\}$ … (1)

Now, we have to show that $A \times (B \cap C)$ equals $(A \times B) \cap (A \times C)$.

$A \times B = \{(1, 2), (1, 4), (1, 10), (1, 11), (9, 2), (9, 4), (9, 10), (9, 11)\}$

$A \times C = \{(1, 2), (1, 4), (1, 6), (1, 10), (9, 2), (9, 4), (9, 6), (9, 10)\}$

$(A \times B) \cap (A \times C) = \{(1,2), (1,4), (1,10), (9,2), (9,4), (9,10)\}$ … (2)

From equations (1) and (2), we obtain

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Example 2: Let $A = \{1. -1\}$ and $B = \{0, 1, 9\}$. How many subsets will $A \times B$ have?

Solution:

We have $n(A) = 2$ and $n(B) = 3$.

Accordingly, $n(A \times B) = 2 \times 3 = 6$

Number of subsets of $A \times B = 2^6 = 64$

Thus, $A \times B$ has 64 subsets.

Relations

- A **relation** R from a non-empty set A to a non-empty set B is a subset of the Cartesian product $A \times B$. The subset is derived by describing a relation between the first element and the second element of the ordered pairs in $A \times B$.

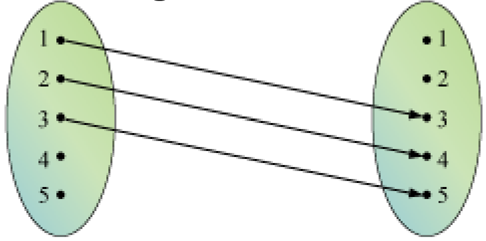
- The second element is called the **image** of the first element.

- A relation R can be represented algebraically by two methods: **Roster method** and **Set-builder method**.
  It can be visually represented by an **arrow diagram.**

- If we have a set $A = \{1, 2, 3, 4, 5\}$ and we define a relation R from A to A such that the second element of the ordered pair is two greater than the first element, then we can represent it in three forms, i.e., Roster form, Set-builder form and arrow diagram as
  Roster form
  R = {(1, 3), (2, 4), (3, 5)}

  Set-builder form
  R =$\{(x, y): x, y \in A$ and $y = x + 2\}$

  Arrow diagram

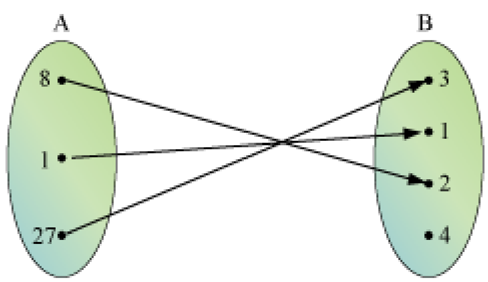


- Few terms associated with any relation R from set A to set B are as follows
- The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the **domain** of R.
- The set of all second elements in a relation R from a set A to a set B is called the **range** of R.
- The whole of set B in a relation R from a set A to a set B is the **codomain** of R.
- Range is a part of set B and codomain is the whole set B. Hence, we can say that range is the subset of codomain, i.e., range ≤ codomain.

Example 1: For the given arrow diagram of a relation from set A to set B, write the relation in

1. Set-builder form
2. Roster form

Also find the domain and range of the relation.



Solution:

It can be seen in the arrow diagram that the images of the elements in set A are their cube roots.

1. The relation R can be written in set-builder form as
$R = \{(x, y): y \text{ is the cube root of } x, x \in A, y \in B\}$
2. The relation R can be written in roster form as
$R = \{(8, 2), (1, 1), (27, 3)\}$

The domain of R is the set of first elements of the ordered pair in R.

∴ Domain = {8, 1, 27)

The range is the set of the second elements of the ordered pair in R.

∴ Range = {2, 1, 3}

Concept of Functions

Consider the relation R from set A to set B, where $A = \{5, 6, 7, 8, 9, 10\}$ and $B = \{7, 8, 9, 10, 11\}$ defined by $R = \{(5, 7), (6, 8), (7, 9), (8, 10)\}$.

Do you observe anything about the images of the elements in set A?

Observe that every element in set A has one and only one image in set B. We say that this type of relation is a **function**.

A relation R from a set A to a set B is said to be a function if for every a in A, there is a unique b in B such that (a, b) ∈ R.

In simple words, we can say that R is a relation from a non-empty set A to a non-empty set B such that the domain of R is A and no two distinct ordered pairs in R have the same first element.

Since a function is a special type of relation, we can define the domain, range, and codomain of the function as we did in the case of relations.

If R is a function from A to B and $(a, b) \in R$, then b is called the **image** of a under the relation R and a is called the **preimage** of b under R.

For a function R from set A to set B, set A is the **domain** of the function; the images of the elements in set A or the second elements in the ordered pairs form the **range**,while the whole of set B is the **codomain** of the function.

Let us now consider a relation R from set X to set Y, where $X = \{1, 2, 3, 4\}$ and $Y = \{1, 8, 27, 64\}$ defined by $R = \{(1, 1), (2, 8), (3, 27), (4, 64)\}$.

Clearly, we can observe that relation R is a function.

Do you notice any rule associated with the elements of the function?

Observe that the rule associated is that “the image b is the cube of the preimage a.”

Hence, we can write $b = a^3$.

This is the **equation form** of the function R.

If we rename the function as f, then we express the function as follows:

$f: A \to B$

$b = f(a) = a^3$

Or

$f: a \to a^3$

Or

$f = \{(x, f(x) = x^3, x \in A\}$

Example 1 Is the relation $R = \{(1, 1), (0, 0), (2, 2), (3, 5), (4, 4)\}$ a function? What are its domain and range?

Solution:

The first elements of the ordered pairs are 1, 0, 2, 3 and 4.

The second elements of the ordered pairs or the images of the first elements are 1, 0, 2, 5 and 4.

For a relation to be a function, each first element should have one and only one image.

It can be seen that in the given relation, all the first elements have one and only one image. Hence, the given relation is a function.

The set of first elements form the domain.

$\therefore$ Domain = $\{1, 0, 2, 3, 4\}$

The set of images or the second elements form the range.

$\therefore$ Range = $\{1, 0, 2, 5, 4\}$

Example 2 For the function $f: x \to 2x^2 - 1, x \in \{-2, -1, 0, 1, 2\}$ (a) draw an arrow diagram (b) write it in roster form, and (c) find the domain and range.

Solution:

$f: x \to 2x^2 - 1, x \in \{-2, -1, 0, 1, 2\}$

We have,

$f(-2) = 2(-2)^2 - 1 = 7$

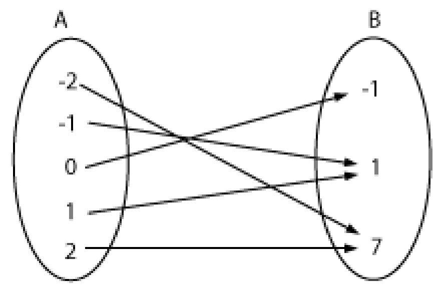
$f(-1) = 2(-1)^2 - 1 = 1$

$f(0) = 2(0)^2 - 1 = -1$

$f(1) = 2(1)^2 - 1 = 1$

$f(2) = 2(2)^2 - 1 = 7$

(a) The arrow diagram for the given function can be drawn as

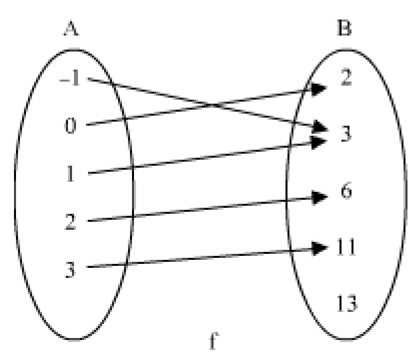


(b) The given function can be written in roster form as

$f = \{(-2, 7), (-1, 1), (0, -1), (1, 1), (2, 7)\}$

(c) Domain of $f = \{-2, -1, 0, 1, 2\}$; Range of $f = \{-1, 1, 7\}$

Example 3 Consider the following arrow diagram.



(i) Write this relation in roster form

(ii) Is it a function? If yes, then write it in equation form.

(iii) Find the domain, co-domain and range.

Solution:

(i) Relation *f* can be written in roster form as

$$f = \{(-1,3)(0,2),(1,3),(2,6),(3,11)\}$$

(ii) Since each element in *A* has a unique image, *f* is a function.

Each image in *B* is 2 more than the square of pre-image.

Hence, the formula for *f* is

$f(x) = x^2 + 2$ Or $f : x \rightarrow x^2 + 2$

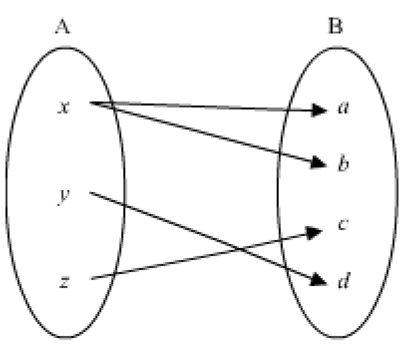
(iii) Domain = {−1, 0, 1, 2, 3}

Co-domain = {2, 3, 6, 11, 13}

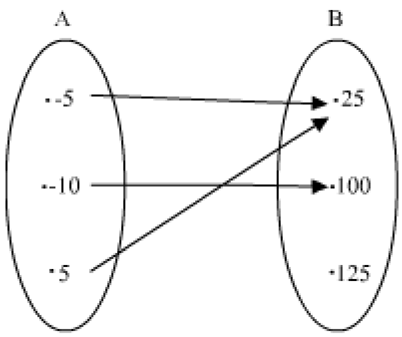
Range = {2, 6, 3, 11}

Example 4 Consider the following arrow diagrams.

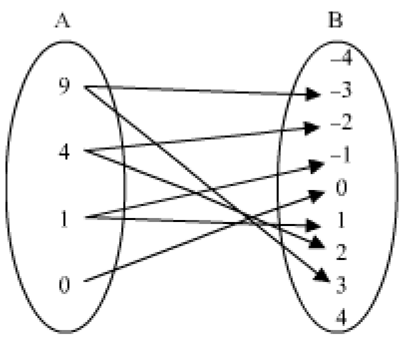
(a)



(b)



(c)



In each case, write the roster form of the relation.

Which of these are not functions?

Solution:

(a) $R = \{(x, a), (x, b), (y, d), (z, c)\}$

Relation R is not a function because element x in set A has two images i.e., a and b in set A.

(b) $R = \{(-5, 25), (-10, 100), (5, 25)\}$

Relation R is a function because every element in set A has a unique image in set B.

(c) $R = \{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

This relation is not a function since elements 9, 4, and 1 in set A do not have a unique image in set B.

Real Valued Functions

Definition of Real Valued function

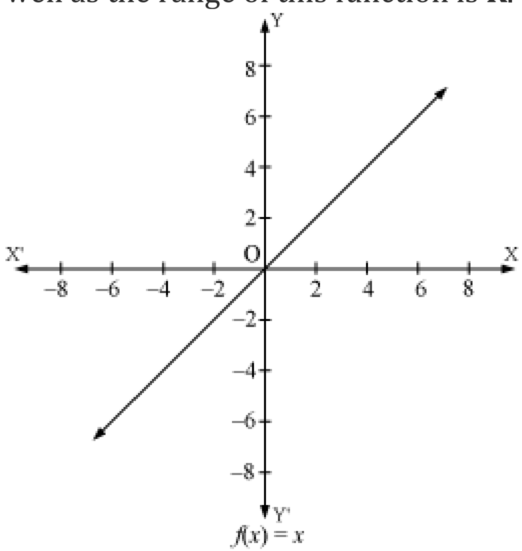
- A function that has either **R** (set of real numbers) or one of its subsets as its range is called a **real valued function**.

- If the domain of a real valued function is also either R or a subset of R, then it is called a **real function**.

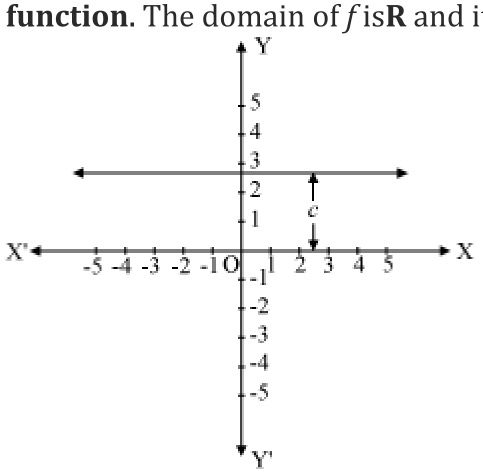
- E.g., a function f: N → N defined by $f(x) = 3x$ is a real function as the domain as well as the range of the function is **N** (set of natural numbers), which is a subset of **R**.

Graphs of Some Common Real Valued Functions

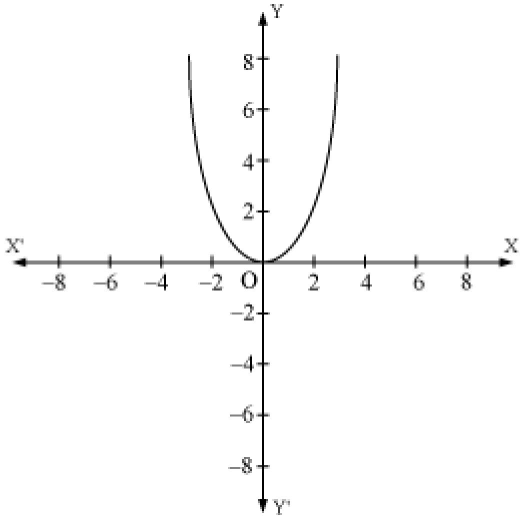
- A real function f: **R** → **R** defined by $y = f(x) = x$ is called an **identity function**. The domain as well as the range of this function is **R**. Its graph is a straight line passing through the origin.



- A function f: **R** → **R** defined by $y = f(x) = c$, $x \in$ **R**,where c is a constant is called a **constant function**. The domain of f is**R** and its range is $\{c\}$.

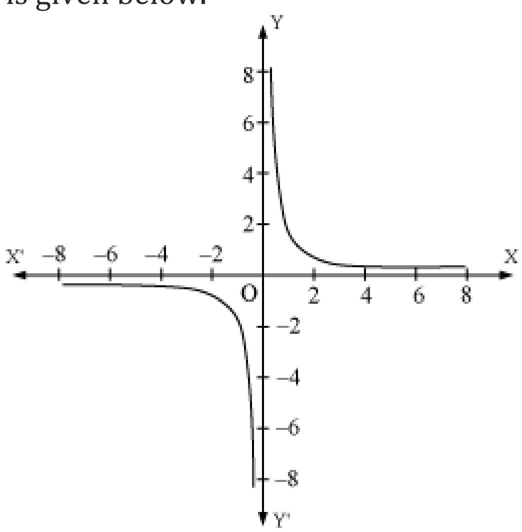


- A function f: **R** → **R** is said to be a **polynomial function**if for each x in **R**, $y = f(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$, where n is a non-negative integer and $a_0, a_1, a_2 \ldots a_n \in$ **R**. E.g., $f(x) = x^2$ is a polynomial function and its graph is given below.



$f(x) = x^2$

- **Rational functions** are of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$. E.g., $f(x) = \frac{1}{x}$ is a rational function and its graph is given below.

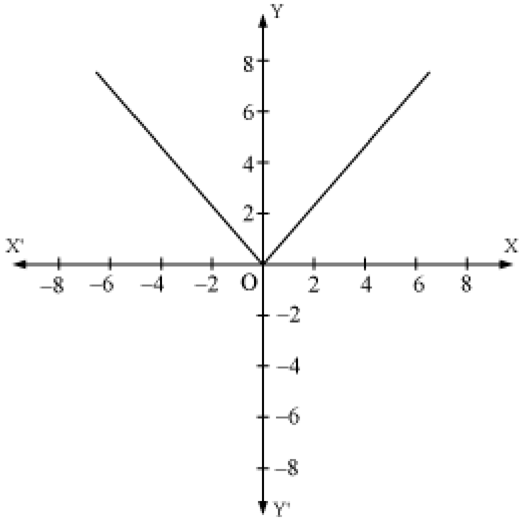


$f(x) = \frac{1}{x}$

- The function f: **R** → **R** defined by $y = f(x) = |x|$ for each $x \in$ **R** is called **modulus function**. For non-negative values of x, $f(x) = x$ and for negative values of x, $f(x) = -x$. Modulus function can also be defined as

$$f(x)=\begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

The graph of the modulus function is given below.

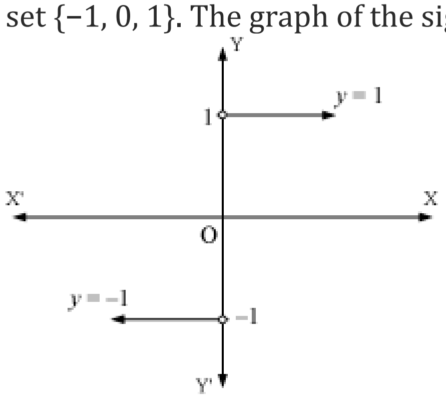


$f(x) = |x|$

- The function f: **R** → **R** defined by

$$f(x)=\begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$$

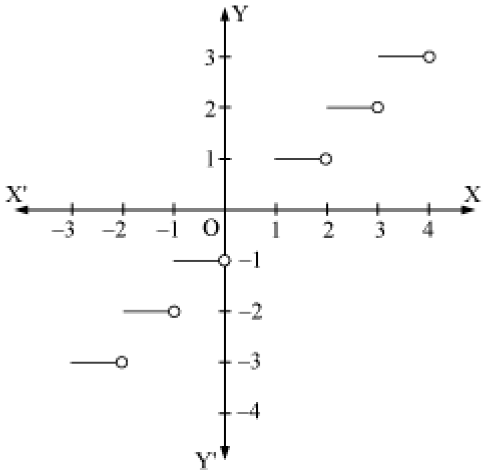
is called the **signum function**. The domain of the signum function is **R** and the range is the set {−1, 0, 1}. The graph of the signum function is given below.



$f(x) = \frac{|x|}{x}$, $x \neq 0$ and 0 for $x = 0$

- The function f: **R** → **R** defined by $f(x) = [x]$, $x \in$ **R** assumes the value of the greatest integer, less than or equal to x. Such a function is called the **greatest integer function**. E.g., $[x] = 3$

for $3 \le x < 4$. The graph of the function is given below.



$f(x) = [x]$

Solved Examples

Example 1: Define the function f: R → R by $y = f(x) = 2x^2 + 1$, $x \in$ R. Complete the given table by using this function. What is the domain and range of this function? Draw the graph of f.

x	−3	−2	−1	0	1	2	3
$y = f(x) = 2x^2 + 1$	-	-	-	-	-	-	-

Solution:

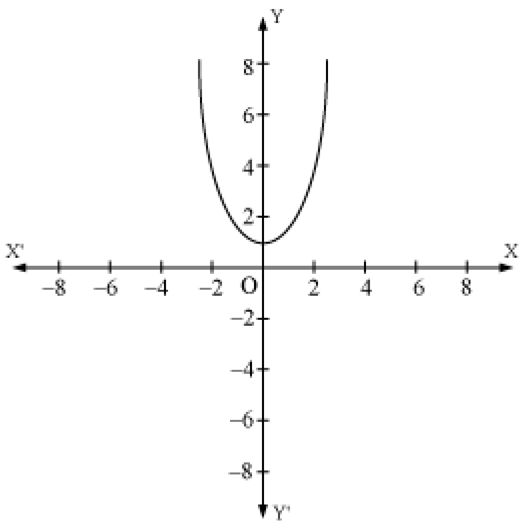
The completed table is as follows:

x	−3	−2	−1	0	1	2	3
$y = f(x) = 2x^2 + 1$	19	9	3	1	3	9	19

Domain of $f = \{x: x \in \mathbf{R}\}$

Range of $f = \{x: x \geq 1, x \in \mathbf{R}\}$

The graph of f is given below.



$f(x) = 2x^2 + 1$

Example 2: Find the domain and range of $f(x) = 4-|x-4|$.

Solution:

We have $f(x) = 4-|x-4|$

Now, $f(x)$ is defined for all $x \in \mathbf{R}$.

Domain of $f = \mathbf{R}$

Now, let us find the range of f.

We know that $|x| \geq 0$ for all $x \in \mathbf{R}$.

Hence, we have

$$|x-4| \geq 0$$
$$-|x-4| \leq 0$$
$$4-|x-4| \leq 4$$
$$f(x) \leq 4$$
$$f(x) \in (-\infty, 4]$$

Hence, range of f = $(-\infty, 4]$

Addition and Subtraction of Functions

- Let f: $X \to \mathbf{R}$ and g: $X \to \mathbf{R}$ be any two real functions, where $X \subset \mathbf{R}$. Then, we define $(f+g)$: $X \to \mathbf{R}$ by
  $(f+g)(x) = f(x) + g(x)$,for all $x \in X$
- For e.g., $f(x) = x^2$and $g(x) = x$, then $(f+g)(x) = f(x) + g(x) = x^2 + x$
- Let f: $X \to \mathbf{R}$ and g: $X \to \mathbf{R}$ be any two real functions, where $X \subset \mathbf{R}$. Then, we define $(f-g)$: $X \to \mathbf{R}$ by
  $(f-g)(x) = f(x) - g(x)$, for all $x \in X$
- For e.g., $f(x) = x^2$, $g(x) = x$, then $(f-g)(x) = f(x) - g(x) = x^2 - x$

Solved Examples

Example 1: Two functions $f(x)$ and $g(x)$ are such that $f(x)=\sqrt{x}$ and $g(x)=-x^2$.

Find $(f+g)(x)$ and $(f-g)(x)$.

Solution:

$(f+g)(x) = f(x) + g(x) = \sqrt{x}+(-x^2) = \sqrt{x}-x^2$

$(f-g)(x) = f(x) - g(x) = \sqrt{x}-\left(-x^2\right)=\sqrt{x}+x^2$

Multiplication of Functions

- **Multiplication of a function by a scalar**
  Let f: $X \to \mathbf{R}$ be a real valued function and α be a scalar that is a real number. Then, the product αf is a function from X to $\mathbf{R}$ defined by $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

- **Multiplication of two real functions**
  The product (or multiplication) of two real functions $f: X \to \mathbf{R}$ and $g: X \to \mathbf{R}$ is a function $fg: X \to \mathbf{R}$ defined by
  $(fg)(x) = f(x)\, g(x)$, for all $x \in X$.

- This multiplication is also called **pointwise multiplication**.

Solved Examples

Example 1: Two functions $f(x)$ and $g(x)$ are such that $f(x) = 3x + 1$ and $g(x) = x^3$. Find $(fg)(x)$.

Solution:

$(fg)(x) = f(x)\, g(x) = (3x + 1)(x^3) = 3x^4 + x^3$

Example 2: If $f(x) = 3x^2$, then find the product $2 \times f$.

Solution:

$(\alpha f)(x) = \alpha f(x) = 2(3x^2) = 6x^2$

Quotient of Two Real Functions

Quotient of Two Real Functions

- Let f and g be two real functions defined from $X \to \mathbf{R}$, where $X \subset \mathbf{R}$.
  The **quotient** of f by g denoted by $\frac{f}{g}$ is a function defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, for all $x \in X$.

- For e.g., $f(x) = x^3$ and $g(x) = x^{\frac{-1}{2}}$, $\left(\frac{f}{g}\right)(x) = \frac{x^3}{x^{\frac{-1}{2}}} = x^{\frac{7}{2}}$

Solved Examples

Example 1: Three functions, *f* (*x*), *g* (*x*) and *h* (*x*) are such that $f(x) = x^2 - x$, $g(x) = 6 - 4x$ and $h(x) = x^2 + 2x - 8$. Find $\frac{(f+g)(x)}{h(x)}$.

Solution:

$(f+g)(x) = x^2 - x + 6 - 4x = x^2 - 5x + 6 = (x-3)(x-2)$

$h(x) = x^2 + 2x - 8 = (x-2)(x+4)$

$\frac{(f+g)(x)}{h(x)} = \frac{(x-3)(x-2)}{(x-2)(x+4)}$, where $x \neq 2,\ x \neq -4$

$= \frac{x-3}{x+4}$