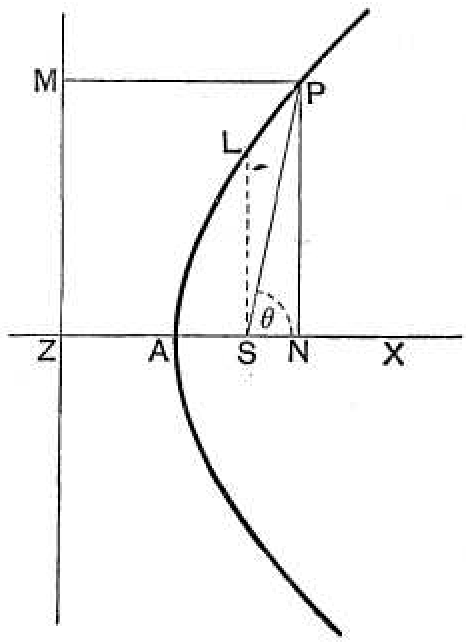
Chapter 14

POLAR EQUATION OF A CONIC SECTION ITS FOCUS BEING THE POLE

335. Let S be the focus, A the vertex, and ZM the directrix; draw SZ perpendicular to ZM.

Let ZS be chosen as the positive direction of the initial line, and produce it to X.



Take any point P on the curve, and let its polar coordinates be r and θ, so that we have

$$SP = r, \text{ and } \angle XSP = \theta.$$

Draw PN perpendicular to the initial line, and PM perpendicular to the directrix.

Let SL be the semi-latus-rectum, and let $SL = l$.

Since $SL = e \,.\, SZ$, we have

$$SZ = \frac{l}{e}.$$

Hence

$$r = SP = e \,.\, PM = e \,.\, ZN$$
$$= e\,(ZS + SN)$$
$$= e\left(\frac{l}{e} + SP \,.\, \cos\theta\right) = l + e \,.\, r \,.\, \cos\theta.$$

Therefore $$\mathbf{r = \frac{1}{1 - e\cos\theta}} \quad \text{.....................(1).}$$

This, being the relation holding between the polar coordinates of *any* point on the curve, is, by Art. 42, the required polar equation.

Cor. If SZ be taken as the positive direction of the initial line and the vectorial angle measured clockwise, the equation to the curve is

$$r = \frac{l}{1 + e\cos\theta}.$$

336. If the conic be a parabola, we have $e = 1$, and the equation is

$$r = \frac{l}{1 - \cos\theta} = \frac{l}{2\sin^2\frac{\theta}{2}} = \frac{l}{2}\operatorname{cosec}^2\frac{\theta}{2}.$$

If the initial line, instead of being the axis, be such that the axis is inclined at an angle γ to it, then, in the previous article, instead of θ we must substitute $\theta - \gamma$.

The equation in this case is then

$$\frac{l}{r} = 1 - e\cos(\theta - \gamma).$$

337. *To trace the curve* $\frac{l}{r} = 1 - e\cos\theta$.

Case I. $\mathbf{e = 1}$, so that the equation is $\frac{l}{r} = 1 - \cos\theta$.

When θ is zero, we have $\frac{l}{r} = 0$, so that r is infinite. As θ increases from $0°$ to $90°$, $\cos\theta$ decreases from 1 to 0, and hence $\frac{l}{r}$ increases from 0 to 1, *i.e.* r decreases from infinity to l.

As θ increases from $90°$ to $180°$, $\cos\theta$ decreases from 0 to -1, and hence $\frac{l}{r}$ increases from 1 to 2, *i.e.* r decreases from l to $\frac{1}{2}l$.

Similarly, as θ changes from $180°$ to $270°$, r increases from $\frac{l}{2}$ to l, and, as θ changes from $270°$ to $360°$, r increases from l to ∞.

The curve is thus the parabola $\infty FPLAL'P'F' \infty$ of Art. 197.

Case II. e < 1. When θ is zero, we have $\frac{l}{r} = 1 - e$, *i.e.* $r = \frac{l}{1-e}$. This gives the point A' in the figure of Art. 247.

As θ increases from $0°$ to $90°$, $\cos\theta$ decreases from 1 to 0, and therefore $1 - e\cos\theta$ increases from $1 - e$ to 1, *i.e.* $\frac{l}{r}$ increases from $1 - e$ to 1, *i.e.* r decreases from $\frac{l}{1-e}$ to l. We thus obtain the portion $A'PBL$.

As θ increases from $90°$ to $180°$, $\cos\theta$ decreases from 0 to -1, and therefore $1 - e\cos\theta$ increases from 1 to $1 + e$, *i.e.* $\frac{l}{r}$ increases from 1 to $1 + e$, *i.e.* r decreases from l to $\frac{l}{1+e}$.

We thus obtain the portion LA of the curve, where

$$SA = \frac{l}{1+e}.$$

Similarly, as θ increases from $180°$ to $270°$ and then to $360°$, we have the portions AL' and $L'B'P'A'$.

Since $\cos\theta = \cos(-\theta) = \cos(360° - \theta)$, the curve is symmetrical about the line SA'.

Case III. e > 1. When θ is zero, $1 - e\cos\theta$ is equal to $1 - e$, *i.e.* $-(e-1)$, and is therefore a negative quantity, since $e > 1$. This zero value of θ gives $r = -l \div (e-1)$.

We thus have the point A' in the figure of Art. 295.

Let θ increase from $0°$ to $\cos^{-1}\left(\frac{1}{e}\right)$. Thus $1 - e\cos\theta$ increases algebraically from $-(e-1)$ to -0,

i.e. $\frac{l}{r}$ increases algebraically from $-(e-1)$ to -0,

i.e. r decreases algebraically from $-\frac{l}{e-1}$ to $-\infty$.

For these values of θ the radius vector is therefore negative and increases in numerical length from $\frac{l}{e-1}$ to ∞.

We thus have the portion $A'P_1'R'\infty$ of the curve. For this portion r is negative.

If θ be very slightly greater than $\cos^{-1}\frac{1}{e}$, then $\cos\theta$ is slightly less than $\frac{1}{e}$, so that $1-e\cos\theta$ is small and positive, and therefore r is very great and is positive. Hence, as θ increases through the angle $\cos^{-1}\frac{1}{e}$, the value of r changes from $-\infty$ to $+\infty$.

As θ increases from $\cos^{-1}\frac{1}{e}$ to π, $1-e\cos\theta$ increases from 0 to $1+e$ and hence r decreases from ∞ to $\frac{l}{1+e}$. Now $\frac{l}{1+e}$ is $< \frac{l}{e-1}$. Hence the point A, which corresponds to $\theta=\pi$, is such that $SA < SA'$.

For values of θ between $\cos^{-1}\frac{1}{e}$ and π we therefore have the portion, ∞RPA, of the curve. For this portion r is positive.

As θ increases from π to $2\pi-\cos^{-1}\frac{1}{e}$, $e\cos\theta$ increases from $-e$ to 1, so that $1-e\cos\theta$ decreases from $1+e$ to 0, and therefore r increases from $\frac{l}{1+e}$ to ∞. Corresponding to these values of θ we have the portion $AL'R_1\infty$ of the curve, for which r is positive.

Finally, as θ increases from $2\pi-\cos^{-1}\frac{1}{e}$ to 2π, $e\cos\theta$ increases from 1 to e, so that $1-e\cos\theta$ decreases algebraically from 0 to $1-e$, *i.e.* $\frac{l}{r}$ is negative and increases numerically from 0 to $e-1$, and therefore r is negative and decreases from ∞ to $\frac{l}{e-1}$. Corresponding to these values of θ we have the portion, $\infty R_1'A'$, of the curve. For this portion r is negative.

r is therefore always positive for the right-hand branch of the curve and negative for the left-hand branch.

It will be noted that the curve is described in the order

$$A'P_1'R' \infty \infty RPAL'R_1 \infty \infty R_1'A'.$$

338. In Case III. of the last article, let any straight line be drawn through S to meet the nearer branch in p, and the further branch in q.

The vectorial angle of p is XSp, and we have

$$Sp = \frac{l}{1 - e\cos XSp}.$$

The vectorial angle of q is not XSq but the angle that qS produced makes with SX, *i.e.* it is $XSq \mp \pi$. Also for the point q the radius vector is negative so that the relation (1) of Art. 335 gives, for the point q,

$$-Sq = \frac{l}{1 - e\cos(XSq \mp \pi)} = \frac{l}{1 + e\cos XSq},$$

i.e.
$$Sq = -\frac{l}{1 + e\cos XSq}.$$

This is the relation connecting the distance, Sq, of any point on the further branch of the hyperbola with the angle XSq that it makes with the initial line.

339. *Equation to the directrices.*

Considering the figure of Art. 295, the numerical values of the distances SZ and SZ' are $\frac{l}{e}$ and $\frac{l}{e} + 2CZ$,

i.e.
$$\frac{l}{e} \text{ and } \frac{l}{e} + 2\frac{l}{e(e^2-1)},$$

since
$$CZ = \frac{a}{e} = \frac{l}{e(e^2-1)}. \qquad \text{[Art. 300.]}$$

The equations to the two directrices are therefore

$$r\cos\theta = -\frac{l}{e},$$

and
$$r\cos\theta = -\left[\frac{l}{e} + \frac{2l}{e(e^2-1)}\right] = -\frac{l}{e}\frac{e^2+1}{e^2-1}.$$

The same equations would be found to hold in the case of the ellipse.

340. *Equation to the asymptotes.*

The perpendicular distance from S upon an asymptote (Fig., Art. 315)

$$= CS \sin ACK_1 = ae \cdot \frac{b}{\sqrt{a^2+b^2}} = b.$$

Also the asymptote CQ makes an angle $\cos^{-1}\frac{1}{e}$ with the axis. The perpendicular on it from S therefore makes an angle $\frac{\pi}{2}+\cos^{-1}\frac{1}{e}$.

Hence, by Art. 88, the polar equation to the asymptote CQ is

$$b = r\cos\left[\theta - \frac{\pi}{2} - \cos^{-1}\frac{1}{e}\right] = r\sin\left[\theta - \cos^{-1}\frac{1}{e}\right].$$

The polar equation to the other asymptote is similarly

$$b = r\cos\left[\theta - \left(\frac{3\pi}{2} - \cos^{-1}\frac{1}{e}\right)\right] = -r\sin\left(\theta + \cos^{-1}\frac{1}{e}\right).$$

341. Ex. 1. *In any conic, prove that*

(1) *the sum of the reciprocals of the segments of any focal chord is constant, and*

(2) *the sum of the reciprocals of two perpendicular focal chords is constant.*

Let PSP' be any focal chord, and let the vectorial angle of P be α, so that the vectorial angle of P' is $\pi+\alpha$.

(1) By equation (1) of Art. 335, we have

$$\frac{l}{SP} = 1 - e\cos\alpha,$$

and $$\frac{l}{SP'} = 1 - e\cos(\pi+\alpha) = 1 + e\cos\alpha.$$

Hence $$\frac{l}{SP} + \frac{l}{SP'} = 2,$$

so that $$\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}.$$

The semi-latus-rectum is therefore the harmonic mean between the segments of any focal chord.

(2) Let QSQ' be the focal chord perpendicular to PSP', so that the vectorial angles of Q and Q' are $\frac{\pi}{2}+\alpha$ and $\frac{3\pi}{2}+\alpha$. We then have

$$\frac{l}{SQ}=1-e\cos\left(\frac{\pi}{2}+\alpha\right)=1+e\sin\alpha,$$

and $$\frac{l}{SQ'}=1-e\cos\left(\frac{3\pi}{2}+\alpha\right)=1+e\cos\left(\frac{\pi}{2}+\alpha\right)=1-e\sin\alpha.$$

Hence

$$PP'=SP+SP'=\frac{l}{1-e\cos\alpha}+\frac{l}{1+e\cos\alpha}=\frac{2l}{1-e^2\cos^2\alpha},$$

and $$QQ'=SQ+SQ'=\frac{l}{1+e\sin\alpha}+\frac{l}{1-e\sin\alpha}=\frac{2l}{1-e^2\sin^2\alpha}.$$

Therefore

$$\frac{1}{PP'}+\frac{1}{QQ'}=\frac{1-e^2\cos^2\alpha}{2l}+\frac{1-e^2\sin^2\alpha}{2l}=\frac{2-e^2}{2l},$$

and is therefore the same for all such pairs of chords.

Ex. 2. *Prove that the locus of the middle points of focal chords of a conic section is a conic section.*

Let PSQ be any chord, the angle PSX being θ, so that

$$SP=\frac{l}{1-e\cos\theta},$$

and $$SQ=\frac{l}{1-e\cos(\pi+\theta)}=\frac{l}{1+e\cos\theta}.$$

Let R be the middle point of PQ, and let its polar coordinates be r and θ.

Then $$r=SP-RP=SP-\frac{SP+SQ}{2}=\frac{SP-SQ}{2}$$

$$=\tfrac{1}{2}l\left[\frac{1}{1-e\cos\theta}-\frac{1}{1+e\cos\theta}\right]=l\,\frac{e\cos\theta}{1-e^2\cos^2\theta},$$

i.e. $$r^2-e^2r^2\cos^2\theta=le\,.\,r\cos\theta.$$

Transforming to Cartesian coordinates this equation becomes

$$x^2+y^2-e^2x^2=lex \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If the original conic be a parabola, we have $e=1$, and equation (1) becomes $y^2=lx$, so that the locus is a parabola whose vertex is S and latus-rectum l.

If e be not equal to unity, equation (1) may be written in the form

$$(1-e^2)\left[x-\tfrac{1}{2}\frac{le}{1-e^2}\right]^2+y^2=\frac{l^2e^2}{4(1-e^2)}$$

and therefore represents an ellipse or a hyperbola according as the original conic is an ellipse or a hyperbola.

342. *To find the polar equation of the tangent at any point P of the conic section $\frac{l}{r} = 1 - e\cos\theta$.*

Let P be the point (r_1, α), and let Q be another point on the curve, whose coordinates are (r_1, β), so that we have

$$\frac{l}{r_1} = 1 - e\cos\alpha \quad \text{.....................(1)},$$

and

$$\frac{l}{r_2} = 1 - e\cos\beta \quad \text{.....................(2)}.$$

By Art. 89, the polar equation of the line PQ is

$$\frac{\sin(\beta-\alpha)}{r} = \frac{\sin(\theta-\alpha)}{r_2} + \frac{\sin(\beta-\theta)}{r_1}.$$

By means of equations (1) and (2) this equation becomes

$$\frac{l}{r}\sin(\beta-\alpha) = \sin(\theta-\alpha)\{1-e\cos\beta\} + \sin(\beta-\theta)\{1-e\cos\alpha\}$$

$$= \{\sin(\theta-\alpha)+\sin(\beta-\theta)\} - e\{\sin(\theta-\alpha)\cos\beta + \sin(\beta-\theta)\cos\alpha\}$$

$$= 2\sin\frac{\beta-\alpha}{2}\cos\frac{2\theta-\alpha-\beta}{2}$$

$$-e\{(\sin\theta\cos\alpha - \cos\theta\sin\alpha)\cos\beta + (\sin\beta\cos\theta - \cos\beta\sin\theta)\cos\alpha\}$$

$$= 2\sin\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) - e\cos\theta\sin(\beta-\alpha),$$

i.e.

$$\frac{l}{r} = \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) - e\cos\theta \quad \text{.........(3)}.$$

This is the equation to the straight line joining two points, P and Q, on the curve whose vectorial angles, α and β, are given.

To obtain the equation of the tangent at P we take Q indefinitely close to P, *i.e.* we put $\beta = \alpha$, and the equation (3) then becomes

$$\frac{1}{r} = \cos(\theta-\alpha) - e\cos\theta \quad \text{............(4)}.$$

This is the required equation to the tangent at the point α.

343. If we assume a suitable form for the equation to the joining chord we can more easily obtain the required equation.

Let the required equation be

$$\frac{l}{r} = L\cos(\theta - \gamma) - e\cos\theta \text{ .... (1).}$$

[On transformation to Cartesian coordinates this equation is easily seen to represent a straight line; also since it contains two arbitrary constants, L and γ, it can be made to pass through any two points.]

If it pass through the point (r_1, α), we have

$$1 - e\cos\alpha = \frac{l}{r_1} = L\cos(\alpha - \gamma) - e\cos\alpha,$$

i.e.

$$L\cos(\alpha - \gamma) = 1 \text{ .... (2).}$$

Similarly, if it pass through the point (r_2, β) on the curve, we have

$$L\cos(\beta - \gamma) = 1 \text{ .... (3).}$$

Solving these, we have, [since α and β are not equal]

$$\alpha - \gamma = -(\beta - \gamma), \ \ i.e. \ \ \gamma = \frac{\alpha + \beta}{2}.$$

Substituting this value in (3), we obtain $L = \sec\frac{\alpha - \beta}{2}$.

The equation (1) is then

$$\frac{l}{r} = \sec\frac{\alpha - \beta}{2}\cos\left(\theta - \frac{\alpha + \beta}{2}\right) - e\cos\theta.$$

As in the last article, the equation to the tangent at the point α is then

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta.$$

***344.** *To find the polar equation of the* **polar** *of any point* (r_1, θ_1) *with respect to the conic section* $\frac{l}{r} = 1 - e\cos\theta$.

Let the tangents at the points whose vectorial angles are α and β meet in the point (r_1, θ_1).

The coordinates r_1 and θ_1 must therefore satisfy equation (4) of Art. 342, so that

$$\frac{l}{r_1} = \cos(\theta_1 - \alpha) - e\cos\theta_1 \text{ .... (1).}$$

Similarly,

$$\frac{l}{r_1} = \cos(\theta_1 - \beta) - e\cos\theta_1 \text{ .... (2).}$$

Subtracting (2) from (1), we have

$$\cos(\theta_1 - \alpha) = \cos(\theta_1 - \beta),$$

and therefore

$$\theta_1 - \alpha = -(\theta_1 - \beta), \text{ [since } \alpha \text{ and } \beta \text{ are not equal]},$$

i.e.
$$\frac{\alpha+\beta}{2} = \theta_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Substituting this value in (1), we have

$$\frac{l}{r_1} = \cos\left\{\frac{\alpha+\beta}{2} - \alpha\right\} - e\cos\theta_1,$$

i.e.
$$\cos\frac{\beta-\alpha}{2} = \frac{l}{r_1} + e\cos\theta_1 \quad\ldots\ldots\ldots\ldots(4).$$

Also, by equation (3) of Art. 342, the equation of the line joining the points α and β is

$$\frac{l}{r} + e\cos\theta = \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right),$$

i.e.
$$\left(\frac{l}{r} + e\cos\theta\right)\cos\frac{\beta-\alpha}{2} = \cos\left(\theta - \frac{\alpha+\beta}{2}\right),$$

i.e.
$$\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r_1} + e\cos\theta_1\right) = \cos(\theta - \theta_1)\ldots\ldots(5).$$

This therefore is the required polar equation to the polar of the point (r_1, θ_1).

***345.** *To find the equation to the* **normal** *at the point whose vectorial angle is* α.

The equation to the tangent at the point α is

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta,$$

i.e., in Cartesian coordinates,

$$x(\cos\alpha - e) + y\sin\alpha = l \quad\ldots\ldots\ldots\ldots(1).$$

Let the equation to the normal be

$$A\cos\theta + B\sin\theta = \frac{l}{r} \quad\ldots\ldots\ldots\ldots(2),$$

i.e.
$$Ax + By = l \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Since (1) and (3) are perpendicular, we have

$$A(\cos\alpha - e) + B\sin\alpha = 0 \;\ldots\ldots\ldots\ldots\ldots(4).$$

Since (2) goes through the point $\left(\frac{l}{1 - e\cos\alpha}, \alpha\right)$ we have

$$A\cos\alpha + B\sin\alpha = 1 - e\cos\alpha \;\ldots\ldots\ldots\ldots(5).$$

Solving (4) and (5), we have

$$A = \frac{1 - e\cos\alpha}{e}, \text{ and } B = \frac{(1 - e\cos\alpha)(e - \cos\alpha)}{e\sin\alpha}.$$

The equation (2) then becomes

$$\sin\alpha\cos\theta + (e - \cos\alpha)\sin\theta = \frac{le\sin\alpha}{r(1 - e\cos\alpha)},$$

i.e.

$$\sin(\theta - \alpha) - e\sin\theta = -\frac{e\sin\alpha}{1 - e\cos\alpha}\cdot\frac{l}{r}.$$

346. If the axis of the conic be inclined at an angle γ to the initial line, so that the equation to the conic is

$$\frac{l}{r} = 1 - e\cos(\theta - \gamma),$$

the equation to the tangent at the point α is obtained by substituting $\alpha - \gamma$ and $\theta - \gamma$ for α and θ in the equation of Art. 342.

The tangent is therefore

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos(\theta - \gamma).$$

The equation of the line joining the two points α and β is, by the same article,

$$\frac{l}{r} = \sec\frac{\beta - \alpha}{2}\cos\left(\theta - \frac{\alpha + \beta}{2}\right) - e\cos(\theta - \gamma).$$

The equation to the polar of the point (r_1, θ_1) is, by Art. 344,

$$\left\{\frac{l}{r} + e\cos(\theta - \gamma)\right\}\left\{\frac{l}{r_1} + e\cos(\theta_1 - \gamma)\right\} = \cos(\theta - \theta_1).$$

Also the equation to the normal at the point α

$$r\{e\sin(\theta - \gamma) + \sin(\alpha - \theta)\} = \frac{el\sin(\alpha - \gamma)}{1 - e\cos(\alpha - \gamma)}.$$

347. Ex. 1. *If the tangents at any two points P and Q of a conic meet in a point T, and if the straight line PQ meet the directrix corresponding to S in a point K, then the angle KST is a right angle.*

If the vectorial angles of P and Q be α and β respectively, the equation to PQ is, by equation (3) of Art. 342,

$$\frac{l}{r}=\sec\frac{\beta-\alpha}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right)-e\cos\theta\ldots\ldots\ldots\ldots\ldots(1).$$

Also the equation to the directrix is, by Art. 339,

$$\frac{l}{r}=-e\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If we solve the equations (1) and (2), we shall obtain the polar coordinates of K.

But, by subtracting (2) from (1), we have

$$0=\sec\frac{\beta-\alpha}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right),\quad i.e.\ \theta-\frac{\alpha+\beta}{2}=\frac{\pi}{2},$$

i.e.

$$\angle KSX=\frac{\pi}{2}+\frac{\alpha+\beta}{2},$$

so that SK bisects the exterior angle between SP and SQ.

Also, by equation (3) of Art. 344, we have the vectorial angle of T equal to $\frac{\alpha+\beta}{2}$, *i.e.* $\angle TSX=\frac{\alpha+\beta}{2}$.

Hence $$\angle KST=\angle KSX-\angle TSX=\frac{\pi}{2}.$$

Ex. 2. *S is the focus and P and Q two points on a conic such that the angle PSQ is constant and equal to 2δ; prove that*

(1) *the locus of the intersection of tangents at P and Q is a conic section whose focus is S,*

and (2) *the line PQ always touches a conic whose focus is S.*

(1) Let the vectorial angles of P and Q be respectively $\gamma+\delta$ and $\gamma-\delta$, where γ is variable.

By equation (4) of Art. 342, the tangents at P and Q are therefore

$$\frac{l}{r}=\cos(\theta-\gamma-\delta)-e\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$\frac{l}{r}=\cos(\theta-\gamma+\delta)-e\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If, between these two equations, we eliminate the variable quantity γ, we shall have the locus of the point of intersection of the two tangents.

Subtracting (2) from (1), we have

$$\cos(\theta-\gamma-\delta)=\cos(\theta-\gamma+\delta).$$

Hence, (since δ is not zero) we have $\gamma=\theta$.

Substituting for γ in (1), we have

$$\frac{l}{r}=\cos\delta-e\cos\theta,$$

i.e. $$\frac{l\sec\delta}{r}=1-e\sec\delta\cos\theta.$$

Hence the required locus is a conic whose focus is S, whose latus rectum is $2l\sec\delta$, and whose eccentricity is $e\sec\delta$.

It is therefore an ellipse, parabola, or hyperbola, according as $e\sec\delta$ is $<=>1$, *i.e.* according as $\cos\delta>=<e$.

(2) The equation to PQ is, by equation (3) of Art. 342,

$$\frac{l}{r}=\sec\delta\cos(\theta-\gamma)-e\cos\theta,$$

i.e. $$\frac{l\cos\delta}{r}=\cos(\theta-\gamma)-e\cos\delta\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Comparing this with equation (4) of Art. 342, we see that it always touches a conic whose latus rectum is $2l\cos\delta$ and whose eccentricity is $e\cos\delta$.

Also the directrix is in each case the same as that of the original conic. For both $\frac{l\sec\delta}{e\sec\delta}$ and $\frac{l\cos\delta}{e\cos\delta}$ are equal to $\frac{l}{e}$.

Ex. 3. *A circle passes through the focus S of a conic and meets it in four points whose distances from S are r_1, r_2, r_3, and r_4. Prove that*

(1) $r_1r_2r_3r_4=\frac{d^2l^2}{e^2}$, *where $2l$ and e are the latus rectum and eccentricity of the conic, and d is the diameter of the circle,*

and (2) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{l}$.

Take the focus as pole, and the axis of the conic as initial line, so that its equation is

$$\frac{l}{r}=1-e\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If the diameter of the circle, which passes through S, be inclined at an angle γ to the axis, its equation is, by Art. 172,

$$r=d\cos(\theta-\gamma)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If, between (1) and (2), we eliminate θ, we shall have an equation in r, whose roots are r_1, r_2, r_3, and r_4.

From (1) we have $\cos\theta=\frac{r-l}{er}$, and hence $\sin\theta=\sqrt{1-\left(\frac{r-l}{er}\right)^2}$, and then (2) gives

$$r=d\cos\gamma\cos\theta+d\sin\gamma\sin\theta,$$

i.e. $$\{er^2-d\cos\gamma(r-l)\}^2=d^2\sin^2\gamma\,[e^2r^2-(r-l)^2],$$

i.e. $$e^2r^4-2ed\cos\gamma\,.\,r^3+r^2(d^2+2eld\cos\gamma-e^2d^2\sin^2\gamma)-2ld^2r+d^2l^2=0.$$

Hence, by Art. 2, we have

$$r_1r_2r_3r_4 = \frac{d^2l^2}{e^2} \quad\text{..............................(3)},$$

and

$$r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2 + r_1r_2r_3 = \frac{2ld^2}{e^2} \quad\text{...............(4)}.$$

Dividing (4) by (3), we have $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} + \dfrac{1}{r_4} = \dfrac{2}{l}$.

EXAMPLES XXXIX

1. In a parabola, prove that the length of a focal chord which is inclined at 30° to the axis is four times the length of the latus-rectum.

The tangents at two points, P and Q, of a conic meet in T, and S is the focus; prove that

2. if the conic be a parabola, then $ST^2 = SP \, . \, SQ$.

3. if the conic be central, then $\dfrac{1}{SP \, . \, SQ} - \dfrac{1}{ST^2} = \dfrac{1}{b^2}\sin^2\dfrac{PSQ}{2}$, where b is the semi-minor axis.

4. The vectorial angle of T is the semi-sum of the vectorial angles of P and Q.

Hence, by reference to Art. 338, prove that, if P and Q be on different branches of a hyperbola, then ST bisects the supplement of the angle PSQ, and that in other cases, whatever be the conic, ST bisects the angle PSQ.

5. A straight line drawn through the common focus S of a number of conics meets them in the points P_1, P_2, ... ; on it is taken a point Q such that the reciprocal of SQ is equal to the sum of the reciprocals of SP_1, SP_2, ... Prove that the locus of Q is a conic section whose focus is S, and shew that the reciprocal of its latus-rectum is equal to the sum of the reciprocals of the latera recta of the given conics.

6. Prove that perpendicular focal chords of a rectangular hyperbola are equal.

7. PSP' and QSQ' are two perpendicular focal chords of a conic; prove that $\dfrac{1}{PS \, . \, SP'} + \dfrac{1}{QS \, . \, SQ'}$ is constant.

8. Shew that the length of any focal chord of a conic is a third proportional to the transverse axis and the diameter parallel to the chord.

9. If a straight line drawn through the focus S of a hyperbola, parallel to an asymptote, meet the curve in P, prove that SP is one quarter of the latus rectum.

10. Prove that the equations $\frac{l}{r}=1-e\cos\theta$ and $\frac{l}{r}=-e\cos\theta-1$ represent the same conic.

11. Two conics have a common focus; prove that two of their common chords pass through the intersection of their directrices.

12. P is any point on a conic, whose focus is S, and a straight line is drawn through S at a given angle with SP to meet the tangent at P in T; prove that the locus of T is a conic whose focus and directrix are the same as those of the original conic.

13. If a chord of a conic section subtend a constant angle 2α at the focus, prove that the locus of the point where it meets the internal bisector of the angle 2α is the conic section

$$\frac{l\cos\alpha}{r}=1-e\cos\alpha\cos\theta.$$

14. Two conic sections have a common focus about which one of them is turned; prove that the common chord is always a tangent to another conic, having the same focus, and whose eccentricity is the ratio of the eccentricities of the given conics.

15. Two ellipses have a common focus; two radii vectores, one to each ellipse, are drawn from the focus at right angles to one another and tangents are drawn at their extremities; prove that these tangents meet on a fixed conic, and find when it is a parabola.

16. Prove that the sum of the distances from the focus of the points in which a conic is intersected by any circle, whose centre is at a fixed point on the transverse axis, is constant.

17. Shew that the equation to the circle circumscribing the triangle formed by the three tangents to the parabola $r=\frac{2a}{1-\cos\theta}$ drawn at the points whose vectorial angles are α, β, and γ, is

$$r=a\operatorname{cosec}\frac{\alpha}{2}\operatorname{cosec}\frac{\beta}{2}\operatorname{cosec}\frac{\gamma}{2}\sin\left(\frac{\alpha+\beta+\gamma}{2}-\theta\right),$$

and hence that it always passes through the focus.

18. If tangents be drawn to the same parabola at points whose vectorial angles are α, β, γ, and δ, shew that the centres of the circles circumscribing the four triangles formed by these four lines all lie on the circle whose equation is

$$r=-a\operatorname{cosec}\frac{\alpha}{2}\operatorname{cosec}\frac{\beta}{2}\operatorname{cosec}\frac{\gamma}{2}\operatorname{cosec}\frac{\delta}{2}\cos\left[\theta-\frac{\alpha+\beta+\gamma+\delta}{2}\right].$$

19. The circle circumscribing the triangle formed by three tangents to a parabola is drawn; prove that the tangent to it at the focus makes with the axis an angle equal to the sum of the angles made with the axis by the three tangents.

20. Shew that the equation to the circle, which passes through the focus and touches the curve $\frac{l}{r}=1-e\cos\theta$ at the point $\theta=\alpha$, is

$$r(1-e\cos\alpha)^2=l\cos(\theta-\alpha)-el\cos(\theta-2\alpha).$$

21. A given circle, whose centre is on the axis of a parabola, passes through the focus S and is cut in four points A, B, C, and D by any conic, of given latus-rectum, having S as focus and a tangent to the parabola for directrix; prove that the sum of the distances of the points A, B, C, and D from S is constant.

22. Prove that the locus of the vertices of all parabolas that can be drawn touching a given circle of radius a and having a fixed point on the circumference as focus is $r=2a\cos^3\frac{\theta}{3}$, the fixed point being the pole and the diameter through it the initial line.

23. Two conic sections have the same focus and directrix. Shew that any tangent from the outer curve to the inner one subtends a constant angle at the focus.

24. Two equal ellipses, of eccentricity e, are placed with their axes at right angles and they have one focus S in common; if PQ be a common tangent, shew that the angle PSQ is equal to $2\sin^{-1}\frac{e}{\sqrt{2}}$.

25. Prove that the two conics $\frac{l_1}{r}=1-e_1\cos\theta$ and $\frac{l_2}{r}=1-e_2\cos(\theta-\alpha)$ will touch one another, if $l_1^2(1-e_2^2)+l_2^2(1-e_1^2)+2l_1l_2e_1e_2\cos\alpha=0.$

26. An ellipse and a hyperbola have the same focus S and intersect in four real points, two on each branch of the hyperbola; if r_1 and r_2 be the distances from S of the two points of intersection on the nearer branch, and r_3 and r_4 be those of the two points on the further branch, and if l and l' be the semi-latera-recta of the two conics, prove that

$$(l+l')\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+(l-l')\left(\frac{1}{r_3}+\frac{1}{r_4}\right)=4.$$ [Make use of Art. 338.]

27. If the normals at three points of the parabola $r=a\operatorname{cosec}^2\frac{\theta}{2}$, whose vectorial angles are α, β, and γ, meet in a point whose vectorial angle is δ, prove that $2\delta=\alpha+\beta+\gamma-\pi$.

ANSWERS

19. Transform the equation of the previous example to Cartesian Coordinates.

SOLUTIONS/HINTS

1. See Art. 336.

Length of chord $= \frac{l}{2}\left(\operatorname{cosec}^2\frac{\theta}{2} + \sec^2\frac{\theta}{2}\right) = 2l\operatorname{cosec}^2\theta$

$$= 8l, \text{ if } \theta = 30°.$$

2. $SP = a\operatorname{cosec}^2\frac{\alpha}{2}$, $SQ = a\operatorname{cosec}^2\frac{\beta}{2}$.

The tangents at α and β intersect where

$$\theta = \frac{\alpha+\beta}{2}. \quad \text{[Art. 344, equation (3).]}$$

$$ST = \frac{2a}{\cos\frac{\beta-\alpha}{2} - \cos\frac{\beta+\alpha}{2}}, \quad \text{[Art. 342, equation (4)],}$$

$$= a\operatorname{cosec}\frac{\alpha}{2}\,.\operatorname{cosec}\frac{\beta}{2}. \quad \therefore\ ST^2 = SP\,.\,SQ.$$

3. Let $\alpha+\beta$, $\alpha-\beta$, be the vectorial angles of P and Q.

Then, as in Art. 344, $\frac{1}{SP\,.\,SQ} - \frac{1}{ST^2}$

$$= \frac{\{1-e\cos(\alpha+\beta)\}\{1-e\cos(\alpha-\beta)\}-\{\cos\beta-e\cos\alpha\}^2}{l^2}$$

$$= \frac{\sin^2\beta\,(1-e^2)}{l^2} = \frac{\sin^2\beta}{b^2} = \frac{1}{b^2}\sin^2\frac{PSQ}{2}.$$

4. The tangents intersect where $\theta = \frac{\alpha+\beta}{2}$. [Art. 344.]

If P and Q are on opposite branches, then, as in Art. 344, we have $\cos(\theta-\alpha) = \cos(\theta-\beta+\pi)$.

$$\therefore\ \theta-\alpha = -\theta+\beta-\pi; \quad \therefore\ \theta = \frac{\alpha+\beta}{2} - \frac{\pi}{2}.$$

5. $\frac{1}{r} = \frac{1-e_1\cos(\theta-\alpha_1)}{l_1} + \frac{1-e_2\cos(\theta-\alpha_2)}{l_2} + \dots$

$$= \Sigma\frac{1}{l_1} - \Sigma\frac{e_1}{l_1}\cos(\theta-\alpha_1) = \frac{1}{L} - \frac{E}{L}\cos(\theta-\gamma),$$

where $\frac{1}{L} = \Sigma\frac{1}{l_1}$, $\frac{E\cos\gamma}{L} = \Sigma\frac{e_1\cos\alpha_1}{l_1}$

and $$\frac{E\sin\gamma}{L} = \Sigma\frac{e_1\sin\alpha_1}{l_1}.$$

This is a conic, of latus rectum L, whose focus is S.

6. The length of the focal chord, inclined at θ, of the conic $\frac{l}{r} = 1 - \sqrt{2}.\cos\theta$,

$$= l\left\{\frac{1}{1-\sqrt{2}.\cos\theta} + \frac{1}{1+\sqrt{2}.\cos\theta}\right\} = \frac{2l}{1-2\cos^2\theta} = -\frac{2l}{\cos 2\theta}.$$

The length of the perpendicular chord

$$= -\frac{2l}{\cos\left[2\left(\theta+\frac{\pi}{2}\right)\right]} = \frac{2l}{\cos 2\theta}.$$

7. $\frac{l}{SP} = 1 - e\cos\alpha,\ \ \frac{l}{SP'} = 1 + e\cos\alpha,$

$$\frac{l}{SQ} = 1 - e\cos\left(\alpha+\frac{\pi}{2}\right)$$

$$= 1 + e\sin\alpha, \text{ and } \frac{l}{SQ'} = 1 - e\sin\alpha.$$

$$\therefore\ \frac{1}{SP.SP'} + \frac{1}{SQ.SQ'} = \frac{1-e^2\cos^2\alpha + 1 - e^2\sin^2\alpha}{l^2} = \frac{2-e^2}{l^2}.$$

8. Let the diameter $= d$, and chord $= c$.

Then $$d^2 = \frac{4a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta},\ \text{[Art. 256]},$$

and $$c = \frac{l}{1-e\cos\theta} + \frac{l}{1+e\cos\theta} = \frac{2l}{1-e^2\cos^2\theta}$$

$$= \frac{2b^2a}{b^2\cos^2\theta + a^2\sin^2\theta};\ \ \therefore\ d^2 = 2ac.$$

9. If $\theta = \pi + \cos^{-1}\frac{1}{e}$, $r = \dfrac{l}{1+e.\frac{1}{e}} = \dfrac{l}{2}$.

10. Change r into $-r$, and θ into $\theta+\pi$ (Art. 32), and the first equation then becomes the second.

11. Let the equation to one conic be, as in the last example,

$$\frac{l_1}{r} = 1 - e_1 \cos\theta, \ldots\text{(i)} \quad \text{or} \quad \frac{l_1}{r} = -1 - e_1 \cos\theta \ldots\text{(ii)}$$

and that of the second $\dfrac{l_2}{r} = 1 - e_2 \cos(\theta - \alpha)$. ...(iii)

Adding (ii) and (iii), and subtracting (i) and (iii), we obtain $\dfrac{l_1}{r} + e_1 \cos\theta \pm \left(\dfrac{l_2}{r} + e_2 \cos(\theta - \alpha)\right) = 0$.

Since this equation represents straight lines, they are the common chords, and they clearly pass through the intersection of the directrices, which, by Art. 339, have as equations $r\cos\theta = -\dfrac{l_1}{e_1}$ and $r\cos(\theta - \alpha) = -\dfrac{l_2}{e_2}$.

12. The intersection of the tangent

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta$$

with the line $\theta = \alpha - \beta$ (where β is a constant) lies on

$$\frac{l}{r} = \cos\beta - e\cos\theta, \ \ i.e. \ \ \frac{l\sec\beta}{r} = 1 - e\sec\beta \,.\, \cos\theta,$$

which is a conic having the same focus and directrix as the given one.

13. The equation of the chord joining the points whose vectorial angles are $\beta - \alpha$, $\beta + \alpha$ is [Art. 342]

$$\frac{l}{r} = \sec\alpha \,.\, \cos(\theta - \beta) - e\cos\theta.$$

This cuts $\theta = \beta$ where

$$\frac{l}{r} = \sec\alpha - e\cos\theta, \ \text{or} \ \frac{l\cos\alpha}{r} = 1 - e\cos\alpha \,.\, \cos\theta.$$

14. The equation to the common chord of the conics

$$\frac{l}{r} = 1 - e\cos\theta, \text{ and } \frac{l'}{r} = 1 - e'\cos(\theta - \alpha) \text{ is}$$

$$\frac{l' - l}{r} = e\cos\theta - e'\cos(\theta - \alpha),$$

or
$$\frac{\frac{1}{e'}(l - l')}{r} = \cos(\theta - \alpha) - \frac{e}{e'}\cos\theta,$$

which, by Art. 342, touches

$$\frac{\frac{1}{e'}(l - l')}{r} = 1 - \frac{e}{e'}\cos\theta.$$

15. Let
$$\frac{l}{r} = 1 - e\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and
$$\frac{l'}{r} = 1 - e'\cos(\theta - \gamma) \;\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

be the equations to the two conics.

The tangent to (1) at the point β is [Art. 342]

$$\frac{l}{r} = \cos(\theta - \beta) - e\cos\theta. \;\ldots\ldots\ldots\ldots\ldots(3)$$

The tangent to (2) at the point $(90^\circ + \beta)$ is [Art. 346]

$$\frac{l'}{r} = \cos(\theta - 90^\circ - \beta) - e'\cos(\theta - \gamma)$$

$$= \sin(\theta - \beta) - e'\cos(\theta - \gamma). \;\ldots\ldots\ldots\ldots\ldots(4)$$

For the required locus we eliminate β between (3) and (4), and have

$$\left(\frac{l}{r} + e\cos\theta\right)^2 + \left(\frac{l'}{r} + e'\cos(\theta - \gamma)\right)^2 = 1,$$

or, in Cartesians, $(l + ex)^2 + (l' + e'x\cos\gamma + e'y\sin\gamma)^2 = x^2 + y^2$.

This is always a conic section; and it is easily seen to be a parabola if the terms of the second degree form a perfect square, *i.e.* if

$$0 = 1 - e^2 - e'^2 + e^2 e'^2 \sin^2 \gamma.$$

16. The intersections of $r^2 - 2r\rho \cos\theta + \rho^2 - a^2 = 0$, [Art. 172],

with $$\frac{l}{r} = 1 - e\cos\theta,$$

are given by $$e(r^2 + \rho^2 - a^2) = 2\rho(r - l);$$

$$\therefore\ r_1 + r_2 = \frac{2\rho}{e}.$$

17 and 18. The intersections of the tangents at the points β, γ to the parabola $\frac{l}{r} = 1 - \cos\theta$ are easily seen to be

$$\theta = \tfrac{1}{2}(\beta + \gamma),\ r = a \operatorname{cosec}\frac{\beta}{2} \operatorname{cosec}\frac{\gamma}{2},$$

and two similar points.

These clearly all lie on the circle

$$r = -a \operatorname{cosec}\frac{\alpha}{2} \operatorname{cosec}\frac{\beta}{2} \operatorname{cosec}\frac{\gamma}{2} \sin\left[\theta - \frac{\alpha + \beta + \gamma}{2}\right],$$

i.e. $$r = a \operatorname{cosec}\frac{\alpha}{2} \operatorname{cosec}\frac{\beta}{2} \operatorname{cosec}\frac{\gamma}{2} \cos\left[\theta - \left(\frac{\alpha + \beta + \gamma}{2} - \frac{\pi}{2}\right)\right].$$

The centre of this circle is, by Art. 171, the point whose vectorial coordinates are

$$\frac{a}{2} \operatorname{cosec}\frac{\alpha}{2} \operatorname{cosec}\frac{\beta}{2} \operatorname{cosec}\frac{\gamma}{2} \text{ and } \frac{\alpha + \beta + \gamma}{2} - \frac{\pi}{2}.$$

This point lies on the circle whose equation is

$$r = -\frac{a}{2} \operatorname{cosec}\frac{\alpha}{2} \operatorname{cosec}\frac{\beta}{2} \operatorname{cosec}\frac{\gamma}{2} \operatorname{cosec}\frac{\delta}{2} \cos\left[\theta - \frac{\alpha + \beta + \gamma + \delta}{2}\right].$$

Similarly for the centres of the other three circles.

19. The centre of the circle lies on the line

$$\theta = \frac{\alpha + \beta + \gamma - \pi}{2};$$

$\therefore$ the equation to the tangent at S is $\theta = \frac{\alpha + \beta + \gamma}{2}$ = sum of the angles made with the axis by the three tangents.

20. Let (ρ, ϕ) be the coordinates of its centre, and its equation

$$r = 2\rho\cos(\theta - \phi). \quad \text{.....................(i)}$$

Since it passes through the point $\theta = \alpha$,

$$\therefore \frac{l}{1 - e\cos\alpha} = 2\rho\cos(\alpha - \phi). \quad \text{.........(ii)}$$

Since (ρ, ϕ) lies on the normal at this point,

$$\therefore \sin(\alpha - \phi) + e\sin\phi = \frac{e\sin\alpha}{1 - e\cos\alpha}\cdot\frac{l}{\rho}, \text{ ...(iii) [Art. 345]},$$

$$= 2e\sin\alpha\cos(\alpha - \phi), \text{ from (ii)}, = e[\sin\phi - \sin(\phi - 2\alpha)],$$

whence
$$\frac{\sin\phi}{\sin\alpha - e\sin 2\alpha} = \frac{\cos\phi}{\cos\alpha - e\cos 2\alpha}.$$

From (i) and (ii), $l\cos(\theta - \phi) = r(1 - e\cos\alpha)\,.\,\cos(\alpha - \phi)$; eliminating ϕ, we have

$$l\{\cos\theta(\cos\alpha - e\cos 2\alpha) + \sin\theta(\sin\alpha - e\sin 2\alpha)\} = r(1 - e\cos\alpha)^2,$$

or
$$l\cos(\theta - \alpha) - el\cos(\theta - 2\alpha) = r(1 - e\cos\alpha)^2.$$

21. Let $\frac{a}{r} = 1 + \cos\theta$ and $r = c\cos\theta$ be the equations of the parabola and the circle and $\frac{l}{r} = 1 + e\cos(\theta - \beta)$ that of the variable conic. Since its directrix $\frac{l}{r} = e\cos(\theta - \beta)$ touches the parabola, then on comparing with

$$\frac{a}{r} = \cos(\theta - \gamma) + \cos\theta = 2\cos\frac{\gamma}{2}\cos\left(\theta - \frac{\gamma}{2}\right),$$

we have
$$\beta = \frac{\gamma}{2}, \text{ and } ae = 2l\cos\beta. \quad \text{............(1)}$$

Also, eliminating θ between the circle and the conic,

$$\{c(l - r) - er^2\cos\beta\}^2 = c^2e^2r^2\sin^2\beta - e^2r^4\sin^2\beta,$$

$$\therefore r_1 + r_2 + r_3 + r_4 = -\frac{2ec\cos\beta}{e^2} = -\frac{ac}{l}, \text{ by (1)}.$$

22. Taking the fixed point for origin, let the equation to the circle be $r = 2a\cos\theta$. Let P be the point where $\theta = \alpha$, and $\theta = \beta$ the equation to the axis of one of the parabolas.

The equation of the tangent at P is, [Art. 174],

$$r\cos(\theta-2\alpha)=2a\cos^2\alpha.$$

Putting $\theta=\beta$, $\dfrac{2a\cos^2\alpha}{\cos(\beta-2\alpha)}=ST=SP=2a\cos\alpha.$

(See Fig. of Art. 211.)

$$\therefore\ \cos(\beta-2\alpha)=\cos\alpha\,;\quad\therefore\ \beta=3\alpha.$$

$$\therefore\ SN=SP\cos(\beta-\alpha)=2a\cos\alpha\,.\cos 2\alpha.$$

$$\therefore\ 2SA=SN+ST=2a\cos\alpha+2a\cos\alpha\cos 2\alpha$$

$$=4a\cos^3\alpha=4a\cos^3\frac{\beta}{3}.$$

Hence the equation to the locus of A is $r=2a\cos^3\dfrac{\theta}{3}$.

23. Let $\qquad\dfrac{ae_1}{r}=1-e_1\cos\theta,$ ………………(i)

and $\qquad\dfrac{ae_2}{r}=1-e_2\cos\theta,$ ………………(ii)

be the equations of the conics.

Any tangent to (1) is $\dfrac{ae_1}{r}=\cos(\theta-\alpha)-e_1\cos\theta.$

If this meets (ii) at the point θ_1, we have

$$\frac{e_1}{e_2}=\frac{\cos(\theta_1-\alpha)-e_1\cos\theta_1}{1-e_2\cos\theta_1};$$

$$\therefore\ \cos(\theta_1-\alpha)=\frac{e_1}{e_2},\text{ which is constant.}$$

$$\therefore\ \theta_1-\alpha\text{ is constant.}$$

24. Let $\dfrac{l}{r}=1-e\cos\theta$, and $\dfrac{l}{r}=1-e\sin\theta$ be the equations of the ellipses.

The equations of tangents to them are

$$\frac{l}{r}=\cos(\theta-\alpha)-e\cos\theta,\ \text{ and }\ \frac{l}{r}=\cos(\theta-\beta)-e\sin\theta.$$

If they are coincident, then

$$\cos\alpha - e = \cos\beta, \text{ and } \sin\beta - e = \sin\alpha;$$

$$\therefore \cos\alpha - \cos\beta = \sin\beta - \sin\alpha,$$

whence $$\tan\frac{\alpha+\beta}{2} = 1; \quad \therefore \frac{\alpha+\beta}{2} = \frac{\pi}{4}.$$

Also $$\cos\alpha - \sin\alpha + \sin\beta - \cos\beta = 2e.$$

$$\therefore \sin\left(\frac{\pi}{4} - \alpha\right) + \sin\left(\beta - \frac{\pi}{4}\right) = \sqrt{2}\,.\,e.$$

$$\therefore \sin\frac{\beta-\alpha}{2}\,.\,\cos\left(\frac{\alpha+\beta}{2} - \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}.$$

$$\therefore \beta - \alpha = 2\sin^{-1}\frac{e}{\sqrt{2}}.$$

25. The tangents at the point β to the two conics are

$$\frac{l_1}{r} = \cos(\theta - \beta) - e_1\cos\theta,$$

and $$\frac{l_2}{r} = \cos(\theta - \beta) - e_2\cos(\theta - \alpha).$$

These are coincident if

$$\frac{l_1}{l_2} = \frac{\cos\beta - e_1}{\cos\beta - e_2\cos\alpha} = \frac{\sin\beta}{\sin\beta - e_2\sin\alpha}.$$

$$\therefore \cos\beta = \frac{l_1e_2\cos\alpha - l_2e_1}{l_1 - l_2} \text{ and } \sin\beta = \frac{l_1e_2\sin\alpha}{l_1 - l_2}.$$

$$\therefore (l_1 - l_2)^2 = (l_1e_2\cos\alpha - l_2e_1)^2 + l_1^2e_2^2\sin^2\alpha.$$

$$\therefore l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) = 2l_1l_2(1 - e_1e_2\cos\alpha).$$

26. Let $\frac{l}{r} = 1 - e_1\cos\theta$ and $\frac{l'}{r} = 1 - e_2\cos(\theta - \alpha)$ be the equations of the conics.

Eliminating θ, we have

$$r^2[\ldots] - 2r\left[le_2^2 + l'e_1^2 - e_1e_2\cos\alpha\,(l + l')\right] + l'^2e_1^2 + l^2e_2^2 - 2e_1e_2ll'\cos\alpha = 0.$$

Also eliminating θ between

$$\frac{l}{r} = 1 - e_1 \cos\theta \text{ and } \frac{l'}{r} = -1 - e_2 \cos(\theta - \alpha), \text{ [Art. 338]},$$

we have

$$r^2[\ldots] - 2r[le_2^2 - l'e_1^2 + e_1e_2 \cos\alpha\,(l - l')] + l'^2 e_1^2 + l^2 e_2^2 - 2e_1e_2 ll' \cos\alpha = 0.$$

$$\therefore\ (l + l')\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + (l - l')\left(\frac{1}{r_3} + \frac{1}{r_4}\right)$$

$$= \frac{2(l + l')\{le_2^2 + l'e_1^2 - e_1e_2 \cos\alpha\,(l + l')\} + 2(l - l')\{le_2^2 - l'e_1^2 + e_1e_2 \cos\alpha\,(l - l')\}}{l^2 e_2^2 + l'^2 e_1^2 - 2e_1e_2 ll' \cos\alpha} = 4.$$

27. If the normal at ϕ [Art. 345], passes through the point whose polar coordinates are r_1 and δ, then

$$r_1 \cos\left(\delta - \frac{\phi}{2}\right) \sin^2\frac{\phi}{2} = \frac{l}{2} \cos\frac{\phi}{2}.$$

$$\therefore\ r_1 \cos\delta \cos\frac{\phi}{2} \sin^2\frac{\phi}{2} + r_1 \sin\delta \sin^3\frac{\phi}{2} - \frac{l}{2}\cos^3\frac{\phi}{2} - \frac{l}{2}\cos\frac{\phi}{2}\sin^2\frac{\phi}{2} = 0,$$

or $$r_1 \sin\delta t^3 + t^2\left(r_1 \cos\delta - \frac{l}{2}\right) - \frac{l}{2} = 0, \text{ where } t \equiv \tan\frac{\phi}{2}.$$

$$\therefore\ \tan\frac{\alpha + \beta + \gamma}{2} = \frac{s_1 - s_3}{1 - s_2} = -\cot\delta = \tan\left(\frac{\pi}{2} + \delta\right).$$

$$\therefore\ \frac{\alpha + \beta + \gamma}{2} - \delta = \frac{\pi}{2}. \quad \therefore\ 2\delta = \alpha + \beta + \gamma - \pi.$$