11

Waves in Optical Systems

Light. Waves or Rays?

Light exhibits a dual nature. In practice, its passage through optical instruments such as telescopes and microscopes is most easily shown by geometrical ray diagrams but the fine detail of the images formed by these instruments is governed by diffraction which, together with interference, requires light to propagate as waves. This chapter will correlate the geometrical optics of these instruments with wavefront propagation. In Chapter 12 we shall consider the effects of interference and diffraction.

The electromagnetic wave nature of light was convincingly settled by Clerk–Maxwell in 1864 but as early as 1690 Huygens was trying to reconcile waves and rays. He proposed that light be represented as a wavefront, each point on this front acting as a source of secondary wavelets whose envelope became the new position of the wavefront, Figure 11.1(a). Light propagation was seen as the progressive development of such a process. In this way, reflection and refraction at a plane boundary separating two optical media could be explained as shown in Figure 11.1(b) and (c).

Huygens' theory was explicit only on those contributions to the new wavefront directly ahead of each point source of secondary waves. No statement was made about propagation in the backward direction nor about contributions in the oblique forward direction. Each of these difficulties is resolved in the more rigorous development of the theory by Kirchhoff which uses the fact that light waves are oscillatory (see Appendix 2, p. 547).

The way in which rays may represent the propagation of wavefronts is shown in Figure 11.2 where spherically diverging, plane and spherically converging wavefronts are moving from left to right. All parts of the wavefront (a surface of constant phase) take the same time to travel from the source and all points on the wavefront are the same *optical distance* from the source. This optical distance must take account of the changes of refractive index met by the wavefront as it propagates. If the physical path length is measured as x in a medium of refractive index n then the *optical path length* in the medium is the product nx. In travelling from one point to another light chooses a unique optical path which may always be defined in terms of Fermat's Principle.

The Physics of Vibrations and Waves, 6th Edition H. J. Pain

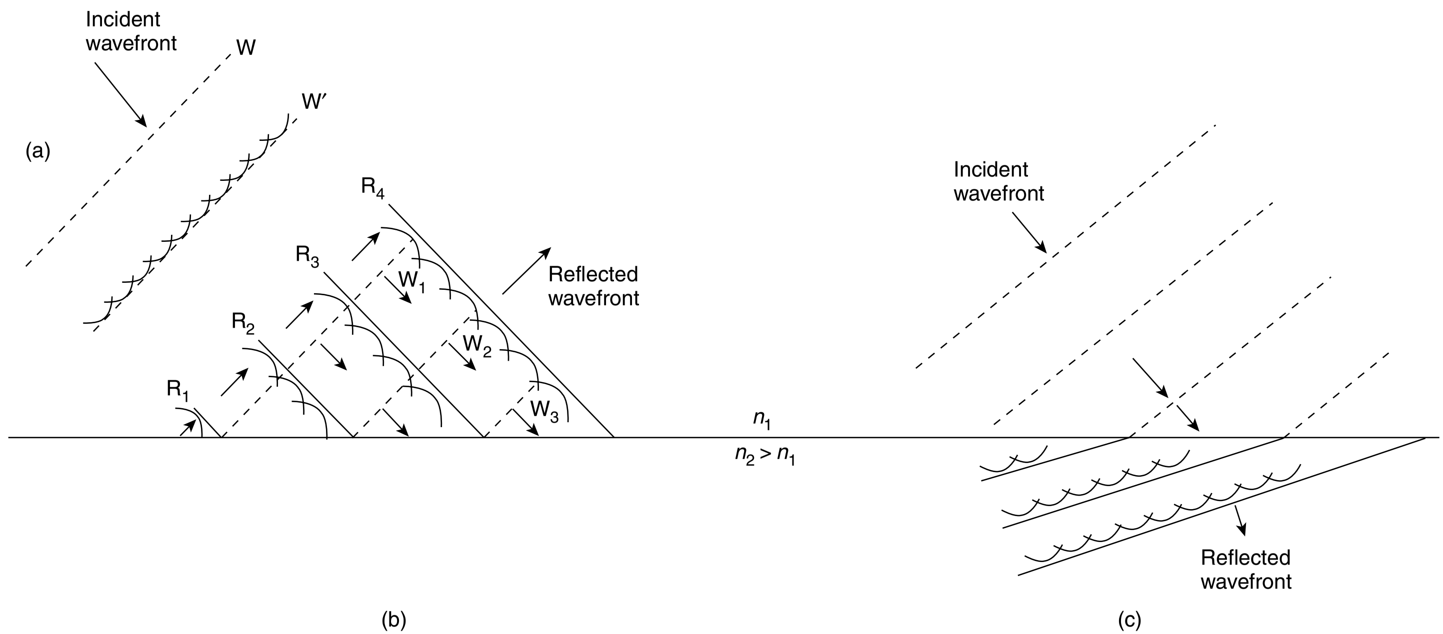


Figure 11.1 (a) Incident plane wavefront W propagates via Huygens wavelets to W′. (b) At the plane boundary between the media (refractive index $n_2 > n_1$) the incident wavefront W_1 has a reflected section R_1. Increasing sections R_2 and R_3 are reflected until the whole wavefront is *reflected* as R_4. (c) An increasing section of the incident wavefront is refracted. Incident wavefronts are shown dashed, and reflected and refracted wavefronts as a continuous line

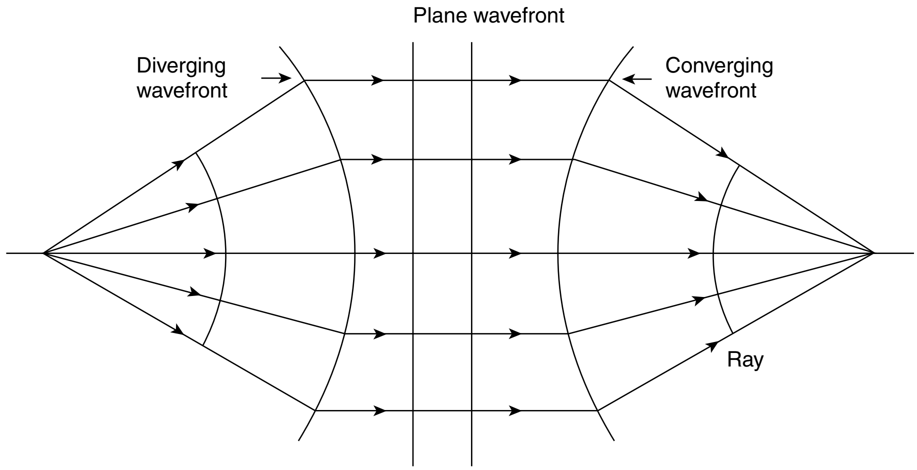


Figure 11.2 Ray representation of spherically diverging, plane and spherically converging wavefronts

Fermat's Principle

Fermat's Principle states that the optical path length has a stationary value; its first order variation or first derivative in a Taylor series expansion is zero. This means that when an optical path lies wholly within a medium of constant refractive index the path is a straight line, the shortest distance between its end points, and the light travels between these points in the minimum possible time. When the medium has a varying refractive index or the path crosses the boundary between media of different refractive indices the direction of the path always adjusts itself so that the time taken between its end points is a minimum. Fermat's Principle is therefore sometimes known as the Principle of Least Time. Figure 11.3 shows examples of light paths in a medium of varying refractive index. As examples of light meeting a boundary between two media we use Fermat's Principle to derive the laws of reflection and refraction.

The Laws of Reflection

In Figure 11.4a Fermat's Principle requires that the optical path length OSI should be a minimum where O is the object, S lies on the plane reflecting surface and I is the point on the reflected ray at which the image of O is viewed. The plane OSI must be perpendicular to the reflecting surface for, if reflection takes place at any other point S′ on the reflecting surface where OSS′ and ISS′ are right angles then evidently OS′ > OS and IS′ > IS, giving OS′I > OSI.

The laws of reflection also require, in Figure 11.4a that the angle of incidence i equals the angle of reflection r. If the coordinates of O, S and I are those shown and the velocity of light propagation is c then the time taken to traverse OS is

$$t = (x^2 + y^2)^{1/2}/c$$

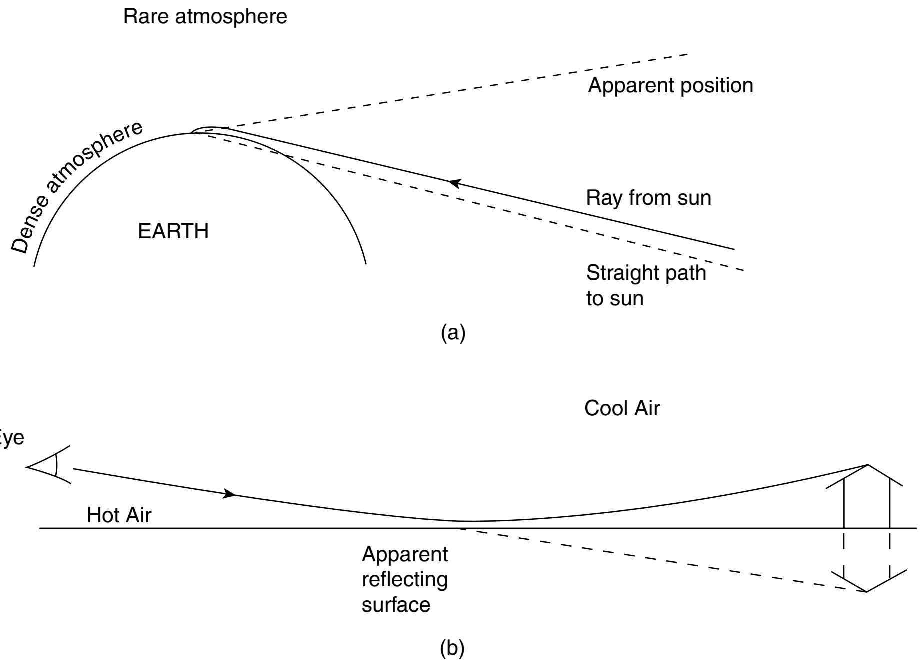


Figure 11.3 Light takes the shortest optical path in a medium of varying refractive index. (a) A light ray from the sun bends towards the earth in order to shorten its path in the denser atmosphere. The sun remains visible after it has passed below the horizon. (b) A light ray avoids the denser atmosphere and the road immediately below warm air produces an apparent reflection

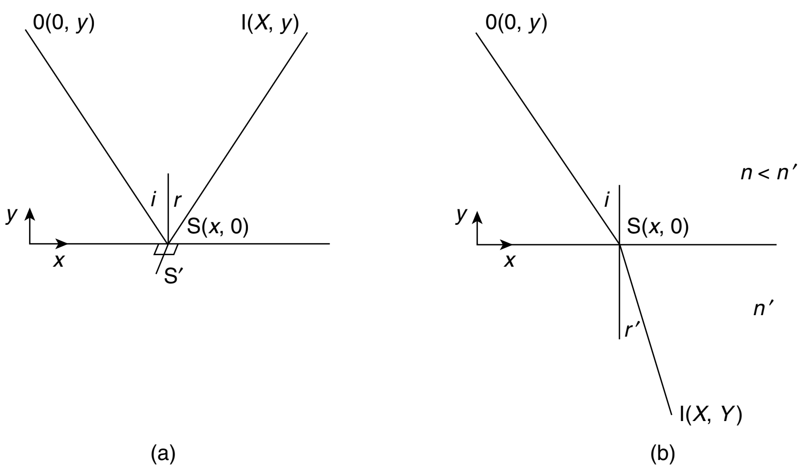


Figure 11.4 The time for light to follow the path OSI is a minimum (a) in reflection, when OSI forms a plane perpendicular to the reflecting surface and $\hat{i} = \hat{r}$; and (b) in refraction, when $n \sin i = n' \sin r'$ (Snell's Law)

and the time taken to traverse SI is

$$t' = [(X - x)^2 + y^2]^{1/2}/c$$

so that the total time taken to travel the path OSI is

$$T = t + t'$$

The position of S is now varied along the x axis and we seek, via Fermat's Principle of Least Time, that value of x which minimizes T, so that

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{x}{c(x^2 + y^2)^{1/2}} - \frac{X - x}{c[(X - x)^2 + y^2]^{1/2}} = 0$$

But

$$\frac{x}{(x^2 + y^2)^{1/2}} = \sin i$$

and

$$\frac{X - x}{[(X - x)^2 + y^2]^{1/2}} = \sin r$$

Hence

$$\sin i = \sin r$$

and

$$\hat{i} = \hat{r}$$

The Law of Refraction

Exactly similar arguments lead to Snell's Law, already derived on p. 256.

Here we express it as

$$n \sin i = n' \sin r'$$

where i is the angle of incidence in the medium of refractive index n and r' is the angle of refraction in the medium of refractive index $n'(n' > n)$. In Figure 11.4b a plane boundary separates the media and light from O (0, y) is refracted at S (x, 0) and viewed at I (X, Y) on the refracted ray. If v and v' are respectively the velocities of light propagation in the media n and n' then OS is traversed in the time

$$t = (x^2 + y^2)^{1/2}/v$$

and SI is traversed in the time

$$t' = [(X - x)^2 + Y^2]^{1/2}/v'$$

The total time to travel from O to I is $T = t + t'$ and we vary the position of S along the x axis which lies on the plane boundary between n and n', seeking that value of x which minimizes T. So

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{1}{v}\frac{x}{(x^2+y^2)^{1/2}} - \frac{1}{v'}\frac{(X-x)}{[(X-x)^2+Y^2]^{1/2}} = 0$$

where

$$\frac{x}{(x^2+y^2)^{1/2}} = \sin i$$

and

$$\frac{(X-x)}{[(X-x)^2+Y^2]^{1/2}} = \sin r'$$

But

$$\frac{1}{v} = \frac{n}{c}$$

and

$$\frac{1}{v'} = \frac{n'}{c}$$

Hence

$$n \sin i = n' \sin r'$$

Rays and Wavefronts

Figure 11.2 showed the ray representation of various wavefronts. In order to reinforce the concept that rays trace the history of wavefronts we consider the examples of a thin lens and a prism.

The Thin Lens

In Figure 11.5 a plane wave in air is incident normally on the plane face of a plano convex glass lens of refractive index n and thickness d at its central axis. Its spherical face has a radius of curvature $R \gg d$. The power of a lens to change the curvature of a wavefront is the inverse of its focal length f. A lens of positive power converges a wavefront, negative power diverges the wavefront.

Simple rays optics gives the power of the plano convex lens as

$$\mathscr{P} = \frac{1}{f} = (n-1)\frac{1}{R}$$

but we derive this result from first principles that is, by considering the way in which the lens modifies the wavefront.

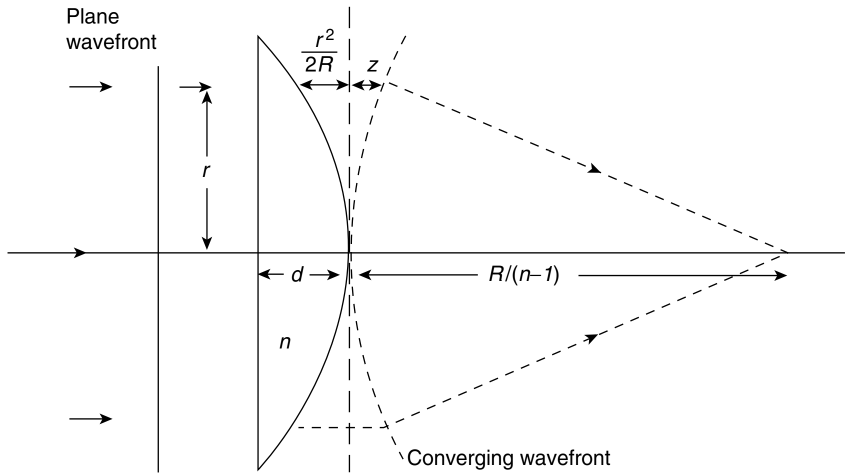


Figure 11.5 A plane wavefront is normally incident on a plano-convex lens of refractive index n and thickness d at the central axis. The radius of the curved surface $R \gg d$. The wavefront is a surface of constant phase and the optical path length is the same for each section of the wavefront. At a radius r from the central axis the wavefront travels a shorter distance in the denser medium and the lens curves the incident wavefront which converges at a distance $R/(n-1)$ from the lens

At the central axis the wavefront takes a time $t = nd/c$ to traverse the thickness d. At a distance r from the axis the lens is thinner by an amount $r^2/2R$ (using the elementary relation between the sagitta, arc and radius of a circle) so that, in the time $t = nd/c$, points on the wavefront at a distance r from the axis travel a distance

$$(d - r^2/2R)$$

in the lens plus a distance $(r^2/2R + z)$ in air as shown in the figure. Equating the times taken by the two parts of the wave front we have

$$nd/c = (n/c)(d - r^2/2R) + (1/c)(z + r^2/2R)$$

which yields

$$z = (n-1)r^2/2R$$

But this is again the relation between the sagitta z, its arc and a circle of radius $R/(n-1)$ so, in three dimensions, the locus of z is a sphere of radius $R/(n-1)$ and the emerging spherical wavefront converges to a focus at a distance

$$f = R/(n-1)$$

(Problems 11.1, 11.2, 11.3)

The Prism

In Figure 11.6 a section, height y, of a plane wavefront in air is deviated through an angle θ when it is refracted through an isosceles glass prism, base l, vertex angle β and refractive

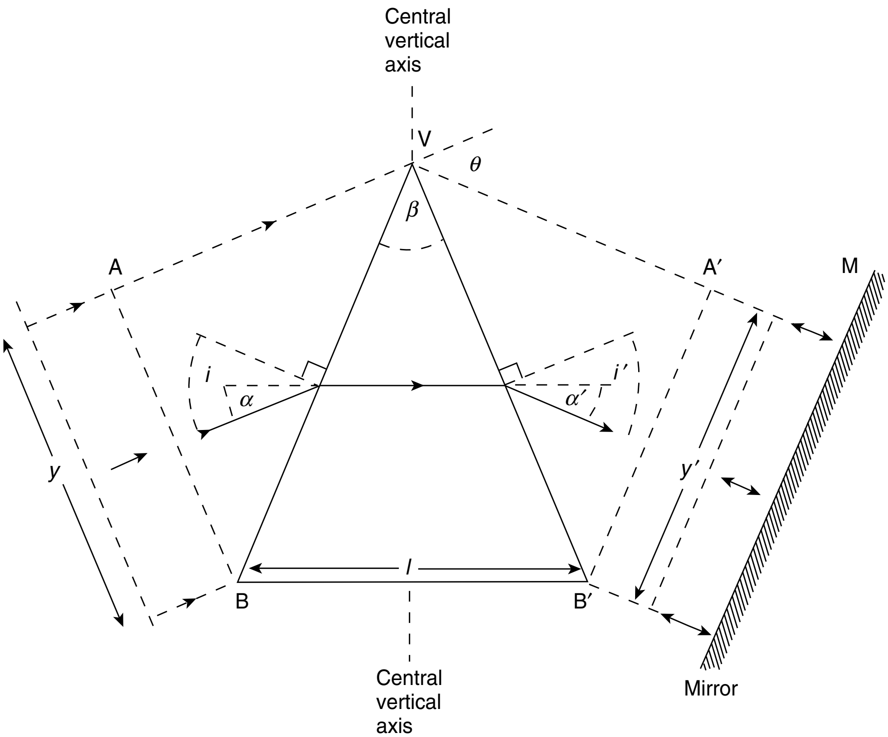


Figure 11.6 A plane wavefront suffers minimum deviation ($\theta_{\min}$) when its passage through a prism is symmetric with respect to the central vertical axis ($i = i'$). The wavefront obeys the Optical Helmholtz Condition that $ny \tan \alpha$ is a constant where n is the refractive index, y is the width of the wavefront and α is shown. (Here $\alpha = \alpha'$)

index n. Experiment shows that there is *one*, and only one, value of the incident angle i for which the angle of deviation is a minimum $= \theta_{\min}$. It is easily shown *using ray optics* that this unique value of i requires the passage of the wavefront through the prism to be symmetric about the central vertical axis as shown in the figure so that the incident angle i equals the emerging angle i'. Equating the lengths of the *optical* paths AVA$'$ and BB$'(= nl)$ followed by the edges of the wavefront section gives the familiar result

$$\sin\left(\frac{\theta_{\min} + \beta}{2}\right) = n \sin \frac{\beta}{2}$$

which is used in the standard experiment to determine n, the refractive index of the prism.

Now there is only *one* value of i which produces minimum deviation and this leads us to expect that the passage of the wavefront will be symmetric about the central vertical axis for if a plane mirror (M in the figure) is placed parallel to the emerging wavefront the wavefront is reflected back along its original path, and if $i \neq i'$ there would be *two* values of incidence, each producing minimum deviation. At i for minimum deviation any rotation increases i'.

However, the real argument for symmetry from a wavefront point of view depends on the optical Helmholtz equation which we shall derive on p. 321. This states that for a plane wavefront the product $ny \tan\alpha$ remains constant as it passes through an optical system irrespective of the local variations of the factors n, y and $\tan\alpha$. Now the wavefront has the same width on entry into and exit from the prism so $y = y'$ and although n changes at the prism faces the initial and final medium for the wavefront is air where $n = 1$.

Hence, from the optical Helmholtz equation $\tan\alpha = \tan\alpha'$ in Figure 11.6. It is evident that as long as its width $y = y'$ the wavefront section will turn through a minimum angle when the physical path length BB$'$ followed by its lower edge is a maximum with respect to AVA$'$, the physical path length of its upper edge.

Ray Optics and Optical Systems

An optical system changes the curvature of a wavefront. It is formed by one or more optical surfaces separating media of different refractive indices. In Fig. 11.7 rays from the object point L_0 pass through the optical system to form an image point L'. When the optical surfaces are spherical the line joining L_0 and L', which passes through the centres of curvature of the surfaces, is called the *optical axis*. This axis cuts each optical surface at its *pole*. If the object lies in a plane normal to the optical axis its perfect image lies in a *conjugate* plane normal to the optical axis. Conjugate planes cut the optical axis at conjugate points, e.g. L_0 and L'. In Figure 11.7 the plane at $+\infty$ has a conjugate focal plane cutting the optical axis at the focal point F. The plane at $-\infty$ has a conjugate focal plane cutting the optical axis at the focal point F'.

Paraxial Rays

Perfect geometrical images require perfect plane and spherical optical surfaces and in a real optical system a perfect spherical optical surface is obtained by using only that part of the wavefront close to the optical axis. This means that all angles between the axis and rays are very small. Such rays are called paraxial rays.

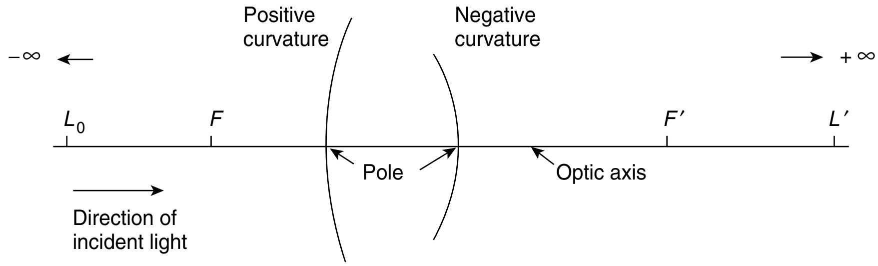


Figure 11.7 Optical system showing direction of incident light from left to right and optical surfaces of positive and negative curvature. Rays from L_0 pass through L' and this defines L_0 and L' as conjugate points. The conjugate point of F is at $+\infty$, the conjugate point of F' is at $-\infty$

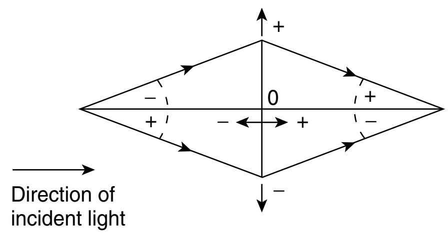


Figure 11.8 Sign convention for lengths is Cartesian measured from the right angles at 0. Angles take the sign of their tangents. 0 is origin of measurements

Sign Convention

The convention used here involves only signs of lengths and angles. The direction of incident light is positive and is always taken from left to right. Signs for horizontal and vertical directions are Cartesian. If AB $= l$ then BA $= -l$. The radius of curvature of a surface is measured from its pole to its centre so that, in Figure 11.7, the convex surface presented to the incident light has a positive radius of curvature and the concave surface has a negative radius of curvature.

The Cartesian convention with origin O at the right angles of Figure 11.8 gives the angle between a ray and the optical axis the sign of its tangent.

If the angle between a ray and the axis is α then, for paraxial rays

$$\sin\alpha = \tan\alpha = \alpha$$

and

$$\cos\alpha = 1$$

so that Snell's Law of Refraction

$$n\sin i = n'\sin r'$$

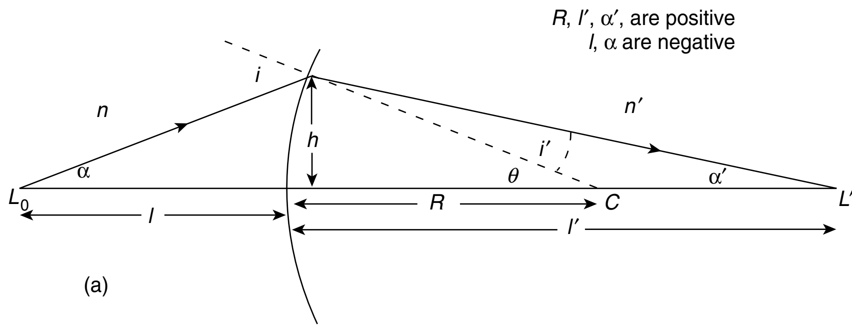
becomes

$$ni = n'r'$$

Power of a Spherical Surface

In Figure 11.9(a) and (b) a spherical surface separates media of refractive indices n and n'. Any ray through L_0 is refracted to pass through its conjugate point L'. The angles are exaggerated so that the base of h is very close to the pole of the optical surface which is taken as the origin. In Figure 11.9(a) the signs of R, l' and α' are positive with l and α negative. In Figure 11.9(b) R, l, l' , α and α' are all positive quantities. In both figures Snell's Law gives

$$ni = n'r'$$



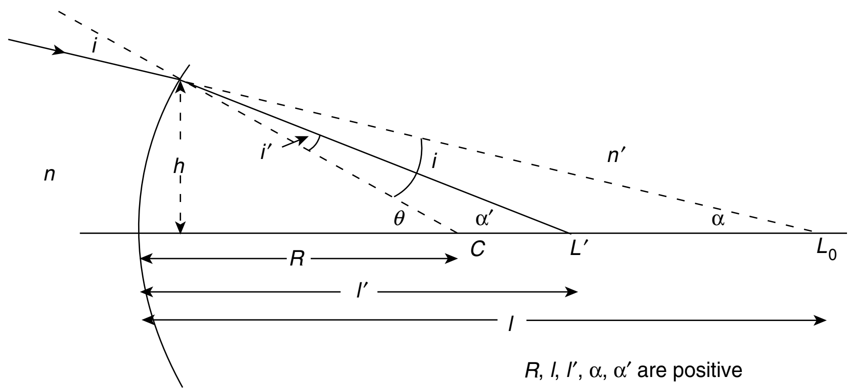


Figure 11.9 Spherical surface separating media of refractive indices n and n'. Rays from L_0 pass through L'. Snell's Law gives the power of the surface as

$$\mathscr{P} = \frac{n'}{l'} - \frac{n}{l} = \frac{n' - n}{R}$$

i.e.

$$n(\theta - \alpha) = n'(\theta - \alpha')$$

or

$$n'\alpha' - n\alpha = (n' - n)\theta = \left(\frac{n' - n}{R}\right)h = \mathscr{P}h \qquad (11.1)$$

Thus

$$\frac{n'}{l'} - \frac{n}{l} = \frac{n' - n}{R} = \mathscr{P} \qquad (11.2)$$

where $\mathscr{P}$ is the power of the surface. For $n' > n$ the power $\mathscr{P}$ is positive and the surface converges the wavefront. For $n' < n$, $\mathscr{P}$ is negative and the wavefront diverges. When the radius of curvature R is measured in metres the units of $\mathscr{P}$ are *dioptres*.

Magnification by the Spherical Surface

In Figure 11.10 the points QQ' form a conjugate pair, as do L_0L'. The ray QQ' passes through C the centre of curvature, L_0Q is the object height y, $L'Q'$ is the image height y' so

$$ni = n'r'$$

gives

$$ny/l = n'y'/l'$$

or

$$nyh/l = n'y'h/l'$$

that is

$$ny\alpha = n'y'\alpha' \tag{11.3}$$

This is the paraxial form of the optical Helmholtz equation.

The Transverse Magnification is defined as

$$M_T = y'/y = nl'/n'l.$$

The image y' is inverted so y and y' (and l and l') have opposite signs.

The Angular Magnification is defined as

$$M_\alpha = \alpha'/\alpha$$

Note that

$$M_T = n/n'M_\alpha$$

which, being independent of y, applies to any point on the object so that the object in the plane L_0Q is similar to the image in the plane $L'Q'$.

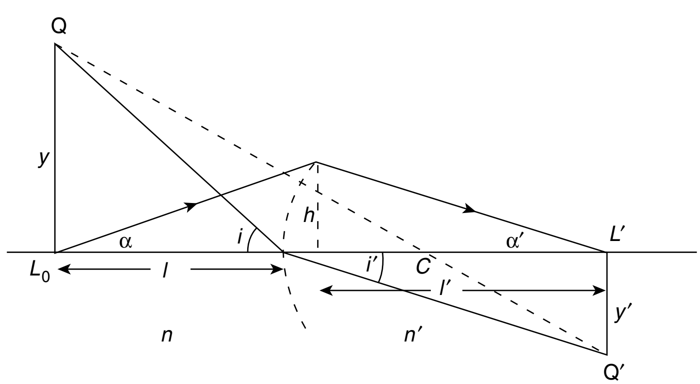


Figure 11.10 Magnification by a spherical surface. The paraxial form of the optical Helmholtz equation is $ny\alpha = n'y'\alpha'$ so Transverse Magnification $M_T = y'/y = nl'/ln'$ Angular Magnification $M_\alpha = \alpha'/\alpha$. Note that the image is inverted so y and y' (and l and l') have opposite signs

A series of optical surfaces separating media of refractive indices n, $n'n''$ yields the expression

$$ny\alpha = n'y'\alpha' = n''y''\alpha''$$

which is the paraxial form of the optical Helmholtz equation.

Power of Two Optically Refracting Surfaces

If Figure 11.11 the refracting surfaces have powers $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. At the first surface equation (11.1) gives

$$n_1\alpha_1 - n\alpha = \mathscr{P}_1 h_1$$

and at the second surface

$$n'\alpha' - n_1\alpha_1 = \mathscr{P}_2 h_2$$

Adding these equations gives

$$n'\alpha' - n\alpha = \mathscr{P}_1 h_1 + \mathscr{P}_2 h_2$$

If the object is located at $-\infty$ so that $\alpha = 0$ we have

$$n'\alpha' = \mathscr{P}_1 h_1 + \mathscr{P}_2 h_2$$

or

$$\alpha' = \frac{1}{n'}(\mathscr{P}_1 h_1 + \mathscr{P}_2 h_2)$$

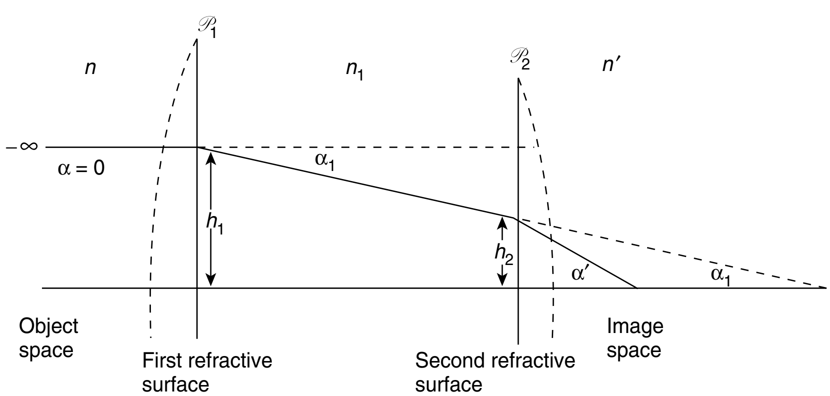


Figure 11.11 Two optically refracting surfaces of power $\mathscr{P}_1$ and $\mathscr{P}_2$ have a combined power of

$$\mathscr{P} = \frac{1}{h_1}(\mathscr{P}_1 h_1 + \mathscr{P}_2 h_2)$$

This gives the same image as a single element of power $\mathscr{P}$ if

$$\alpha' = \frac{1}{n'}(\mathscr{P}_1 h_1 + \mathscr{P}_2 h_2) = \frac{1}{n'}\mathscr{P} h_1$$

where

$$\mathscr{P} = \frac{1}{h_1}(\mathscr{P}_1 h_1 + \mathscr{P}_2 h_2) \tag{11.4}$$

is the total power of the system. *This is our basic equation* and we use it first to find the power of a thin lens in air.

Power of a Thin Lens in Air (Figure 11.12)

Equation (11.2) gives

$$\frac{n'}{l'} - \frac{n}{l} = \frac{n' - n}{R} = \mathscr{P}$$

for each surface, so that in Figure 11.12

$$\mathscr{P}_1 = (n_1 - 1)/R_1$$

and

$$\mathscr{P}_2 = (1 - n_1)/R_2$$

From equation (11.4)

$$\mathscr{P} = \frac{1}{h_1}(\mathscr{P}_1 h_1 + \mathscr{P}_2 h_2)$$

with

$$h_1 = h_2$$

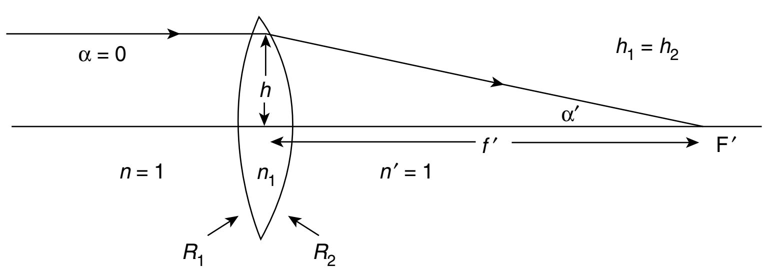


Figure 11.12 A thin lens of refractive index n_1, and radii of surface curvatures R_1 and R_2 has a power

$$\mathscr{P} = (n_1 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{f'}$$

where f' is the focal length. In the figure R_1 is positive and R_2 is negative we have

$$\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2$$

so the expression for the thin lens in air with surfaces of power $\mathscr{P}_1$ and $\mathscr{P}_2$ becomes

$$\mathscr{P} = \frac{1}{l'} - \frac{1}{l} = (n_1 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{f'}$$

where f' is the focal length.

Applying this result to the plano convex lens of p. 311 we have $R_1 = \infty$ and R_2 negative from our sign convention. This gives a positive power which we expect for a converging lens.

Effect of Refractive Index on the Power of a Lens

Suppose, in Figure 11.13, that the object space of the lens remains in air $(n = 1)$ but that the image space is a medium of refractive index $n_2' \neq 1$. How does this affect the focal length in the medium n_2'?

If $\mathscr{P}$ is the power of the lens in air we have

$$n_2'\alpha' - n\alpha = \mathscr{P}h_1 \tag{11.5}$$

and for

$$\alpha = 0$$

we have

$$\alpha' = \mathscr{P}h_1/n_2' = h_1/n_2'f'$$

where f' is the focal length in air.

If f_2' is the focal length in the medium n_2' then

$$f_2'\alpha' = h_1$$

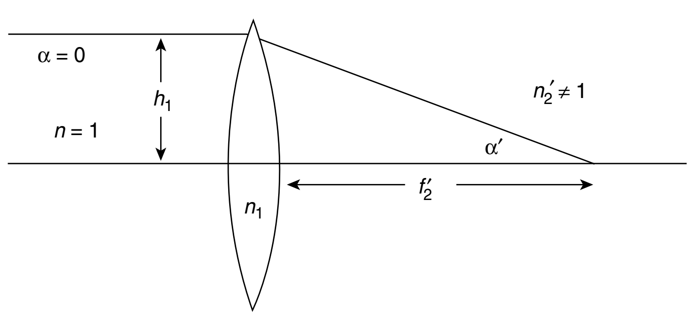


Figure 11.13 The focal length of a thin lens measured in the medium n_2' is given by $f_2' = n_2'f'$ where f' is the focal length of the lens measured in air

so

$$\alpha' = h_1/f_2' = h_1/n_2' f'$$

giving

$$f_2' = n_2' f'$$

Thus, the focal length changes by a factor equal to the refractive index of the medium in which it is measured and the power is affected by the same factor.

If the lens has a medium n_0 in its object space and a medium n_i in its image space then the respective focal lengths f_0 and f_i in these spaces are related by the expression

$$f_i/f_0 = -n_i/n_0 \tag{11.6}$$

where the negative signs shows that f_0 and f_i are measured in opposite directions (f_0 is negative and f_i is positive).

Principal Planes and Newton's Equation

There are two particular planes normal to the optic axis associated with every lens element of an optical system. These planes are called principal planes or unit planes because between these planes there is unit transverse magnification so the path of every ray between them is parallel to the optic axis. Moreover, any complex optical system has two principal planes of its own. In a thin lens the principal planes coincide.

The principal planes of a single lens do not, in general, coincide with its optical surfaces; focal lengths, object and image distances are measured from the principal planes and not from the optical surfaces. In Figure 11.14, PH and $P'H'$ define the first and second

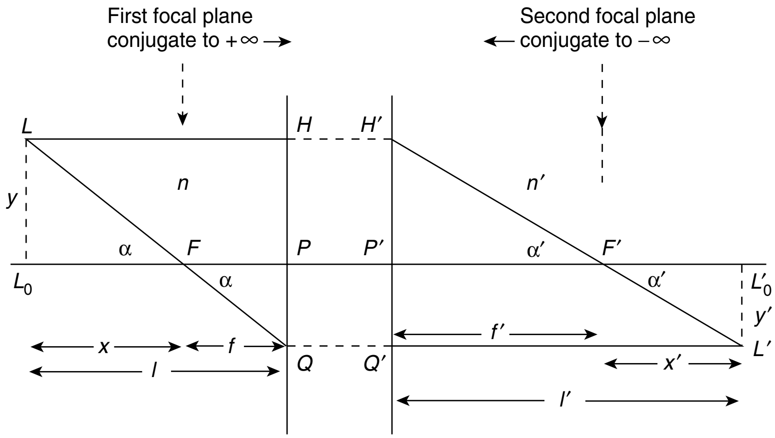


Figure 11.14 Between the principal planes PH and $P'H'$ of a lens or lens system there is unit magnification and rays between these planes are parallel to the optic axis. Newton's equation defines $xx' = ff'$. The optical Helmholtz equation is $ny\alpha =$ constant for paraxial rays and $ny \tan\alpha =$ constant for rays from ∞

principal planes, respectively, of a lens or optical system and PF and $P'F'$ are respectively the first and second focal lengths. The object and image planes cut the optic axis in L_0 and L_0', respectively.

The ray LH parallel to the optic axis proceeds to H' and thence through F' the second focal point. The rays LH and $H'F'$ meet at H' and therefore define the position of the second principal plane, $P'H'$. The position of the first principal plane may be found in a similar way.

If Figure 11.14, the similar triangles FL_0L and FPQ give $y/y' = x/f$ where, measured from P, only y is algebraically positive. The similar triangles $F'L_0'L'$ and $F'P''H'$ give

$$y/y' = f'/x',$$

where, measured from P', only y' is algebraically negative.

We have, therefore,

$$x/f = f'/x',$$

where x and f are negative and x' and f' are positive.
Thus,

$$xx' = ff'$$

This is known as Newton's equation.

If l, the object distance, and l', the image distance, are measured from the principal planes as in Figure 11.14, then

$$l = f + x \quad \text{and} \quad l' = f' + x'$$

and Newton's equation gives

$$xx' = (l - f)(l' - f') = ll' - l'f - lf' + ff' = ff'$$

so that

$$\frac{f'}{l'} + \frac{f}{l} = 1$$

But from $nf' = -n'f$ (equation (11.6)) we have

$$\frac{n'}{l'} - \frac{n}{l} = \frac{n'}{f'} = \frac{-n}{f} = \mathscr{P}$$

the power of the lens.

Optical Helmholtz Equation for a Conjugate Plane at Infinity

Suppose now that the object is no longer located at L_0L but at infinity so that the ray LH originates at one point from the distant object while the ray LFQ comes from a point on the object much more distant from the optic axis.

We still have from triangle $F'P'H'$ that

$$y = f' \tan \alpha'$$

and from triangle FPQ that

$$y' = f \tan \alpha$$

so

$$\frac{f \tan \alpha}{f' \tan \alpha'} = \frac{y}{y'} \quad \text{and} \quad \frac{f}{f'} y \tan \alpha = y' \tan \alpha'$$

But

$$\frac{f}{f'} = \frac{-n}{n'}$$

so

$$ny \tan \alpha = -n'y' \tan \alpha'$$

(Note that α, α' and y' are negative.)

This form of the Helmholtz equation applies when one of the conjugate planes is at infinity and is to be compared with the general unrestricted form of the Helmholtz equation for paraxial rays

$$ny\alpha = n'y'\alpha'$$

The infinite conjugate form $ny \tan \alpha =$ constant is valid when applied to the prism of p. 312 because the plane wavefront originated at infinity.

(Problems 11.4, 11.5, 11.6, 11.7, 11.8)

The Deviation Method for (a) Two Lenses and (b) a Thick Lens

Figure 11.11 illustrated how the deviation of a ray through two optically refracting surfaces could be used to find the power of a thin lens. We now apply this process to (a) a combination of two lenses and (b) a thick lens in order to find the power of these systems and the location of their principal planes. We have already seen in equation (11.5), which may be written

$$n_1'\alpha' - n_1\alpha = \mathscr{P}_1 y \tag{11.7}$$

where $\mathscr{P}_1$ is the power of the first lens in Figure 11.15a or the power of the first refracting surface in Figure 11.15b. If the incident ray is parallel to the optic axis, then $\alpha = 0$ and we have

$$n_1'\alpha' = \mathscr{P}_1 y_1 \tag{11.8}$$

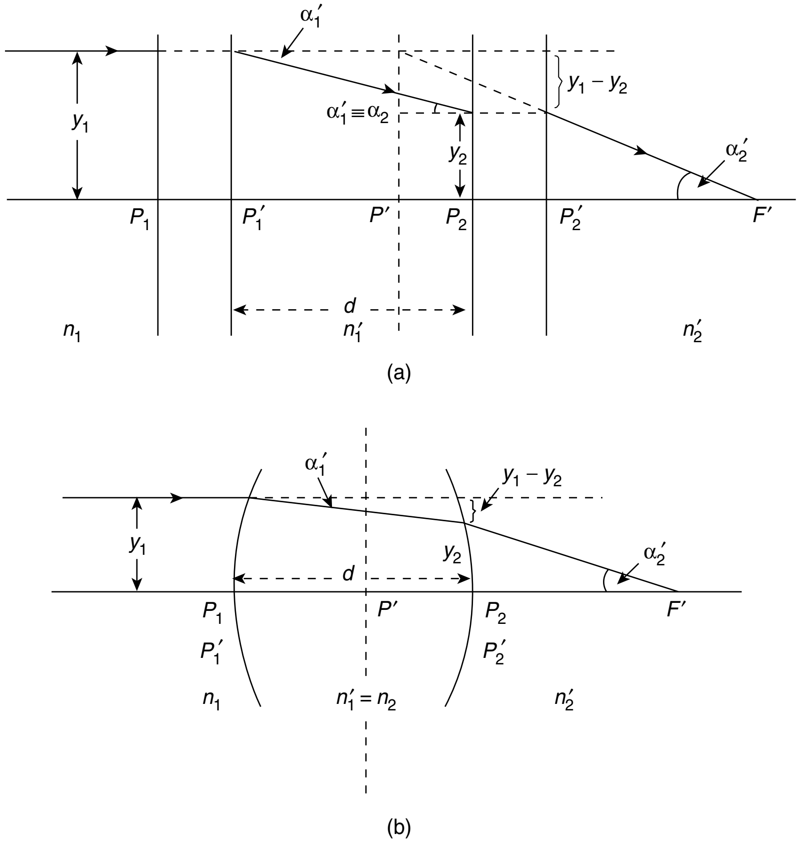


Figure 11.15 Deviation of a ray through (a) a system of two lenses and (b) a single thick lens. P' is a principal plane of the system. All the significant optical properties may be derived via this method

At the second lens or refracting surface

$$n_2\alpha_2 \equiv n'_1\alpha'_1$$

so

$$n'_2\alpha'_2 - n'_1\alpha'_1 = \mathscr{P}_2 y_2 \tag{11.9}$$

Equation (11.8) plus equation (11.9) gives

$$n'_2\alpha'_2 = \mathscr{P}_1 y_1 + \mathscr{P}_2 y_2 \tag{11.10}$$

Now the incident ray strikes the principal plane P' at a height y_1 so, extrapolating the ray from F', the focal point of the system, through the plane P'_2 to the plane P', we have

$$n'_2\alpha'_2 = \mathscr{P} y_1 \tag{11.11}$$

where $\mathscr{P}$ is the power of the complete system.

From equations (11.10) and (11.11) we have

$$\mathscr{P} y_1 = \mathscr{P}_1 y_1 + \mathscr{P}_2 y_2 \tag{11.12}$$

Moreover, Figure 11.15 shows that, algebraically

$$y_2 = y_1 - d\alpha_1'$$

which, with equation (11.8) gives

$$y_2 = y_1 - \frac{d}{n_1'} \mathscr{P}_1 y_1 = y_1 - \bar{d}\mathscr{P}_1 y_1, \tag{11.13}$$

where

$$\bar{d} = d/n_1'$$

This, with equation (11.12), gives

$$\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2 - \bar{d}\mathscr{P}_1 \mathscr{P}_2 \tag{11.14}$$

where $\mathscr{P}$ is the power of the whole system.

From Figure 11.15 we have algebraically

$$P_2' P' = -\frac{y_1 - y_2}{\alpha_2'}$$

which with equations (11.11) and (11.13) gives

$$P_2' P' = \frac{-n_2' \bar{d} \mathscr{P}_1}{\mathscr{P}} \tag{11.15}$$

For a similar ray incident from the right we can find

$$P_1 P = \frac{n_1 \bar{d} \mathscr{P}_2}{\mathscr{P}}$$

where P is the first principal plane (not shown in the figures).

A more significant distance for the thick lens of Figure 11.15(b) is $P_2 F'$ the distance between the second refracting surface and the focal point F'.
Now

$$P_2 F' = P' F' - P' P_2'$$

which with

$$P' F' = n_2' / \mathscr{P} \tag{11.16}$$

gives

$$P_2F' = \frac{n_2'}{\mathscr{P}} - \frac{n_2'\bar{d}\mathscr{P}_1}{\mathscr{P}}$$
$$= \frac{n_2'}{\mathscr{P}}(1 - \bar{d}\mathscr{P}_1) \tag{11.17}$$

We shall see in the following section that the factor $1 - \bar{d}\mathscr{P}_1$ and the power $\mathscr{P}$ of the system arise quite naturally in the matrix treatment of this problem.

The Matrix Method

Tracing paraxial rays through an optical system involves the constant repetition of two consecutive processes and is particularly suited to matrix methods.

A refracting R process carries the ray from one medium across a refracting surface into a second medium from where it is taken by a transmitting T process through the second medium to the next refracting surface for R to be repeated. Both R and T processes and their products are represented by 2×2 matrices.

An R process is characterized by

$$n'\alpha' - n\alpha = \mathscr{P}_1 y \tag{11.7}$$

which changes $n\alpha$ but which leaves y unaffected.

We write this in the form

$$\bar{\alpha}' - \bar{\alpha} = \mathscr{P}_1 y \tag{11.18}$$

where

$$\bar{\alpha}_i = n_i\alpha_i$$

The reader should review Figure 11.8 for the sign convention for angles.

A T process is characterized by

$$y' = y - \bar{d}'\bar{\alpha}' \tag{11.19}$$

which changes y but leaves $\bar{\alpha}$ unaffected. The thick lens of the last section demonstrates the method particularly well and reproduces the results we have already found.

In Figure 11.16 note that

$$n_2\alpha_2 \equiv n_1'\alpha_1'$$

that is

$$\bar{\alpha}_2 = \bar{\alpha}_1'$$

We express equations (11.18) and (11.19) in a suitable 2×2 matrix form by writing them as separate pairs.

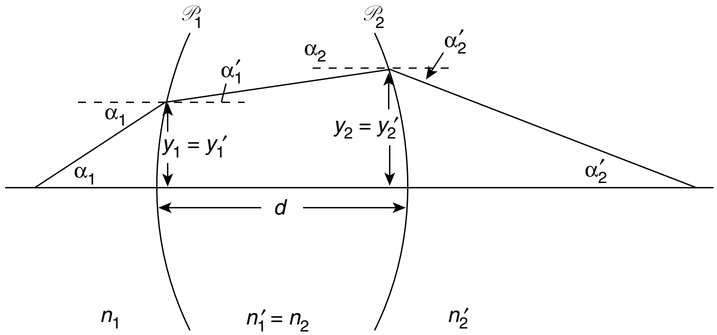


Figure 11.16 The single lens of Figure 11.15 is used to demonstrate the equivalence of the deviation and matrix methods for determining the important properties of a lens system. The matrix method is easily extended to a system of many optical elements

For R we have

$$\bar{\alpha}_1' = \bar{\alpha}_1 + \mathscr{P}_1 y_1$$

where $\mathscr{P}_1$ is the power of the first refracting surface and

$$y_1' = 0\bar{\alpha}_1 + 1y_1$$

so, in matrix form we have

$$\begin{bmatrix} \bar{\alpha}_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} 1 & \mathscr{P}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1 \\ y_1 \end{bmatrix} = R_1 \begin{bmatrix} \bar{\alpha}_1 \\ y_1 \end{bmatrix}$$

This carries the ray across the first refracting surface.

For T we have

$$\bar{\alpha}_2 = 1\bar{\alpha}_1' + 0y_1'$$
$$y_2 = -\bar{d}_1'\bar{\alpha}_1' + 1y_1'$$

where $\bar{\alpha}_2 = \bar{\alpha}_1'$, so

$$\begin{bmatrix} \bar{\alpha}_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\bar{d}_1' & 1 \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1' \\ y_1' \end{bmatrix} = T_{12} \begin{bmatrix} \bar{\alpha}_1' \\ y_1' \end{bmatrix}$$

This carries the ray through the lens between its two refracting surfaces.

At the second refracting surface we repeat R to give

$$\bar{\alpha}_2' = 1\bar{\alpha}_2 + \mathscr{P}_2 y_2$$
$$y_2' = 0\bar{\alpha}_2 + 1y_2$$

or

$$\begin{bmatrix} \bar{\alpha}_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & \mathscr{P}_2 \\ 0 & y_2 \end{bmatrix} = R_2 \begin{bmatrix} \bar{\alpha}_2 \\ y_2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix}\bar{\alpha}'_2\\ y'_2\end{bmatrix} = R_2\begin{bmatrix}\bar{\alpha}_2\\ y_2\end{bmatrix} = R_{12}T_{12}\begin{bmatrix}\bar{\alpha}'_1\\ y'_1\end{bmatrix} = R_2T_{12}R_1\begin{bmatrix}\bar{\alpha}_1\\ y_1\end{bmatrix}$$
$$= \begin{bmatrix}1 & \mathscr{P}_2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\bar{d}'_1 & 1\end{bmatrix}\begin{bmatrix}1 & \mathscr{P}_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\bar{\alpha}_1\\ y_1\end{bmatrix}$$

which, after matrix multiplication, gives

$$\begin{bmatrix}\bar{\alpha}'_2\\ y'_2\end{bmatrix} = \begin{bmatrix}1-\bar{d}'_1\mathscr{P}_2 & \mathscr{P}_1+\mathscr{P}_2-\bar{d}'_1\mathscr{P}_1\mathscr{P}_2\\ -\bar{d}'_1 & 1-\bar{d}'_1\mathscr{P}_1\end{bmatrix}\begin{bmatrix}\bar{\alpha}_1\\ y_1\end{bmatrix}$$

Writing

$$R_2T_{12}R_1 = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

we see that a_{12} is the power $\mathscr{P}$ of the thick lens (equation (11.14)) and that a_{22} apart from the factor $n'_2/\mathscr{P}$ is the distance between the second refracting surface and the second focal point. The product of the coefficient a_{11} and $n_1/\mathscr{P}$ gives the separation between the first focal point and the first refracting surface. Note, too, that a_{11} and a_{22} enable us to locate the principal planes with respect to the refracting surfaces.

The order of the matrices for multiplication purposes is the reverse of the progress of the ray through $R_1T_{12}R_2$, etc.

If the ray experiences a number J of such transformations, the general result is

$$\begin{bmatrix}\bar{\alpha}'_J\\ y'_J\end{bmatrix} = R_JT_{J-1,J}R_{J-1}\ldots R_2T_{12}R_1\begin{bmatrix}\bar{\alpha}_1\\ y_1\end{bmatrix}$$

The product of all these 2×2 matrices is itself a 2×2 matrix.

It is important to note that the determinant of each matrix and of their products is unity, which implies that the column vector represents a property which is invariant in its passage through the system.

The components of the column vector are, of course, $\bar{\alpha}_1 y_1$; that is, $n\alpha$ and y and we already know that for paraxial rays the Helmholtz equation states that the product $ny\alpha$ remains constant throughout the system.

(Problems 11.9, 11.10, 11.11)

Problem 11.1

Apply the principle of p. 311 to a thin bi-convex lens of refractive index n to show that its power is

$$\mathscr{P} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where R_1 and R_2, the radii of curvature of the convex faces, are both much greater than the thickness of the lens.

Problem 11.2

A plane parallel plate of glass of thickness d has a non-uniform refractive index n given by $n = n_0 - \alpha r^2$ where n_0 and α are constants and r is the distance from a certain line perpendicular to the sides of the plate. Show that this plate behaves as a converging lens of focal length $1/2\alpha d$.

Problem 11.3

For oscillatory waves the focal point F of the converging wavefront of Figure 11.17 is located where Huygens secondary waves all arrive in phase: the point F′ vertically above F receives waves whose total phase range $\Delta\phi$ depends on the path difference AF′−BF′. When F′ is such that $\Delta\phi$ is 2π the resultant amplitude tends to zero. Thus,

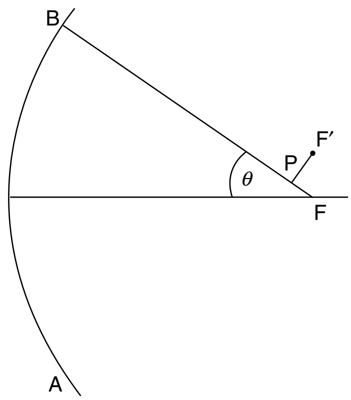


Figure 11.17

the focus is not a point but a region whose width x depends on the wavelength λ and the angle θ subtended by the spherical wave. If PF′ is perpendicular to BF the phase at F′ and P may be considered the same. Show that the width of the focal spot is given by $x = \lambda/\sin\theta$. Note that $\sin\theta$ is directly related to the f/d ratio for a lens (focal length/diameter) so that x defines the minimum size of the image for a given wavelength and a given lens.

Problem 11.4

As an object moves closer to the eye its apparent size grows with the increasing angle it subtends at the eye. A healthy eye can accommodate (that is, focus) objects from infinity to about 25 cm, the closest 'distance of distinct vision'. Beyond this 'near point' the eye can no longer focus and a magnifying glass is required. A healthy eye has a range of accommodation of 4 dioptres ($1/\infty$ to 1/0.25 m). If a man's near point is 40 cm from his eye, show that he needs spectacles of power equal to 1.5 dioptres. If another man is unable to focus at distances greater than 2 m, show that he needs diverging spectacles with a power of −0.5 dioptres.

Problem 11.5

Figure 11.18 shows a magnifying glass of power P with an erect and virtual image at l'. The angular magnification

$$M_\alpha = \beta/\gamma$$

$$= \frac{\text{angular size of image seen through the glass at distance } l'}{\text{angular size of object seen by the unaided eye at } d_o}$$

where d_o is the distance of distinct vision. Show that the transverse magnification $M_T = l'/l$ where l is the actual distance (not d_o) at which the object O is held. Hence show that $M_\alpha = d_o/l$ and use the thin lens power equation, p. 318, to show that

$$M_\alpha = d_o(P + 1/l') = Pd_o + 1$$

when $l' = d_o$. Note that M_α reduces to the value Pd_o when the eye relaxes by viewing the image at ∞.

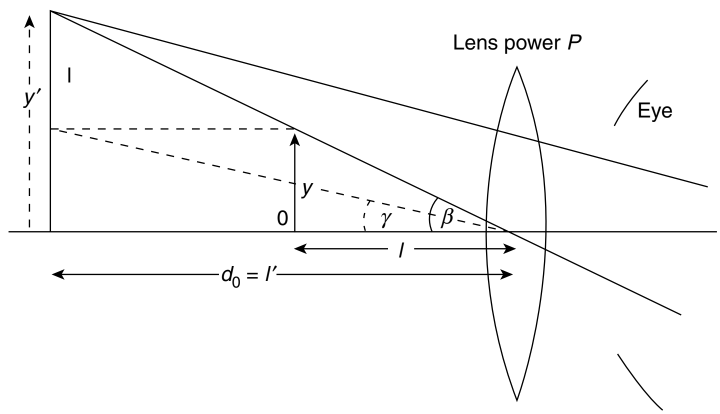


Figure 11.18

Problem 11.6

A telescope resolves details of a distant object by accepting plane wavefronts from individual points on the object and amplifying the very small angles which separate them. In Figure 11.19, α is the angle between two such wavefronts one of which propagates along the optical axis. In normal adjustment the astronomical telescope has both object and image at ∞ so that the total power of the system is zero. Use equation (11.14) to show that the separation of the lenses must be $f_o + f_e$ where f_o and f_e are respectively the focal lengths of the object and eye lenses.

If $2y$ is the width of the wavefront at the objective and $2y'$ is the width of the wavefront at the eye ring show that

$$M_\alpha = \left|\frac{\alpha'}{\alpha}\right| = \left|\frac{f_o}{f_e}\right| = \frac{D}{d}$$

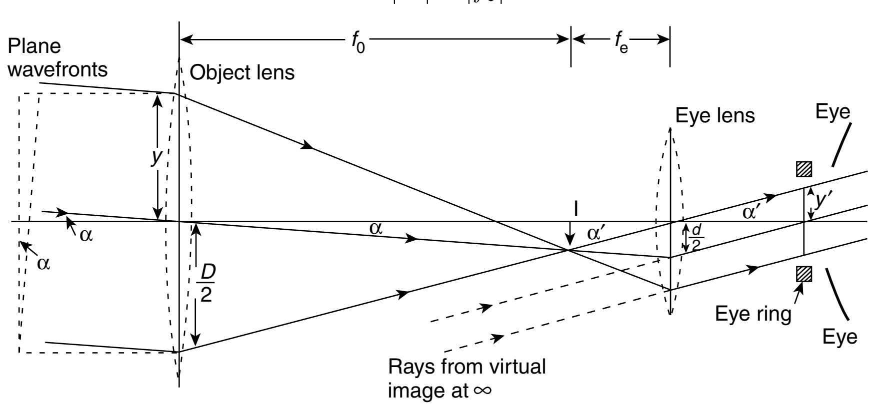


Figure 11.19

where D is the effective diameter of the object lens and d is the effective diameter of the eye lens. Note that the image is inverted.

Problem 11.7

The two lens microscope system of Figure 11.20 has a short focus objective lens of power P_o and a magnifying glass eyepiece of power P_e. The image is formed at the near point of the eye (the distance d_o of Problems 11.4 and 11.5). Show that the magnification by the object lens is $M_o = -P_o x'$ where the minus sign shows that the image is inverted. Hence use the expression for the magnifying glass in Problem 11.5 to show that the total magnification is

$$M = M_o M_e = -P_o P_e d_o x'$$

The length x' is called the optical tube length and is standardized for many microscopes at 0.14 m.

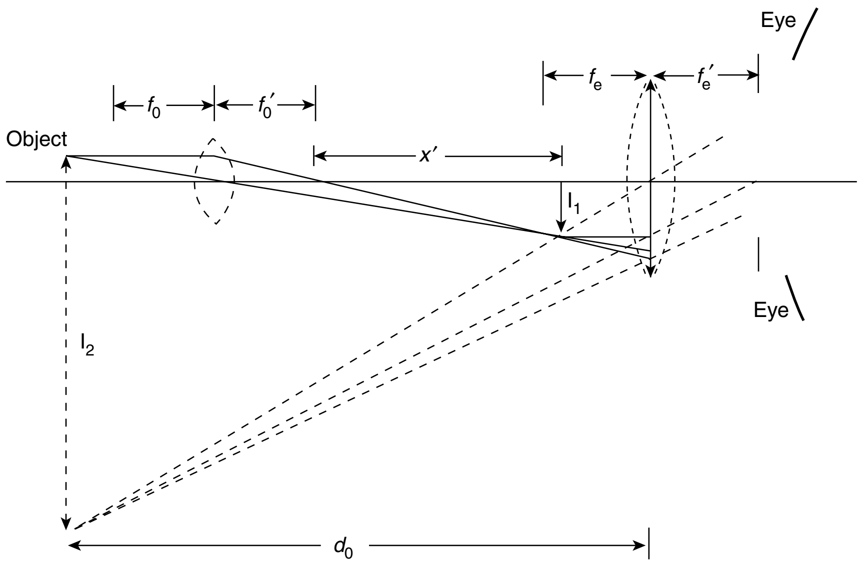


Figure 11.20

Problem 11.8

Microscope objectives are complex systems of more than one lens but the principle of the oil immersion objective is illustrated by the following problem. In Figure 11.21 the object O is embedded a distance R/n from the centre C of a glass sphere of radius

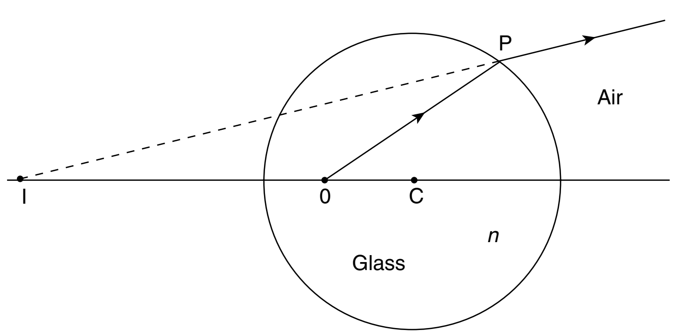
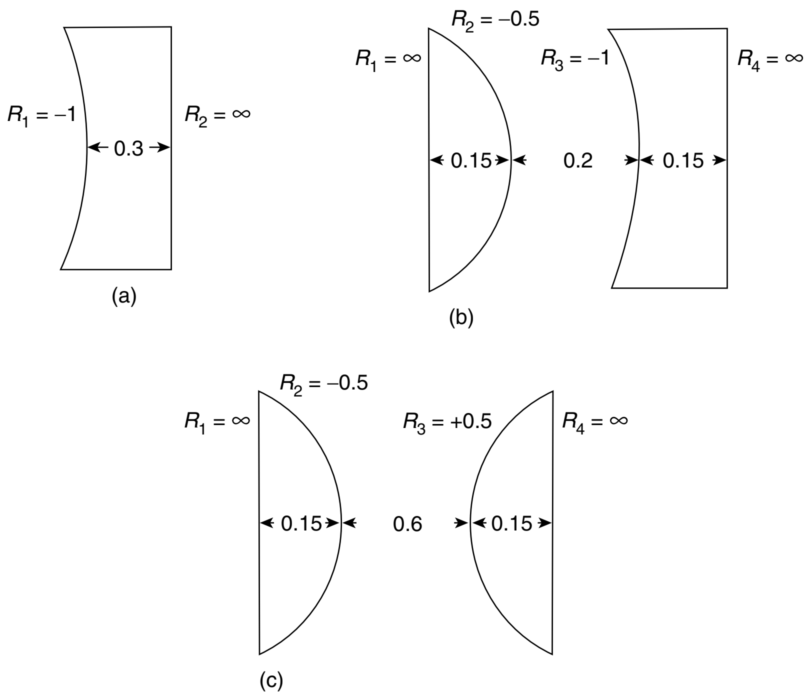


Figure 11.21

R and refractive index n. Any ray OP entering the microscope is refracted at the surface of the sphere and, when projected back, will always meet the axis CO at the point I. Use Snell's Law to show that the distance $IC = nR$.

Problems 11.9, 11.10, 11.11

Using the matrix method or otherwise, find the focal lengths and the location of the principal plane for the following lens systems (a), (b) and (c). The glass in all lenses has a refractive index of $n = 1.5$ and all measurements have the same units. R_i is a radius of curvature.



Summary of Important Results

Power of a Thin Lens

$$\mathscr{P} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{f}$$

where n is the refractive index of the lens material, R_1 and R_2 are the radii of curvature of the lens surfaces and f is the focal length.

Power of two thin lenses separated a distance d in Air

$$\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2 - d\mathscr{P}_1\mathscr{P}_2$$

where $\mathscr{P}_1$ and $\mathscr{P}_2$ are the powers of the thin lenses.

Power of a thick lens of thickness d and refractive index n

$$\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2 - d/n\,\mathscr{P}_1\mathscr{P}_2$$

where $\mathscr{P}_1$ and $\mathscr{P}_2$ are the powers of the refracting surfaces of the lens.

Optical Helmholtz Equation

For a plane wavefront (source at ∞) passing through an optical system the product

$$ny \tan\alpha = \text{constant}$$

where n is the refractive index, y is the width of the wavefront section and α is the angle between the optical axis and the normal to the wavefront.

For a source at a finite distance, this equation becomes, for paraxial rays,

$$ny\alpha = \text{constant}$$