INTEGRATION USING PARTIAL FRACTIONS (XII, R. S. AGGARWAL)

EXERCISE 15A (Pg. no.: 762)

Evaluate

1. $\int \frac{dx}{x(x+2)}$

Sol. Let $I = \int \frac{dx}{x(x+2)}$, Put $\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

...(1)

$\Rightarrow A(x+2) + Bx = 1$, Put $x+2=0 \quad \therefore x=-2,\ B=-\frac{1}{2}$

Put $x=0$, $A=\frac{1}{2}$, from equation (1), we get, $\frac{1}{x(x+2)} = \frac{1}{2}.\frac{1}{x} - \frac{1}{2}.\frac{1}{x+2}$

$$\Rightarrow \int \frac{1}{x(x+2)}dx = \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{x+2}dx = \frac{1}{2}\log x - \frac{1}{2}\log(x+2) + c$$

$$= \frac{1}{2}\left[\log x - \log(x+2)\right] + c = \frac{1}{2}\log\frac{x}{x+2} + c$$

2. $\int \frac{2x+1}{(x+2)(x-3)}dx$

Sol. Let $I = \int \frac{2x+1}{(x+2)(x-3)}dx$, Put $\frac{2x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$ ...(1)

$\Rightarrow 2x+1 = A(x-3) + B(x+2)$

Put $x-3=0$, $x=3$; $2\times3+1 = A(0) + B(3+2) \quad \Rightarrow B = \frac{7}{5}$

Put $x+2=0 \Rightarrow x=-2$; $-4+1 = A(-2-3) + B(0) \quad \Rightarrow A = \frac{-3}{-5} = \frac{3}{5}$

Now, From equation (1), we get, $\frac{2x+1}{(x+2)(x-3)} = \frac{3}{5}.\frac{1}{x+2} + \frac{7}{5}.\frac{1}{x-3}$

$$\Rightarrow \int \frac{2x+1}{(x+2)(x-3)}dx = \frac{3}{5}\int \frac{1}{x+2}dx + \frac{7}{5}\int \frac{1}{x-3}dx = \frac{3}{5}\log(x+2) + \frac{7}{5}\log(x-3) + c$$

3. $\int \frac{x}{(x+2)(3-2x)}dx$

Sol. Let $I = \int \frac{x}{(x+2)(3-2x)}dx$, Put $\frac{x}{(x+2)(3-2x)} = \frac{A}{x+2} + \frac{B}{3-2x}$ ...(1)

$\Rightarrow A(3-2x) + B(x+2) = x$

Put $3-2x=0 \Rightarrow x=\frac{3}{2}$; $A(0) + B\left(\frac{3}{2}+2\right) = \frac{3}{2} \quad \Rightarrow B\left(\frac{7}{2}\right) = \frac{3}{2} \quad \therefore B = \frac{3}{7}$

Put $x+2=0 \Rightarrow x=-2$; $A(7) + B(0) = -2 \quad \Rightarrow A = \frac{-2}{7}$

Now,From equation (1), we get, $\frac{x}{(x+2)(3-2x)} = \frac{-2}{7}.\frac{1}{x+2} + \frac{3}{7}.\frac{1}{3-2x}$

$$\Rightarrow \int \frac{x}{(x+2)(3-2x)}dx = \frac{-2}{7}\int \frac{1}{x+2}dx + \frac{3}{7}\int \frac{1}{3-2x}dx$$

$$= \frac{-2}{7}\log(x+2) + \frac{3}{7}.\frac{1}{-2}\log(3-2x) + c = \frac{-2}{7}\log(x+2)\frac{-3}{14}\log(3-2x) + c$$

4. $\int \frac{dx}{x(x-2)(x-4)}$

Sol. Let $I = \int \frac{dx}{x(x-2)(x-4)}$, Put $\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4}$ ...(1)

$\Rightarrow A(x-2)(x-4) + Bx(x-4) + Cx(x-2) = 1$

Put $x-2=0 \Rightarrow x=2$; $A(0) + B.2(2-4) + C(0) = 1 \Rightarrow B.2(-2) = 1 \therefore B = -\frac{1}{4}$

Put $x-4=0 \Rightarrow x=4$; $A(0) + B(0) + C4(4-2) = 1 \Rightarrow C.4(2) = 1, C = \frac{1}{8}$

Put $x=0 \Rightarrow A(0-2)(0-4) + B(0) + C(0) = 1 \Rightarrow A = \frac{1}{8}$

Now,From equation (1), $\frac{1}{x(x-2)(x-4)} = \frac{1}{8}.\frac{1}{x} - \frac{1}{4}.\frac{1}{x-2} + \frac{1}{8}.\frac{1}{x-4}$

$$\Rightarrow \int \frac{1}{x(x-2)(x-4)}dx = \frac{1}{8}\int \frac{1}{x}dx - \frac{1}{4}\int \frac{1}{x-2}dx + \frac{1}{8}\int \frac{1}{x-4}dx = \frac{1}{8}\log x - \frac{1}{4}\log(x-2) + \frac{1}{8}\log(x-4) + c$$

5. $\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Sol. Let $I = \int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$, Put $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$ ...(1)

$\Rightarrow A(x+2)(x-2) + B(x-1)(x-3) + C(x-1)(x+2) = 2x-1$

Put $x+2=0 \Rightarrow x=-2$; $A(0) + B(-2-1)(-2-3) + C(0) = 2x-2-1$

$\Rightarrow B(-3)(-5) = -5, \quad B = \frac{-1}{3}$

Put $x-3=0 \Rightarrow x=3$; $A(0) + B(0) + C(2)(5) = 5 \Rightarrow C = \frac{1}{2}$

Put $x-1=0 \Rightarrow x=1$; $A(3)(-2) + B(0) + C(0) = 2-1 = 1 \Rightarrow A = -\frac{1}{6}$

Now,From equation (1), we get, $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{-1}{6}.\frac{1}{(x-1)} - \frac{1}{3}.\frac{1}{x+2} + \frac{1}{2}.\frac{1}{x-3}$

$$\Rightarrow \int \frac{2x-1}{(x-1)(x+2)(x-3)}dx = \frac{-1}{6}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{1}{x+2}dx + \frac{1}{2}\int \frac{1}{x-3}dx$$

$$= \frac{-1}{6}\log(x-1) - \frac{1}{3}\log(x+2) + \frac{1}{2}\log(x-3) + c$$

6. $\int \frac{2x-3}{(x^2-1)(2x+3)}dx$

Sol. Let $I = \int \frac{2x-3}{(x^2-1)(2x+3)}dx$

Put $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+3)}$ ...(1)

$\Rightarrow A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1) = 2x-3$

Put $x+1=0 \Rightarrow x=-1;\ A(0)+B(-1-1)(-2+3)+C(0) = -2-3 \quad \therefore B = \frac{-5}{-2} = \frac{5}{2}$

Put $x-1=0 \Rightarrow x=1;\ A(2)(2+3)+B(0)+C(0) = 2-3 = -1 \quad \therefore A = \frac{-1}{10}$

Put $2x+3=0 \Rightarrow x=\frac{-3}{2};\ A(0)+B(0)+C\left(\frac{-3}{2}-1\right)\left(-\frac{3}{2}+1\right) = 2\left(\frac{-3}{2}\right)-3$

$\therefore\ C\left(\frac{-5}{2}\right)\left(\frac{-1}{2}\right) = -3-3 \quad \Rightarrow C(5) = -24 \quad \Rightarrow C = \frac{-24}{5}$

Now, From equation (1), we get, $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{-1}{10}.\frac{1}{x-1} + \frac{5}{2}.\frac{1}{x+1} - \frac{24}{5}.\frac{1}{2x+3}$

$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)}dx = \frac{-1}{10}\int \frac{1}{x-1}dx + \frac{5}{2}\int \frac{1}{x+1}dx - \frac{24}{5}\int \frac{1}{2x+3}dx$

$= \frac{-1}{10}\log(x-1) + \frac{5}{2}\log(x+1) - \frac{24}{5}\frac{\log(2x+3)}{2} + c = \frac{-1}{10}\log(x-1) + \frac{5}{2}\log(x+1) - \frac{12}{5}\log(2x+3) + c$

7. $\int \frac{2x+5}{x^2-x-2}dx$

Sol. Let $I = \int \frac{2x+5}{x^2-x-2}dx = \int \frac{2x+5}{(x-2)(x+1)}dx$,

Put $\frac{2x+5}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} \quad \Rightarrow A(x+1) + B(x-2) = 2x+5$ ...(1)

Put $x+1=0 \Rightarrow x=-1;\ A(0)+B(-1-2) = 2(-1)+5,\ B(-3) = 3 \quad \therefore B = -1$

Put $x-2=0 \Rightarrow x=2;\ A(2+1)+B(0) = 2\times 2+5 = 9,\ A = 3$

Now, From equation (1), we get, $\frac{2x+5}{(x-2)(x+1)} = \frac{3}{x-2} + \frac{-1}{x+1}$

$\Rightarrow \int \frac{2x+5}{(x-2)(x+1)} = \int \frac{3}{x-2}dx + \int \frac{-1}{x+1}dx = 3\log(x-2) - \log(x+1) + c$

8. $\int \frac{x^2+5x+3}{x^2+3x+2}dx$

Sol. Let $I = \int \frac{x^2+5x+3}{x^2+3x+2}dx = \int \frac{x^2+3x+2+2x+1}{x^2+3x+2}dx \quad \Rightarrow I = \int \frac{x^2+3x+2}{x^2+3x+2}dx + \int \frac{2x+1}{x^2+3x+2}dx$

$\Rightarrow I = \int dx + \int \frac{2x+1}{(x+1)(x+2)}dx \quad \Rightarrow I = x + I_1$, where $I_1 = \int \frac{2x+1}{(x+1)(x+2)}dx$

Put $\frac{2x+1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2} \Rightarrow A(x+2)+B(x+1)=2x+1$

Put $x+2=0 \Rightarrow x=-2;\ A(0)+B(-1)=2(-2)+1 \Rightarrow B=3$

Put $x+1=0 \Rightarrow x=-1;\ A(-1+2)+B(0)=2(-1)+1 \Rightarrow A=-1$

$\therefore \frac{2x+1}{(x+1)(x+2)}=\frac{-1}{x+1}+\frac{3}{x+2}$

$\Rightarrow \int\frac{2x+1}{(x+1)(x+2)}dx=-\int\frac{1}{x+1}dx+3\int\frac{1}{x+2}dx=-\log(x+1)+3\log(x+2)+C$

So, $I=x-\log(x+1)+3\log(x+2)+C$

9. $\int\frac{x^2+1}{x^2-1}dx$

Sol. Let $I=\int\frac{x^2+1}{x^2-1}dx \Rightarrow \int\left(1+\frac{2}{x^2-1}\right)dx \Rightarrow I=\int dx+2\int\frac{1}{x^2-1}dx$

$\Rightarrow I=x+2.\frac{1}{2.1}\log\left|\frac{x-1}{x+1}\right|+c \quad \therefore I=x+\log\left|\frac{x-1}{x+1}\right|+c$

10. $\int\frac{x^3}{x^2-4}dx$

Sol. Let $I=\int\frac{x^3}{x^2-4}dx \Rightarrow I=\int\left(x+\frac{4x}{x^2-4}\right)dx$

$\Rightarrow I=\int x\,dx+\int\frac{4x}{(x-2)(x+2)}dx=\frac{x^2}{2}+\int\frac{4x}{(x-2)(x+2)}dx \Rightarrow I=x+I_1 \qquad \ldots(1)$

$\therefore I_1=\int\frac{4x}{x^2-4}dx$, Put $x^2-4=t \Rightarrow 2x\,dx=dt \Rightarrow I_1=2\int\frac{dt}{t} \Rightarrow I_1=2\log\left|x^2-4\right|+c$

Putting the value of I_1 in equation (1), $I=\frac{x^2}{2}+2\log\left|x^2-4\right|+c$

11. $\int\frac{3+4x-x^2}{(x+2)(x-1)}dx$

Sol. Let $I=\int\frac{3+4x-x^2}{(x+2)(x-1)}$

$=\int\left(-1+\frac{5x+1}{(x+2)(x-1)}\right)dx=\int-dx+\int\frac{5x+1}{(x+2)(x-1)}dx=-x+I_1 \qquad \ldots(1)$

$\Rightarrow I_1=\int\frac{5x+1}{(x+2)(x-1)}dx$, Put $\frac{5x+1}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1} \qquad \ldots(2)$

$\Rightarrow A(x-1)+B(x+2)=5x+1$

Put $x-1=0 \Rightarrow x=1;\ A(0)+B(1+2)=5+1=6 \Rightarrow B=2$

Put $x+2=0 \Rightarrow x=-2;\ A(-2-1)+B(0)=5\times(-2)+1 \Rightarrow A(-3)=-9 \Rightarrow A=3$

Now, From equation (2), we get, $\frac{5x+1}{(x+2)(x-1)}=\frac{3}{x+2}+\frac{2}{x-1}$

$$\Rightarrow I=\int\frac{5x+1}{(x+2)(x-1)}dx=3\int\frac{1}{x+2}dx+2\int\frac{1}{x-1}dx=3\log(x+2)+2\log(x-1)+c$$

$\therefore$ From equation (1), we get, $I=-x+3\log(x+2)+2\log(x-1)+c$

12. $\int\frac{x^3}{(x-1)(x-2)}dx$

Sol. Let, $I=\int\frac{x^3}{(x-1)(x-2)}dx=\int\left\{(x+3)+\frac{7x-6}{(x-1)(x-2)}\right\}dx$

$$=\frac{x^2}{2}+3x+\int\frac{7x-6}{(x-1)(x-2)}dx=\frac{x^2}{2}+3x+I_1 \quad \ldots(1), \qquad \text{where, } I_1=\int\frac{7x-6}{(x-1)(x-2)}dx$$

Put $\frac{7x-6}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$ ...(2)

$\Rightarrow A(x-2)+B(x-1)=7x-6$

Put $x-2=0,\ x=2;\ A(0)+B(2-1)=7\times2-6 \Rightarrow B=8$

Put $x-1=0 \Rightarrow x=1;\ A(1-2)+B(0)=7-6=1 \Rightarrow A=-1$

$\therefore$ From equation (2), we get, $\frac{7x-6}{(x-1)(x-2)}=\frac{-1}{x-1}+\frac{8}{x-2}$

$$I_1=\int\frac{7x-6}{(x-1)(x-2)}dx=-\int\frac{1}{x-1}dx+8\int\frac{1}{x-2}dx=-\log(x-1)+8\log(x-2)+C$$

$\therefore$ From equation (1), we get, $I=\frac{x^2}{2}+3x-\log(x-1)+8\log(x-2)+c$

13. $\int\left(\frac{x^3-x-2}{1-x^2}\right)dx$

Sol. Let $I=\int\left(\frac{x^3-x-2}{1-x^2}\right)dx \Rightarrow I=\int\left(-x+\frac{-2}{1-x^2}\right)dx \Rightarrow I=\int -x\,dx+(-2)\int\frac{1}{1-x^2}dx$

$$\Rightarrow I=\frac{-x^2}{2}-2.\frac{1}{2.1}\log\left|\frac{1+x}{1-x}\right|+c \quad \therefore I=-\frac{x^2}{2}+\log\left|\frac{1-x}{1+x}\right|+c$$

14. $\int\frac{2x+1}{(4-3x-x^2)}dx$

Sol. Let $I=\int\frac{2x+1}{4-3x-x^2}dx \Rightarrow I=\int\frac{2x+1}{4-3x-x^2}dx=\int\frac{2x+1}{(1-x)(4+x)}dx$

Put $\frac{2x+1}{(1-x)(4+x)}=\frac{A}{1-x}+\frac{B}{4+x}$ ...(1)

$\therefore\ A(4+x)+B(1-x)=2x+1$

Put $1-x=0\ \therefore\ x=1;\ A(5)+B(0)=3\ \therefore\ A=\frac{3}{5}$

Put $4+x=0\ \therefore\ x=-4;\ A(0)+B(5)=-8+1=-7\ \therefore\ B=-\frac{7}{5}$

From equation (1), $\dfrac{2x+1}{(1-x)(4+x)}=\dfrac{3/5}{1-x}-\dfrac{7/5}{4+x}$

$$\Rightarrow \int\frac{2x+1}{(1-x)(4+x)}dx=\frac{3}{5}\int\frac{1}{1-x}dx-\frac{7}{5}\int\frac{1}{4+x}dx=\frac{-3}{5}\log(1-x)-\frac{7}{5}\log(4+x)+c$$

$$=-\frac{1}{5}\left[3\log(1-x)+7\log(4+x)\right]+c$$

15. $\displaystyle\int\frac{2x}{(x^2+1)(x^2+3)}dx$

Sol. Let $I=\displaystyle\int\frac{2x}{(1+x^2)(3+x^2)}dx$

Put $x^2=t \Rightarrow 2xdx=dt$

$\therefore\ I=\displaystyle\int\frac{dt}{(1+t)(3+t)}=\frac{1}{2}\int\left[\frac{1}{1+t}-\frac{1}{3+t}\right]dt$ [Resolving into partial fractions]

$$=\frac{1}{2}\left[\log|1+t|-\log|3+t|\right]+c=\frac{1}{2}\log\left|\frac{1+t}{3+t}\right|+c$$

$$=\frac{1}{2}\log\left|\frac{1+x^2}{3+x^2}\right|+c=\frac{1}{2}\log\left(\frac{1+x^2}{3+x^2}\right)+c \qquad \left[\because 1+x^2>0,\ 3+x^2>0\right]$$

16. $\displaystyle\int\frac{\cos x}{(1+\sin x)(2+\sin x)}dx$

Sol. Let $I=\displaystyle\int\frac{\cos x.dx}{(1+\sin x)(2+\sin x)}$

Put $t=\sin x \Rightarrow dt=\cos x\,dx$; $I=\displaystyle\int\frac{dt}{(1+t)(2+t)}$,

Put $\dfrac{1}{(1+t)(2+t)}=\dfrac{A}{1+t}+\dfrac{B}{2+t}$ ...(1)

$\Rightarrow \dfrac{1}{(1+t)(2+t)}=\dfrac{A(2+t)+B(1+t)}{(1+t)(2+t)}$ $\therefore\ A(2+t)+B(1+t)=1$

Put $t+1=0 \Rightarrow t=-1$; $A(2-1)+B(0)=1 \Rightarrow A=1$

Put $t+2=0 \Rightarrow t=-2$; $A(0)+B(-2+1)=1 \Rightarrow B=-1$

$\therefore$ From equation (1), we get, $\dfrac{1}{(1+t)(2+t)}=\dfrac{1}{1+t}+\dfrac{-1}{2+t}$

$$\Rightarrow \int\frac{1}{(1+t)(2+t)}dt=\int\frac{1}{1+t}dt-\int\frac{1}{t+2}dt=\log(1+t)-\log(t+2)+c=\log\frac{(1+t)}{t+2}+c$$

So, $I=\displaystyle\int\frac{\cos x.dx}{(1+\sin x)(2+\sin x)}=\log\frac{(1+\sin x)}{2+\sin x}+c$

17. $\displaystyle\int\frac{\sec^2 x}{(2+\tan x)(3x+\tan x)}dx$

Sol. Let $I=\displaystyle\int\frac{\sec^2 x}{(2+\tan x)(3x+\tan x)}dx$

Put $t=\tan x \Rightarrow \frac{dt}{dx}=\sec^2 x,\ dt=\sec^2 x\,dx$; $I=\int\frac{dt}{(2+t)(3+t)}$

Put $\frac{1}{(2+t)(3+t)}=\frac{A}{2+t}+\frac{B}{3+t}$ ...(1)

$\Rightarrow \frac{1}{(2+t)(3+t)}=\frac{A(3+t)+B(2+t)}{(2+t)(3+t)}$ $\therefore\ A(3+t)+B(2+t)=1$

Put $t+2=0 \Rightarrow t=-2$; $A(3-2)+B(0)=1\ \therefore\ A=1$

Put $t+3=0 \Rightarrow t=-3$; $A(0)+B(2-3)=1\ \therefore B=-1$

From equation (1), we get, $\frac{1}{(2+t)(3+t)}=\frac{1}{2+t}-\frac{1}{3+t}$

$$\Rightarrow \int\frac{1}{(2+t)(3+t)}dt=\int\frac{1}{2+t}dt-\int\frac{1}{3+t}dt=\log(2+t)-\log(3+t)+c=\log\left(\frac{2+t}{3+t}\right)+c$$

$$\Rightarrow I=\log\left(\frac{2+\tan x}{3+\tan x}\right)+C$$

18. $\int\frac{\sin x\cos x}{\cos^2 x-\cos x-2}dx$

Sol. Let $I=\int\frac{\sin x.\cos x}{\cos^2 x-\cos x-2}dx$

Put $t=\cos x \Rightarrow dt=-\sin x\,dx$; $I=\int\frac{(-dt)t}{t^2-t-2} \Rightarrow I=-\int\frac{t\,dt}{(t+1)(t-2)}$

Put $\frac{-t}{(t+1)(t-2)}=\frac{A}{t+1}+\frac{B}{t-2}$ ...(1)

$\therefore\ A(t-2)+B(t+1)=-t$

Put $t-2=0 \Rightarrow t=2;\ A(0)+B(2+1)=-2 \Rightarrow B=\frac{-2}{3}$

Put $t+1=0 \Rightarrow t=-1;\ A(-1-2)+B(0)=1 \Rightarrow A=\frac{1}{-3}=-\frac{1}{3}$

From equation (1), we get, $\frac{-t}{(t+1)(t-2)}=-\frac{1}{3}.\frac{1}{t+1}-\frac{2}{3}.\frac{1}{t-2}$

$$\Rightarrow I=\int\frac{-t}{(t+1)(t-2)}dt=-\frac{1}{3}.\int\frac{1}{t+1}dt-\frac{2}{3}\int\frac{1}{t-2}dt$$

$$=\frac{-1}{3}\log(t+1)\frac{2}{-3}\log|(t-2)|+c=-\frac{1}{3}\log(\cos x+1)\frac{-2}{3}\log|(\cos x-2)|+c$$

19. $\int\frac{e^x}{(e^{2x}+5e^x+6)}dx$

Sol. Let $I=\int\frac{e^x}{e^{2x}+5e^x+6}dx$

Put $t=e^x \Rightarrow dt=e^x dx$; $I=\int\frac{dt}{t^2+5t+6}=\int\frac{dt}{(t+2)(t+3)}$

Put $\frac{1}{(t+2)(t+3)} = \frac{A}{t+2} + \frac{B}{t+3}$ ...(1)

$A(t+3) + B(t+2) = 1$

Put $t+3=0 \Rightarrow t=-3$; $A(0)+B(-3+2)=1 \Rightarrow B=-1$

Put $t+2=0 \Rightarrow t=-2$; $A(-2+3)+B(0)=1 \Rightarrow A=1$

$\therefore$ From equation (1), we get, $\frac{1}{(t+2)(t+3)} = \frac{1}{t+2} - \frac{1}{t+3}$

$$\Rightarrow \int \frac{dt}{(t+2)(t+3)} = \int \frac{1}{t+2} dt - \int \frac{1}{t+3} dt = \log(t+2) - \log(t+3) + c = \log \frac{t+2}{t+3} + c$$

$$\therefore \int \frac{e^x}{e^{2x}+5e^x+6} dx = \log \frac{e^x+2}{e^x+3} + c$$

20. $\int \frac{e^x}{e^{3x} - 3e^{2x} - e^x + 3} dx$

Sol. Let $I = \int \frac{e^x}{e^{3x} - 3e^{2x} - e^x + 3} dx$

Put $t = e^x$, $dt = e^x dx$; $I = \int \frac{dt}{t^3 - 3t^2 - t + 3} \Rightarrow I = \int \frac{dt}{t^2(t-3) - (t-3)} = \int \frac{dt}{(t^2-1)(t-3)}$

Put $\frac{1}{(t-1)(t+1)(t-3)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{t-3}$ ...(1)

$\Rightarrow A(t+1)(t-3) + B(t-1)(t-3) + C(t-1)(t+1) = 1$

Put $t+1=0 \Rightarrow t=-1$; $A(0)+B(-1-1)(-1-3)+C(0)=1 \Rightarrow B(-2)(-4)=1 \Rightarrow B=\frac{1}{8}$

Put $t-1=0 \Rightarrow t=1$; $A(1+1)(1-3)+B(0)+C(0)=1 \Rightarrow A=\frac{-1}{4}$

Put $t-3=0 \Rightarrow t=3$; $A(0)+B(0)+C(3-1)(3+1)=1 \Rightarrow C=\frac{1}{8}$

$\therefore$ From equation (1), we get, $\frac{1}{(t-1)(t+1)(t-3)} = \frac{-1}{4} . \frac{1}{t-1} + \frac{1}{8} . \frac{1}{t+1} + \frac{1}{8} . \frac{1}{t-3}$

$$\Rightarrow \int \frac{dt}{(t-1)(t+1)(t-3)} = -\frac{1}{4} \int \frac{1}{t-1} dt + \frac{1}{8} \int \frac{1}{t+1} dt + \frac{1}{8} \int \frac{1}{t-3} dt$$

$$= \frac{-1}{4} \log(t-1) + \frac{1}{8} \log(t+1) + \frac{1}{8} \log(t-3) + c$$

$$\Rightarrow \int \frac{e^x dx}{(e^x-1)(e^x+1)(e^x-3)} = -\frac{1}{4} \log(e^x-1) + \frac{1}{8} \log(e^x+1) + \frac{1}{8} \log(e^x-3) + c$$

21. $\int \frac{2\log x}{x\left[2(\log x)^2 - \log x - 3\right]} dx$

Sol. Let $I = \int \frac{2\log x}{x\left[2(\log x)^2 - \log x - 3\right]} dx$

Put $t=\log x \Rightarrow dt=\frac{1}{x}dx$; $I=\int\frac{2t}{2t^2-t-3}dt$

Put $\frac{2t}{2t^2-t-3}=\frac{2t}{(2t-3)(t+1)} \Rightarrow \frac{2t}{(2t-3)(t+1)}=\frac{A}{2t-3}+\frac{B}{t+1}$ ...(1)

$\therefore \quad A(t+1)+B(2t-3)=2t$

Put $2t-3=0 \Rightarrow t=\frac{3}{2}$; $A\left(\frac{3}{2}+1\right)+B(0)=2.\frac{3}{2}=3 \Rightarrow A\left(\frac{5}{2}\right)=3 \quad \therefore A=\frac{6}{5}$

Put $t+1=0 \Rightarrow t=-1$; $A(0)+B(-2-3)=-2 \quad \therefore B=\frac{-2}{-5}=\frac{2}{5}$

$\therefore$ From equation (1), we get, $\frac{2t}{(2t-3)(t+1)}=\frac{6}{5}.\frac{1}{2t-3}+\frac{2}{5}.\frac{1}{t+1}$

$$\Rightarrow \int\frac{2t}{(2t-3)(t+1)}dt=\frac{6}{5}\int\frac{1}{2t-3}dt+\frac{2}{5}\int\frac{1}{t+1}dt=\frac{6}{5}\log\frac{6}{5}\frac{\log(2t-3)}{2}+\frac{2}{5}\log(\log x+1)$$

$$\Rightarrow \int\frac{2\log x\,dx}{x\left[2(\log x)^2-\log x-3\right]}=\frac{3}{5}\log(2\log x-3)+\frac{2}{5}\log(\log x+1)+c$$

22. $\int\frac{\operatorname{cosec}^2 x}{1-\cot^2 x}dx$

Sol. Let $I=\int\frac{\operatorname{cosec}^2 x}{1-\cot^2 x}dx$. Put $\cot x=t \Rightarrow -\operatorname{cosec}^2 x=\frac{dt}{dx} \Rightarrow \operatorname{cosec}^2 x\,dx=-dt$

$$I=\int\frac{-dt}{1-t^2} \Rightarrow I=-\int\frac{1}{1-t^2}dt \Rightarrow I=-\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right|+c=-\frac{1}{2}\log\left|\frac{1+\cot x}{1-\cot x}\right|+c$$

23. $\int\frac{\sec^2 x}{\tan^3 x+4\tan x}dx$

Sol. Let $I=\int\frac{\sec^2 x}{\tan^3 x+4\tan x}dx$

Put $\tan x=t \Rightarrow dt=\sec^2 x\,dx$; $I=\int\frac{dt}{t^3+4t}$, $I=\int\frac{dt}{t(t^2+4)}$

Put $\frac{1}{t(t^2+4)}=\frac{A}{t}+\frac{Bt+C}{t^2+4}$ ...(1)

$\Rightarrow A(t^2+4)+(Bt+C)t=1$. Put $t=0$, $A(0+4)\times B(0)=1 \Rightarrow A=\frac{1}{4}$

By, Equating the co-efficient of t^2 and constant here, $A+B=0 \Rightarrow \frac{1}{4}+B=0 \quad \therefore B=\frac{-1}{4}$, $C=0$

$\therefore$ From equation (1), we get, $\int\frac{1}{t(t^2+4)}dt=\int\left(\frac{1}{4}.\frac{1}{t}+\frac{-\frac{1}{4}t+0}{t^2+4}\right)dt=\frac{1}{4}\int\frac{1}{t}dt-\frac{1}{4}\int\frac{t}{t^2+4}dt$

$=\frac{1}{4}\log t-\frac{1}{4}.\frac{1}{2}\log(z)+C$, where $Z=t^2+4$, $\frac{dZ}{dt}=2t\,dt$

$=\frac{1}{4}\log t-\frac{1}{8}\log(t^2+4)+c=\frac{1}{4}\log\tan x-\frac{1}{8}\log(\tan^2 x+4)+c$

24. $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}dx$

Sol. Let $I = \int \frac{2\sin x.\cos x}{(1+\sin x)(2+\sin x)}dx$

Put $t = \sin x \Rightarrow dt = \cos x\,dx$; $I = \int \frac{2t}{(1+t)(2+t)}dt$

Put $\frac{2t}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$ ...(1) $\therefore A(2+t) + B(1+t) = 2t$

Put $2+t = 0 \Rightarrow t = -2$; $A(0) + B(1-2) = -4 \Rightarrow B = 4$

Put $1+t = 0 \Rightarrow t = -1$; $A(2-1) + B(0) = -2 \Rightarrow A = -2$

$\therefore$ From equation (1), we get, $\frac{2t}{(1+t)(2+t)} = \frac{-2}{1+t} + \frac{4}{2+t}$

$\Rightarrow \int \frac{2t}{(1+t)(2+t)}dt = -2\int \frac{1}{1+t}dt + 4\int \frac{1}{2+t}dt = -2\log(1+t) + 4\log(2+t) + c$

So, $\int \frac{\sin 2x.dx}{(1+\sin x)(2+\sin x)} = 4\log(2+\sin x) - 2\log(1+\sin x) + c$

25. $\int \frac{e^x}{e^x(e^x-1)}dx$

Sol. Let $I = \int \frac{e^x}{e^x(e^x-1)}dx$, Put $t = e^x \Rightarrow dt = e^x\,dx$; $I = \int \frac{dt}{t(t-1)}$

Put $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$ ...(1) $\therefore A(t-1) + Bt = 1$

Put $t-1 = 0 \Rightarrow t = 1$; $A(0) + B.1 = 1 \Rightarrow B = 1$

Put $t = 0$; $A(0-1) + B(0) = 1 \Rightarrow A = -1$

$\therefore$ From equation (1), we get, $\frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$

$\Rightarrow \int \frac{1}{t(t-1)}dt = -\int \frac{1}{t}dt + \int \frac{1}{t-1}dt = -\log t + \log(t-1) + c = \log\frac{t-1}{t} + c$

So, $\int \frac{e^x}{e^x(e^x-1)}dx = \log\left(\frac{e^x-1}{e^x}\right) + c = \log\left(1 - \frac{1}{e^x}\right) + c$

26. $\int \frac{dx}{x(x^4-1)}$

Sol. Let $I = \int \frac{dx}{x(x^4-1)}$, Put $t = x^4 - 1 \Rightarrow 4x^3dx = dt$

$$I = \int \frac{x^3dx}{x^4(x^4-1)} = \frac{1}{4}\int \frac{dt}{(t+1)t}$$

Put $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$ ...(1) $\Rightarrow A(t+1) + Bt = 1$

Put $t+1=0 \Rightarrow t=-1;\ A(0)+B(-1)=1 \quad \Rightarrow B=-1$

Put $t=0;\ A(0+1)+0=1 \Rightarrow A=1$

$\therefore$ From equation (1), we get, $\dfrac{1}{t(t+1)}=\dfrac{1}{t}-\dfrac{1}{t+1}$

$$\Rightarrow \frac{1}{4}\int\frac{1}{t(t+1)}dt=\frac{1}{4}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt=\frac{1}{4}\left[\log t-\log(t+1)\right]+c$$

$$\Rightarrow \int\frac{dx}{x(x^4-1)}=\frac{1}{4}\left[\log(x^4-1)-\log(x^4-1+1)\right]+c$$

$$=\frac{1}{4}\left[\log(x^4-1)-\log x^4\right]+c=\frac{1}{4}\log(x^4-1)-\frac{1}{4}4\log x+c$$

So, $\displaystyle\int\frac{dx}{x(x^4-1)}=\frac{1}{4}\log(x^4-1)-\log x+c$

27. $\displaystyle\int\frac{1-x^2}{x(1-2x)}dx$

Sol. Let $\displaystyle I=\int\frac{-x^2+1}{x(-2x+1)}dx=\int\frac{-x^2+1}{-x(2x-1)}dx=\int\frac{(x^2-1)}{x(2x-1)}dx$

$$=\int\left(\frac{1}{2}+\frac{\frac{1}{2}x-1}{x(2x-1)}\right)dx=\frac{1}{2}\int dx+\frac{1}{2}\int\frac{x}{x(2x-1)}dx-\int\frac{1}{x(2x-1)}dx$$

$$\Rightarrow I=\frac{1}{2}x+\frac{1}{2}\int\frac{1}{2x-1}dx-I_1=\frac{1}{2}x+\frac{1}{2}\frac{\log(2x-1)}{2}-I_1 \qquad \ldots(1)$$

$\therefore\ \ I_1=\displaystyle\int\frac{1}{x(2x-1)}dx$, put $\dfrac{1}{x(2x-1)}=\dfrac{A}{x}+\dfrac{B}{2x-1}$ $\qquad \ldots(2)$

$\Rightarrow A(2x-1)+Bx=1$, Put $2x-1=0 \Rightarrow x=\dfrac{1}{2},\ \ A(0)+B\left(\dfrac{1}{2}\right)=1 \Rightarrow B=2$

Put $x=0,\ A(0-1)+B(0)=1 \Rightarrow A=-1$

$\therefore$ From equation (2), we get, $\dfrac{1}{x(2x-1)}=\dfrac{-1}{x}+\dfrac{2}{2x-1}$

$$\Rightarrow \int\frac{1}{x(2x-1)}dx=-\int\frac{1}{x}dx+2\int\frac{1}{2x-1}dx=-\log x+\frac{2\log(2x-1)}{2}+c$$

$$=\log(2x-1)-\log x+c$$

From equation (1),

$$I=\frac{1}{2}x+\frac{1}{4}\log(2x-1)-\log(2x-1)+\log x+c=\frac{1}{2}x-\frac{3}{4}\log(1-2x)+\log x+c$$

28. $\displaystyle\int\frac{x^2+x+1}{(x+2)(x+1)^2}dx$

Sol. Let $\displaystyle I=\int\frac{x^2+x+1}{(x+2)(x+1)^2}dx$, Put $\dfrac{x^2+x+1}{(x+2)(x+1)^2}=\dfrac{A}{(x+2)}+\dfrac{B}{(x+1)}+\dfrac{C}{(x+1)^2}$ $\qquad \ldots(1)$

$\Rightarrow A(x+1)^2+B(x+2)(x+1)+C(x+2)=x^2+x+1$

Put $x+1=0 \Rightarrow x=-1;\ A(0)+B(0)+C(-1+2)=1-1+1=1 \Rightarrow C=1$

Put $x+2=0 \Rightarrow x=-2;\ A(-2+1)^2+B(0)+C(0)=4-2+1=3 \Rightarrow A=3$

Equating the coefficient of x^2, $A+B=1 \Rightarrow 3+B=1 \Rightarrow B=-2$

$\therefore$ From equation (1), we get, $\dfrac{x^2+x+1}{(x+2)(x+1)^2}=\dfrac{3}{(x+2)}-\dfrac{2}{(x+1)}+\dfrac{1}{(x+1)^2}$

So, $\int\dfrac{x^2+x+1}{(x+2)(x+1)^2}dx=\int\dfrac{3}{(x+2)}dx-2\int\dfrac{1}{x+1}dx+\int\dfrac{1}{(x+1)^2}dx$

$$=3\log(x+2)-2\log(x+1)-\frac{1}{x+1}+c$$

29. $\int\dfrac{2x+9}{(x+2)(x-3)^2}dx$

Sol. Let $I=\int\dfrac{2x+9}{(x+2)(x-3)^2}dx$, Put $\dfrac{2x+9}{(x+2)(x-3)^2}=\dfrac{A}{(x+2)}+\dfrac{B}{(x-3)}+\dfrac{C}{(x-3)^2}$ ...(1)

$\Rightarrow A(x-3)^2+B(x+2)(x-3)+C(x+2)=2x+9$

Put $x-3=0 \Rightarrow x=3;\ A(0)+B(0)+C(3+2)=6+9=15 \Rightarrow C=3$

Put $x+2=0 \Rightarrow x=-2;\ A(-2-3)^2+B(0)+C(0)=-4+9=5 \Rightarrow A=\dfrac{5}{25}=\dfrac{1}{5}$

By, Equating the coefficient of x^2, we get, $A+B=0 \Rightarrow \dfrac{1}{5}+B=0 \Rightarrow B=-\dfrac{1}{5}$

$\therefore$ From equation (1), we get, $\dfrac{2x+9}{(x+2)(x-3)^2}=\dfrac{1}{5}.\dfrac{1}{x+2}-\dfrac{1}{5}.\dfrac{1}{x-3}+\dfrac{3}{(x-3)^2}$

$\Rightarrow\int\dfrac{2x+9}{(x+2)(x+3)^2}dx=\dfrac{1}{5}\int\dfrac{1}{x+2}dx-\dfrac{1}{5}\int\dfrac{1}{x-3}dx+3\int\dfrac{1}{(x-3)^2}dx$

$=\dfrac{1}{5}\log(x+2)-\dfrac{1}{5}\log(x-3)-\dfrac{3}{x-3}+c$

30. $\int\dfrac{x^2+1}{(x-2)^2(x+3)}dx$

Sol. Let $I=\int\dfrac{x^2+1}{(x-2)^2(x+3)}dx$, Put $\dfrac{x^2+1}{(x-2)^2(x+3)}=\dfrac{A}{x+3}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$ ...(1)

$\Rightarrow A(x-2)^2+B(x+3)(x-2)+C(x+3)=x^2+1$

Put $x-2=0 \Rightarrow x=2;\ A(0)+B(0)+C(5)=5 \Rightarrow C=1$

Put $x+3=0 \Rightarrow x=-3;\ A(-3-2)^2+B(0)+C(0)=9+1=10 \Rightarrow A=\dfrac{10}{25}=\dfrac{2}{5} \Rightarrow A=\dfrac{2}{5}$

By, Equating the coefficient of x^2, we get, $A+B=1 \Rightarrow \dfrac{2}{5}+B=1 \Rightarrow B=1-\dfrac{2}{5}=\dfrac{3}{5}$

$\therefore$ From equation (1), we get, $\dfrac{x^2+1}{(x-2)^2(x+3)}=\dfrac{2}{5}.\dfrac{1}{x+3}+\dfrac{3}{5}.\dfrac{1}{x-2}+\dfrac{1}{(x-2)^2}$

$$\Rightarrow \int \frac{x^2+1}{(x-2)^2(x+3)}dx = \frac{2}{5}\int \frac{1}{x+3}dx + \frac{3}{5}\int \frac{1}{x-2}dx + \int \frac{1}{(x-2)^2}dx$$

$$= \frac{2}{5}\log(x+3) + \frac{3}{5}\log(x-2) - \frac{1}{x-2} + c$$

31. $\int \frac{(x^2+1)}{(x+3)(x-1)^2}dx$

Sol. $\int \frac{x^2+1}{(x-3)(x-1)^2}dx$. Put $\frac{x^2+1}{(x-3)(x-1)^2} = \frac{A}{x-3} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$ ......(i)

$\Rightarrow A(x-1)^2 + B(x-3)(x-1) + C(x-3) = x^2+1$. **Put** $x-1=0, x=1$

$\Rightarrow A(0) + B(0) + C(1-3) = 1+1 \Rightarrow C = \frac{2}{-2} = -1$.

Put $x-3=0, x=3$. $A(3-1)^2 + B(0) + C(0) = 9+1$.

$A(4) = 10 \ \therefore A = \frac{10}{4} = \frac{5}{2}$. Equating the coefficient of x^2

$A+B=1 \Rightarrow \frac{5}{2} + B = 1 \Rightarrow B = 1 - \frac{5}{2} = \frac{-3}{2}$.

From (i) $\int \frac{x^2+1}{(x-3)(x-1)^2}dx = \frac{5}{2}\int \frac{1}{x-3}\,dx - \frac{3}{2}\int \frac{1}{x-1}dx - \int \frac{1}{(x-1)^2}dx$

$$= \frac{5}{2}\log|x-3| - \frac{3}{2}\log|x-1| + \frac{1}{x-1} + C$$

32. $\int \frac{x^2+x+1}{(x+2)(x^2+1)}dx$

Sol. $\int \frac{x^2+x+1}{(x+2)(x^2+1)}dx$. Let $I = \int \frac{x^2+x+1}{(x+2)(x^2+1)}dx \Rightarrow \frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+C}{x^2+1}$

$\Rightarrow x^2+x+1 = (x^2+1)A + (Bx+C)(x+2) \Rightarrow x^2+x+1 = Ax^2 + A + Bx^2 + Cx + 2Bx + 2C$

$\Rightarrow x^2+x+1 = (A+B)x^2 + (C+2B)x + (A+2C)$.

$\therefore$ **Equating coefficients** $A+B=1$..... (i) $A+2C=1 \Rightarrow A=1-2C$ **....(ii)**

$2B+C=1 \Rightarrow 2B=1-C \Rightarrow B=\frac{1-C}{2}$ **....(iii). Putting in equation (i)**

we have $(1-2C) + \frac{1-C}{2} = 1 \Rightarrow 2-4C+1-C=2 \Rightarrow 3-5C=2 \Rightarrow -5C=-1 \Rightarrow C=\frac{1}{5}$

and $2B = 1-\frac{1}{5} = \frac{4}{5} \Rightarrow B = \frac{2}{5}$ and $A = 1-2\times\frac{1}{5} = 1-\frac{2}{5} = \frac{3}{5}$.

$$\therefore I = \int \frac{x^2+x+1}{(x+2)(x^2+1)}\,dx = \int \frac{A}{(x-2)}dx + \int \frac{Bx+C}{x^2+1}dx = \frac{3}{5}\int \frac{1}{(x+2)}dx + \int \frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1}dx$$

$$=\frac{3}{5}\int\frac{1}{(x+2)}+\frac{1}{5}\int\frac{2x+1}{x^2+1}dx=\frac{3}{5}\log(x+2)+\frac{1}{5}I_1$$

$$\therefore\ I_1=\int\frac{4x+1}{x^2+1}=\int\frac{2x}{x^2+1}dx+\int\frac{1}{x^2+1}dx=\log\left|x^2+1\right|+\frac{1}{1}\tan^{-1}\frac{x}{1}+C_1=\log\left|x^2+1\right|+\tan^{-1}x+C_1$$

$$\therefore\int\frac{x^2+x+1}{(x+2)(x^2+1)}dx=\frac{3}{5}\log|x+2|+\frac{1}{5}\left[\log\left|x^2+1\right|+\tan^{-1}x\right]+C$$

$$=\frac{3}{5}\log|x+2|+\frac{1}{5}\log\left|x^2+1\right|+\frac{1}{5}\tan^{-1}x+C$$

33. $I=\int\frac{2x}{(2x+1)^2}dx\ 5$

Sol. Let $I=\int\frac{2x}{(2x+1)^2}dx$, Put $\frac{2x}{(2x+1)^2}=\frac{A}{2x+1}+\frac{B}{(2x+1)^2}$ ...(1) $\therefore A(2x+1)+B=2x$

Put $2x+1=0\ \Rightarrow x=-\frac{1}{2};\ \ A(0)+B=-1\ \Rightarrow B=-1$

By, Equating the coefficient of x, $2A=2\ \Rightarrow A=1$

$\therefore$ From equation (1), we get, $\frac{2x}{(2x+1)^2}=\frac{1}{2x+1}-\frac{1}{(2x+1)^2}$

$$\Rightarrow\int\frac{2x}{(2x+1)^2}dx=\int\frac{1}{2x+1}dx-\int\frac{1}{(2x+1)^2}dx$$

$$=\frac{\log(2x+1)}{2}+\frac{1}{2(2x+1)}+c=\frac{1}{2}\left[\log(2x+1)+\frac{1}{2x+1}\right]+c$$

34. $\int\frac{3x+1}{(x+2)(x-2)^2}dx$

Sol. Let $I=\frac{3x+1}{(x+2)(x-2)^2}=\frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$

$\therefore\ \ A(x-2)^2+B(x+2)(x-2)+C(x+2)=3x+1$

Put $x-2=0\ \Rightarrow x=2;\ \ A(0)+B(0)+C(2+1)=3\times2+1\ \Rightarrow C=\frac{7}{4}$

Put $x+2=0\ \Rightarrow x=-2;\ \ A(-4)^2+B(0)+C(0)=-6+1=-5\ \ \Rightarrow A=\frac{-5}{16}$

By equation the coefficient of x^2, we get, $A+B=0\ \Rightarrow\frac{-5}{16}+B=0\ \ \Rightarrow B=\frac{5}{16}$

$$\therefore I=-\frac{5}{6}\log|x+2|+\frac{5}{16}\log|x-2|-\frac{7}{4(x-2)}+c$$

35. $\int\frac{5x+8}{x^2(3x+8)}dx$

Sol. Let $I=\int\frac{5x+8}{x^2(3x+8)}dx$, Put $\frac{5x+8}{x^2(3x+8)}=\frac{A}{(3x+8)}+\frac{Bx+C}{x^2}$ ...(1)

$\therefore\ \ Ax^2+(Bx+C)(3x+8)=5x+8$

Put $3x+8=0 \Rightarrow x=-\frac{8}{3}$; $A\left(\frac{64}{9}\right)+B(0)=5\left(-\frac{8}{3}\right)+8=\frac{-40+24}{3}=\frac{-16}{3}$

$\Rightarrow A\left(\frac{64}{9/3}\right)=\frac{-16}{3} \Rightarrow A=\frac{-16\times 3}{64}=-\frac{3}{4} \Rightarrow A=-\frac{3}{4}$

By, Equating the coefficient of x^2 and constant term,

$A+3B=0 \Rightarrow -\frac{3}{4}+3B=0 \Rightarrow 3B=\frac{3}{4} \Rightarrow B=\frac{1}{4},\ 8C=8 \Rightarrow C=1$

$\therefore$ From equation (1), we get, $\frac{5x+8}{x^2(3x+8)}=\frac{-3}{4}.\frac{1}{3x+8}+\frac{\frac{1}{4}x+1}{x^2}$

$\Rightarrow \int\frac{5x+8}{x^2(3x+8)}dx=\frac{-3}{4}\frac{\log(3x+8)}{3}+\frac{1}{4}\int\frac{x}{x^2}dx+\int\frac{1}{x^2}dx=\frac{-1}{4}\log(3x+8)+\frac{1}{4}\log x-\frac{1}{x}+C$

36. $\int\frac{5x^2-18x+17}{(x-1)^2(2x-3)}dx$

Sol. Let $I=\int\frac{5x^2-18x+17}{(x-1)^2(2x-3)}dx$, Put $\frac{5x^2-18x+17}{(x-1)^2(2x-3)}=\frac{A}{2x-3}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$ ...(1)

$\Rightarrow A(x-1)^2+B(2x-3)(x-1)+C(2x-3)=5x^2-18x+17$

Put $x-1=0 \Rightarrow x=1$; $A(0)+B(0)+C(2-3)=5-18+17 \Rightarrow C(-1)=4 \quad \therefore C=-4$

Put $2x-3=0 \Rightarrow x=\frac{3}{2}$; $A\left(\frac{3}{2}-1\right)^2+B(0)+C(0)=5\left(\frac{3}{2}\right)^2-18\left(\frac{3}{2}\right)+17$

$\Rightarrow A\left(\frac{1}{4}\right)+0=5.\frac{9}{4}-27+17 \Rightarrow A\left(\frac{1}{4}\right)=\frac{45}{4}-10=\frac{5}{4} \Rightarrow A=5$

By, Equating the coefficient of x^2, we get, $A+2B=5 \Rightarrow 5+2B=5 \Rightarrow 2B=0 \Rightarrow B=0$

$\therefore$ From equation (1), we get, $\frac{5x^2-18x+17}{(x-1)^2(2x-3)}=5.\frac{1}{2x-3}+0-4.\frac{1}{(x-1)^2}$

$\Rightarrow \int\frac{5x^2-18x+17}{(x-1)^2(2x-3)}=5\int\frac{1}{2x-3}dx-4\int\frac{1}{(x-1)^2}dx=\frac{5}{2}\log(2x-3)+\frac{4}{x-1}+C$

37. $\int\frac{8}{(x+2)(x^2+4)}dx$

Sol. Let $I=\int\frac{8}{(x+2)(x^2+4)}dx$, Put $\frac{8}{(x+2)(x^2+4)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+4}$ ...(1)

$\Rightarrow A(x^2+4)+(Bx+C)(x+2)=8$

Put $x+2=0 \Rightarrow x=-2$; $A(4+4)+0=8 \Rightarrow A=1$

By equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow 1+B=0 \Rightarrow B=-1$

$4A+2C=8 \Rightarrow 4.1+2C=8 \Rightarrow 2C=4 \Rightarrow C=2$

From equation (1), we get, $\frac{8}{(x+2)(x^2+4)}=\frac{1}{x+2}+\frac{-x+2}{x^2+4}$

$$\Rightarrow \int \frac{8}{(x+2)(x^2+4)}dx = \int \frac{1}{x+2}dx - \int \frac{x}{x^2+4}dx + 2\int \frac{1}{x^2+4}dx$$

$$\text{I} = \log(x+2) - \frac{1}{2}\log(x^2+4) + 2.\frac{1}{2}\tan^{-1}\frac{x}{2}, \text{ where } t = x^2+4$$

$$\therefore \int \frac{8}{(x+2)(x^2+4)}dx = \log(x+2) - \frac{1}{2}\log(x^2+4) + \tan^{-1}\frac{x}{2}$$

38. $\int \frac{3x+5}{x^3-x^2+x-1}dx$

Sol. Let $I = \int \frac{3x+5}{x^2(x-1)+(x-1)}dx = \int \frac{3x+5}{(x-1)(x^2+1)}dx$

Put $\frac{3x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ ...(1)

$\Rightarrow A(x^2+1) + (Bx+C)(x-1) = 3x+5$

Put $x-1=0 \Rightarrow x=1$; $A(2)+B(0) = 3+5 = 8 \Rightarrow A = 4$

By, Equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow 4+B=0 \Rightarrow B=-4$

$A-C=5 \Rightarrow 4-C=5 \Rightarrow C=-1$

$\therefore$ From equation (1), we get, $\frac{3x+5}{(x-1)(x^2+1)} = \frac{4}{x-1} + \frac{-4x-1}{x^2+1}$

$$\Rightarrow \int \frac{3x+5}{(x-1)(x^2+1)}dx = 4\int \frac{1}{x-1}dx - 4\int \frac{x}{x^2+1}dx - \int \frac{1}{1+x^2}dx$$

$$= 4\log(x-1) - \frac{4}{2}\log(x^2+1) - \tan^{-1}x + C = 4\log(x-1) - 2\log(x^2+1) - \tan^{-1}x + C$$

39. $\int \frac{2x}{(x^2+1)(x^2+3)}dx$

Sol. Let $I = \int \frac{2x}{(x^2+1)(x^2+3)}dx$, Put $t = x^2 \Rightarrow dt = 2x\,dx$; $\int \frac{dt}{(t+1)(t+3)}$

$\frac{1}{(t+1)(t+3)} = \frac{A}{t+1} + \frac{B}{t+3}$ ...(1)

$\Rightarrow A(t+3) + B(t+1) = 1$

Put $t+3=0 \Rightarrow t=-3$; $A(0)+B(-3+1)=1 \Rightarrow B = -\frac{1}{2}$

Put $t+1=0 \Rightarrow t=-1$; $A(-1+3)+B(0)=1 \Rightarrow A = \frac{1}{2}$

From equation (1), we get, $\frac{1}{(t+1)(t+3)} = \frac{1}{2}.\frac{1}{t+1} - \frac{1}{2}.\frac{1}{t+3}$

$$\Rightarrow \int \frac{1}{(t+1)(t+3)}dt = \frac{1}{2}\int \frac{1}{t+1}dt - \frac{1}{2}\int \frac{1}{t+3}dt = \frac{1}{2}\log(t+1) - \frac{1}{2}\log(t+3) + C$$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)}dx = \frac{1}{2}\log(x^2+1) - \frac{1}{2}\log(x^2+3) + C$$

40. $\int \frac{x^2}{(x^4-1)}dx$

Sol. Let $I = \int \frac{x^2}{(x^4-1)}dx$, Put $\frac{x^2}{(x^2-1)(x^2+1)} = \frac{t}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}$ ...(1)

$\Rightarrow A(t+1) + B(t-1) = t$

Put $t+1=0 \Rightarrow t=-1;\ A(0)+B(-1-1)=-1 \Rightarrow B = \frac{1}{2}$

Put $t-1=0 \Rightarrow t=1;\ A(1+1)+B(0)=1 \Rightarrow A = \frac{1}{2}$

From equation (1), we get, $\frac{t}{(t-1)(t+1)} = \frac{1}{2}.\frac{1}{t-1} + \frac{1}{2}.\frac{1}{t+1}$

$$\Rightarrow \int \frac{x^2}{(x^2-1)(x^2+1)} = \frac{1}{2}\int \frac{1}{x^2-1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}.\frac{1}{2.1}\log\frac{x-1}{x+1} + \frac{1}{2}\tan^{-1}x + C = \frac{1}{4}\log\frac{x-1}{x+1} + \frac{1}{2}\tan^{-1}x + C$$

41. $\int \frac{dx}{x^3-1}$

Sol. Let $I = \int \frac{dx}{x^3-1}$, Put $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$ ...(1)

$\Rightarrow A(x^2+x+1) + (Bx+C)(x-1) = 1$

Put $x-1=0 \Rightarrow x=1;\ A(1+1+1)+0=1 \ \therefore\ A = \frac{1}{3}$

By, Equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow \frac{1}{3}+B=0 \ \therefore\ B = -\frac{1}{3}$

$A-C=1 \quad \Rightarrow \frac{1}{3}-C=1 \quad \Rightarrow C = \frac{1}{3}-1 = -\frac{2}{3}$

From equation (1), we get, $\frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3}.\frac{1}{(x-1)} + \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}$

$$\Rightarrow I = \int \frac{1}{(x-1)(x^2+x+1)}dx = \frac{1}{3}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{x}{x^2+x+1}dx - \frac{2}{3}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\int \frac{2x+1-1}{x^2+x+1}dx - \frac{2}{3}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{1}{3}\int (x-1)dx - \frac{1}{6}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{6}\int \frac{1}{x^2+x+1}dx - \frac{2}{3}\int \frac{1}{x^2+x+1}dx$$

Put $t = x^2+x+1 \Rightarrow dt = (2x+1)dx$

$$I = \frac{1}{3}\log(x-1) - \frac{1}{6}\int \frac{dt}{t} + \left(\frac{1}{6}-\frac{2}{3}\right)\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log t + \left(\frac{1-4}{6}\right)\int \frac{dx}{x^2 + 2.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) + \frac{1}{2}\int \frac{dx}{(x+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{2}.\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{x+1/2}{\sqrt{3}/2} + c$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

42. $\int \frac{dx}{x^3+1}$

Sol. Let $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$, Put $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ ...(1)

$\Rightarrow A(x^2-x+1) + (Bx+C)(x+1) = 1$

Put $x+1=0 \Rightarrow x=-1;\quad A(1+1+1)+C(0)=1 \Rightarrow A=\frac{1}{3}$

By, Equating the coefficient of x^2 and constant term, $A+B=0 \Rightarrow \frac{1}{3}+B=0 \Rightarrow B=-\frac{1}{3}$,

$A+C=1 \Rightarrow \frac{1}{3}+C=1 \Rightarrow C=1-\frac{1}{3}=\frac{2}{3}$

$\therefore$ From equation (1), we get, $\frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3}.\frac{1}{(x+1)} + \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}$

$$\Rightarrow \int \frac{1}{(x+1)(x^2-x+1)}dx = \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{x}{x^2-x+1}dx + \frac{2}{3}\int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\int \frac{2x-1+1}{x^2-x+1}dx + \frac{2}{3}\int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx - \frac{1}{6}\int \frac{1}{x^2-x+1}dx + \frac{2}{3}\int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \left(\frac{2}{3}-\frac{1}{6}\right)\int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int \frac{dx}{x^2 - 2.\frac{1}{2}.x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}.\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{x-\frac{1}{2}}{\sqrt{3}/2} + c$$

$$=\frac{1}{3}\log(x+1)-\frac{1}{6}\log\left(x^2-x+1\right)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c$$

43. $\int\frac{dx}{(x+1)^2\left(x^2+1\right)}$

Sol. Let $I=\int\frac{dx}{(x+1)^2\left(x^2+1\right)}$, Put $\frac{1}{(x+1)^2\left(x^2+1\right)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+1}$ ...(1)

$\Rightarrow A(x+1)\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)(x+1)^2=1$

Put $x+1=0 \Rightarrow x=-1;\ A(0)+B(1+1)+0=1 \quad \Rightarrow B=\frac{1}{2}$

By, Equating the coefficient of x^3, x^2 and constant term.

$A+C=0$ ...(2)

$A+B+2C=0 \quad \Rightarrow A+2C=-\frac{1}{2}$ ...(3)

$A+B+D=1$

$\therefore$ Solving (2) and (3), we get, $-C=\frac{1}{2} \quad \Rightarrow C=-\frac{1}{2},\ A-\frac{1}{2}=0 \quad \therefore A-\frac{1}{2}$

$A+B+D=1 \quad \Rightarrow \frac{1}{2}+\frac{1}{2}+D=1 \quad \Rightarrow 1+D=1 \quad \therefore D=0$

$\therefore$ From equation (1), we get, $\frac{1}{(x+1)^2\left(x^2+1\right)}=\frac{1}{2}.\frac{1}{x+1}+\frac{1}{2}.\frac{1}{(x+1)^2}+\frac{-\frac{1}{2}x+0}{x^2+1}$

$$\Rightarrow \int\frac{1}{(x+1)^2\left(x^2+1\right)}dx=\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{1}{(x+1)^2}dx-\frac{1}{2}\int\frac{x}{x^2+1}dx$$

$$=\frac{1}{2}\log(x+1)-\frac{1}{2}.\frac{1}{x+1}-\frac{1}{4}\log\left(x^2+1\right)+C$$

44. $\int\frac{17}{(2x+1)\left(x^2+4\right)}dx$

Sol. Let $I=\int\frac{17}{(2x+1)\left(x^2+4\right)}dx$, Put $\frac{17}{(2x+1)\left(x^2+4\right)}=\frac{A}{2x+1}+\frac{Bx+C}{x^2+4}$ ...(1)

$\Rightarrow A\left(x^2+4\right)+(Bx+C)(2x+1)=17$

Put $2x+1=0 \Rightarrow x=-\frac{1}{2};\ A(1/4+4)+0=17 \ \Rightarrow A(17/4)=17 \quad \therefore A=4$

By Equating the coefficient of x^2 and constant term, $A+2B=0 \ \Rightarrow 4+2B=0 \ \Rightarrow B=-2$

$4A+C=17 \ \Rightarrow 4\times4+C=17 \quad \therefore C=1$

$\therefore$ From equation (1), we get, $\frac{17}{(2x+1)+\left(x^2+4\right)}=\frac{4}{2x+1}+\frac{-2x+1}{x^2+4}$

$$\Rightarrow \int\frac{17}{(2x+1)\left(x^2+4\right)}dx=4\int\frac{1}{2x+1}dx-2\int\frac{2x}{x^2+4}dx+\int\frac{1}{x^2+2^2}dx$$

$$=\frac{4\log(2x+1)}{2}-\log\left(x^2+4\right)+\frac{1}{2}\tan^{-1}\frac{x}{2}+C=2\log(2x+1)-\log\left(x^2+4\right)+\frac{1}{2}\tan^{-1}\frac{x}{2}+C$$

45. $\int \frac{1}{(x^2+2)(x^2+4)} dx$

Sol. Let $I = \int \frac{1}{(x^2+2)(x^2+4)} dx$, Put $\frac{1}{(x^2+2)(x^2+4)} = \frac{1}{(t+2)(t+4)} = \frac{A}{t+2} + \frac{B}{t+4}$ ...(1)

$\Rightarrow A(t+4) + B(t+2) = 1$

Put $t+4=0 \quad \Rightarrow t=-4;\ A(0)+B(-4+2)=1 \ \Rightarrow B = -\frac{1}{2}$

Put $t+2=0 \ \Rightarrow t=-2;\ A(-2+4)+B(0)=1 \ \Rightarrow A = \frac{1}{2}$

From equation (1), we get, $\frac{1}{(t+2)(t+4)} = \frac{1}{2}.\frac{1}{t+2} - \frac{1}{2}.\frac{1}{t+4}$

$$\Rightarrow \int \frac{1}{(x^2+2)(x^2+4)} dx = \frac{1}{2}\int \frac{1}{x^2+2} dx - \frac{1}{2}\int \frac{1}{x^2+4} dx = \frac{1}{2}\int \frac{dx}{x^2+(\sqrt{2})^2} - \frac{1}{2}\int \frac{dx}{x^2+2^2}$$

$$= \frac{1}{2}.\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{2}.\frac{1}{2}.\tan^{-1}\frac{x}{2} + C = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{4}\tan^{-1}\frac{x}{2} + C$$

46. $\int \frac{x^2}{(x^2+2)(x^2+3)} dx$

Sol. Let $I = \int \frac{x^2}{(x^2+2)(x^2+3)} dx$, Put $\frac{x^2}{(x^2+2)(x^2+3)} dx = \frac{t}{(t+2)(t+3)}$, where $t = x^2$

$\Rightarrow \frac{t}{(t+2)(t+3)} = \frac{A}{t+2} + \frac{B}{t+3}$ ...(1)

$\Rightarrow A(t+3) + B(t+2) = t$

Put $t+3=0 \ \Rightarrow t=-3;\ A(0)+B(-3+2)=-3 \ \Rightarrow B=3$

Put $t+2=0 \ \Rightarrow t=-2;\ A(-2+3)+B(0)=-2 \ \Rightarrow A=-2$

From equation (1), $\frac{t}{(t+2)(t+3)} = \frac{-2}{t+2} + \frac{3}{t+3}$

$$\Rightarrow \int \frac{x^2}{(x^2+2)(x^2+3)} dx = \int \left(\frac{-2}{x^2+(\sqrt{2})^2} + \frac{3}{x^2+(\sqrt{3})^2} \right) dx$$

$$= -2.\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C = -\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}} + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

47. $\int \frac{dx}{(e^x-1)^2}$

Sol. Put $t = e^x - 1 \ \Rightarrow e^x = t+1 \ \Rightarrow dt = e^x dx \ \Rightarrow \frac{dt}{e^x} = dx \ \Rightarrow \frac{dt}{t+1} = dx \ \Rightarrow \int \frac{dt}{(t+1)t^2} = dx$

Put $\frac{1}{(t+1)t^2} = \frac{A}{t+1} + \frac{Bt+C}{t^2}$ ...(1)

$\Rightarrow A(t^2) + (Bt+C)(t+1) = 1$, Put $t+1=0 \ \Rightarrow t=-1;\ A=1$

Equating the coefficient of t^2 and constant term, $A+B=0 \quad \Rightarrow 1+B=0 \quad \therefore B=-1,\ C=1$

From equation (1), we get, $\frac{1}{(t+1)t^2}dt=\frac{1}{t+1}+\frac{-1t+1}{t^2}$

$$\Rightarrow \int\frac{1}{(t+1)t^2}dt=\int\frac{1}{t+1}dt-\int\frac{t}{t^2}dt+\int\frac{1}{t^2}dt$$

$$=\log(t+1)-\int\frac{1}{t}dt+\int\frac{1}{t^2}dt=\log(t+1)-\log t-\frac{1}{t}+C$$

$$\Rightarrow \int\frac{1}{\left(e^x-1\right)^2}dx=\log e^x-\log\left(e^x-1\right)-\frac{1}{e^x-1}+C=x-\log\left(e^x-1\right)-\frac{1}{e^x-1}+C$$

48. $\int\frac{dx}{x\left(x^5+1\right)}$

Sol. Let $I=\int\frac{dx}{x\left(x^5+1\right)}$, Put $t=x^5 \Rightarrow dt=5x^4dx \quad \Rightarrow \frac{dt}{5x^4}=dx$

$$\int\frac{\frac{dt}{5x^4}}{x(t+1)}=\frac{1}{5}\int\frac{dt}{x^5(t+1)}=\frac{1}{5}\int\frac{dt}{t(t+1)}$$

Put $\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{t+1} \qquad \ldots(1) \qquad \therefore\ A(t+1)+Bt=1$

Put $t+1=0 \Rightarrow t=-1;\ A(0)+B(-1)=1 \Rightarrow B=-1$

Put $t=0;\ A(0+1)=1 \Rightarrow A=1,$

$$\frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{t+1} \quad \Rightarrow \int\frac{1}{t(t+1)}dt=\int\frac{1}{t}dt-\int\frac{1}{t+1}dt=\log t-\log(t+1)+C=\log\frac{t}{t+1}+C$$

$$\Rightarrow \frac{1}{5}\int\frac{1}{t(t+1)}dt=\frac{1}{5}\log\frac{t}{t+1}+C \quad \therefore\ \int\frac{1}{x\left(x^5+1\right)}dx=\frac{1}{5}\log\frac{x^5}{x^5+1}+C$$

$\log x-\frac{1}{5}\log\left|x^2+1\right|+c$

49. $\int\frac{dx}{x\left(x^6+1\right)}$

Sol. Let $I=\int\frac{dx}{x\left(x^6+1\right)}$, Put $t=x^6 \Rightarrow \frac{dt}{dx}=6x^5 \Rightarrow \frac{dt}{6x^5}=dx$

$$I=\int\frac{\frac{dt}{6x^5}}{x(t+1)}=\frac{1}{6}\int\frac{dt}{x^6(t+1)}=\frac{1}{6}\int\frac{dt}{t(t+1)}$$

Put $\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{t+1} \qquad \ldots(1) \qquad \Rightarrow A(t+1)+Bt=1$

Put $t+1=0 \Rightarrow t=-1;\ A(0)+B(-1)=1 \quad \therefore B=-1$

Put $t=0;\ A(0+1)+0=1 \Rightarrow A=1$

From equation (1), we get, $\frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{t+1}$

$$\Rightarrow \int \frac{1}{t(t+1)}dt = \int \frac{1}{t}dt - \int \frac{1}{t+1}dt = \log t - \log(t+1) + C$$

$$\Rightarrow \int \frac{1}{t(t+1)}dt = \log \frac{t}{t+1} + C \quad \therefore \int \frac{1}{x(x^6+1)}dx = \frac{1}{6}\log \frac{x^6}{x^6+1} + C$$

$$\therefore I = \log|x| - \frac{1}{6}\log|x^6+1| + c$$

50. $\int \frac{dx}{\sin x(3+2\cos x)}$

Sol. Let $I = \int \frac{dx}{\sin x(3+2\cos x)}$, Put $t = \cos x \Rightarrow dt = -\sin x\, dx \Rightarrow \frac{dt}{-\sin x} = dx$

$$I = \int \frac{\frac{dt}{-\sin x}}{\sin x(3+2t)} = -\int \frac{dt}{\sin^2 x(3+2t)} = -\int \frac{dt}{(1-\cos^2 x)(3+2t)} = -\int \frac{dt}{(1-t^2)(3+2t)}$$

Put $\frac{1}{(1-t^2)(3+2t)} = \frac{1}{(1-t)(1+t)(3+2t)}$

$$\frac{1}{(1-t)(1+t)(3+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{3+2t} \qquad \ldots(1)$$

$$\Rightarrow A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t^2) = 1$$

Put $1+t=0 \Rightarrow t=-1;\ A(0)+B(2)(3-2)+(0)=1,\ B=\frac{1}{2}$

Put $1-t=0 \Rightarrow t=1;\ A(2)(5)+B(0)+C(0)=1 \Rightarrow A=\frac{1}{10}$

Put $3+2t=0 \Rightarrow t=-\frac{3}{2};\ A(0)+B(0)+C\left(1-\frac{9}{4}\right)=1 \Rightarrow C\left(\frac{-5}{4}\right)=1 \quad \therefore C=-\frac{4}{5}$

$\therefore$ From equation (1), we get $\frac{1}{(1-t)(1+t)(3+2t)} = \frac{1}{10}.\frac{1}{1-t} + \frac{1}{2}.\frac{1}{1+t} - \frac{4}{5}.\frac{1}{3+2t}$

$$\Rightarrow \int \frac{1}{(1-t)(1+t)(3+2t)}dt = \frac{1}{10}\int \frac{1}{1-t}dt + \frac{1}{2}\int \frac{1}{1+t}dt - \frac{4}{5}\int \frac{1}{3+2t}dt$$

$$= \frac{-1}{10}\log(1-t) + \frac{1}{2}\log(1+t) - \frac{4}{5}\frac{\log(3+2t)}{2} + C = \int \frac{dx}{\sin x(3+2\cos x)}$$

$$= \frac{-1}{10}(1-\cos x) + \frac{1}{2}\log(1+\cos x) - \frac{2}{5}\log(3+2\cos x) + C$$

51. $\int \frac{dx}{\cos x(5-4\sin x)}$

Sol. Let $I = \int \frac{dx}{\cos x(5-4\sin x)}$, Put $t = \sin x,\ dt = \cos x\, dx$; $I = \int \frac{dt}{\cos^2(5-4t)}$

$$\Rightarrow I = \int \frac{dt}{(1-\sin^2 x)(5-4t)} = \int \frac{dt}{(1-t^2)(5-4t)}$$

Put $\frac{1}{(1-t)(1+t)(5-4t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{5-4t} \qquad \ldots(1)$

$$\Rightarrow A(1+t)(5-4t)+B(1-t)(5-4t)+C(1-t^2)=1$$

Put $1+t=0 \Rightarrow t=-1$; $A(0)+B(2)(9)=1 \Rightarrow B=\dfrac{1}{18}$

Put $1-t=0 \Rightarrow t=1$; $A(2)+B(0)+C(0)=1 \Rightarrow A=\dfrac{1}{2}$

Put $5-4t=0 \Rightarrow t=\dfrac{5}{4}$; $A(0)+B(0)+C\left(1-\dfrac{25}{16}\right)=1 \Rightarrow C\left(\dfrac{-9}{16}\right)=1 \quad \therefore C=\dfrac{-16}{9}$

From equation (1), we get, $\dfrac{1}{(1-t^2)(5-4t)}=\dfrac{1}{2}.\dfrac{1}{1-t}+\dfrac{1}{18}.\dfrac{1}{1+t}+\dfrac{-16}{9}.\dfrac{1}{5-4t}$

$$\Rightarrow \int\frac{1}{(1-t^2)(5-4t)}dt=\frac{1}{2}\int\frac{1}{1-t}dt+\frac{1}{18}\int\frac{1}{1+t}dt-\frac{16}{9}\int\frac{1}{5-4t}dt$$

$$=\frac{-1}{2}\log(1-t)+\frac{1}{18}\log(1+t)-\frac{16}{9}\frac{\log(5-4t)}{-4}+C$$

$$=\frac{-1}{2}\log(1-t)+\frac{1}{18}\log(1+t)+\frac{4}{9}\log(5-4t)+C$$

$$\Rightarrow \int\frac{1}{\cos x(5-4\sin x)}dx=\frac{-1}{2}\log(1-\sin x)+\frac{1}{18}\log(1+\sin x)+\frac{4}{9}\log(5-4\sin x)+C$$

52. $\displaystyle\int\frac{dx}{\sin x\cos^2 x}$

Sol. Let $I=\displaystyle\int\frac{1}{\sin x.\cos^2 x}dx=\int\frac{\sin^2 x+\cos^2 x}{\sin x.\cos^2 x}dx=\int\frac{\sin^2 x}{\sin x.\cos^2 x}dx+\int\frac{\cos^2 x}{\sin x.\cos^2 x}dx$

$$=\int\frac{\sin x}{\cos x.\cos x}dx+\int\frac{1}{\sin x}dx=\int(\tan x\sec x+\operatorname{cosec} x)dx$$

$$=\sec x-\frac{1}{2}\log\cot^2\frac{x}{2}=\sec x-\frac{1}{2}\log\left(\frac{1+\cos x}{1-\cos x}\right)+C$$

53. $\displaystyle\int\frac{\tan x}{(1-\sin x)}dx$

Sol. Let $I=\displaystyle\int\frac{\tan x}{1-\sin x}dx \Rightarrow I=\int\frac{\sin x}{\cos x(1-\sin x)}dx$

Put $t=\sin x \Rightarrow dt=\cos x\,dx$; $I=\displaystyle\int\frac{\sin x.\cos x\,dx}{\cos^2 x(1-\sin x)}=\int\frac{t\,dt}{(1-\sin^2 x)(1-t)}=\int\frac{t\,dt}{(1-t^2)(1-t)}$

Put $\dfrac{t}{(1-t)(1+t)(1-t)}=\dfrac{t}{(1-t)^2(1+t)}=\dfrac{A}{1+t}+\dfrac{B}{1-t}+\dfrac{C}{(1-t)^2}$ ...(1)

$$\Rightarrow A(1-t)^2+B(1-t)(1+t)+C(1+t)=t$$

Put $1-t=0 \Rightarrow t=1$; $A(0)+B(0)+C(1+1)=1 \Rightarrow C=\dfrac{1}{2}$

Put $1+t=0 \Rightarrow t=-1$; $A(2)^2+B(0)+C(0)=-1 \Rightarrow A=-\dfrac{1}{4}$

By, Equating the coefficient of t^2, we get, $A-B=0 \Rightarrow \dfrac{-1}{4}-B=0 \Rightarrow B=\dfrac{-1}{4}$

$\therefore$ From equation (1), we get, $\dfrac{t}{(1-t)^2(1+t)}=\dfrac{-1}{4}.\dfrac{1}{1+t}-\dfrac{1}{4}.\dfrac{1}{1-t}+\dfrac{1}{2}.\dfrac{1}{(1-t)^2}$

$$\Rightarrow \int\frac{t}{(1-t)^2(1+t)}dt=-\frac{1}{4}\int\frac{1}{1+t}dt-\frac{1}{4}\int\frac{1}{1-t}dt+\frac{1}{2}\int\frac{1}{(1-t)^2}dt$$

$$=-\frac{1}{4}\log(1+t)+\frac{1}{4}\log(1-t)-\frac{1}{2}.\frac{1}{1-t}+C$$

$$\Rightarrow \int\frac{\tan x}{1-\sin x}dx=-\frac{1}{4}\log(1+\sin x)+\frac{1}{4}\log(1-\sin x)-\frac{1}{2}.\frac{1}{1-\sin x}+C$$

54. $\displaystyle\int\frac{dx}{\sin x+\sin 2x}$

Sol. Let $I=\displaystyle\int\frac{dx}{\sin x+2\sin x.\cos x}=\int\frac{dx}{\sin x(1+2\cos x)}$

Now, Put $t=\cos x \Rightarrow dt=-\sin x\,dx \Rightarrow \dfrac{-dt}{\sin x}=dx$

$$I=\int\frac{-dt}{\sin^2 x(1+2t)}=-\int\frac{dt}{(1-\cos^2 t)(1+2t)} \quad \Rightarrow I=-\int\frac{dt}{(1-t^2)(1+2t)}$$

Put $\dfrac{1}{(1-t)(1+t)(1+2t)}=\dfrac{A}{1-t}+\dfrac{B}{1+t}+\dfrac{C}{1+2t}$ ...(1)

$\Rightarrow A(1+t)(1+2t)+B(1-t)(1+2t)+C(1-t^2)=1$

Put $1+t=0 \Rightarrow t=-1;\ A(0)+B(2)(1-2)+C(0)=1 \Rightarrow B=\dfrac{-1}{2}$

Put $1-t=0 \Rightarrow t=1;\ A(2)(3)+B(0)+C(0)=1,\ \Rightarrow A=\dfrac{1}{6}$

Put $1+2t=0 \Rightarrow t=-\dfrac{1}{2},\ A(0)+B(0)+C\left(1-\dfrac{1}{4}\right)=1 \Rightarrow C\left(\dfrac{3}{4}\right)=1 \Rightarrow C=\dfrac{4}{3}$

From equation (1), we get, $\dfrac{1}{(1-t^2)(1+2t)}=\dfrac{1}{6}.\dfrac{1}{1-t}-\dfrac{1}{2}.\dfrac{1}{1+t}+\dfrac{4}{3}.\dfrac{1}{1+2t}$

$$\Rightarrow \int\frac{1}{(1-t)^2(1+2t)}dt=\frac{1}{6}\int\frac{1}{1-t}dt-\frac{1}{2}\int\frac{1}{1+t}dt+\frac{4}{3}\int\frac{1}{1+2t}dt$$

$$=\frac{-1}{6}\log(1-t)-\frac{1}{2}\log(1+t)+\frac{4}{3}\frac{\log(1+2t)}{2}+C$$

$$=\frac{-1}{6}\log(1-t)-\frac{1}{2}\log(1+t)+\frac{2}{3}\log(1+2t)+C$$

$$\Rightarrow \int\frac{1}{\sin x+\sin 2x}dx=\frac{-1}{6}\log(1-\cos x)-\frac{1}{2}\log(1+\cos x)+\frac{2}{3}\log(1+\cos x)+C$$

55. $\displaystyle\int\frac{x^2}{x^4-x^2-12}dx$

Sol. Let $I=\displaystyle\int\frac{x^2}{x^4-x^2-12}dx$, Put $\dfrac{x^2}{x^4-x^2-12}=\dfrac{t}{t^2-t-12}$, where $t=x^2$

$$\frac{t}{t^2-t-12}=\frac{t}{(t-4)(t+3)}=\frac{A}{t-4}+\frac{B}{t+3} \quad ...(1)$$

$\therefore \quad A(t+3)+B(t-4)=t$

Put $t+3=0 \Rightarrow t=-3$; $A(0)+B(-7)=-3 \Rightarrow B=\frac{3}{7}$

Put $t-4=0 \Rightarrow t=4$; $A(4+3)+B(0)=4 \Rightarrow A=\frac{4}{7}$

From (1), $\frac{t}{(t-4)(t+3)}=\frac{4}{7}.\frac{1}{t-4}+\frac{3}{7}.\frac{1}{t+3}$

$$\Rightarrow \int \frac{x^2}{(x^2-4)(x^2+3)}dx=\frac{4}{7}\int \frac{1}{x^2-2^2}dx+\frac{3}{7}\int \frac{1}{x^2+\left(\sqrt{3}\right)^2}$$

$$=\frac{4}{7}.\frac{1}{2.2}\log\frac{x-2}{x+2}+\frac{3}{7}.\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C=\frac{1}{7}\log\left(\frac{x-2}{x+2}\right)+\frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}+C$$

56. $\int \frac{x^4}{(x^2+1)(x^2+9)(x^2+16)}dx$

Sol. Let $I=\int \frac{x^4}{(x^2+1)(x^2+9)(x^2+16)}dx$

$$I=\int \frac{(x^2)^2}{(x^2+1)(x^2+9)(x^2+16)}dx$$

Again Let $x^2=t$

$$I=\int \frac{t^2}{(t+1)(t+9)(t+16)}dt$$

$$\frac{t^2}{(t+1)(t+9)(t+16)}=\frac{A}{(t+1)}+\frac{B}{(t+9)}+\frac{C}{(t+16)} \qquad \text{.... (i)}$$

$$\frac{t^2}{(t+1)(t+9)(t+16)}=\frac{A(t+9)(t+16)+B(t+1)(t+16)+C(t+1)(t+9)}{(t+1)(t+9)(t+16)}$$

$$t^2=A(t+9)(t+16)+B(t+1)(t+16)+C(t+1)(t+9)$$

Putting $t=-1$

$$(-1)^2=A(-1+9)(-1+16)+B(-1+1)(-1+16)+C(-1+1)(-1+9)$$

$$1=A(8)(15)+0+0$$

$$I=120A$$

$$\therefore A=\frac{1}{120}$$

Putting $t=-9$

$$(-9)^2=A(-9+9)(-9+16)+B(-9+1)(-9+16)+C(-9+1)(-9+9)$$

$$81=0+B(-8)(7)+0$$

$$81=-56B$$

$$\therefore B=\frac{-81}{56}$$

Putting $t=-16$

$$(-16)^2 = A(-16+9)(-16+16)+B(-8)B(-16+1)(-16+16)+C(-16+1)(-16+9)$$

$$256 = 0+0+C(-15)(-7)$$

$$256 = C(105)$$

$$C = \frac{256}{105}$$

Putting the value of A, B and C in equation (i) we get

$$\frac{t^2}{(t+1)(t+9)(t+16)} = \frac{A}{(t+1)} + \frac{B}{(t+9)} + \frac{C}{(t+16)}$$

$$\int \frac{t^2}{(t+1)(t+9)(t+16)} dt = \int \left[\frac{A}{(t+1)} + \frac{B}{(t+9)} + \frac{C}{(t+16)} \right] dt$$

$$I = \int \left[\frac{\frac{1}{120}}{(t+1)} + \left(\frac{\frac{-81}{56}}{(t+9)} \right) + \left(\frac{\frac{256}{105}}{t+16} \right) \right] dt$$

$$I = \frac{1}{120} \int \left(\frac{1}{(t+1)} \right) dt - \frac{81}{56} \int \frac{1}{(t+9)} dt + \frac{256}{105} \int \frac{1}{(t+16)} dt$$

Putting $t = x^2$

$$I = \frac{1}{120} \int \frac{1}{x^2+1} dx - \frac{81}{56} \int \frac{1}{x^2+9} dx + \frac{256}{105} \int \frac{1}{x^2+16} dx$$

$$I = \frac{1}{120} \int \frac{1}{x^2+1} dx - \frac{81}{56} \int \frac{1}{x^2+(3)^2} dx + \frac{256}{105} \int \frac{1}{x^2+(4)^2} dx$$

$$I = \frac{1}{120} \tan^{-1}(x) = \frac{81}{56} \times \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + \frac{256}{105} \times \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + c$$

$$I = \frac{1}{120} \tan^{-1} x - \frac{27}{56} \tan^{-1}\left(\frac{x}{3}\right) + \frac{64}{105} \tan^{-1}\left(\frac{x}{4}\right) + c$$

57. $\int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)} dx$

Sol. Let $I = \int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)} dx$

Put $t = \cos 2x \Rightarrow dt = -\sin 2x\, dx$; $\int \frac{-dt/2}{(1-t)(2-t)} = \frac{1}{2} \int \frac{dt}{(t-2)(1-t)}$

Put $\frac{1}{(t-2)(1-t)} = \frac{A}{t-2} + \frac{B}{1-t}$ ...(1) $\therefore A(1-t)+B(t-2)=1$

Put $1-t=0 \Rightarrow t=1$; $A(0)+B(1-2)=1 \Rightarrow B=-1$

Put $t-2=0 \Rightarrow t=2$; $A(1-2)+B(0)=1 \Rightarrow A=-1$

From (1), $\frac{1}{(t-2)(1-t)} = \frac{-1}{t-2} + \frac{-1}{1-t}$

$$\Rightarrow \int \frac{1}{(t-2)(1-t)} dt = \int \frac{1}{2-t} dt + \int \frac{1}{t-1} dt$$

$$= -\log(2-t) + \log(t-1) + C = \log(t-1) - \log(2-t) + C$$

$$\Rightarrow \int \frac{\sin 2x}{(1-\cos 2x)(2-\cos 2x)} dx = \log(\cos 2x - 1) - \log(2 - \cos 2x) + C$$

58. $\int \frac{2}{(1-x)(1+x^2)} dx$

Sol. Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx}{1+x^2} + \frac{C}{1+x^2}$

$\Rightarrow 2 = A(1+x^2) + Bx(1-x) + C(1-x)$

Putting $x = 1$, we have $2 = 2A + 0 + 0$

$\Rightarrow A = 1$

Putting $x = 0$ we have $2 = A + C$

$\Rightarrow C = 2 - A \Rightarrow C = 2 - 1 = 1$

Putting $x = 2$ we have $2 = 5A - 2B - C$

$\Rightarrow 2 = 5 \times 1 - 2B - 1 \quad \Rightarrow 2B = 2 \Rightarrow B = 1$

$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}$

Integrating both sides we have

$$\int \frac{2\,dx}{(1-x)(1+x^2)} = \int \frac{dx}{1-x} + \int \frac{x}{1+x^2} + \int \frac{dx}{1+x^2}$$

$$= -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1} x + C$$

59. $\int \frac{2x^2+1}{x^2(x^2+4)} dx$

Sol. Let $I = \int \frac{2x^2+1}{x^2(x^2+4)} dx$

Again Let $x^2 = t$

$\frac{2t+1}{t(t+4)} = \frac{A}{t} + \frac{B}{(t+4)}$ ... (i)

$\frac{2t+1}{t(t+4)} = \frac{A(t+4) + B(t)}{t(t+4)}$

$2t + 1 = A(t+4) + B(t)$

Putting $t = -4$

$2(-4) + 1 = A(-4+4) + B(-4)$

$-8 + 1 = 0 - 4B$

$-7 = -4B$

$\therefore B = \frac{7}{4}$

Putting $t=0$

$$2(0)+1=A(0+4)+B(0)$$

$$1=4A$$

$$\therefore A=\frac{1}{4}$$

Now $\frac{2t+1}{t(t+4)}=\frac{A}{t}+\frac{B}{(t+4)}$

Putting the value of A and B in equation (i) we get $\frac{2t+1}{t(t+4)}=\frac{\frac{1}{4}}{t}+\frac{\frac{7}{4}}{t+4}$

$$\int\frac{2t+1}{t(t+4)}dt=\int\left(\frac{\frac{1}{4}}{t}+\frac{\frac{7}{4}}{t+4}\right)dt$$

Putting $t=x^2$

$$\int\frac{2x^2+1}{x^2(x^2+4)}dx=\int\left(\frac{1/4}{x^2}+\frac{7/4}{x^2+4}\right)dx$$

$$I=\frac{1}{4}\int x^{-2}dx+\frac{7}{4}\int\frac{1}{x^2+2^2}dx$$

$$I=\frac{1}{4}\times\frac{x^{-1}}{-1}+\frac{7}{4}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$$

$$I=\frac{-1}{4x}+\frac{7}{8}\tan^{-1}\frac{x}{2}+c$$

EXERCISE 4B (Pg. No.: 770)

Very-short Answer question

Evaluate:

1. $\int x^{-6}\, dx$

Sol. Let $I=\int x^{-6}\, dx \quad \Rightarrow I=\frac{x^{-6+1}}{-6+1}+c \quad \therefore\ I=\frac{x^{-5}}{-5}+c=-\frac{1}{5x^{5}}+c$

2. $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) dx$

Sol. Let $I=\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) dx \quad \Rightarrow I=\int x^{1/2}\, dx+\int x^{-\frac{1}{2}}\, dx \quad \Rightarrow I=\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+c$

$\therefore\ I=\frac{2}{3}x^{3/2}+2x^{1/2}+c=\frac{2}{3}x^{3/2}+2\sqrt{x}+c$

3. $\int \sin 3x\, dx$

Sol. Let $I=\int \sin 3x\, dx \quad \therefore\ I=-\frac{\cos 3x}{3}+c$

4. $\int \frac{x^{2}}{1+x^{3}}\, dx$

Sol. Let $I=\int \frac{x^{2}}{1+x^{3}}\, dx$, Put $1+x^{3}=t \quad \Rightarrow 3x^{2}\, dx=dt \quad \Rightarrow x^{2}\, dx=\frac{dt}{3}$

$\Rightarrow I=\int \frac{1}{3}\cdot\frac{dt}{t} \quad \Rightarrow I=\frac{1}{3}\log|t|+c \quad \therefore\ I=\frac{1}{3}\log\left|1+x^{3}\right|+c$

5. $\int \frac{2\cos x}{3\sin^{2}x}\, dx$

Sol. Let $I=\int \frac{2\cos x}{3\sin^{2}x}\, dx \quad \Rightarrow I=\frac{2}{3}\int \cot x \operatorname{cosec} x\, dx \quad \Rightarrow I=\frac{2}{3}(-\operatorname{cosec} x)+c$

$\therefore\ I=\frac{-2}{3}\operatorname{cosec} x+c$

6. $\int \frac{(3\sin\phi-2)\cos\phi}{\left(5-\cos^{2}\phi-4\sin\phi\right)}\, d\phi$

Sol. Let $I=\int \frac{(3\sin\phi-2)\cos\phi}{\left(5-\cos^{2}\phi-4\sin\phi\right)}\, d\phi \quad =\int \frac{(3\sin\phi-2)\cos\phi}{\left\{5-\left(1-\sin^{2}\phi\right)-4\sin\phi\right\}}\, d\phi$

$\Rightarrow I=\int \frac{(3\sin\phi-2)\cos\phi}{\left(5-1+\sin^{2}\phi-4\sin\phi\right)}\, d\phi \quad \Rightarrow I=\int \frac{(3\sin\phi-2)\cos\phi}{\sin^{2}\phi-4\sin\phi+4}\, d\phi$

Put $\sin\phi=t \quad \Rightarrow \cos\phi\, d\phi=dt$

$\Rightarrow I=\int \frac{(3t-2)}{t^{2}-4t+4}\, dt$

Let $3t-2=A.\frac{d}{dt}\left(t^{2}-4t+4\right)+B \quad \Rightarrow 3t-2=A(2t-4)+B \quad \Rightarrow 3t-2=2At-4A+B$

Equating co-efficient we get, $2A=3 \quad \therefore\ A=\frac{3}{2}$ & $-4A+B=-2 \quad \Rightarrow B=-2+4A \quad \therefore\ B=4$

$$\Rightarrow I=\int\frac{A(2t-4)+B}{t^2-4t+4}\,dt \quad \Rightarrow I=A\int\frac{2t-4}{t^2-4t+4}dt+B\int\frac{1}{t^2-4t+4}\,dt$$

$$\Rightarrow I=\frac{3}{2}I_1+4I_2 \qquad \ldots(1)$$

$$I_1=\int\frac{2t-4}{t^2-4t+4}\,dt,\ \text{Put } t^2-4t+4=y \quad \Rightarrow (2t-4)\,dt=dy$$

$$\Rightarrow I_1=\int\frac{dy}{y} \quad \Rightarrow I_1=\log|y|+c_1 \quad \Rightarrow I_1=\log\left|t^2-4t+4\right|+c_1$$

$$\Rightarrow I_1=\log\left|\sin^2\phi-4\sin\phi+4\right|+c_1$$

$$I_2=\int\frac{1}{t^2-4t+4}\,dt \quad \Rightarrow I_2=\int\frac{1}{(t-2)^2}\,dt \quad \Rightarrow I_2=-\frac{1}{t-2}+c_2$$

$$\therefore\ I_2=-\frac{1}{\sin\phi-2}+c_2$$

Putting the value of I_1 & I_2 in equation (1)

$$\Rightarrow I=\frac{3}{2}\log\left|\sin^2\phi-4\sin\phi+4\right|-\frac{4}{\sin\phi-2}+c$$

$$I=\frac{3}{2}\log\left|\sin^2\Phi-2.2.\sin\Phi+2^2\right|-\frac{-4}{\sin\Phi-2}+c$$

$$=\frac{3}{2}\log|\sin\Phi-2|^2-\frac{4}{\sin\Phi-2}+c$$

$$=3\log|\sin\Phi-2|-\frac{4}{\sin\Phi-2}+c$$

$$=3\log|\sin\Phi-2|-\frac{4}{\sin\Phi-2}+c$$

$$I=\frac{3}{2}\log\left|\sin^2\phi-2\sin\phi+2^2\right|-\frac{4}{\sin\phi-2}+c$$

$$\therefore\ I=3\log|\sin\phi-2|-\frac{4}{\sin\phi-2}+c$$

7. $\int\sin^2x\,dx$

Sol. Let $I=\int\sin^2x\,dx \quad \Rightarrow I=\int\frac{1-\cos 2x}{2}\,dx$

$$\Rightarrow I=\frac{1}{2}\int(1-\cos 2x)\,dx \quad \Rightarrow I=\frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]+c \quad \therefore\ I=\frac{x}{2}-\frac{\sin 2x}{4}+c$$

8. $\int\frac{(\log x)^2}{x}\,dx$

Sol. Let $I=\int\frac{(\log x)^2}{2}\,dx$, Put $\log x=t \quad \Rightarrow \frac{1}{x}dx=dt$

$$\Rightarrow I = \int t^2\, dt \quad \Rightarrow I = \frac{t^3}{3} + c \quad \therefore\ I = \frac{(\log x)^3}{3} + c$$

9. $\int \frac{(x+1)(x+\log x)^2}{x}\, dx$

Sol. Let $I = \int \frac{(x+1)(x+\log x)^2}{x}\, dx$, Put $x + \log x = t \quad \Rightarrow \left(1 + \frac{1}{x}\right) dx = dt \quad \Rightarrow \frac{x+1}{x}\, dx = dt$

$$\Rightarrow I = \int t^2 . dt \quad \Rightarrow I = \frac{t^3}{3} + c \quad \therefore\ I = \frac{(x+\log x)^3}{3} + c$$

10. $\int \frac{\sin x}{1+\cos x}\, dx$

Sol. Let $I = \int \frac{\sin x}{1+\cos x}\, dx$, Put $1 + \cos x = t \quad \Rightarrow -\sin x\, dx = dt \quad \Rightarrow \sin x\, dx = -dt$

$$\Rightarrow I = -\int \frac{dt}{t} \quad \Rightarrow I = -\log|t| + c \quad \therefore\ I = -\log|1+\cos x| + c$$

11. $\int \frac{1+\tan x}{1-\tan x}\, dx$

Sol. Let $I = \int \frac{1+\tan x}{1-\tan x}\, dx \quad \Rightarrow I = \int \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}\, dx \quad \Rightarrow I = \int \frac{\cos x + \sin x}{\cos x - \sin x}\, dx$

Put $\cos x - \sin x = t \quad \Rightarrow (-\sin x - \cos x) dx = dt \quad \Rightarrow (\sin x + \cos x) dx = -dt$

$$\Rightarrow I = -\int \frac{dt}{t} \quad \Rightarrow I = -\log|t| + c \quad \therefore\ I = -\log|\cos x - \sin x| + c$$

12. $\int \frac{1-\cot x}{1+\cot x}\, dx$

Sol. Let $I = \int \frac{1-\cot x}{1+\cot x}\, dx \quad \Rightarrow I = \int \frac{1-\frac{\cos x}{\sin x}}{1+\frac{\cos x}{\sin x}}\, dx \quad \Rightarrow I = \int \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$

Put $\sin x + \cos x = t \quad \Rightarrow (\cos x - \sin x) dx = dt \quad \Rightarrow -(\sin x - \cos x) dx = dt \quad \Rightarrow (\sin x - \cos x)\, dx = -dt$

$$\Rightarrow I = -\int \frac{dt}{t} \quad \Rightarrow I = -\log|t| + c \quad \therefore\ I = -\log|\sin x + \cos x| + c$$

13. $\int \frac{1+\cot x}{x + \log(\sin x)}\, dx$

Sol. Let $I = \int \frac{1+\cot x}{x+\log(\sin x)}\, dx$

Put $x + \log(\sin x) = t \quad \Rightarrow \left(1 + \frac{1}{\sin x} . \cos x\right) dx = dt \quad \Rightarrow (1 + \cot x)\, dx = dt$

$$\Rightarrow I = \int \frac{dt}{t} \quad \Rightarrow I = \log|t| + c \quad \therefore\ I = \log|x + \log(\sin x)| + c$$

14. $\int \frac{1-\sin 2x}{x+\cos^2 x}\,dx$

Sol. Let $I=\int \frac{1-\sin 2x}{x+\cos^2 x}\,dx$, Put $x+\cos^2 x=t \Rightarrow 1+2\cos x(-\sin x)=\frac{dt}{dx} \Rightarrow (1-\sin 2x)\,dx=dt$

$\Rightarrow I=\int \frac{dt}{t} \Rightarrow I=\log|t|+c \quad \therefore I=\log\left|x+\cos^2 x\right|+c$

15. $\int \frac{\sec^2(\log x)}{x}\,dx$

Sol. Let $I=\int \frac{\sec^2(\log x)}{x}\,dx$, Put $\log x=t \Rightarrow \frac{1}{x}\,dx=dt$

$\Rightarrow I=\int \sec^2(t)\,dt \Rightarrow I=\tan(t)+c \quad \therefore I=\tan(\log x)+c$

16. $\int \frac{\sin\left(2\tan^{-1}x\right)}{1+x^2}\,dx$

Sol. Let $I=\int \frac{\sin\left(2\tan^{-1}x\right)}{1+x^2}\,dx$, Put $2\tan^{-1}x=t \Rightarrow 2.\frac{1}{1+x^2}\,dx=dt \Rightarrow \frac{1}{1+x^2}\,dx=\frac{dt}{2}$

$\Rightarrow I=\int \sin(t)\frac{dt}{2} \Rightarrow I=-\frac{1}{2}\cos\left(2\tan^{-1}x\right)+c \quad \therefore I=-\frac{1}{2}\cos\left(2\tan^{-1}x\right)+c$

17. $\int \frac{\tan x\sec^2 x}{\left(1-\tan^2 x\right)}dx$

Sol. Let $I=\int \frac{\tan x\sec^2 x}{1-\tan^2 x}\,dx$, Put $1-\tan^2 x=t \Rightarrow -2\tan x\sec^2 x=\frac{dt}{dx} \Rightarrow \tan x\sec^2 x\,dx=-\frac{dt}{2}$

$\Rightarrow I=\int \frac{1}{t}.\left(\frac{dt}{-2}\right) \Rightarrow I=-\frac{1}{2}\log|t|+c \quad \therefore I=-\frac{1}{2}\log\left|1-\tan^2 x\right|+c$

18. $\int \frac{x^4+1}{x^2+1}\,dx$

Sol. Let $I=\int \frac{x^4+1}{x^2+1}\,dx \Rightarrow I=\int \frac{x^4-1+2}{x^2+1}\,dx \Rightarrow I=\int \frac{x^4-1}{x^2+1}\,dx+2\int \frac{1}{x^2+1}\,dx$

$\Rightarrow I=\int \frac{\left(x^2-1\right)\left(x^2+1\right)}{\left(x^2+1\right)}\,dx+2\tan^{-1}x \Rightarrow I=\int\left(x^2-1\right)dx+2\tan^{-1}x \quad \therefore I=\frac{x^3}{3}-x+2\tan^{-1}x+c$

19. $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$

Sol. Let $I=\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$

$\Rightarrow I=\int \tan^{-1}\sqrt{\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)^2}\,dx \Rightarrow I=\int \tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)dx$

Dividing numerator and denominator by $\cos\frac{x}{2}$ we get,

$$I=\int \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)dx \quad \Rightarrow I=\int \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}dx$$

$$\Rightarrow I=\int\left(\frac{\pi}{4}-\frac{x}{2}\right)dx \quad \Rightarrow I=\frac{\pi}{4}.x-\frac{1}{2}.\frac{x^2}{2}+c \quad \therefore\ I=\frac{\pi x}{4}-\frac{x^2}{4}+c$$

20. $\int \log\left(1+x^2\right)dx$

Sol. Let $I=\int \log\left(1+x^2\right)dx$

$$\Rightarrow I=\log\left(1+x^2\right)\int dx-\int\left[\frac{d\left\{\log\left(1+x^2\right)\right\}}{d\left(1+x^2\right)}\times\frac{d\left(1+x^2\right)}{dx}\int dx\right]dx$$

$$\Rightarrow I=\log\left(1+x^2\right).x-\int\frac{1}{1+x^2}.2x.x\,dx \quad \Rightarrow I=x\log\left(1+x^2\right)-2\int\frac{x^2}{x^2+1}dx$$

$$I=x\log\left(1+x^2\right)-2\int\frac{\left(x^2+1\right)-1}{x^2+1}dx \quad \Rightarrow I=x\log\left(1+x^2\right)-2\int\frac{x^2+1}{x^2+1}dx+2\int\frac{1}{x^2+1}dx$$

$$\Rightarrow x\log\left(1+x^2\right)-2\int dx+2\tan^{-1}x+c \quad \therefore\ I=x\log\left(1+x^2\right)-2x+2\tan^{-1}x+c$$

21. $\int \cos x\cos 3x\,dx$

Sol. Let $I=\int \cos x.\cos 3x\,dx \quad \Rightarrow I=\frac{1}{2}\int 2\cos x\cos 3x\,dx$

$$\Rightarrow I=\frac{1}{2}\int\left\{\cos\left(x+3x\right)+\cos\left(x-3x\right)\right\}dx \quad \Rightarrow I=\frac{1}{2}\int\left(\cos 4x+\cos 2x\right)dx$$

$$\Rightarrow I=\frac{1}{2}\left[\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right]+c \quad \therefore\ I=\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+c$$

22. $\int \sin 3x\sin x\,dx$

Sol. Let $I=\int \sin 3x\sin x\,dx \quad \Rightarrow I=\frac{1}{2}\int 2\sin 3x\sin x\,dx$

$$\Rightarrow I=\frac{1}{2}\int\left\{\cos\left(3x-x\right)-\cos\left(3x+x\right)\right\}dx \quad \Rightarrow I=\frac{1}{2}\int\left(\cos 2x-\cos 4x\right)dx$$

$$\Rightarrow I=\frac{1}{2}\left[\frac{\sin 2x}{2}-\frac{\sin 4x}{4}\right]+c \quad \therefore\ I=\frac{\sin 2x}{4}-\frac{\sin 4x}{8}+c$$

23. $\int\frac{x\,e^x}{\left(x+1\right)^2}dx$

Sol. Let $I=\int\frac{x\,e^x}{\left(x+1\right)^2}dx \quad \Rightarrow I=\int e^x\left\{\frac{\left(x+1\right)-1}{\left(x+1\right)^2}\right\}dx \quad \Rightarrow I=\int e^x\left\{\frac{1}{x+1}-\frac{1}{\left(x+1\right)^2}\right\}dx$

where $f\left(x\right)=\frac{1}{x+1},\ f'\left(x\right)=-\frac{1}{\left(x+1\right)^2}$

$$\Rightarrow I=\int e^x\left[f\left(x\right)+f'\left(x\right)\right]dx \quad \Rightarrow I=e^x f\left(x\right)+c \quad \therefore\ I=e^x.\frac{1}{x+1}+c$$

24. $\int e^x(\tan x - \log\cos x)\,dx$

Sol. Let $I = \int e^x(\tan x - \log\cos x)\,dx$

where $f(x) = -\log\cos x,\ \ f'(x) = \tan x$

$\Rightarrow I = \int e^x\left[f(x) + f'(x)\right]dx \quad \Rightarrow e^x f(x) + c \qquad \therefore\ I = -e^x\log(\cos x) + c$

25. $\int \frac{1}{1-\sin x}\,dx$

Sol. Let $I = \int \frac{1}{1-\sin x}\,dx \ \Rightarrow I = \int \frac{1}{1-\sin x}\times\frac{1+\sin x}{1+\sin x}\,dx \ \Rightarrow I = \int \frac{1+\sin x}{1-\sin^2 x}\,dx$

$\Rightarrow I = \int \frac{1+\sin x}{\cos^2 x}\,dx \ \Rightarrow I = \int \frac{1}{\cos^2 x}\,dx + \int \frac{\sin x}{\cos^2 x}\,dx$

$\Rightarrow I = \int \sec^2 x\,dx + \int \tan x\,\sec x\,dx \quad \therefore\ I = \tan x + \sec x + c$

26. $\int x\cos(x^2)\,dx$

Sol. Let $I = \int x\cos(x^2)\,dx$, Put $x^2 = t \ \Rightarrow 2x\,dx = dt \ \Rightarrow x\,dx = \frac{dt}{2}$

$\Rightarrow I = \int \cos(t).\frac{dt}{2} \ \Rightarrow I = \frac{1}{2}\int\cos(t)\,dt \ \Rightarrow I = \frac{1}{2}\sin(t) + c \quad \therefore\ I = \frac{1}{2}\sin(x^2) + c$

27. $\int \frac{\cot x}{\sqrt{\sin x}}\,dx$

Sol. Let $I = \int \frac{\cot x}{\sqrt{\sin x}}\,dx \ \Rightarrow I = \int \frac{\cos x}{\sin x\sqrt{\sin x}}\,dx \ \Rightarrow I = \int \frac{\cos x}{(\sin x)^{3/2}}\,dx$

Put $\sin x = t \ \Rightarrow \cos x\,dx = dt$

$\Rightarrow I = \int \frac{dt}{t^{3/2}} \ \Rightarrow I = \int t^{-3/2}\,dt \ \Rightarrow I = \frac{t^{-\frac{1}{2}}}{-1/2} + c \ \Rightarrow I = \frac{-2}{\sqrt{t}} + c \quad \therefore\ I = \frac{-2}{\sqrt{\sin x}} + c$

28. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\,dx$

Sol. Let $I = \int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\,dx \ \Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}\,dx \ \Rightarrow I = \int \frac{\sin^2 x}{\cos^2 x}\,dx \ \Rightarrow I = \int \tan^2 x\,dx$

$\Rightarrow I = \int(\sec^2 x - 1)\,dx \qquad \therefore\ I = \tan x - x + c$

29. $\int \sin^{-1}(\cos x)\,dx$

Sol. Let $I = \int \sin^{-1}(\cos x)\,dx \ \Rightarrow I = \int \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}dx$

$\Rightarrow I = \int\left(\frac{\pi}{2} - x\right)dx \ \Rightarrow I = \frac{\pi}{2}x - \frac{x^2}{2} + c \quad \therefore\ I = \frac{\pi}{2}x - \frac{x^2}{2} + c$

30. $\int \frac{dx}{\sqrt{x+2} + \sqrt{x+1}}$

Sol. Let $I=\int \frac{1}{\sqrt{x+2}+\sqrt{x+1}}\,dx \quad \Rightarrow I=\int \frac{1}{\sqrt{x+2}+\sqrt{x+1}}\times\frac{\sqrt{x+2}-\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+1}}\,dx$

$\Rightarrow I=\int \frac{\sqrt{x+2}-\sqrt{x+1}}{\left(\sqrt{x+2}\right)^2-\left(\sqrt{x+1}\right)^2}\,dx \quad \Rightarrow I=\int \frac{\sqrt{x+2}-\sqrt{x+1}}{(x+2)-(x+1)}\,dx$

$\Rightarrow I=\int \sqrt{x+2}\,dx-\int \sqrt{x+1}\,dx \quad \Rightarrow I=\frac{(x+2)^{3/2}}{3/2}-\frac{(x+1)^{3/2}}{3/2}+c$

$\therefore\ I=\frac{2}{3}\left[(x+2)^{3/2}-(x+1)^{3/2}\right]+c$

31. $\int 2^x\,dx$

Sol. Let $I=\int 2^x\,dx \quad \therefore\ I=\frac{2^x}{\log 2}+c$

32. $\int \frac{1+\tan x}{(x+\log\sec x)}\,dx$

Sol. Let $I=\int \frac{1+\tan x}{x+\log(\sec x)}\,dx$

Put $x+\log(\sec x)=t \quad \Rightarrow 1+\frac{1}{\sec x}.\sec x\tan x=\frac{dt}{dx} \quad \Rightarrow (1+\tan x)\,dx=dt$

$\Rightarrow I=\int \frac{dt}{t} \quad \Rightarrow I=\log|t|+c \quad \therefore\ I=\log\left|x+\log(\sec x)\right|+c$

33. $\int \frac{\sec^2(\log x)}{x}\,dx$

Sol. Let $I=\int \frac{\sec^2(\log x)}{x}\,dx$, Put $\log x=t \quad \Rightarrow \frac{1}{x}\,dx=dt$

$\Rightarrow I=\int \sec^2(t)\,dt \quad \Rightarrow I=\tan(t)+c \quad \therefore\ I=\tan(\log x)+c$

34. $\int (2x+1)\sqrt{x^2+x+1}\,dx$

Sol. Let $I=\int (2x+1)\sqrt{x^2+x+1}\,dx$, Put $x^2+x+1=t \quad \Rightarrow (2x+1)\,dx=dt$

$\Rightarrow I=\int \sqrt{t}\,dt \quad \Rightarrow I=\frac{t^{3/2}}{3/2}+c \quad \therefore\ I=\frac{2}{3}\left(x^2+x+1\right)^{3/2}+c$

35. $\int \frac{dx}{\sqrt{9x^2+16}}$

Sol. Let $I=\int \frac{dx}{\sqrt{9x^2+16}} \quad \Rightarrow I=\int \frac{dx}{\sqrt{9\left(x^2+\frac{16}{9}\right)}} \quad \Rightarrow I=\frac{1}{3}\int \frac{1}{\sqrt{(x)^2+\left(\frac{4}{3}\right)^2}}\,dx$

$\Rightarrow I=\frac{1}{3}\log\left|x+\sqrt{(x)^2+\left(\frac{4}{3}\right)^2}\right|+c \quad \therefore\ I=\frac{1}{3}\log\left|3x+\sqrt{9x^2+16}\right|+c$

36. $\int \frac{dx}{\sqrt{4-9x^2}}$

Sol. Let $I = \int \frac{dx}{\sqrt{4-9x^2}} \quad \Rightarrow I = \int \frac{1}{\sqrt{9\left(\frac{4}{9}-x^2\right)}}\, dx$

$\Rightarrow I = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - (x)^2}}\, dx \quad \Rightarrow I = \frac{1}{3}.\sin^{-1}\left(\frac{x}{2/3}\right) + c \quad \therefore\ I = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$

37. $\int \frac{dx}{\sqrt{4x^2-25}}$

Sol. Let $I = \int \frac{1}{\sqrt{4x^2-25}}\, dx \quad \Rightarrow I = \int \frac{1}{\sqrt{4\left(x^2-\frac{25}{4}\right)}\, dx} \quad \Rightarrow I = \frac{1}{2}\int \frac{1}{\sqrt{(x)^2-\left(\frac{5}{2}\right)^2}}\, dx$

$\Rightarrow I = \frac{1}{2}\log\left| x + \sqrt{(x)^2 - \left(\frac{5}{2}\right)^2} \right| + c \quad \therefore\ I = \frac{1}{2}\log\left| 2x + \sqrt{4x^2-25} \right| + c$

38. $\int \sqrt{4-x^2}\ dx$

Sol. Let $I = \int \sqrt{4-x^2}\ dx \quad \Rightarrow I = \int \sqrt{(2)^2-(x)^2}\ dx$

$\Rightarrow I = \frac{x}{2}\sqrt{(2)^2-(x)^2} + \frac{(2)^2}{2}\sin^{-1}\left(\frac{x}{2}\right) + c \quad \therefore\ I = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + c$

39. $\int \sqrt{9+x^2}\ dx$

Sol. Let $I = \int \sqrt{9+x^2}\ dx \quad \Rightarrow I = \int \sqrt{(3)^2+(x)^2}\ dx$

$\Rightarrow I = \frac{x}{2}\sqrt{(3)^2+(x)^2} + \frac{(3)^2}{2}\log\left| x + \sqrt{(3)^2+x^2} \right| + c \quad \therefore\ I = \frac{x}{2}\sqrt{9+x^2} + \frac{9}{2}\log\left| x+\sqrt{9+x^2} \right| + c$

40. $\int \sqrt{x^2-16}\ dx$

Sol. Let $I = \int \sqrt{x^2-16}\ dx \quad \Rightarrow I = \int \sqrt{(x)^2-(4)^2}\ dx$

$\Rightarrow I = \frac{x}{2}\sqrt{x^2-16} - \frac{(4)^2}{2}\log\left| x+\sqrt{x^2-4^2} \right| + c = \frac{x}{2}\sqrt{x^2-16} - 8\log\left| x+\sqrt{x^2-16} \right| + c$