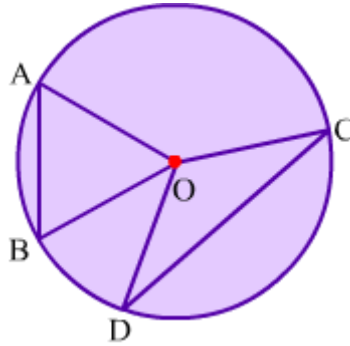


# Circles

## Angles Subtended by Chords at The Centre

We know that the chord of a circle is a line segment having its endpoints on the circumference of the circle. There can be several chords of the same or different lengths in a circle. Observe, for example, two unequal chords of a circle with centre O.



In the circle, chord CD is longer than chord AB. Note the angles subtended by the chords at the centre. While AB subtends an acute angle (i.e.,  $\angle AOB$ ), CD subtends an obtuse angle (i.e.,  $\angle COD$ ). Thus, in a circle, chords of different lengths subtend different angles at the centre.

On the other hand, **equal (or congruent) chords subtend equal (or congruent) angles at the centre**. This is a very useful property of circle. In this lesson, we will learn more about this property and solve some examples based on the same.

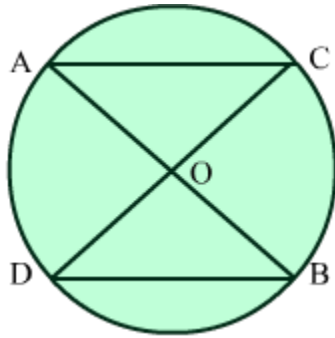
### Know More

The diameter of a circle is the longest chord of the circle.

### Solved Examples

#### Easy

**Example 1:** AB and CD are diameters of the given circle. Show that AC and BD are equal chords of the circle.



**Solution:**

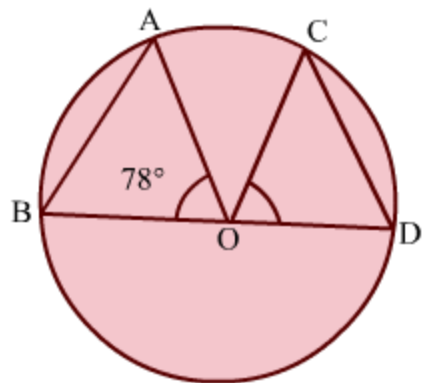
Since AB and CD are diameters of the circle, their point of intersection is the centre of the circle. Thus, O is the centre of the circle.

Chord AC subtends  $\angle AOC$  at the centre, while chord BD subtends  $\angle BOD$  at the centre.

Now,  $\angle AOC = \angle BOD$  (Vertically opposite angles)

$\therefore AC = BD$  ( $\because$  Chords subtending equal angles at the centre are equal in length)

**Example 2:** AB and CD are equal chords of the given circle with centre O. Find the measure of  $\angle COA$ .



**Solution:**

It is given that AB and CD are equal chords of the circle.

$\therefore \angle AOB = \angle COD = 78^\circ$  ( $\because$  Equal chords subtend equal angles at the centre)

Now,

$\angle AOB + \angle COA + \angle COD = 180^\circ$  ( $\because$  BOD is the diameter of the circle)

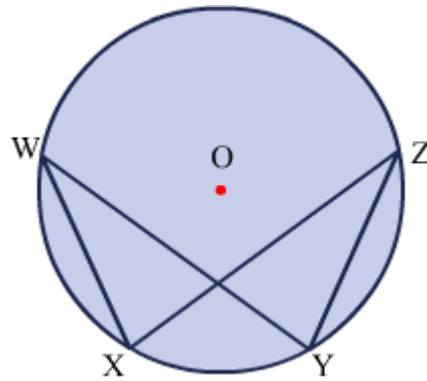
$$\Rightarrow 78^\circ + \angle COA + 78^\circ = 180^\circ$$

$$\Rightarrow \angle COA = 180^\circ - (78^\circ + 78^\circ)$$

$$\Rightarrow \angle COA = 24^\circ$$

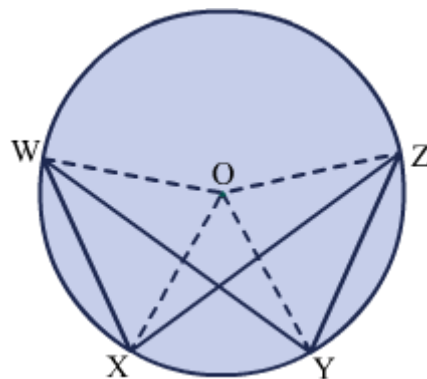
### Medium

**Example 1:** If WY and ZX are equal chords of the given circle with centre O, then show that WX and ZY are also equal chords.



**Solution:**

**Construction:** Join O to W, X, Y and Z.



It is given that WY and ZX are equal chords.

$\therefore \angle WOY = \angle ZOX$  ( $\because$  Equal chords subtend equal angles at the centre)

On subtracting  $\angle XOY$  from both sides of the equation, we obtain:

$$\angle WOY - \angle XOY = \angle ZOX - \angle XOY$$

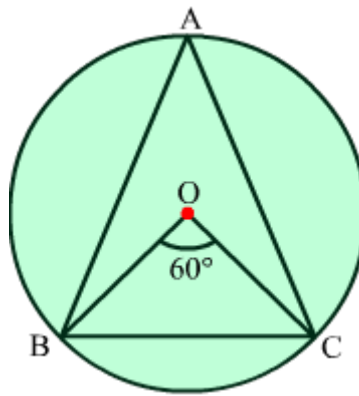
$$\Rightarrow \angle WOX = \angle ZOY$$

Now,  $\angle WOX$  and  $\angle ZOY$  are the angles subtended by chords  $WX$  and  $ZY$  at the centre.

$\therefore WX = ZY$  ( $\because$  Chords subtending equal angles at the centre are equal in length)

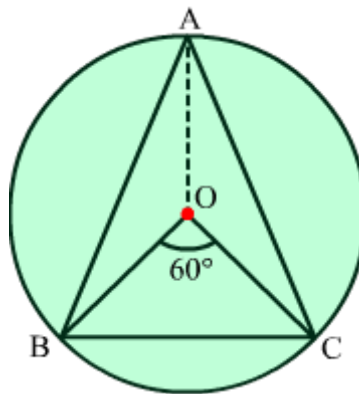
### Hard

**Example 1:** In the given circle with centre  $O$ ,  $AB = AC$ . Find the measure of  $\angle BAC$ .



**Solution:**

**Construction:** Join  $O$  to  $A$ .



It is given that  $AB = AC$ .

$\angle AOB = \angle AOC$  ( $\because$  Equal chords subtend equal angles at the centre)

Let  $\angle AOB = \angle AOC = x$

Now,

$$\angle AOB + \angle BOC + \angle AOC = 360^\circ \text{ (Complete angle)}$$

$$\Rightarrow x + 60^\circ + x = 360^\circ$$

$$\Rightarrow 2x + 60^\circ = 360^\circ$$

$$\Rightarrow 2x = 360^\circ - 60^\circ$$

$$\Rightarrow 2x = 300^\circ$$

$$\Rightarrow x = 150^\circ \dots (1)$$

Now, in  $\triangle OAB$ , we have:

$$OA = OB \text{ (Radii of the circle)}$$

$$\Rightarrow \angle OBA = \angle OAB \text{ (Angles opposite equal sides are equal)}$$

Using the angle sum property in  $\triangle OAB$ , we get:

$$\angle OAB + \angle OBA + \angle AOB = 180^\circ$$

$$\Rightarrow 2\angle OAB + 150^\circ = 180^\circ \text{ (By equation 1)}$$

$$\Rightarrow 2\angle OAB = 180^\circ - 150^\circ$$

$$\Rightarrow 2\angle OAB = 30^\circ$$

$$\Rightarrow \angle OAB = 15^\circ$$

Similarly, in  $\triangle OAC$ , we get  $\angle OAC = 15^\circ$

$$\text{Now, } \angle BAC = \angle OAB + \angle OAC$$

$$\Rightarrow \angle BAC = 15^\circ + 15^\circ$$

$$\Rightarrow \angle BAC = 30^\circ$$

**Perpendicular from The Centre of a Circle to a Chord Bisects The Chord**

**Perpendicular from the Centre to the Chord**

Observe the clock shown below.



Note how the progression of the second hand from '5' to '7' has been marked on the clock face. We know that the second hand takes five seconds to cover the distance between '5' and '6'. Similarly, it takes five seconds to cover the distance between '6' and '7'. Clearly, '6' lies in the middle of '5' and '7'. Observe how the line marking the position of the second hand at '6' is perpendicular to the straight line marking the distance between '5' and '7'. What we have here is a system similar to that obtained on drawing a perpendicular from the centre of a circle to one of its chords.

In this lesson, we will learn about the property of a circle relating to perpendiculars drawn to chords from the centre of the circle. We will also solve some problems based on this property.

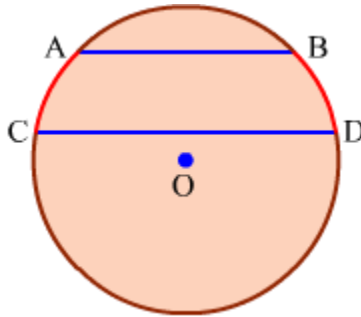
### Know More

The circle that is centred at the origin with radius 1 is called the **unit circle**.

### Whiz Kid

- For a given length of perimeter, the circle is the shape with the largest area.
- The circle is a highly symmetric shape as every line passing through its centre forms a line of reflection symmetry and every angle around the centre has rotational symmetry.
- In a circle, parallel chords always cut congruent arcs.

Here, chord AB is parallel to chord CD and arc AC and arc BD are congruent arcs.



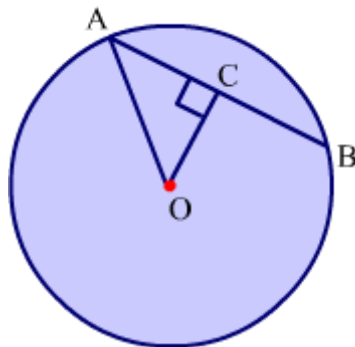
### Whiz Kid

If the intersection of any two chords divides one chord into lengths  $a$  and  $b$  and the other into lengths  $c$  and  $d$ , then  $ab = cd$ .

### Solved Examples

#### Easy

**Example 1:** In the given circle with centre O, if  $AB = 6$  cm and  $OC = 4$  cm, then find the perimeter of  $\triangle OCA$ .



### Solution:

We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

O is the centre of the given circle with chord AB and  $OC \perp AB$ ; therefore, OC bisects AB.

$$\therefore AC = CB = \frac{AB}{2} = \left(\frac{6}{2}\right) \text{ cm} = 3 \text{ cm}$$

On using the Pythagoras theorem in the right-angled  $\triangle OCA$ , we get:

$$OA^2 = OC^2 + AC^2$$

$$\Rightarrow OA^2 = (4^2 + 3^2) \text{ cm}^2$$

$$\Rightarrow OA^2 = (16 + 9) \text{ cm}^2$$

$$\Rightarrow OA^2 = 25 \text{ cm}^2$$

$$\Rightarrow OA = 5 \text{ cm}$$

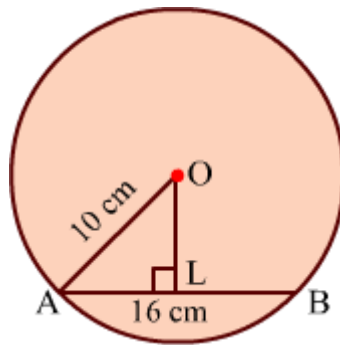
So,

$$\text{Perimeter of } \triangle OCA = OA + AC + OC$$

$$\Rightarrow \text{Perimeter of } \triangle OCA = (5 + 3 + 4) \text{ cm}$$

$$\Rightarrow \text{Perimeter of } \triangle OCA = 12 \text{ cm}$$

**Example 2:** In the given circle centred at O, find the distance of chord AB from the centre.



**Solution:**

We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

O is the centre of the given circle with chord AB and  $OL \perp AB$ ; therefore, OL bisects AB.

$$\Rightarrow AL = LB = \frac{AB}{2} = \left( \frac{16}{2} \right) \text{ cm} = 8 \text{ cm}$$

On using the Pythagoras theorem in the right-angled  $\triangle OLA$ , we get:

$$OA^2 = OL^2 + AL^2$$

$$\Rightarrow OL^2 = OA^2 - AL^2$$



$$\Rightarrow OL^2 = (10^2 - 8^2) \text{ cm}^2$$

$$\Rightarrow OL^2 = (100 - 64) \text{ cm}^2$$

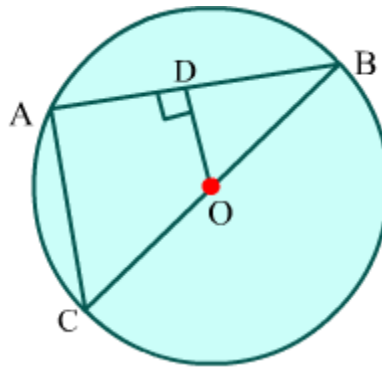
$$\Rightarrow OL^2 = 36 \text{ cm}^2$$

$$\Rightarrow OL = 6 \text{ cm}$$

Thus, chord AB is at a distance of 6 cm from the centre of the circle.

### Medium

**Example 1:** In the given circle with centre O, prove that  $AC = 2OD$ .



### Solution:

We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

O is the centre of the given circle with chord AB and  $OD \perp AB$ ; therefore, OD bisects AB.

Thus, D is the midpoint of AB.

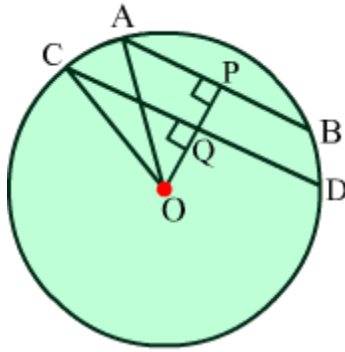
Since O is the centre, it is the midpoint of diameter BC.

So, in  $\triangle ABC$ , D and O are the midpoints of sides AB and BC respectively.

$$\therefore OD = \frac{AC}{2} \text{ and } OD \parallel AC \text{ (By the midpoint theorem)}$$

$$\Rightarrow AC = 2OD$$

**Example 2:** The given circle centred at O has a radius of 5 cm and two parallel chords AB and CD. If  $AB = 6 \text{ cm}$  and  $CD = 8 \text{ cm}$ , then find the length of PQ.



**Solution:**

We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

O is the centre of the given circle with chord AB and  $OP \perp AB$ ; therefore, OP bisects AB.

Thus, P is the midpoint of AB.

$$\Rightarrow AP = PB = \frac{AB}{2} = \left(\frac{6}{2}\right)\text{cm} = 3\text{ cm}$$

Similarly, Q is the midpoint of CD.

$$\Rightarrow CQ = QD = \frac{CD}{2} = \left(\frac{8}{2}\right)\text{cm} = 4\text{ cm}$$

On using the Pythagoras theorem in right-angled  $\triangle OPA$ , we get:

$$OA^2 = OP^2 + AP^2$$

$$\Rightarrow OP^2 = OA^2 - AP^2$$

$$\Rightarrow OP^2 = (5^2 - 3^2)\text{ cm}^2$$

$$\Rightarrow OP^2 = (25 - 9)\text{ cm}^2$$

$$\Rightarrow OP^2 = 16\text{ cm}^2$$

$$\Rightarrow OP = 4\text{ cm}$$

Similarly, in right-angled  $\triangle OQC$ , we get:

$$OC^2 = OQ^2 + CQ^2$$

$$\Rightarrow OQ^2 = OC^2 - CQ^2$$

$$\Rightarrow OQ^2 = (5^2 - 4^2) \text{ cm}^2$$

$$\Rightarrow OQ^2 = (25 - 16) \text{ cm}^2$$

$$\Rightarrow OQ^2 = 9 \text{ cm}^2$$

$$\Rightarrow OQ = 3 \text{ cm}$$

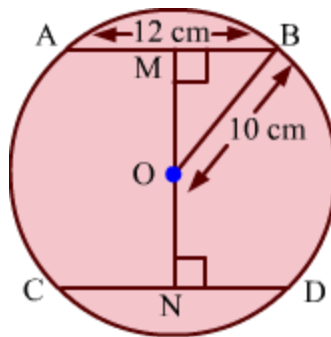
$$\text{Now, } PQ = OP - OQ$$

$$\Rightarrow PQ = (4 - 3) \text{ cm}$$

$$\Rightarrow PQ = 1 \text{ cm}$$

### Hard

**Example 1:** In the given circle with centre O, find the distance between the parallel and equal chords AB and CD.



### Solution:

We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

O is the centre of the given circle with chord AB and  $OM \perp AB$ ; therefore, OM bisects AB.

$$\Rightarrow AM = MB = \frac{AB}{2} = \left( \frac{12}{2} \right) \text{ cm} = 6 \text{ cm}$$

Similarly, ON bisects CD. Since  $AB = CD$ , we get:

$$CN = ND = 6 \text{ cm}$$

On using the Pythagoras theorem in right-angled  $\triangle OMB$ , we get:

$$OB^2 = OM^2 + BM^2$$

$$\Rightarrow OM^2 = OB^2 - BM^2$$

$$\Rightarrow OM^2 = (10^2 - 6^2) \text{ cm}^2$$

$$\Rightarrow OM^2 = (100 - 36) \text{ cm}^2$$

$$\Rightarrow OM^2 = 64 \text{ cm}^2$$

$$\Rightarrow OM = 8 \text{ cm}$$

On joining point O to point D and calculating as above, we get  $ON = 8 \text{ cm}$ .

Now, MN is the perpendicular distance between the given chords.

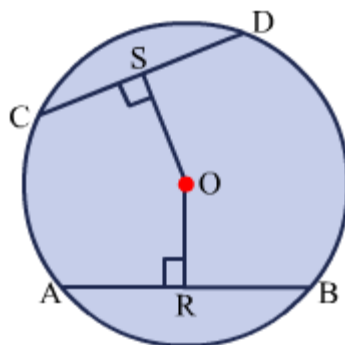
$$MN = OM + ON$$

$$\Rightarrow MN = (8 + 8) \text{ cm}$$

$$\Rightarrow MN = 16 \text{ cm}$$

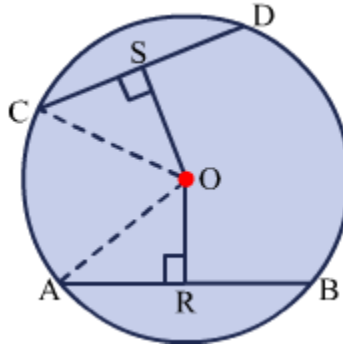
Thus, the distance between the parallel and equal chords AB and CD is 16 cm.

**Example 2:** In the given circle centred at O,  $AB = 8 \text{ cm}$ ,  $OR = 3 \text{ cm}$  and  $OS = 4 \text{ cm}$ . Find the length of CD.



**Solution:**

**Construction:** Join point O to points A and C.



We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

O is the centre of the circle with chord AB and  $OR \perp AB$ ; therefore, OR bisects AB.

$$\Rightarrow AR = RB = \frac{AB}{2} = \left(\frac{8}{2}\right) \text{ cm} = 4 \text{ cm}$$

On using the Pythagoras theorem in right-angled  $\triangle ORA$ , we get:

$$OA^2 = OR^2 + AR^2$$

$$\Rightarrow OA^2 = (3^2 + 4^2) \text{ cm}^2$$

$$\Rightarrow OA^2 = (9 + 16) \text{ cm}^2$$

$$\Rightarrow OA^2 = 25 \text{ cm}^2$$

$$\Rightarrow OA = 5 \text{ cm}$$

Clearly, OA and OC are radii of the same circle.

$$\therefore OC = OA = 5 \text{ cm}$$

On using the Pythagoras theorem in right-angled  $\triangle OSC$ , we get:

$$OC^2 = OS^2 + CS^2$$

$$\Rightarrow CS^2 = OC^2 - OS^2$$

$$\Rightarrow CS^2 = (5^2 - 4^2) \text{ cm}^2$$

$$\Rightarrow CS^2 = (25 - 16) \text{ cm}^2$$

$$\Rightarrow CS^2 = 9 \text{ cm}^2$$

$$\Rightarrow CS = 3 \text{ cm}$$

Since  $OS \perp CD$ , we have:

$$CS = SD = 3 \text{ cm}$$

$$\text{Now, } CD = CS + SD$$

$$\Rightarrow CD = (3 + 3) \text{ cm}$$

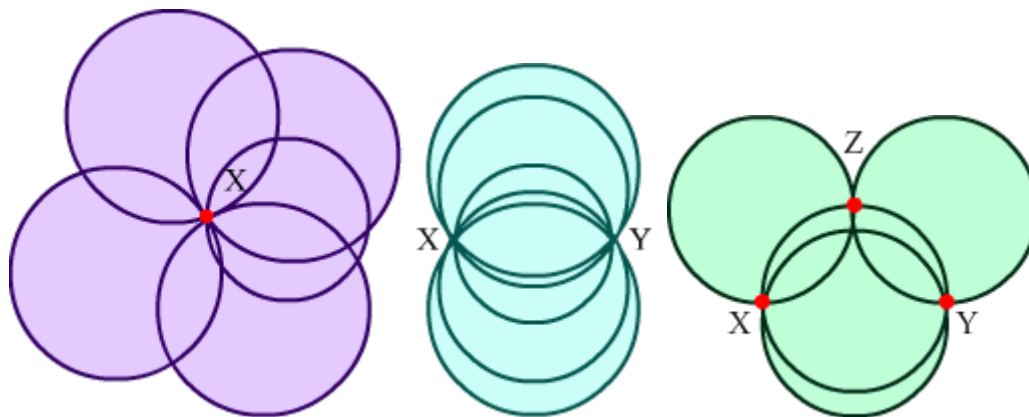
$$\Rightarrow CD = 6 \text{ cm}$$

### **Only One Circle can Pass Through Three Non-Collinear Points**

#### **Limitation of Points Shared by Circles**

We know that the most important point required to draw a circle is its centre which is equidistant from all other points lying on the boundary of the circle. We can also draw infinitely many circles of different radii with the same centre.

Now, let us observe some points shared by circles on their boundaries. A few circles passing through common points X, Y and Z are shown below.



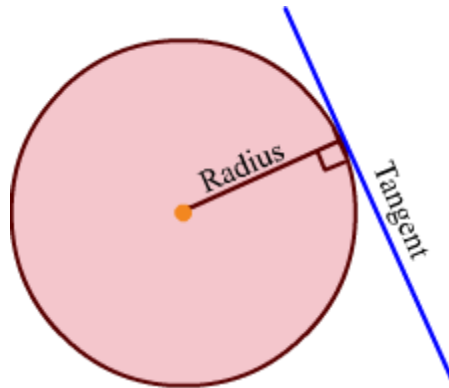
It can be observed that when point X is taken alone, we can draw infinitely many circles passing through it. Similarly, when X and Y are taken together, we can get infinitely many circles passing through them. However, when we take the three points X, Y and Z together, we obtain only one circle passing through them. Thus, we can conclude that to draw a unique circle, we require at least three non-collinear points. In this lesson, we will study more about this conclusion.

#### **Did You Know?**

- Circles were worshiped in ancient Rome as they were thought to be divine and holy.
- A circle is a shape that does not exist in nature. It is a mental construct and a symbolic representation that was invented in a manner similar to how the alphabet and language were invented.

### Whiz Kid

A tangent is a line that touches a circle at only one point. It always forms a right angle with a radius of that circle.



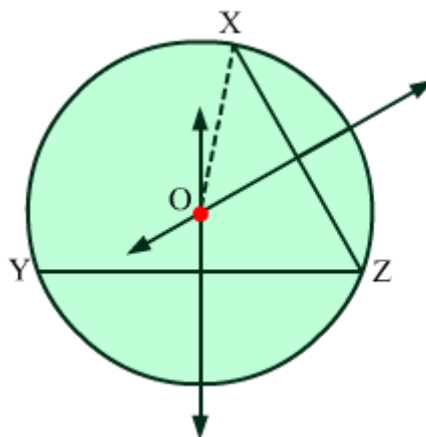
### Know More

Two lines cannot intersect at more than one point.

### Verification of the Uniqueness of a Circle

Let us verify the fact that only one circle passes through three non-collinear points.

The given circle with centre O passes through points X, Y and Z.



Let us assume that another circle with centre  $O'$  and the same radius can pass through  $X$ ,  $Y$  and  $Z$ . Then,  $O'$  must lie on the perpendicular bisectors of  $XZ$  and  $YZ$ .

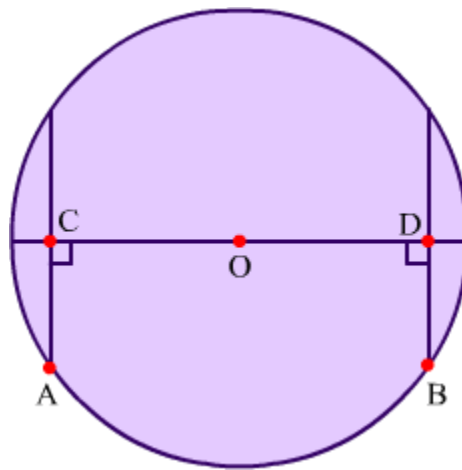
We know that two lines cannot intersect at more than one point. So,  $O'$  must coincide with  $O$ .

Hence, one and only one circle can pass through three non-collinear points.

## **Distance of Chords from The Centre of The Circle**

### **Equal Chords and Their Distance from the Centre**

Consider a big circular ground in which three ropes are tied to the circumference. Two ropes are of the same length and the third is perpendicular to them, as is shown in the following figure.



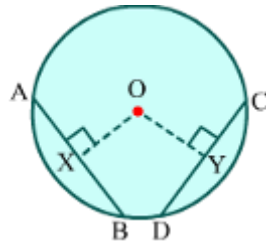
Anita, Bindu and Omkar are standing at points  $A$ ,  $B$  and  $O$  respectively. The girls begin walking along the lengths of the shorter ropes to reach points  $C$  and  $D$  at the same time. Thereafter, they continue in a similar manner toward point  $O$  with the same speed as before. Who will reach Omkar first?

The answer to the above question is based on an important property of chords which we will study in this lesson. This property shows the relation between equal chords in terms of their distances from the centre of a circle.

### **Equal Chords Are Equidistant from the Centre of a Circle**

Consider the given circle centred at  $O$  with two equal chords  $AB$  and  $CD$ .

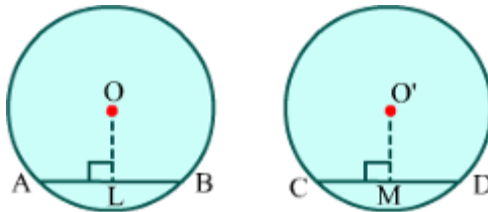




OX and OY are the perpendicular distances of chords AB and CD respectively from centre O. Now, the property that relates these perpendicular distances of equal chords is stated as follows:

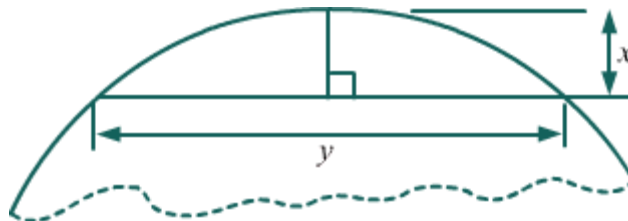
***Equal chords are equidistant from the centre of a circle.***

So, according to this property, since AB is equal to CD, their distances from the centre are also equal, i.e.,  $OX = OY$ . This property is also true in case of congruent circles. Consider, for example, the following congruent circles with centres O and O', and chords AB and CD. Now, if AB and CD are equal, then AB and CD are equidistant from O and O', i.e.,  $OL = O'M$ .



### Know More

**Sagitta:** It is the perpendicular drawn from an arc of a circle to a chord such that it bisects the chord.

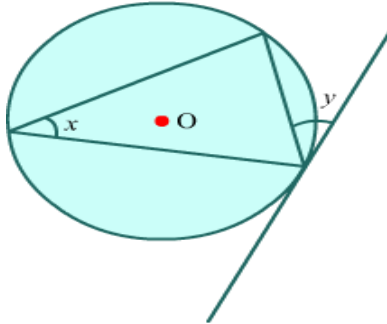


Here,  $x = r - \sqrt{r^2 - \left(\frac{y}{2}\right)^2}$ , where  $r$  is the radius of the circle

### Whiz Kid

### Alternate segment theorem

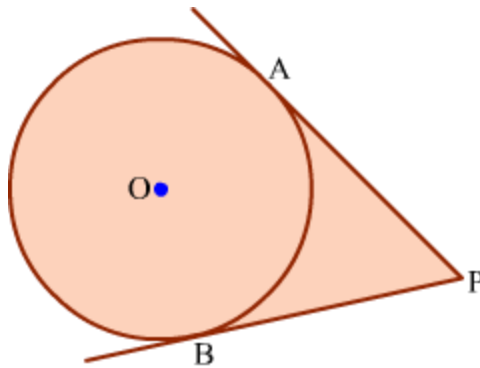
If a line touches a circle and a chord is drawn from the point of contact, then the angle between the tangent and the chord is equal to the angles in the corresponding alternate segments.



Here,  $x = y$

### Whiz Kid

The lengths of tangents drawn from a point outside a circle are equal.



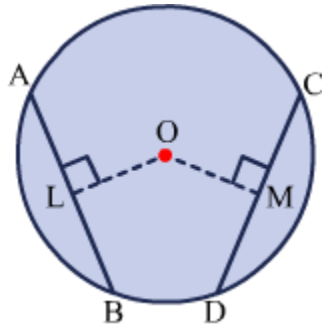
Here,  $PA = PB$

### Converse of the Property

The converse of the property can be stated as follows:

***Chords that are equidistant from the centre of a circle are equal in length.***

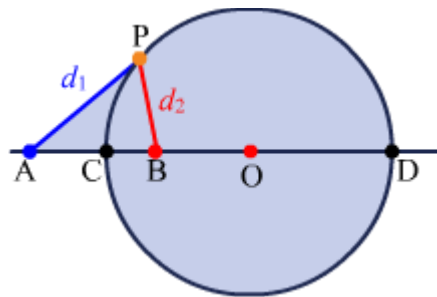
Consider, for example, the given circle with centre O and chords AB and CD that are equidistant from O, i.e.,  $OL = OM$ .



Using the converse of the property, we can say that AB and CD are equal in length.

### Did You Know?

### Circle of Apollonius



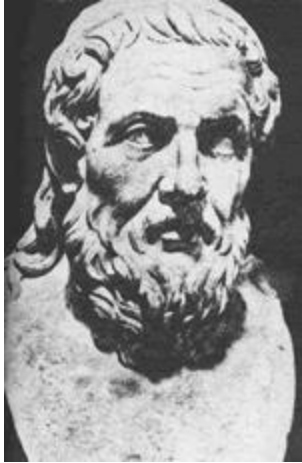
Apollonius of Perga, an ancient Greek geometer, showed that a circle can be defined as a set of points in a plane having a constant ratio of distances to two fixed foci.

In the given figure, points A and B are two fixed foci.

Therefore, by the above definition, we have  $\frac{AP}{BP} = \frac{AC}{BC}$

Note that the constant ratio cannot be equal to 1.

### Know Your Scientist



Apollonius (262 BC–190 BC) was an ancient Greek geometer and astronomer. He is known as ‘the Great Geometer’. His book *Conics* is one of the greatest scientific works from the ancient world. Terms such as ‘parabola’, ‘ellipse’ and ‘hyperbola’ were introduced in this book. Famous scholars like René Descartes and Isaac Newton were influenced by his innovative methods and the terms he used in the field of conics.

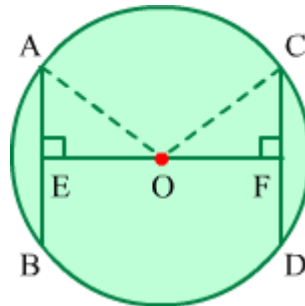
### Proof of the Converse

**Statement:** Chords that are equidistant from the centre of a circle are equal in length.

**Given:** A circle centred at O with chords AB and CD and  $OE = OF$ , where  $OE \perp AB$  and  $OF \perp CD$ .

**To Prove:**  $AB = CD$

**Construction:** Join point O to points A and C.



**Proof:** It is given that  $OE \perp AB$  and  $OF \perp CD$

We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord. So, we have

$$AE = \frac{AB}{2} \quad \dots (1)$$

$$CF = \frac{CD}{2} \quad \dots (2)$$

In  $\triangle AEO$  and  $\triangle CFO$ , we have:

$OA = OC$  (Radii of the circle)

$\angle AEO = \angle CFO = 90^\circ$  ( $\because$  OE and OF are perpendiculars)

$OE = OF$  (Given)

$\therefore \triangle AEO \cong \triangle CFO$  (By the RHS congruence rule)

$\Rightarrow AE = CF$  (By CPCT)

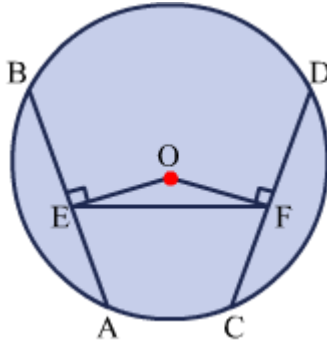
$\Rightarrow \frac{1}{2} AB = \frac{1}{2} CD$  (Using equations (1) and (2))

$\Rightarrow AB = CD$

### Solved Examples

#### Easy

**Example 1:** In the given circle centred at O, AB and CD are two equal chords. Prove that  $\angle AEF = \angle CFE$ .



#### Solution:

It is given that  $AB = CD$ . We know that equal chords are equidistant from the centre of a circle.

$\therefore OE = OF$

In  $\triangle OEF$ , we have:

$OE = OF$

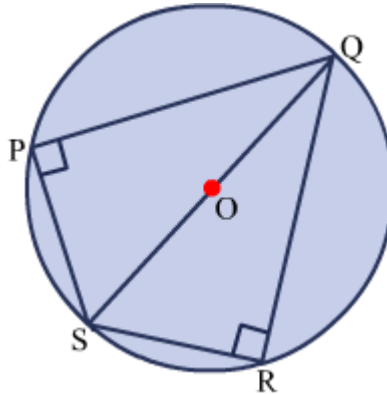
$\therefore \angle OFE = \angle OEF$  ( $\because$  Angles opposite equal sides are equal)

$$\Rightarrow 90^\circ - \angle AEF = 90^\circ - \angle CFE (\because OE \text{ and } OF \text{ are perpendiculars})$$

$$\Rightarrow \angle AEF = \angle CFE$$

### Medium

**Example 1:** In the given circle with centre O, chords PQ and RQ are equidistant from the centre. Prove that diameter SQ bisects  $\angle PQR$  and  $\angle PSR$ .



### Solution:

We know that chords which are equidistant from the centre of a circle are equal in length.

$$\therefore PQ = RQ$$

In  $\triangle QPS$  and  $\triangle QRS$ , we have:

$$PQ = RQ \text{ (Proved above)}$$

$$\angle QPS = \angle QRS = 90^\circ (\because SP \text{ and } SR \text{ are perpendiculars})$$

$$QS = QS \text{ (Common side)}$$

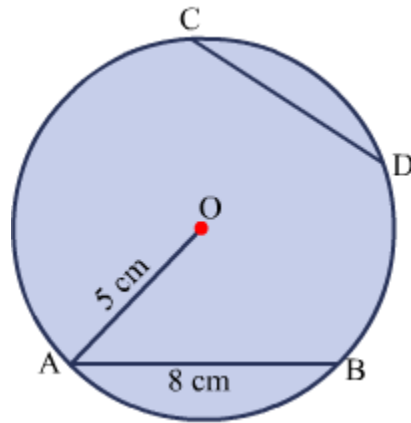
$$\therefore \triangle QPS \cong \triangle QRS \text{ (By the RHS congruence rule)}$$

$$\Rightarrow \angle PQS = \angle RQS \text{ and } \angle PSQ = \angle RSQ \text{ (By CPCT)}$$

$$\text{Now, } \angle PQR = \angle PQS + \angle RQS \text{ and } \angle PSR = \angle PSQ + \angle RSQ$$

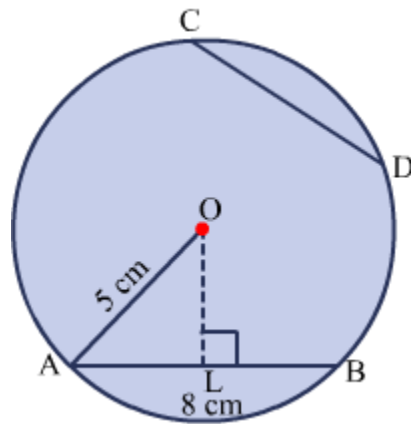
Thus, diameter SQ bisects  $\angle PQR$  and  $\angle PSR$ .

**Example 2:** In the given circle, find the length of chord CD which is 3 cm away from centre O.



**Solution:**

**Construction:** Draw a perpendicular OL from centre O to chord AB.



We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

$$\therefore AL = LB = \frac{AB}{2} = \left(\frac{8}{2}\right) \text{ cm} = 4 \text{ cm}$$

On using the Pythagoras theorem in right-angled  $\triangle OLA$ , we obtain:

$$OA^2 = OL^2 + AL^2$$

$$\Rightarrow OL^2 = OA^2 - AL^2$$

$$\Rightarrow OL^2 = (5^2 - 4^2) \text{ cm}^2$$

$$\Rightarrow OL^2 = (25 - 16) \text{ cm}^2$$

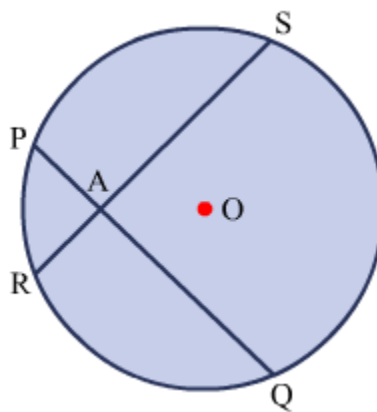
$$\Rightarrow OL^2 = 9 \text{ cm}^2$$

$\Rightarrow OL = 3 \text{ cm}$ , which is the distance of chord AB from O

It is given that the distance of chord CD from the centre is also 3 cm. We know that chords which are equidistant from the centre are equal in length. Therefore, the length of chord CD is also 8 cm.

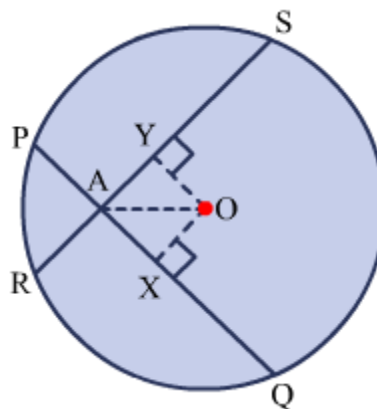
### Hard

**Example 1:** In the given circle, PQ and RS are two chords equidistant from centre O and A is the point of intersection of the chords. Prove that  $AR = AP$ .



**Solution:**

**Construction:** Draw perpendiculars OX and OY to chords PQ and RS respectively. Join O to A.



In  $\triangle OXA$  and  $\triangle OYA$ , we have:



$OX = OY$  ( $\because$  PQ and RS are equidistant from the centre)

$\angle OXA = \angle OYA = 90^\circ$  ( $\because$  OX and OY are perpendiculars)

$OA = OA$  (Common side)

$\therefore \triangle OXA \cong \triangle OYA$  (By the RHS congruence rule)

$\Rightarrow AX = AY \dots (1)$  [By CPCT]

We know that chords which are equidistant from the centre are equal in length.

$\therefore PQ = RS \dots (2)$

We also know that the perpendicular drawn from the centre of a circle to a chord bisects the chord.

$$\therefore PX = \frac{PQ}{2} \text{ and } RY = \frac{RS}{2}$$

Using equation 2, we obtain:

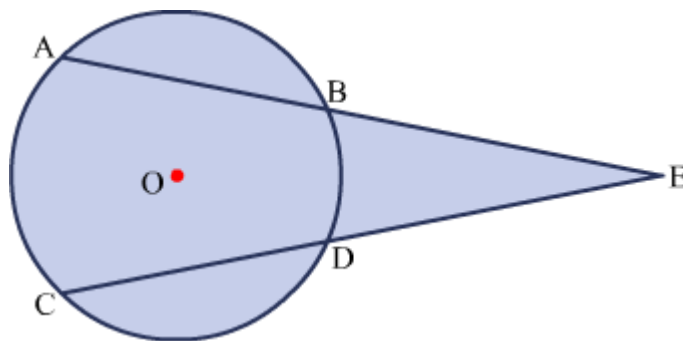
$$PX = RY \dots (3)$$

On subtracting equation 1 from equation 3, we obtain:

$$PX - AX = RY - AY$$

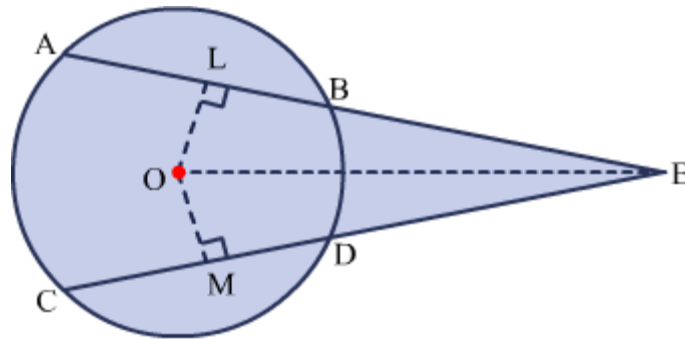
$$\Rightarrow AP = AR$$

**Example 2:** When two equal chords AB and CD of a circle with centre O are extended, they meet at a point E as is shown in the figure. Prove that  $BE = DE$  and  $AE = CE$ .



**Solution:**

**Construction:** Join O to E and construct  $OL \perp AB$  and  $OM \perp CD$ .



It is given that  $AB = CD$ . We know that equal chords are equidistant from the centre.

$$\therefore OL = OM$$

In  $\triangle OLE$  and  $\triangle OME$ , we have:

$$OL = OM \text{ (Proved above)}$$

$$\angle OLE = \angle OME = 90^\circ (\because OL \text{ and } OM \text{ are perpendiculars})$$

$$OE = OE \text{ (Common side)}$$

$$\therefore \triangle OLE \cong \triangle OME \text{ (By the RHS congruence rule)}$$

$$\Rightarrow LE = ME \dots (1) \text{ [By CPCT]}$$

We know that the perpendicular drawn from the centre of a circle to a chord bisects the chord. Thus L and M are the midpoints of AB and CD respectively.

$$\therefore BL = DM \dots (2) [\because AB = CD]$$

On subtracting equation 2 from equation 1, we get:

$$LE - BL = ME - DM$$

$$\Rightarrow BE = DE$$

Now,  $AB = CD$  and  $BE = DE$

$$\therefore AB + BE = CD + DE$$

$$\Rightarrow AE = CE$$

## Relation Between The Lengths of Chords and Their Corresponding Arcs

### Observing Chords and Their Corresponding Arcs

Mathematics is an inseparable part of our life. It is hidden in various things around us; we just need to observe the same. For instance, look at the top of a table shown below. Two rulers of equal length are placed near the edge.



Clearly, the top of the table is circular and the two rulers near the edge resemble two chords drawn on the opposite sides of a circle. Can you notice the two arcs formed by the rulers? What do you observe about the lengths of these arcs? Finally, what can be concluded from this about the relation between the lengths of chords of a circle and their corresponding arcs?

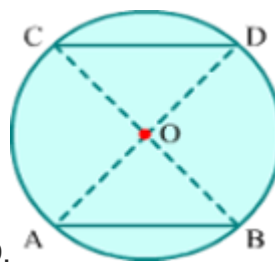
Let us go through this lesson to learn about the property of chords that relates their lengths to their corresponding arcs. We will also test the validity of this property and solve some problems based on it.

### Proof of the Property

**Statement:** If two chords of a circle are equal, then their corresponding arcs are congruent.

**Given:** A circle with centre O and two equal chords AB and CD

**To prove:**  $\widehat{AB} \cong \widehat{CD}$



**Construction:** Join point O to points A, B, C and D.

**Proof:** In  $\triangle OAB$  and  $\triangle OCD$ , we have:

$AB = CD$  (Given)

$OA = OD$  (Radii of the circle)

$OB = OC$  (Radii of the circle)

$\therefore \triangle OAB \cong \triangle OCD$  (By the SSS congruence rule)

$\Rightarrow \angle AOB = \angle COD$  (By CPCT)

$\Rightarrow \text{Length of } \widehat{AB} = \text{Length of } \widehat{CD}$

$\Rightarrow \widehat{AB} \cong \widehat{CD}$

### Know More

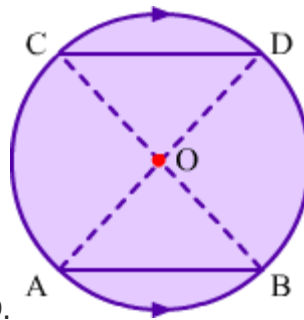
Length of an arc =  $\frac{\text{Central angle}}{360^\circ} \times \text{Circumference}$

### Proof of the Converse

**Statement:** If two arcs of a circle are congruent, then their corresponding chords are equal.

**Given:** A circle with centre O and two congruent arcs AB and CD

**To prove:** Chord AB = Chord CD



**Construction:** Join point O to points A, B, C and D.

**Proof:** It is given that  $\widehat{AB} \cong \widehat{CD}$

$\Rightarrow \text{Length of } \widehat{AB} = \text{Length of } \widehat{CD}$

$\Rightarrow \angle AOB = \angle COD$

In  $\triangle OAB$  and  $\triangle OCD$ , we have

$OA = OD$  (Radii of the circle)

$\angle AOB = \angle COD$  (Proved above)

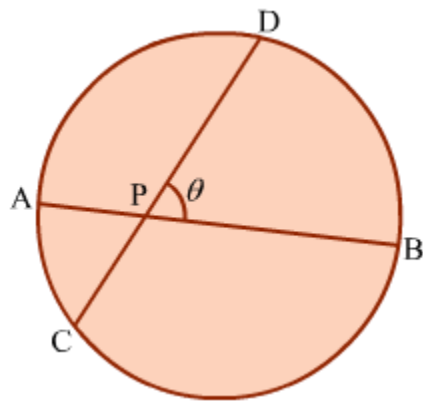
$OB = OC$  (Radii of the circle)

$\therefore \triangle OAB \cong \triangle OCD$  (By the SAS congruence rule)

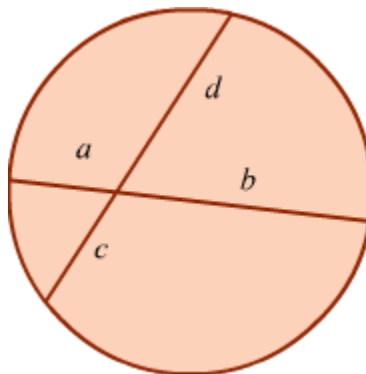
$\Rightarrow AB = CD$  (By CPCT)

### Whiz Kid

- Two intersecting chords determine four arcs in a circle. The relation between the angle of intersection of the two chords and the lengths of these arcs is given as  $\theta = \frac{m\widehat{AC} + m\widehat{BD}}{2}$ .



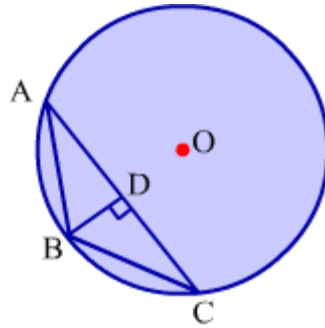
- If two chords of a circle intersect such that one chord has sub-segment lengths  $a$  and  $b$  and the other has sub-segment lengths  $c$  and  $d$ , then  $ab = cd$ .



### Solved Examples

Easy

**Example 1:** In the given circle, find the length of AD if arcs AB and BC are congruent and AC = 5 cm.



**Solution:**

We know that if two arcs are congruent, then their corresponding chords are equal.

Since arcs AB and BC are congruent, chords AB and BC are equal.

$\Rightarrow \triangle ABC$  is isosceles.

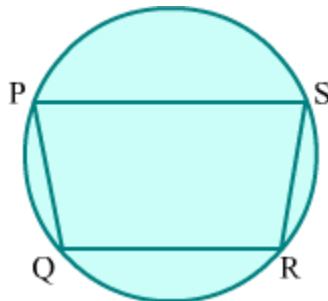
We also know that the perpendicular drawn from the vertex to the base of an isosceles triangle bisects the base.

$$\therefore AD = DC = \frac{1}{2}AC$$

$$\Rightarrow AD = \frac{5}{2} \text{ cm}$$

$$\Rightarrow AD = 2.5 \text{ cm}$$

**Example 2:** In the given circle, arc QPS is congruent to arc RSP. Prove that PQ = SR.



**Solution:** It is given that  $\widehat{QPS} \cong \widehat{RSP}$

$$\Rightarrow \widehat{QPS} = \widehat{RSP}$$

On subtracting  $\widehat{PS}$  from both sides, we get:

$$\widehat{QPS} - \widehat{PS} = \widehat{RSP} - \widehat{PS}$$

$$\Rightarrow \widehat{PQ} = \widehat{SR}$$

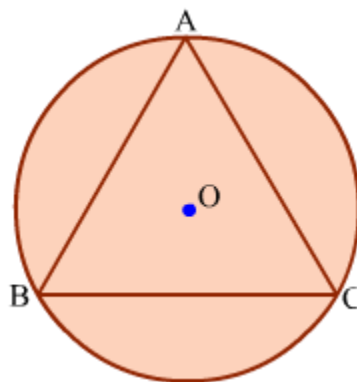
$$\Rightarrow \widehat{PQ} \cong \widehat{SR}$$

We know that if two arcs are congruent, then their corresponding chords are equal.

$$\therefore PQ = SR$$

### Medium

**Example 1:** An equilateral triangle ABC is inscribed in the given circle with centre O and radius 7 cm. Find the lengths of arcs AB, BC and CA.



### Solution:

It is given that  $\triangle ABC$  is equilateral.

$$\therefore AB = BC = CA$$

We know that if two chords of a circle are equal, then their corresponding arcs are congruent.

$$\therefore \widehat{AB} = \widehat{BC} = \widehat{CA}$$

Thus, points A, B and C divide the circle into three equal parts.

$$\Rightarrow \widehat{AB} = \widehat{BC} = \widehat{CA} = \frac{1}{3} \times \text{Circumference}$$

Now,

$$\text{Circumference} = 2\pi r$$

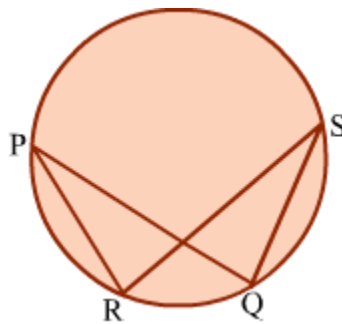
$$\Rightarrow \text{Circumference} = 2 \times \frac{22}{7} \times 7 \text{ cm}$$

$$\Rightarrow \text{Circumference} = 44 \text{ cm}$$

$$\Rightarrow \widehat{AB} = \widehat{BC} = \widehat{CA} = \frac{44}{3} \text{ cm}$$

$$\Rightarrow \widehat{AB} = \widehat{BC} = \widehat{CA} = 14\frac{2}{3} \text{ cm}$$

**Example 2:** If PQ and SR are two equal chords of the given circle, then show that PR and SQ are also equal chords.



**Solution:** It is given that PQ and SR are equal chords of the given circle. We know that if two chords of a circle are equal, then their corresponding arcs are congruent.

$$\therefore \widehat{PRQ} \cong \widehat{SQR}$$

$$\Rightarrow \widehat{PRQ} = \widehat{SQR}$$

On subtracting  $\widehat{RQ}$  from both sides, we get:



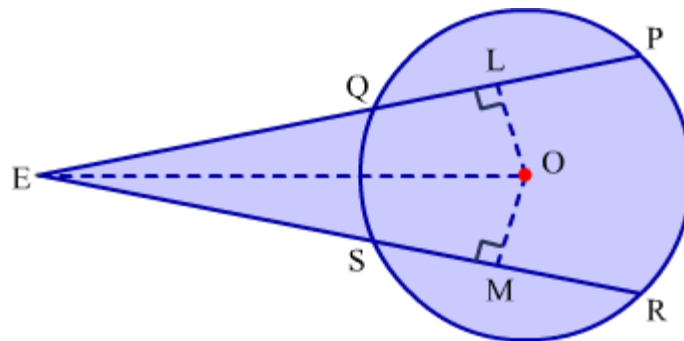
$$\widehat{PR} = \widehat{SQ}$$

By using the converse of the above property, we get:

$$PR = SQ$$

**Hard**

**Example 1:** In the given circle, arcs PQ and RS are congruent. Show that SE = QE.



**Solution:**

We know that equal chords are equidistant from the centre of a circle.

$$\therefore OL = OM$$

In  $\triangle OLE$  and  $\triangle OME$ , we have:

$$\angle OLE = \angle OME = 90^\circ \quad (\because OL \text{ and } OM \text{ are perpendiculars})$$

$$OL = OM \quad (\text{Proved above})$$

$$OE = OE \quad (\text{Common side})$$

$$\therefore \triangle OLE \cong \triangle OME \quad (\text{By the RHS congruence rule})$$

$$\Rightarrow LE = ME \dots (1) \quad [\text{By CPCT}]$$

We know that if two arcs are congruent, then their corresponding chords are equal. Since arcs PQ and RS are congruent, chords PQ and RS are equal.

$$\Rightarrow \frac{1}{2} PQ = \frac{1}{2} RS$$

$$\Rightarrow QL = SM \dots (2) \quad [\because \text{Perpendicular from centre to chord bisects chord}]$$

On subtracting equation (2) from equation (1), we get:

$$LE - QL = ME - SM$$

$$\Rightarrow QE = SE$$

### Angles Subtended by Congruent Arcs

The given figure shows five children playing ball in a circular park. They are positioned at points A, B, C, D and E. The child standing at point A is at the centre of the circle, while the others are at its circumference.

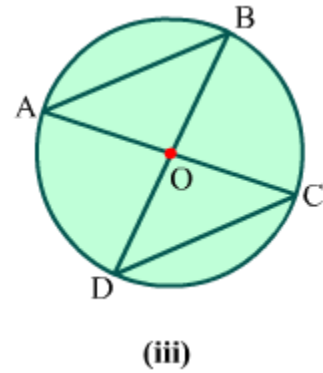
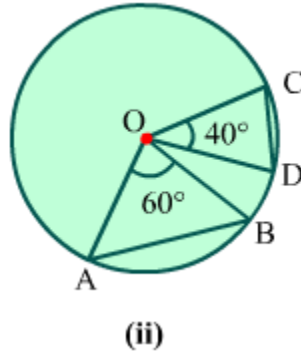
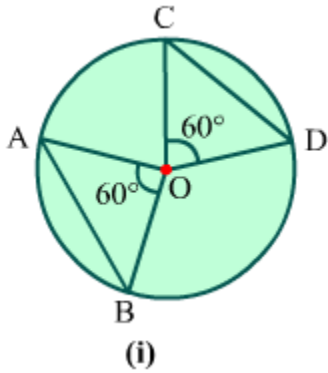


The lines joining the different points represent the paths followed by the ball during play. Note how paths BC and DE act as chords of the circle. If we assume that the distances BC and DE are equal, then the arcs corresponding to them will also be equal or congruent. Clearly, the angles subtended at centre A by arcs BC and DE are equal, i.e.,  $\angle BAC$  and  $\angle DAE$  are equal. By this, we can conclude that congruent arcs subtend equal angles at the centre of a circle. Let us understand this property and solve some problems based on it.

### Solved Examples

#### Easy

**Example 1:** For each figure, state whether or not the arcs AB and CD are equal.



**Solution:**

We know that arcs subtending equal angles at the centre of a circle are congruent.

**In figure (i):**

$$\angle AOB = \angle COD = 60^\circ$$

$$\Rightarrow \text{Arc AB} = \text{Arc CD}$$

**In figure (ii):**

$$\angle AOB = 60^\circ \text{ and } \angle COD = 40^\circ$$

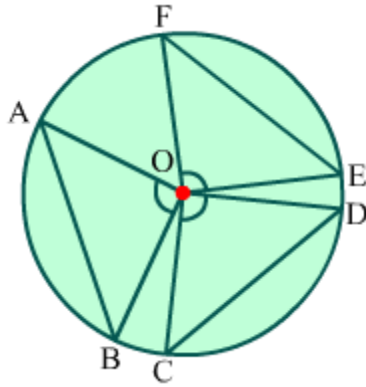
$$\Rightarrow \text{Arc AB} \neq \text{Arc CD}$$

**In figure (iii):**

$$\angle AOB = \angle COD \text{ (Vertically opposite angles)}$$

$$\Rightarrow \text{Arc AB} = \text{Arc CD}$$

**Example 2:** In the given circle with centre O, arcs AB and CD are equal and arcs AB and EF are equal. If  $\angle AOB = 55^\circ$ , then find the measure of  $\angle FOE$  and  $\angle COD$ .



**Solution:**

It is given that:

$$\text{Arc AB} = \text{Arc CD}$$

$$\text{Arc AB} = \text{Arc EF}$$

$$\therefore \text{Arc CD} = \text{Arc EF}$$

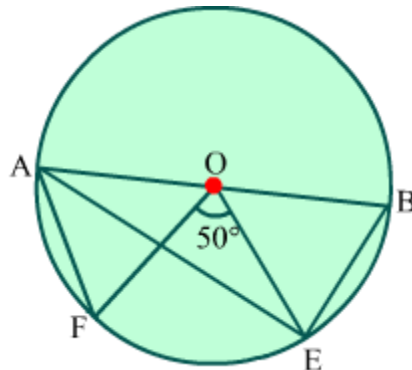
We know that congruent arcs subtend equal angles at the centre of a circle.

$$\therefore \angle AOB = \angle COD = \angle EOF$$

$$\text{Now, } \angle AOB = 55^\circ$$

$$\therefore \angle COD = \angle EOF = 55^\circ$$

**Example 3:** In the given circle with centre O,  $AF = BE$  and  $\angle FOE = 50^\circ$ . Find the measure of  $\angle AOF$ .



**Solution:**

It is given that:

Chord AF = Chord BE

We know that congruent arcs subtend equal angles at the centre of a circle.

$$\therefore \angle AOF = \angle BOE \dots (1)$$

AOB is a straight line; therefore, we have:

$$\angle AOF + \angle FOE + \angle BOE = 180^\circ$$

$$\Rightarrow 2\angle AOF + 50^\circ = 180^\circ \text{ (By equation 1)}$$

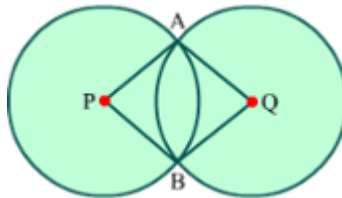
$$\Rightarrow 2\angle AOF = 180^\circ - 50^\circ$$

$$\Rightarrow 2\angle AOF = 130^\circ$$

$$\Rightarrow \angle AOF = 65^\circ$$

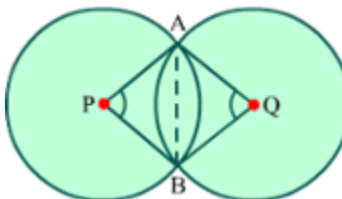
### Medium

**Example 1:** The given figure shows two congruent circles with centres P and Q, and intersecting each other at points A and B. Show that  $\angle APB = \angle AQB$ .



**Solution:**

**Construction:** Draw a chord AB that is common to the given circles.



We know that if two chords of congruent circles are equal, then their corresponding arcs are congruent.

We have AB as the common chord of the two given circles. Therefore, the length of arc AB is the same in both circles.

We know that congruent arcs subtend equal angles at the centre of a circle.

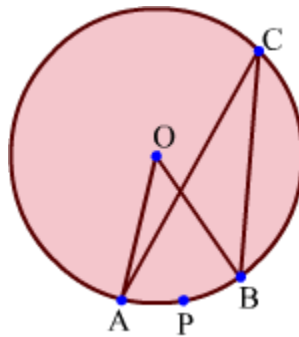
$$\therefore \angle APB = \angle AQB$$

### **Relation between Angles Subtended by an Arc at The Centre and Anywhere on the Circle**

#### **Observing the Angles Subtended by an Arc at the Centre and on the Circle**

We know that an infinite number of points lie on the circumference of a circle. The portion of circumference between any two such points is known as an **arc**. Every arc subtends an angle at the centre and a particular angle at any point on the circle.

Let us consider any angle  $\angle ACB$  inscribed in the major arc ACB of a circle having centre at point O as shown below.



It can be seen that the arc APB is intercepted by  $\angle ACB$ .

Also, the arc APB subtends  $\angle AOB$  at the centre. Thus,  $\angle AOB$  is the measure of arc APB.

In other words,  $\angle AOB$  and  $\angle ACB$  are subtended by the same arc APB at the centre O and at any point C on the circle respectively.

There is a relation between  $\angle AOB$  (measure of intercepted arc) and  $\angle ACB$  (inscribed angle).

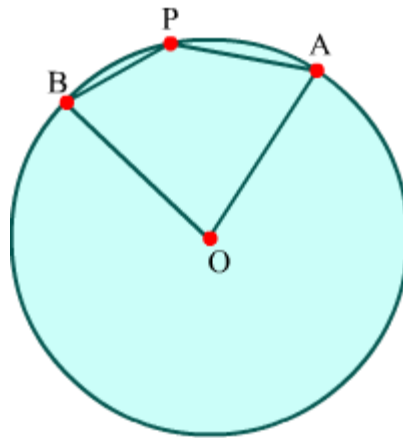
In this lesson, we will learn the theorem defining the relation between these two types of angles. We will also solve some examples related to the same.

#### **Know More**

The relation between the angles subtended by an arc at the centre and on the circumference of a circle is known as the **central angle theorem**.

This relation holds true only when the inscribed angle (i.e., the angle subtended at the circumference) is in the major arc. If, however, the inscribed angle is in the minor arc (as is  $\angle BPA$  in the following figure), then its relation with the central angle (i.e., the angle at the centre) is given by the formula:

$$\text{Inscribed angle} = 180^\circ - \left( \frac{\text{Central angle}}{2} \right)$$

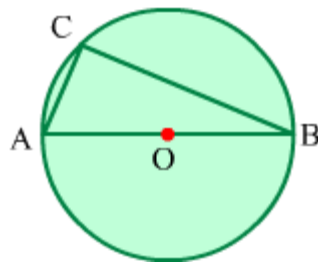


### Angle in a Semicircle is a Right Angle

**Statement:** Angle in a semicircle is a right angle.

**Given:** A circle with centre O and diameter AB

**To prove:**  $\angle ACB = 90^\circ$



**Proof:** We know that the angle subtended by an arc at the centre of a circle is twice the angle subtended by it at the circumference of the circle.

$$\therefore \angle AOB = 2\angle ACB$$

$$\Rightarrow 2\angle ACB = 180^\circ (\because AOB \text{ is a straight line})$$

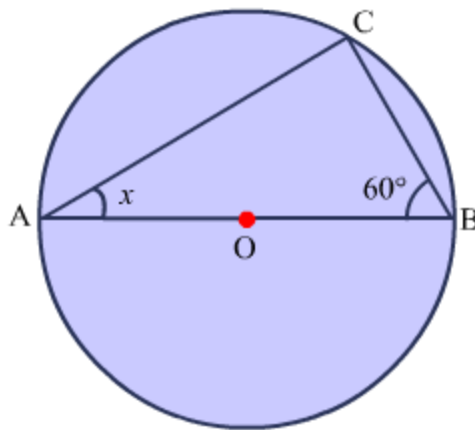
$$\Rightarrow \angle ACB = 90^\circ$$

Now, AB is the diameter of the circle and it divides the circle into two semicircles.  $\angle ACB$  is inscribed in the semicircle. Hence, an angle in a semicircle is a right angle.

### Solved Examples

#### Easy

**Example 1:** Find the value of  $x$  in the given circle with centre O and diameter AB.



#### Solution:

We know that an angle in a semicircle is a right angle.

$$\therefore \angle ACB = 90^\circ (\because AB \text{ is the diameter of the circle})$$

On using the angle sum property in  $\triangle ACB$ , we get:

$$\angle ACB + \angle CBA + \angle BAC = 180^\circ$$

$$\Rightarrow 90^\circ + 60^\circ + x = 180^\circ$$

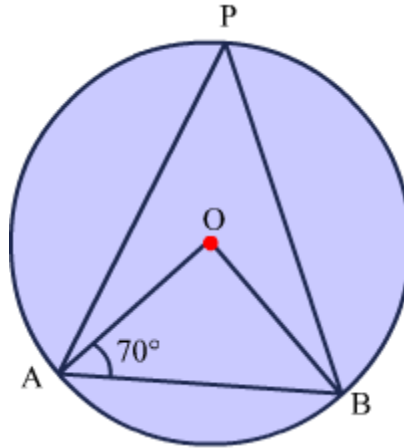
$$\Rightarrow 150^\circ + x = 180^\circ$$

$$\Rightarrow x = 180^\circ - 150^\circ$$

$$\Rightarrow x = 30^\circ$$

**Example 2:** Find the measure of  $\angle APB$  in the given circle.





**Solution:**

In  $\triangle OAB$ , we have:

$OA = OB$  (Radii of the circle)

$\Rightarrow \angle OBA = \angle OAB = 70^\circ$  ( $\because$  Angles opposite equal sides are equal)

Using the angle sum property in  $\triangle OAB$ , we get:

$$\angle OAB + \angle OBA + \angle AOB = 180^\circ$$

$$\Rightarrow 70^\circ + 70^\circ + \angle AOB = 180^\circ$$

$$\Rightarrow 140^\circ + \angle AOB = 180^\circ$$

$$\Rightarrow \angle AOB = 180^\circ - 140^\circ$$

$$\Rightarrow \angle AOB = 40^\circ$$

We know that the angle subtended by an arc at the centre of a circle is double the angle subtended by it at the circumference of the circle. In the given circle, arc AB subtends  $\angle AOB$  at the centre and  $\angle APB$  at the circumference.

So,  $\angle AOB = 2\angle APB$

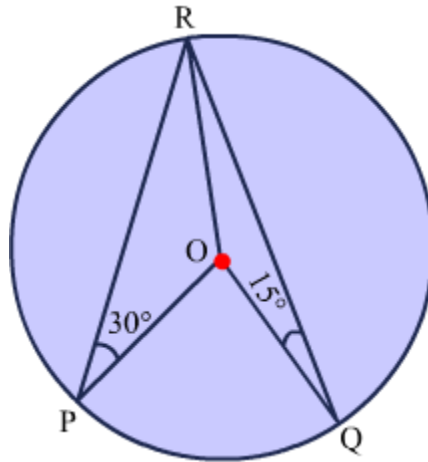
$$\Rightarrow 40^\circ = 2\angle APB$$

$$\Rightarrow \angle APB = \frac{40^\circ}{2}$$

$$\Rightarrow \angle APB = 20^\circ$$

### Medium

**Example 1:** Find the measure of  $\angle POQ$  in the given circle.



### Solution:

In the given circle,  $\angle OPR = 30^\circ$  and  $\angle OQR = 15^\circ$ .

In  $\triangle OPR$ , we have:

$OP = OR$  (Radii of the circle)

$\therefore \angle ORP = \angle OPR = 30^\circ$  ( $\because$  **Angles opposite equal sides are equal**)

Similarly, we can find that  $\angle ORQ = \angle OQR = 15^\circ$

Now,  $\angle PRQ = \angle ORP + \angle ORQ$

$$\therefore \angle PRQ = 30^\circ + 15^\circ$$

$$\therefore \angle PRQ = 45^\circ$$

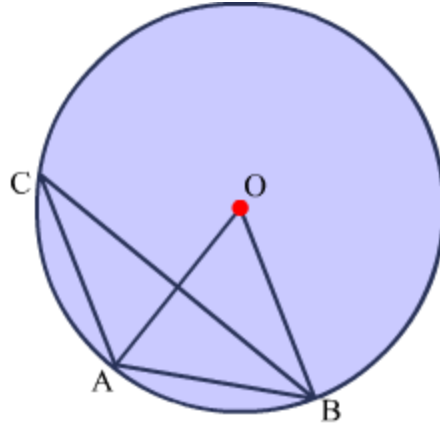
We know that the angle subtended by an arc at the centre of a circle is double the angle subtended by it at the circumference of the circle. In the given circle, arc PQ subtends  $\angle POQ$  at the centre and  $\angle PRQ$  at the circumference.

$$\text{So, } \angle POQ = 2\angle PRQ$$

$$\Rightarrow \angle POQ = 2 \times 45^\circ$$

$$\Rightarrow \angle POQ = 90^\circ$$

**Example 2:** In the given circle with centre O, chord AB is equal to the radius of the circle. Find the measure of  $\angle ACB$ .



**Solution:**

It is given that chord AB is equal to the radius of the circle.

So,  $AB = OA = OB$  ( $\because$  OA and OB are radii of the circle)

Thus,  $\triangle OAB$  is equilateral.

$\therefore \angle AOB = 60^\circ$  ( $\because$  Each angle of an equilateral triangle measures  $60^\circ$ )

We know that the angle subtended by an arc at the centre of a circle is double the angle subtended by it at the circumference of the circle. In the given circle, arc AB subtends  $\angle AOB$  at the centre and  $\angle ACB$  at the circumference.

So,  $\angle AOB = 2\angle ACB$

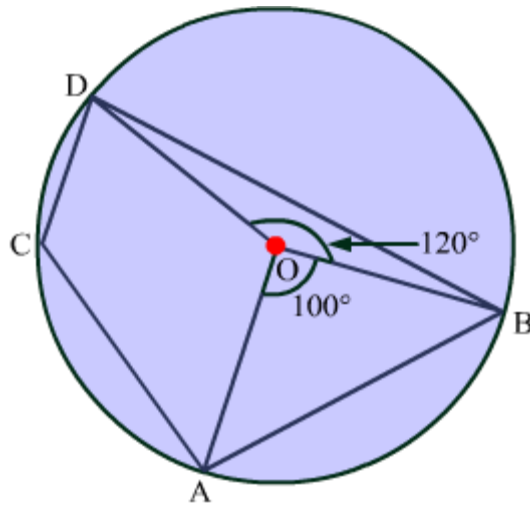
$$\Rightarrow 60^\circ = 2\angle ACB$$

$$\Rightarrow \angle ACB = \frac{60^\circ}{2}$$

$$\Rightarrow \angle ACB = 30^\circ$$

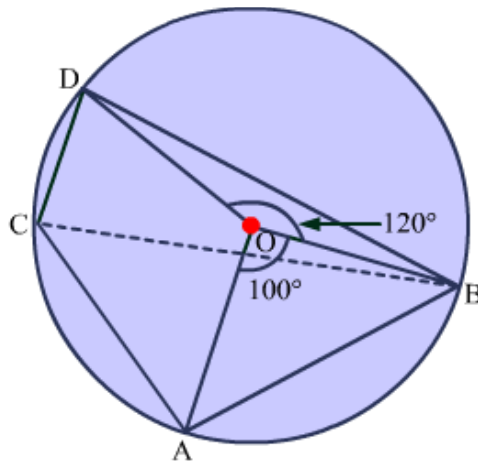
**Hard**

**Example 1:** Find the measure of  $\angle ACD$  in the given circle.



**Solution:**

**Construction:** Join B to C.



We know that the angle subtended by an arc at the centre of a circle is double the angle subtended by it at the circumference of the circle.

In the given circle, arc AB subtends  $\angle AOB$  at the centre and  $\angle ACB$  at the circumference.

$$\text{So, } \angle AOB = 2\angle ACB$$

$$\Rightarrow 100^\circ = 2\angle ACB$$

$$\Rightarrow \angle ACB = \frac{100^\circ}{2}$$

$$\Rightarrow \angle ACB = 50^\circ$$

Also, arc BD subtends  $\angle BOD$  at the centre and  $\angle BCD$  at the circumference.

$$\text{So, } \angle BOD = 2\angle BCD$$

$$\Rightarrow 120^\circ = 2\angle BCD$$

$$\Rightarrow \angle BCD = \frac{120^\circ}{2}$$

$$\Rightarrow \angle BCD = 60^\circ$$

$$\text{Now, } \angle ACD = \angle ACB + \angle BCD$$

$$\Rightarrow \angle ACD = 50^\circ + 60^\circ$$

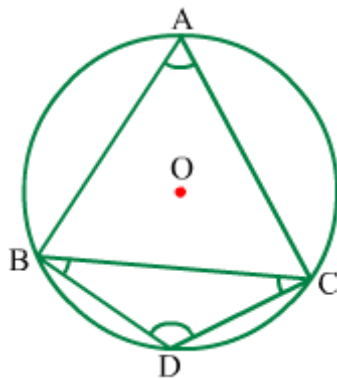
$$\Rightarrow \angle ACD = 110^\circ$$

Angles in The Same Segment of a Circle

### Angles in the Major and Minor Segments

We know that the chord of a circle divides it into two regions. These regions are called **segments of the circle** and are classified as the **major segment** and the **minor segment**.

Observe the given circle.



In this circle,  $\angle BAC$  lies in the major segment whereas  $\angle BDC$  lies in the minor segment. It can be seen that  $\angle BAC$  is an acute angle while  $\angle BDC$  is an obtuse angle.

So, it can be concluded that the angle lying in the major segment is an acute angle and the angle lying in the minor segment is an obtuse angle. This statement is true for all major and minor segments in a circle.

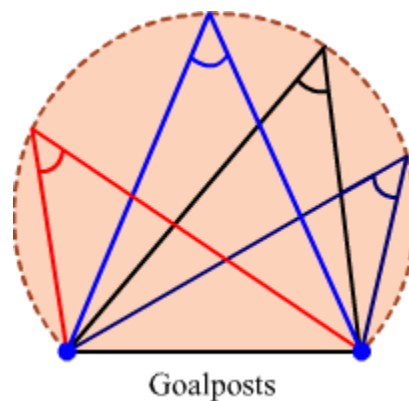
**There is no relation between angles in different segments, but what about the angles in the same segment?**

In this lesson, we will learn about the angles in the same segment of a circle and the relation between them. We will also solve some examples dealing with the same.

**Did You Know?**

**Angle for scoring a goal in soccer**

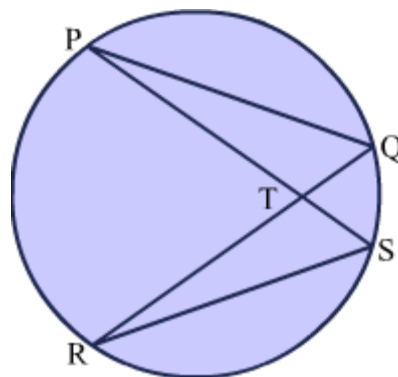
The angle of every possible shot to score a goal is constant for all positions on the same arc of a circle; however, the distance of a shot changes with change in position.



**Solved Examples**

**Easy**

**Example 1:** In the given circle, chords PQ and RS are equal and chords PS and QR intersect at point T. Show that  $PT = RT$  and  $TQ = TS$ .



**Solution:**

In  $\Delta PQT$  and  $\Delta RST$ , we have:

$$PQ = RS \text{ (Given)}$$

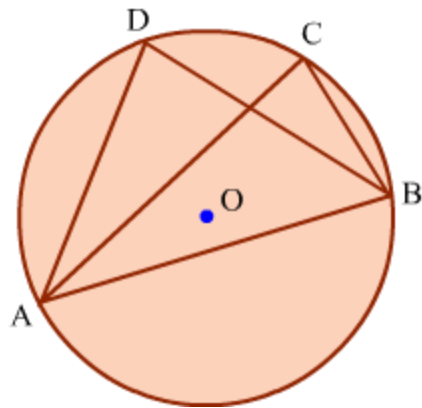
$$\angle TPQ = \angle TRS (\because \text{Angles in the same segment of a circle are equal})$$

$$\text{Similarly, } \angle TQP = \angle TSR$$

$$\therefore \Delta PQT \cong \Delta RST \text{ (By the ASA congruence criterion)}$$

$$\Rightarrow PT = RT \text{ and } TQ = TS \text{ (By CPCT)}$$

**Example 2:** In the given circle, find the value of  $\angle DAB$  if  $\angle BCA = 80^\circ$  and  $DA = DB$ .



**Solution:**

From the figure, we have:

$$\angle BCA = \angle BDA = 80^\circ (\because \text{Angles in the same segment of a circle are equal})$$

$$DA = DB \text{ (Given)}$$

$$\Rightarrow \angle DBA = \angle DAB \dots (1) [\because \text{Angles opposite equal sides are equal}]$$

On using the angle sum property in  $\Delta ADB$ , we get:

$$\angle DAB + \angle DBA + \angle BDA = 180^\circ$$

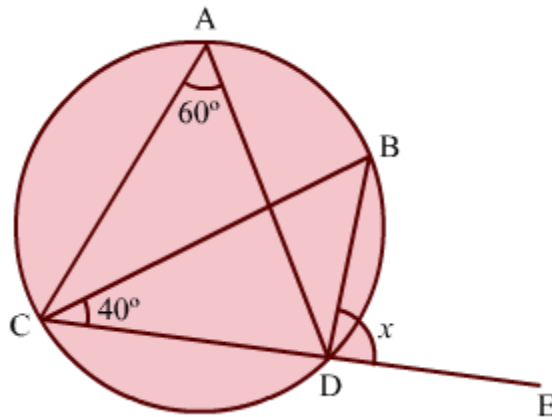
$$\Rightarrow 2\angle DAB + 80^\circ = 180^\circ \text{ (By equation 1)}$$

$$\Rightarrow 2\angle DAB = 180^\circ - 80^\circ$$

$$\Rightarrow 2\angle DAB = 100^\circ$$

$$\Rightarrow \angle DAB = 50^\circ$$

**Example 3:** What is the value of  $x$  in the given figure?



**Solution:**

We know that angles in the same segment are equal.

$$\therefore \angle CAD = \angle CBD = 60^\circ$$

Now,  $\angle BDE$  is an exterior angle of  $\triangle BCD$ .

So,  $\angle BDE = \angle CBD + \angle DCB$  ( $\because$  Exterior angle equals sum of interior opposite angles)

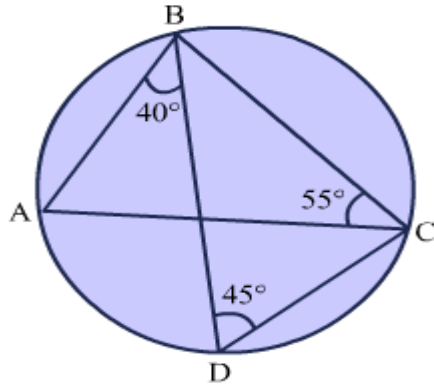
$$\Rightarrow x = 60^\circ + 40^\circ$$

$$\Rightarrow x = 100^\circ$$

**Medium**

**Example 1:** What are the measures of  $\angle BAC$ ,  $\angle ACD$ ,  $\angle ABC$ , and  $\angle DBC$ ?





**Solution:**

We know that angles in the same segment are equal.

So,  $\angle BAC = \angle BDC = 45^\circ$

Similarly,  $\angle ABD = \angle ACD = 40^\circ$

On using the angle sum property in  $\triangle ABC$ , we get:

$$\angle ABC + \angle BAC + \angle ACB = 180^\circ$$

$$\Rightarrow \angle ABC + 45^\circ + 55^\circ = 180^\circ$$

$$\Rightarrow \angle ABC + 100^\circ = 180^\circ$$

$$\Rightarrow \angle ABC = 180^\circ - 100^\circ$$

$$\Rightarrow \angle ABC = 80^\circ$$

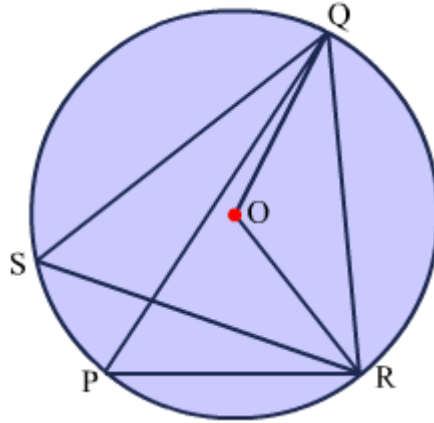
$$\text{Now, } \angle ABC = \angle ABD + \angle DBC$$

$$\Rightarrow 80^\circ = 40^\circ + \angle DBC$$

$$\Rightarrow \angle DBC = 80^\circ - 40^\circ$$

$$\Rightarrow \angle DBC = 40^\circ$$

**Example 2:** In the given circle with centre O,  $\angle PQR = 37^\circ$  and  $\angle QRP = 83^\circ$ . What are the measures of  $\angle RSQ$  and  $\angle ROQ$ ?



**Solution:**

On using the angle sum property in  $\Delta PQR$ , we get:

$$\angle PQR + \angle QRP + \angle RPQ = 180^\circ$$

$$\Rightarrow 37^\circ + 83^\circ + \angle RPQ = 180^\circ$$

$$\Rightarrow 120^\circ + \angle RPQ = 180^\circ$$

$$\Rightarrow \angle RPQ = 180^\circ - 120^\circ$$

$$\Rightarrow \angle RPQ = 60^\circ$$

We know that angles in the same segment are equal.

$$\therefore \angle RPQ = \angle RSQ = 60^\circ$$

We also know that the angle subtended by an arc at the centre of a circle is double the angle subtended by it at the circumference of the circle.

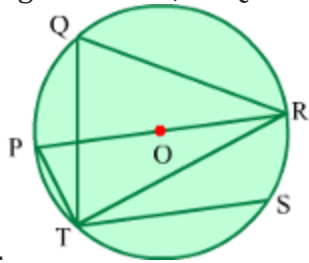
$$\text{So, } \angle ROQ = 2\angle RSQ$$

$$\Rightarrow \angle ROQ = 2 \times 60^\circ$$

$$\Rightarrow \angle ROQ = 120^\circ$$

**Hard**

**Example 1:** In the given circle,  $\angle TQR = 70^\circ$  and PR is the diameter. If  $TS \parallel PR$ , then find the



measure of  $\angle STR$ .

**Solution:**

We know that angles in the same segment are equal.

$$\therefore \angle TQR = \angle TPR = 70^\circ$$

We also know that an angle in a semicircle is a right angle.

$$\therefore \angle RTP = 90^\circ (\because PR \text{ is the diameter})$$

On using the angle sum property in  $\triangle RTP$ , we obtain:

$$\angle TPR + \angle RTP + \angle PRT = 180^\circ$$

$$\Rightarrow 70^\circ + 90^\circ + \angle PRT = 180^\circ$$

$$\Rightarrow 160^\circ + \angle PRT = 180^\circ$$

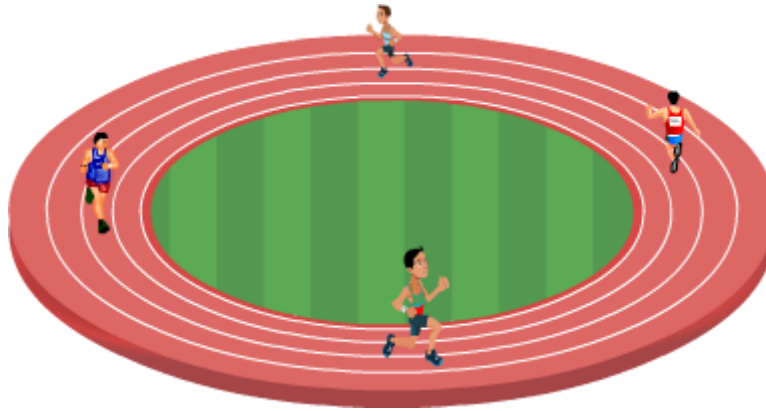
$$\Rightarrow \angle PRT = 20^\circ$$

Now,  $PR \parallel TS$  and  $RT$  is the transversal.

$$\text{So, } \angle STR = \angle PRT = 20^\circ (\because \text{Alternate angles are equal})$$

### Concyclic Points

You must have seen circular running tracks like the one shown below. You can see concentric circles that divide the track into different paths. Four runners can be seen at different positions on the track.



Note how the runners are on the circumference of the same circle on the track. Things which lie on the same circle are referred to as concyclic; so, these runners are also concyclic. Let us go through this lesson to know more about concyclic points and the theorem related to them.

### Know More

A set of more than four points is concyclic if and only if every four-point subset is concyclic. This property is the analogue of concyclicity.

### Whiz Kid

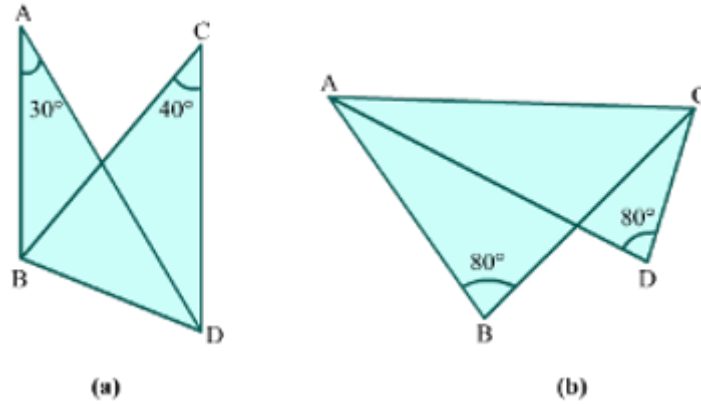
#### General condition for concyclicity

If  $n$  distinct points lie on a circle and we join any two points, then  $\frac{n(n-1)}{2}$  perpendicular bisectors of the line segments should meet at a single point that is the centre of the circle.

### Solved Examples

#### Easy

**Example 1:** For each figure, state whether or not the points A, B, C and D are concyclic.



**Solution:**

**In figure (a):**

Points A and C are present on the same side of the line joining points B and D.

We have  $\angle BAD = 30^\circ$  and  $\angle BCD = 40^\circ$ .

Clearly, line segment BD does not subtend equal angles at points A and C that lie on the same side of BD. Therefore, points A, B, C and D do not lie on a circle, i.e., they are not concyclic.

**In figure (b):**

Points B and D are present on the same side of the line joining points C and A.

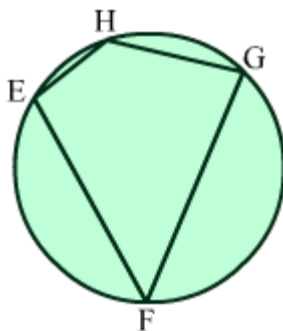
We have  $\angle ABC = \angle ADC = 80^\circ$ .

Clearly, line segment CA subtends equal angles at points B and D that lie on the same side of CA. Therefore, points A, B, C and D lie on a circle, i.e., they are concyclic.

## **Properties of Cyclic Quadrilaterals**

### **Cyclic Quadrilaterals**

We know that points lying on the same circle are called concyclic points. Let us consider four concyclic points, say E, F, G and H, and the circle passing through them. If we join the four points, then we get a quadrilateral as is shown in the figure below.



A quadrilateral whose vertices lie on a circle or through whose vertices it is possible to draw a circle is known as a **cyclic quadrilateral**. In the given figure, the vertices E, F, G and H lie on a circle; hence, EFGH is a cyclic quadrilateral. The circle on which the quadrilateral lies is called a circumcircle.

Cyclic quadrilaterals are a little different from regular quadrilaterals as they exhibit a few special properties. In this lesson, we will discuss these properties of cyclic quadrilaterals and solve some problems based on them.

### Did You Know?

If a cyclic quadrilateral has unequal rational sides in either arithmetic or geometric progression, then there does not exist any cyclic quadrilateral with rational area.

### Know More

- A cyclic quadrilateral is also called **chordal quadrilateral** because the sides of the quadrilateral are chords of the circumcircle. Another name for this quadrilateral is **concyclic quadrilateral**.
- If the opposite sides of a cyclic quadrilateral are extended to meet, say at points E and F, then the internal angle bisectors of the angles formed at points E and F are perpendicular.
- The opposite sides and the diagonals of a cyclic quadrilateral ABCD are related as:  $AC \cdot BD = AD \cdot BC + AB \cdot CD$ . This relationship is known as **Ptolemy's theorem**.
- The area of a cyclic quadrilateral is  $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ , where  $a, b, c$  and  $d$  are the lengths of the sides of the cyclic quadrilateral and  $s = \frac{a+b+c+d}{2}$ .

### Whiz Kid

- In a cyclic quadrilateral ABCD with circumcentre O, if the diagonals AC and BD intersect at point P, then  $\angle APB$  is the arithmetic mean of  $\angle AOB$  and  $\angle COD$ .
- Four line segments are concurrent if each is perpendicular to one side of a cyclic quadrilateral and passes through the midpoint of the opposite side. These line segments are called **multitudes**, which means 'midpoint altitudes'.

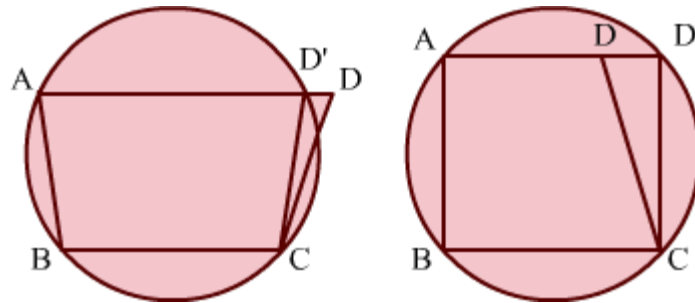
### Proving the Converse of Property

**Statement:** If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

**Given:** A quadrilateral ABCD with  $\angle ABC + \angle ADC = 180^\circ$  and  $\angle BAD + \angle BCD = 180^\circ$

**To prove:** ABCD is a cyclic quadrilateral.

**Proof:** Let us assume that ABCD is not a cyclic quadrilateral. Suppose a circle passes through the three non-collinear points A, B and C and meets AD or AD produced, at D'.



Now, on joining D' to C, we get the cyclic quadrilateral ABCD'.

In ABCD', we have:

$\therefore \angle ABC + \angle AD'C = 180^\circ$  ( $\because$  Opposite angles of a cyclic quadrilateral are supplementary)

But  $\angle ABC + \angle ADC = 180^\circ$  (Given)

$\therefore \angle AD'C = \angle ADC$ , which can be possible only if D and D' coincide

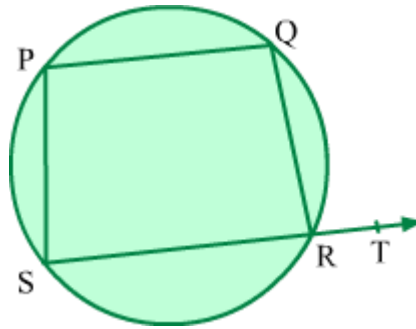
Thus, the circle passing through points A, B and C also passes through point D. Therefore, ABCD is a cyclic quadrilateral.

**Proving that the Exterior Angle of a Cyclic Quadrilateral Is Equal to the Interior Opposite Angle**

**Statement:** The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

**Given:** A cyclic quadrilateral PQRS with side SR extended up to point T

**To prove:**  $\angle QRT = \angle QPS$ .



**Proof:** We know that the opposite angles of a cyclic quadrilateral are supplementary.

$$\therefore \angle QPS + \angle QRS = 180^\circ \dots (1)$$

$$\text{Also, } \angle QRT + \angle QRS = 180^\circ \dots (2) \text{ [Linear pair of angles]}$$

From equations (1) and (2), we obtain:

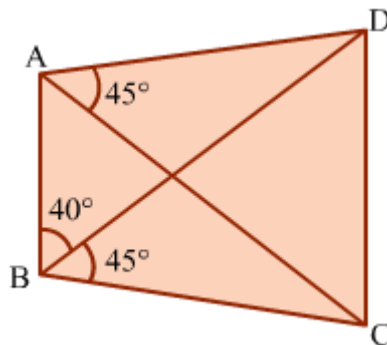
$$\angle QPS + \angle QRS = \angle QRT + \angle QRS$$

$$\Rightarrow \angle QPS = \angle QRT$$

### Solved Examples

#### Easy

**Example 1:** What is the measure of  $\angle ADC$  in the given figure?





**Solution:**

In the figure,  $\angle CAD = \angle CBD = 45^\circ$

We know that if a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, then the four points are concyclic.

Therefore, A, B, C and D are concyclic points and ABCD is a cyclic quadrilateral.

We know that in a cyclic quadrilateral, opposite angles are supplementary.

So,  $\angle ABC + \angle ADC = 180^\circ$

$$\Rightarrow (40^\circ + 45^\circ) + \angle ADC = 180^\circ$$

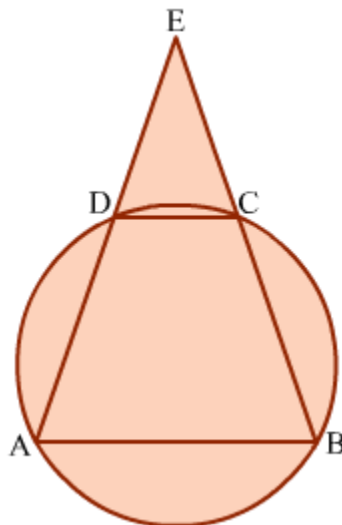
$$\Rightarrow 85^\circ + \angle ADC = 180^\circ$$

$$\Rightarrow \angle ADC = 180^\circ - 85^\circ$$

$$\Rightarrow \angle ADC = 95^\circ$$

**Example 2:** In  $\triangle AEB$ ,  $AE = BE$ . A circle passing through points A and B intersects AE and BE at points D and C respectively. Prove that the line segment DC is parallel to AB.

**Solution:** The figure for the given problem can be drawn as is shown.



In  $\triangle AEB$ , we have:

$AE = BE$  (Given)

$$\Rightarrow \angle EBA = \angle EAB \dots (1) \quad [\because \text{Angles opposite equal sides of a triangle are equal}]$$

Now, ABCD lies on a circle; so, it is a cyclic quadrilateral. We know that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

$$\text{So, } \angle EDC = \angle CBA$$

$$\Rightarrow \angle EDC = \angle EBA \dots (2) \quad [\because \angle CBA = \angle EBA]$$

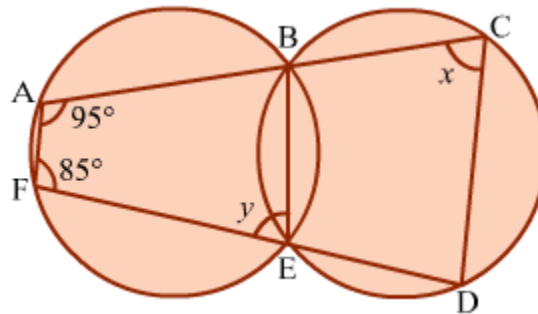
From equations (1) and (2), we obtain:

$$\angle EDC = \angle EAB$$

We can see that line segments DC and AB are cut by the transversal EA.  $\angle EDC$  and  $\angle EAB$  are equal corresponding angles. Therefore, by the converse of the corresponding angles axiom, we can say that DC is parallel to AB.

### Medium

**Example 1:** Find the values of  $x$  and  $y$  in the given figure.



### Solution:

In the figure, we have two cyclic quadrilaterals ABEF and BCDE.

In ABEF, we have:

$$\angle BAF + \angle BEF = 180^\circ \quad (\because \text{Opposite angles of a cyclic quadrilateral are supplementary})$$

$$\Rightarrow 95^\circ + y = 180^\circ$$

$$\Rightarrow y = 180^\circ - 95^\circ$$

$$\Rightarrow y = 85^\circ$$

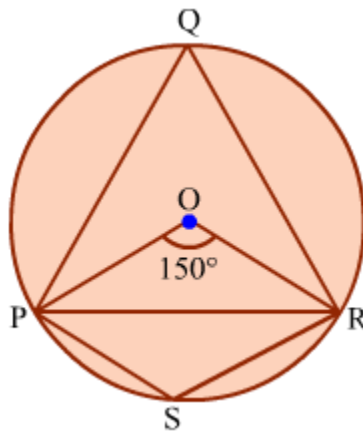
In BCDE, we have:

$\angle BEF = \angle BCD$  ( $\because$  Exterior angle of a cyclic quadrilateral equals interior opposite angle)

$$\Rightarrow y = x$$

$$\Rightarrow x = 85^\circ$$

**Example 2:** What is the measure of  $\angle PSR$  in the given figure?



**Solution:** We know that the angle subtended by an arc at the centre of a circle is double the angle subtended by it at the circumference of the circle.

$$\text{So, } \angle POR = 2\angle PQR$$

$$\Rightarrow \angle PQR = \frac{1}{2} \angle POR$$

$$\Rightarrow \angle PQR = \frac{1}{2} \times 150^\circ$$

$$\Rightarrow \angle PQR = 75^\circ \dots (1)$$

Now, quadrilateral PQRS is cyclic.

So,  $\angle PQR + \angle PSR = 180^\circ$  ( $\because$  Opposite angles of a cyclic quadrilateral are supplementary)

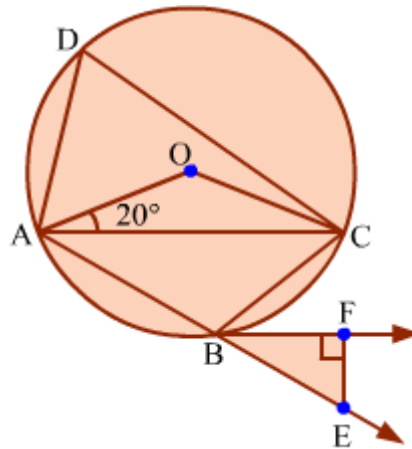
$$\Rightarrow 75^\circ + \angle PSR = 180^\circ \text{ (By equation 1)}$$

$$\Rightarrow \angle PSR = 180^\circ - 75^\circ$$

$$\Rightarrow \angle PSR = 105^\circ$$

**Hard**

**Example 1:** In the given figure, find the value of  $\angle BEF$  if  $BF$  is the bisector of  $\angle CBE$ .



**Solution:**

In  $\triangle OAC$ , we have:

$$OA = OC \text{ (Radii of the circle)}$$

$$\Rightarrow \angle OCA = \angle OAC = 20^\circ \text{ (}\because \text{Angles opposite equal sides of a triangle are equal)}$$

On using the angle sum property in  $\triangle OAC$ , we obtain:

$$\angle AOC + \angle OAC + \angle OCA = 180^\circ$$

$$\Rightarrow \angle AOC + 20^\circ + 20^\circ = 180^\circ$$

$$\Rightarrow \angle AOC + 40^\circ = 180^\circ$$

$$\Rightarrow \angle AOC = 180^\circ - 40^\circ$$

$$\Rightarrow \angle AOC = 140^\circ$$

We know that the angle subtended by an arc at the centre of a circle is double the angle subtended by it at the circumference of the circle.

$$\text{So, } \angle AOC = 2\angle ADC$$

$$\Rightarrow \angle ADC = \frac{1}{2} \angle AOC$$

$$\Rightarrow \angle ADC = \frac{1}{2} \times 140^\circ$$

$$\Rightarrow \angle ADC = 70^\circ$$

Since ABCD is a cyclic quadrilateral, we have:

$$\angle CBE = \angle ADC \quad (\because \text{Exterior angle of a cyclic quadrilateral equals interior opposite angle})$$

$$\Rightarrow \angle CBE = 70^\circ$$

It is given that BF bisects  $\angle CBE$ .

$$\text{So, } \angle EBF = \frac{1}{2} \angle CBE$$

$$\Rightarrow \angle EBF = \frac{1}{2} \times 70^\circ$$

$$\Rightarrow \angle EBF = 35^\circ$$

On using the angle sum property in  $\triangle BEF$ , we obtain:

$$\angle EBF + \angle BEF + \angle BFE = 180^\circ$$

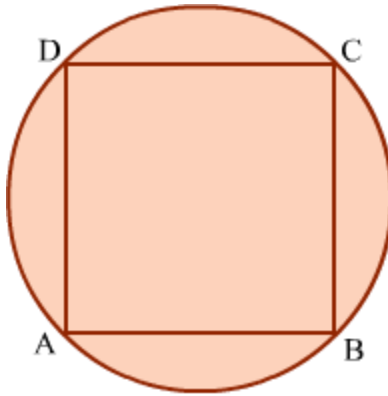
$$\Rightarrow 35^\circ + \angle BEF + 90^\circ = 180^\circ$$

$$\Rightarrow \angle BEF + 125^\circ = 180^\circ$$

$$\Rightarrow \angle BEF = 180^\circ - 125^\circ$$

$$\Rightarrow \angle BEF = 55^\circ$$

**Example 2:** If points A, B, C and D divide the circumference of the given circle into four equal parts, then show that ABCD is a square.



**Solution:**

It is given that points A, B, C and D divide the circle into four equal parts.

$$\therefore \text{Arc AB} = \text{Arc BC} = \text{Arc CD} = \text{Arc DA}$$

We know that if the arcs in a circle are congruent, then their corresponding chords are equal.

$$\therefore \text{Chord AB} = \text{Chord BC} = \text{Chord CD} = \text{Chord DA}$$

Thus, all sides of quadrilateral ABCD are equal in length. Therefore, ABCD is a rhombus.

Points A, B, C and D are concyclic; so, ABCD is a cyclic quadrilateral.

Now, we know that in a cyclic quadrilateral, opposite angles are supplementary.

$$\text{So, } \angle \text{BAD} + \angle \text{BCD} = 180^\circ$$

$$\text{But } \angle \text{BAD} = \angle \text{BCD} (\because \text{Opposite angles of a rhombus are equal})$$

$$\Rightarrow 2\angle \text{BAD} = 180^\circ$$

$$\Rightarrow \angle \text{BAD} = 90^\circ$$

Similarly,  $\angle \text{ABC}$ ,  $\angle \text{BCD}$  and  $\angle \text{CDA}$  measure  $90^\circ$ .

Hence, ABCD is a square.