DEFINITE INTEGRALS (XII, R. S. AGGARWAL)

EXERCISE 16A (Pg. No.: 799)

Very-Short-Answer Questions

Evaluate :

1. $\int_1^3 x^4\,dx$

Sol. Let $I=\int_1^3 x^4\,dx \Rightarrow I=\left[\frac{x^5}{5}\right]_1^3 \Rightarrow I=\frac{1}{5}\left[(3)^5-(1)^5\right]$

$\Rightarrow I=\frac{1}{5}(243-1) \Rightarrow I=\frac{1}{5}\times 242 \quad \therefore I=\frac{242}{5}$

2. $\int_1^4 \sqrt{x}\,dx$

Sol. Let $I=\int_1^4 \sqrt{x}\,dx \Rightarrow I=\int_1^4 (x)^{1/2}\,dx \Rightarrow I=\left[\frac{x^{3/2}}{3/2}\right]_1^4 \Rightarrow I=\frac{2}{3}\left[x^{3/2}\right]_1^4$

$\Rightarrow I=\frac{2}{3}\left[(4)^{3/2}-(1)^{3/2}\right] \Rightarrow I=\frac{2}{3}\left[\left(2^2\right)^{3/2}-\left(1^2\right)^{3/2}\right] \Rightarrow I=\frac{2}{3}\left[(2)^3-1\right]$

$\Rightarrow I=\frac{2}{3}(8-1) \Rightarrow I=\frac{2}{3}\times 7 \quad \therefore I=\frac{14}{3}$

3. $\int_1^2 x^{-5}\,dx$

Sol. Let $I=\int_1^2 x^{-5}\,dx \Rightarrow I=\left[\frac{x^{-4}}{-4}\right]_1^2 \Rightarrow I=\frac{-1}{4}\left[\frac{1}{(2)^4}-\frac{1}{(1)^4}\right]$

$\Rightarrow I=\frac{-1}{4}\left[\frac{1}{16}-\frac{1}{1}\right] \Rightarrow I=\frac{-1}{4}\left[\frac{-15}{16}\right] \quad \therefore I=\frac{15}{64}$

4. $\int_0^{16} x^{3/4}\,dx$

Sol. Let $I=\int_0^{16} x^{3/4}\,dx \Rightarrow I=\left[\frac{x^{7/4}}{7/4}\right]_0^{16} \Rightarrow I=\frac{4}{7}\left[x^{7/4}\right]_0^{16} \Rightarrow I=\frac{4}{7}\left[(16)^{7/4}-(0)^{7/4}\right]$

$\Rightarrow I=\frac{4}{7}\left[\left(2^4\right)^{7/4}\right] \Rightarrow I=\frac{4}{7}(2)^7 \Rightarrow I=\frac{4}{7}\times 128 \quad \therefore I=\frac{512}{7}$

5. $\int_{-4}^{-1} \frac{dx}{x}$

Sol. Let $I=\int_{-4}^{-1} \frac{dx}{x} \Rightarrow I=\left[\log|x|\right]_{-4}^{-1} \Rightarrow I=\log|-1|-\log|-4|$

$\Rightarrow I=\log|1|-\log|4| \Rightarrow I=0-\log 4 \quad \therefore I=-\log 4$

6. $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

Sol. Let $I=\int_1^4 \frac{1}{\sqrt{x}}\,dx \quad \Rightarrow I=\int_1^4 x^{-1/2}\,dx \quad \Rightarrow I=\left[\frac{x^{1/2}}{1/2}\right]_1^4 \quad \Rightarrow I=2\left[x^{1/2}\right]_1^4$

$\Rightarrow I=2\left[(4)^{1/2}-(1)^{1/2}\right] \quad \Rightarrow I=2\left[\left(2^2\right)^{1/2}-(1)^{1/2}\right] \quad \Rightarrow I=2[2-1] \quad \therefore I=2$

7. $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$

Sol. Let $I=\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx \quad \Rightarrow I=\int_0^1 x^{-1/3}\,dx \quad \Rightarrow I=\left[\frac{x^{2/3}}{2/3}\right]_0^1 \quad \Rightarrow I=\frac{3}{2}\left[x^{2/3}\right]_0^1$

$\Rightarrow I=\frac{3}{2}\left[(1)^{2/3}-(0)^{2/3}\right] \quad \Rightarrow I=\frac{3}{2}\times 1 \quad \therefore I=\frac{3}{2}$

8. $\int_1^8 \frac{1}{x^{2/3}}\,dx$

Sol. Let $I=\int_1^8 \frac{1}{x^{2/3}}\,dx \quad \Rightarrow I=\int_1^8 x^{-2/3}\,dx \quad \Rightarrow I=\left[\frac{x^{1/3}}{1/3}\right]_1^8 \quad \Rightarrow I=3\left[x^{1/3}\right]_1^8$

$\Rightarrow I=3\left[(8)^{1/3}-(1)^{1/3}\right] \quad \Rightarrow I=3\left[\left(2^3\right)^{1/3}-(1)^{1/3}\right] \quad \Rightarrow I=3(2-1) \quad \therefore I=3$

9. $\int_2^4 3\,dx$

Sol. Let $I=\int_2^4 3\,dx \quad \Rightarrow I=3\int_2^4 dx \quad \Rightarrow I=3[x]_2^4 \quad \Rightarrow I=3(4-2) \quad \Rightarrow I=3\times 2 \quad \therefore I=6$

10. $\int_0^1 \frac{1}{\left(1+x^2\right)}dx$

Sol. Let $I=\int_0^1 \frac{1}{\left(1+x^2\right)}dx \quad \Rightarrow I=\left[\tan^{-1}x\right]_0^1 \quad \Rightarrow I=\left[\tan^{-1}(1)-\tan^{-1}(0)\right] \quad \therefore I=\frac{\pi}{4}$

11. $\int_0^\infty \frac{1}{\left(1+x^2\right)}dx$

Sol. Let $I=\int_0^\infty \frac{1}{\left(1+x^2\right)}dx \quad \Rightarrow I=\left[\tan^{-1}x\right]_0^\infty \quad \Rightarrow I=\left[\tan^{-1}(\infty)-\tan^{-1}0\right]$

$\Rightarrow I=\left(\frac{\pi}{2}-0\right) \quad \therefore I=\frac{\pi}{2}$

12. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Sol. Let $I=\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx \quad \Rightarrow I=\left[\sin^{-1}x\right]_0^1 \quad \Rightarrow I=\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$

$\Rightarrow I=\left(\frac{\pi}{2}-0\right) \quad \therefore I=\frac{\pi}{2}$

13. $\int_{0}^{\pi/6} \sec^2 x \, dx$

Sol. Let $I = \int_{0}^{\pi/6} \sec^2 x \, dx \quad \Rightarrow I = [\tan x]_0^{\pi/6} \quad \Rightarrow I = \left[\tan\frac{\pi}{6} - \tan 0\right]$

$\Rightarrow I = \left(\frac{1}{\sqrt{3}} - 0\right) \quad \therefore I = \frac{1}{\sqrt{3}}$

14. $\int_{-\pi/4}^{\pi/4} \text{cosec}^2 x \, dx$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \text{cosec}^2 x \, dx \quad \Rightarrow I = -[\cot x]_{-\pi/4}^{\pi/4}$

$\Rightarrow I = -[\cot(\pi/4) - \cot(-\pi/4)] \quad \Rightarrow I = -[1+1] \quad \therefore I = -2$

15. $\int_{\pi/4}^{\pi/2} \cot^2 x \, dx$

Sol. Let $I = \int_{\pi/4}^{\pi/2} \cot^2 x \, dx \quad \Rightarrow \int_{\pi/4}^{\pi/2} (\text{cosec}^2 x - 1) dx \quad \Rightarrow I = \int_{\pi/4}^{\pi/2} \text{cosec}^2 x \, dx - \int_{\pi/4}^{\pi/2} dx$

$\Rightarrow I = [-\cot x]_{\pi/4}^{\pi/2} - [x]_{\pi/4}^{\pi/2} \quad \Rightarrow I = -\left[\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right] - \left[\frac{\pi}{2} - \frac{\pi}{4}\right]$

$\Rightarrow I = \left[-0 + 1 - \frac{\pi}{2} + \frac{\pi}{4}\right] \quad \therefore I = \left(1 - \frac{\pi}{4}\right)$

16. $\int_{0}^{\pi/4} \tan^2 x dx$

Sol. Let $I = \int_{0}^{\pi/4} \tan^2 x dx \quad \Rightarrow I = \int_{0}^{\pi/4} (\sec^2 x - 1) dx \quad \Rightarrow I = \int_{0}^{\pi/4} \sec^2 x \, dx - \int_{0}^{\pi/4} dx$

$\Rightarrow I = [\tan x]_0^{\pi/4} - [x]_0^{\pi/4} \quad \Rightarrow I = \left[\tan\frac{\pi}{4} - \tan 0\right] - \left[\frac{\pi}{4} - 0\right]$

$\Rightarrow I = (1-0) - \left(\frac{\pi}{4}\right) \quad \therefore I = \left[1 - \frac{\pi}{4}\right]$

17. $\int_{0}^{\pi/2} \sin^2 x \, dx$

Sol. Let $I = \int_{0}^{\pi/2} \sin^2 x \, dx \quad \Rightarrow I = \int_{0}^{\pi/2} \frac{1-\cos 2x}{2} dx \quad \Rightarrow I = \frac{1}{2}\int_{0}^{\pi/2} (1 - \cos 2x) dx$

$\Rightarrow I = \frac{1}{2}\left[\int_{0}^{\pi/2} dx - \int_{0}^{\pi/2} \cos 2x \, dx\right] \quad \Rightarrow I = \frac{1}{2}\left\{[x]_0^{\pi/2} - \left[\frac{\sin 2x}{2}\right]_0^{\pi/2}\right\}$

$\Rightarrow I = \frac{1}{2}\left\{\left(\frac{\pi}{2} - 0\right) - \left(\frac{\sin(\pi/2)}{2} - \frac{\sin 2(0)}{2}\right)\right\} \quad \Rightarrow I = \frac{1}{2}\left\{\frac{\pi}{2} - 0 - 0\right\} \quad \therefore I = \frac{\pi}{4}$

18. $\int_{0}^{\pi/4} \cos^2 x \, dx$

Sol. Let $I = \int_0^{\pi/4} \cos^2 x\, dx \quad \Rightarrow I = \int_0^{\pi/4} \frac{1+\cos 2x}{2} dx \quad \Rightarrow I = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/4}$

$$\Rightarrow I = \frac{1}{2}\left[\left\{\frac{\pi}{4} + \frac{\sin\left(2.\frac{\pi}{4}\right)}{2}\right\} - \left(0 + \frac{\sin 2.0}{2}\right)\right] \quad \Rightarrow I = \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] \quad \therefore\ I = \frac{\pi}{8} + \frac{1}{4}$$

19. $\int_0^{\pi/3} \tan x\, dx$

Sol. Let $I = \int_0^{\pi/3} \tan x\, dx \quad \Rightarrow I = \left[-\log|\cos x|\right]_0^{\pi/3} \quad \Rightarrow I = \left[-\log|\cos \pi/3| - \left(-\log|\cos 0|\right)\right]$

$$\Rightarrow I = \left(-\log\frac{1}{2} + \log 1\right) \quad \Rightarrow I = -(\log 1 - \log 2) + \log 1$$

$$\Rightarrow I = -\log 1 + \log 2 + \log 1 \quad \therefore\ I = \log 2$$

20. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx$

Sol. Let $I = \int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx \quad \Rightarrow I = \left[\log|\operatorname{cosec} x - \cot x|\right]_{\pi/6}^{\pi/4}$

$$\Rightarrow I = \left[\log\left(\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right) - \log\left(\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right)\right]$$

$$\Rightarrow I = \log\left(\sqrt{2} - 1\right) - \log\left(2 - \sqrt{3}\right) \quad \therefore\ I = \log\left[\frac{\sqrt{2}-1}{2-\sqrt{3}}\right]$$

21. $\int_0^{\pi/3} \cos^3 x\, dx$

Sol. Let $I = \int_0^{\pi/3} \cos^3 x\, dx$

We know that $\cos 3x = 4\cos^3 x - 3\cos x \quad \Rightarrow \cos 3x + \cos x = 4\cos^3 x \quad \therefore\ \cos^3 x = \frac{\cos 3x + 3\cos x}{4}$

$$\Rightarrow I = \int_0^{\pi/3}\left(\frac{\cos 3x + 3\cos x}{4}\right)dx \quad \Rightarrow I = \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\pi/3}$$

$$\Rightarrow I = \frac{1}{4}\left[\left\{\frac{\sin\left(3.\frac{\pi}{3}\right)}{3} + 3\sin\left(\frac{\pi}{3}\right)\right\} - (0)\right] \quad \Rightarrow I = \frac{1}{4}\left[0 + 3.\frac{\sqrt{3}}{2}\right] \quad \therefore\ I = \frac{3\sqrt{3}}{8}$$

22. $\int_0^{\pi/2} \sin^3 x\, dx$

Sol. Let $I = \int_0^{\pi/2} \sin^3 x\, dx$

As, we know that $\sin 3x = 3\sin x - 4\sin^3 x \quad \Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$ (general trigonometric formula)

$$\therefore\ I=\int_0^{\pi/2}\sin^3 x\,dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{3\sin x-\sin 3x}{4}dx \quad \Rightarrow I=\frac{1}{4}\int_0^{\pi/2}3\sin x-\sin 3x\,dx$$

$$\Rightarrow I=\frac{1}{4}\left[-3\cos x+\frac{\cos 3x}{3}\right]_0^{\pi/2} \quad \Rightarrow I=\frac{1}{4}\left[\left(-3\cos\frac{\pi}{2}+\cos\frac{3(\pi/2)}{3}\right)-\left(-3\cos(0)+\frac{\cos(0)}{3}\right)\right]$$

$$\Rightarrow I=\frac{1}{4}\left[(0+0)-\left(-3.1+\frac{1}{3}\right)\right] \quad \Rightarrow I=\frac{1}{4}\left[-\left(\frac{-8}{3}\right)\right] \quad \Rightarrow I=\frac{1}{4}\times\frac{8}{3} \quad \therefore\ I=\frac{2}{3}$$

23. $\int_{\pi/4}^{\pi/2}\frac{1-3\cos x}{\sin^2 x}\,dx$

Sol. Let $I=\int_{\pi/4}^{\pi/2}\frac{1-3\cos x}{\sin^2 x}\,dx \quad \Rightarrow \int_{\pi/4}^{\pi/2}\left(\frac{1}{\sin^2 x}-\frac{3\cos x}{\sin^2 x}\right)dx \quad \Rightarrow I=\int_{\pi/4}^{\pi/2}\left(\text{cosec}^2 x-3\,\text{cosec}\ x\ \cot x\right)dx$

$$\Rightarrow I=\left[-\cot\ x-(-3\,\text{cosec}\ x)\right]_{\pi/4}^{\pi/2} \quad \Rightarrow I=\left[-\cot x+3\,\text{cosec}\,x\right]_{\pi/4}^{\pi/2}$$

$$\Rightarrow I=\left[\left(-\cot\frac{\pi}{2}+3\,\text{cosec}\frac{\pi}{2}\right)-\left(-\cot\frac{\pi}{4}+3\,\text{cosec}\frac{\pi}{4}\right)\right]$$

$$\Rightarrow I=\left[(0+3.1)-\left(-1+3.\sqrt{2}\right)\right] \quad \Rightarrow I=3+1-3\sqrt{2} \quad \therefore\ I=4-3\sqrt{2}$$

24. $\int_0^{\pi/4}\sqrt{1+\cos 2x}\,dx$

Sol. Let $I=\int_0^{\pi/4}\sqrt{1+\cos 2x}\,dx \quad \Rightarrow I=\int_0^{\pi/4}\sqrt{2\cos^2 x}\,dx \quad \Rightarrow I=\sqrt{2}\int_0^{\pi/4}\cos x\,dx$

$$\Rightarrow I=\sqrt{2}\left[\sin x\right]_0^{\pi/4} \quad \Rightarrow I=\sqrt{2}\left[\sin\frac{\pi}{4}-\sin 0\right] \quad \Rightarrow I=\sqrt{2}\left[\frac{1}{\sqrt{2}}-0\right] \quad \Rightarrow I=\sqrt{2}\times\frac{1}{\sqrt{2}}$$

$\therefore\ I=1$

25. $\int_0^{\pi/4}\sqrt{1-\sin 2x}\,dx$

Sol. Let $I=\int_0^{\pi/4}\sqrt{1-\sin 2x}\,dx \quad \Rightarrow I=\int_0^{\pi/4}\sqrt{(\cos x-\sin x)^2}\,dx$

$$\Rightarrow I=\int_0^{\pi/4}(\cos x-\sin x)dx \quad \Rightarrow\left[\sin x+\cos x\right]_0^{\pi/4} \quad \Rightarrow I=\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)-(\sin 0+\cos 0)$$

$$\Rightarrow I=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-(0+1) \quad \Rightarrow I=\frac{2}{\sqrt{2}}-1 \quad \therefore\ I=\sqrt{2}-1$$

26. $\int_{-\pi/4}^{\pi/4}\frac{dx}{(1+\sin x)}$

Sol. Let $I=\int_{-\pi/4}^{\pi/4}\frac{1}{(1+\sin x)}dx \quad \Rightarrow \int_{-\pi/4}^{\pi/4}\frac{1}{(1+\sin x)}\times\frac{(1-\sin x)}{(1-\sin x)}dx \quad \Rightarrow I=\int_{-\pi/4}^{\pi/4}\frac{(1-\sin x)}{(1-\sin^2 x)}dx$

$$\Rightarrow I=\int_{-\pi/4}^{\pi/4}\frac{(1-\sin x)}{(1-\sin^2 x)}dx \quad \Rightarrow I=\int_{-\pi/4}^{\pi/4}\frac{1}{(1-\sin^2 x)}-\frac{\sin x}{1-\sin^2 x}\,dx$$

$$\Rightarrow I=\int_{-\pi/4}^{\pi/4}\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\,dx \quad \Rightarrow I=\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx-\int_{-\pi/4}^{\pi/4}\tan x\sec x\,dx$$

$$\Rightarrow I=[\tan x-\sec x]_{-\pi/4}^{\pi/4} \quad \Rightarrow I=\left[\left(\tan\frac{\pi}{4}-\sec\frac{\pi}{4}\right)-\left(\tan\left(\frac{-\pi}{4}\right)-\sec\left(\frac{-\pi}{4}\right)\right)\right]$$

$$\Rightarrow I=\left(1-\sqrt{2}\right)-\left(-1-\sqrt{2}\right) \quad \Rightarrow I=1-\sqrt{2}+1+\sqrt{2} \quad \therefore I=2$$

27. $\int_0^{\pi/4} \frac{dx}{1+\cos 2x}$

Sol. Let $I=\int_0^{\pi/4} \frac{dx}{1+\cos 2x} \quad \Rightarrow I=\int_0^{\pi/4} \frac{1}{2\cos^2 x}dx \quad \Rightarrow I=\frac{1}{2}\int_0^{\pi/4} \sec^2 x\, dx$

$$\Rightarrow I=\frac{1}{2}[\tan x]_0^{\pi/4} \quad \Rightarrow I=\frac{1}{2}\left[\tan\frac{\pi}{4}-\tan 0\right] \quad \Rightarrow I=\frac{1}{2}[1-0] \quad \therefore I=\frac{1}{2}$$

28. $\int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos x}$

Sol. Let $I=\int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos x}=\int_{\pi/4}^{\pi/2} \frac{dx}{2\sin^2 x}=\frac{1}{2}\int_{\pi/4}^{\pi/2} \operatorname{cosec}^2 x\, dx$

$$=-\frac{1}{2}[\cot x]_{\pi/4}^{\pi/2}=-\frac{1}{2}\left[\cot\frac{\pi}{2}-\cot\frac{\pi}{4}\right]=-\frac{1}{2}[0-1]=\frac{1}{2}$$

29. $\int_0^{\pi/4} \sin 2x \sin 3x\, dx$

Sol. Let $I=\int_0^{\pi/4} \sin 2x\sin 3x\, dx \quad \Rightarrow I=\frac{1}{2}\int_0^{\pi/4} 2\sin 2x\sin 3x\, dx$

$$\Rightarrow I=\frac{1}{2}\int_0^{\pi/4}\left[\cos(2x-3x)-\cos(2x+3x)\right]dx \quad \Rightarrow I=\frac{1}{2}\int_0^{\pi/4}\left(\cos(-x)-\cos 5x\right)dx$$

$$\Rightarrow I=\frac{1}{2}\int_0^{\pi/4}\left(\cos x-\cos 5x\right)dx \quad \Rightarrow I=\frac{1}{2}\left[\sin x-\frac{\sin 5x}{5}\right]_0^{\pi/4}$$

$$\Rightarrow I=\frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right)-\frac{\sin 5\left(\frac{\pi}{4}\right)}{5}\right]-\left[\sin(0)-\frac{\sin 5(0)}{5}\right] \quad \Rightarrow I=\frac{1}{2}\left[\frac{1}{\sqrt{2}}+\frac{1}{5\sqrt{2}}\right]$$

$$\Rightarrow I=\frac{1}{2}\left(\frac{5+1}{5\sqrt{2}}\right) \quad \Rightarrow I=\frac{1}{2}\times\frac{6}{5\sqrt{2}} \quad \therefore I=\frac{3}{5\sqrt{2}}$$

30. $\int_0^{\pi/6} \cos x\cos 2x\, dx$

Sol. Let $I=\int_0^{\pi/6} \cos x\cos 2x\, dx \quad \Rightarrow I=\frac{1}{2}\int_0^{\pi/6} 2\cos x\cos 2x\, dx$

$$\Rightarrow I=\frac{1}{2}\int_0^{\pi/6} \cos(x+2x)+\cos(x-2x)dx \quad \Rightarrow I=\frac{1}{2}\int_0^{\pi/6} \cos 3x+\cos(-x)dx$$

$$\Rightarrow I=\frac{1}{2}\left[\frac{\sin 3x}{3}+\frac{\sin x}{1}\right]_0^{\pi/6} \quad \Rightarrow I=\frac{1}{2}\left[\frac{\sin 3\left(\frac{\pi}{6}\right)}{3}+\frac{\sin\left(\frac{\pi}{6}\right)}{1}-0\right] \quad \Rightarrow I=\frac{1}{2}\left[\frac{\sin\frac{\pi}{2}}{3}+\frac{1}{2}\right]$$

$$\Rightarrow I=\frac{1}{2}\left[\frac{1}{3}+\frac{1}{2}\right] \quad \Rightarrow I=\frac{1}{2}\left[\frac{2+3}{6}\right] \quad \therefore I=\frac{5}{12}$$

31. $\int_0^{\pi}\sin 2x\cos 3x\,dx$

Sol. Let $I=\int_0^{\pi}\sin 2x\cos 3x\,dx \quad \Rightarrow I=\frac{1}{2}\int_0^{\pi}2\sin 2x\cos 3x\,dx$

$$\Rightarrow I=\frac{1}{2}\int_0^{\pi}\left\{\sin(2x+3x)+\sin(2x-3x)\right\}dx \quad \Rightarrow I=\frac{1}{2}\int_0^{\pi}(\sin 5x-\sin x)\,dx$$

$$\Rightarrow I=\frac{1}{2}\left[\frac{-\cos 5x}{5}+\cos x\right]_0^{\pi} \quad \Rightarrow I=\frac{1}{2}\left[\left\{\frac{-\cos 5\pi}{5}+\cos\pi\right\}-\left\{\frac{-\cos 5.0}{5}+\cos 0\right\}\right]$$

$$\Rightarrow I=\frac{1}{2}\left[\left(\frac{1}{5}-1\right)-\left(\frac{-1}{5}+1\right)\right] \quad \Rightarrow I=\frac{1}{2}\left[\left(\frac{-4}{5}\right)-\left(\frac{4}{5}\right)\right] \quad \Rightarrow I=\frac{1}{2}\left[\frac{-8}{5}\right] \quad \therefore I=\frac{-4}{5}$$

32. $\int_0^{\pi/2}\sqrt{1+\sin x}\,dx$

Sol. Let $I=\int_0^{\pi/2}\sqrt{1+\sin x}\,dx \quad \Rightarrow I=\int_0^{\pi/2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx \quad \Rightarrow I=\left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}-\frac{\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2}$

$$\Rightarrow I=2\left[\sin\frac{x}{2}-\cos\frac{x}{2}\right]_0^{\pi/2} \quad \Rightarrow I=2\left[\left\{\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right\}-\left\{\sin 0-\cos 0\right\}\right]$$

$$\Rightarrow I=2\left[\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-(0-1)\right] \quad \Rightarrow I=2[0+1] \quad \therefore I=2$$

33. $\int_0^{\pi/2}\sqrt{1+\cos x}\,dx$

Sol. Let $I=\int_0^{\pi/2}\sqrt{1+\cos x}\,dx \quad \Rightarrow I=\int_0^{\pi/2}\sqrt{2\cos^2\frac{x}{2}} \quad \Rightarrow I=\sqrt{2}\int_0^{\pi/2}\cos\frac{x}{2}dx$

$$\Rightarrow I=\sqrt{2}.\left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} \quad \Rightarrow I=2\sqrt{2}\left[\left(\sin\frac{\pi}{4}\right)-(\sin 0)\right] \quad \Rightarrow I=2\sqrt{2}\left[\frac{1}{\sqrt{2}}-0\right] \quad \therefore I=2$$

34. $\int_0^{2}\frac{x^4+1}{x^2+1}dx$

Sol. Let $I=\int_0^{2}\frac{x^4+1}{x^2+1}dx \quad \Rightarrow I=\int_0^{2}\left\{(x^2-1)+\frac{2}{x^2+1}\right\}dx \quad \Rightarrow I=\left[\frac{x^3}{3}-x+2\tan^{-1}(x)\right]_0^{2}$

$$\Rightarrow I=\left[\left\{\frac{(2)^3}{3}-2+2\tan^{-1}(2)\right\}-\left\{0-0+2\tan^{-1}(0)\right\}\right]$$

$$\Rightarrow I=\left[\left(\frac{8}{3}-2\right)+2\tan^{-1}(2)\right] \quad \therefore I=\frac{2}{3}+2\tan^{-1}(2)$$

35. $\int_1^2 \frac{dx}{(x+1)(x+2)}dx$

Sol. Let $I=\int_1^2 \frac{dx}{(x+1)(x+2)}dx \Rightarrow I=[I_1]_1^2$ ...(1)

$I_1=\int \frac{1}{(x+1)(x+2)}dx \Rightarrow \frac{1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$

$\Rightarrow \frac{1}{(x+1)(x+2)}=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)} \Rightarrow 1=Ax+2A+Bx+B \Rightarrow 1=x(A+B)+2A+B$

Now, Equating co-efficient both side we get

$A+B=0$ ...(1)

$2A+B=1$ ...(2)

By, Solving equation (1) & (2) then we get $A=1, B=-1$

$I_1=\int\left(\frac{A}{x+1}+\frac{B}{x+2}\right)dx \Rightarrow I_1=1.\int\frac{1}{x+1}dx-1.\int\frac{1}{x+2}dx \Rightarrow I_1=\log\left|\frac{x+1}{x+2}\right|$

$\therefore\ I=\left[\log\left|\frac{x+1}{x+2}\right|\right]_1^2 \Rightarrow I=\log\left(\frac{3}{4}\right)-\log\left(\frac{2}{3}\right) \Rightarrow I=\log\left(\frac{9}{8}\right) \Rightarrow I=\log 9-\log 8$

$\therefore\ I=2\log 3-3\log 2$

36. $\int_1^2 \frac{(x+3)}{x(x+2)}dx$

Sol. Let $I=\int_1^2 \frac{(x+3)}{x(x+2)}dx \Rightarrow I=[I_1]_1^2$

$I_1=\int\frac{x+3}{x(x+2)}dx \Rightarrow \frac{x+3}{x(x+2)}=\frac{A}{x}+\frac{B}{x+2} \Rightarrow \frac{x+3}{x(x+2)}=\frac{A(x+2)+Bx}{x(x+2)}$

$\Rightarrow x+3=Ax+2A+Bx \Rightarrow x+3=x(A+B)+2A$

By, Equating co-efficient both side we get

$A+B=1$ ...(1)

$2A=3$ ...(2)

Now, solving equation (1) & (2) then we get $A=\frac{3}{2}, B=\frac{-1}{2}$

$I_1=\int\left(\frac{A}{x}+\frac{B}{x+2}\right)dx \Rightarrow I_1=A\int\frac{1}{x}dx+B\int\frac{1}{x+2}dx \Rightarrow I_1=\frac{3}{2}\log|x|-\frac{1}{2}\log|x+2|$

$\therefore\ I=\left[\frac{3}{2}\log|x|-\frac{1}{2}\log|x+1|\right]_1^2 \Rightarrow I=\left\{\frac{3}{2}\log 2-\frac{1}{2}\log 3\right\}-\left\{\frac{3}{2}\log 1-\frac{1}{2}\log 2\right\}$

$\Rightarrow I=\frac{3}{2}\log 2-\frac{1}{2}\log 3+\frac{1}{2}\log 2 \quad \therefore\ I=2\log 2-\frac{1}{2}\log 3$

37. $\int_3^4 \frac{dx}{x^2-4}$

Sol. Let $I=\int_3^4\frac{dx}{x^2-4} \Rightarrow I=\int_3^4\frac{dx}{(x)^2-(2)^2} \Rightarrow I=\left[\frac{1}{2.2}\log\left|\frac{x-2}{x+2}\right|\right]_3^4$

$$\Rightarrow I=\frac{1}{4}\left[\log\left(\frac{1}{3}\right)-\log\left(\frac{1}{5}\right)\right] \quad \Rightarrow I=\frac{1}{4}\left[\log\frac{5}{3}\right] \quad \therefore I=\frac{1}{4}[\log 5-\log 3]$$

38. $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$

Sol. Let $I=\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}} \quad \Rightarrow I=\int_0^4 \frac{dx}{\sqrt{(x)^2+2.x.1+(1)^2-(1)^2+3}}$

$$\Rightarrow I=\int_0^4 \frac{dx}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}} \quad \Rightarrow I=\left[\log\left|(x+1)+\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}\right|\right]_0^4$$

$$\Rightarrow I=\left[\log\left|(x+1)+\sqrt{x^2+2x+3}\right|\right]_0^4$$

$$\Rightarrow I=\left[\log\left|(4+1)+\sqrt{(4)^2+2.4+3}\right|-\log\left|(0+1)+\sqrt{0+0+3}\right|\right]$$

$$\Rightarrow I=\log\left(5+3\sqrt{3}\right)-\log\left(1+\sqrt{3}\right) \quad \therefore I=\log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$$

39. $\int_1^2 \frac{dx}{\sqrt{x^2+4x+3}}$

Sol. Let $I=\int_1^2 \frac{dx}{\sqrt{x^2+4x+3}} \quad \Rightarrow I=\int_1^2 \frac{dx}{\sqrt{(x)^2+2.x.2+(2)^2-(2)^2+3}} \quad \Rightarrow I=\int_1^2 \frac{dx}{\sqrt{(x+2)^2-1}}dx$

$$\Rightarrow I=\left[\log\left|(x+2)+\sqrt{(x+2)^2-1}\right|\right]_1^2 \quad \Rightarrow I=\left[\log\left|(x+2)+\sqrt{x^2+4x+3}\right|\right]_1^2$$

$$\Rightarrow I=\log\left(4+\sqrt{15}\right)-\log\left(3+\sqrt{8}\right)$$

40. $\int_0^1 \frac{dx}{1+x+2x^2}$

Sol. Let $I=\int_0^1 \frac{dx}{1+x+2x^2} \quad \Rightarrow I=\int_0^1 \frac{dx}{2\left(x^2+\frac{x}{2}+\frac{1}{2}\right)} \quad \Rightarrow I=\frac{1}{2}\int_0^1 \frac{dx}{x^2+\frac{x}{2}+\frac{1}{2}}$

$$\Rightarrow I=\frac{1}{2}\int_0^1 \frac{dx}{(x)^2+2.x.\frac{1}{4}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2+\frac{1}{2}} \quad \Rightarrow I=\frac{1}{2}\int_0^1 \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}$$

$$\Rightarrow I=\frac{1}{2}\int_0^1 \frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2} \quad \Rightarrow I=\frac{1}{2}.\frac{1}{\frac{\sqrt{7}}{4}}\left[\tan^{-1}\left(\frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}}\right)\right]_0^1$$

$$\Rightarrow I=\frac{2}{\sqrt{7}}\left[\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)\right]_0^1 \quad \therefore I=\frac{2}{\sqrt{7}}\left[\tan^{-1}\left(\frac{5}{\sqrt{7}}\right)-\tan^{-1}\left(\frac{1}{\sqrt{7}}\right)\right]$$

Short-Answer Questions

Evaluate:

41. $\int_0^{\pi/2}\left(a\cos^2 x+b\sin^2 x\right)dx$

Sol. Let $I=\int_0^{\pi/2}\left(a\cos^2 x+b\sin^2 x\right)dx \quad \Rightarrow I=a\int_0^{\pi/2}\frac{1+\cos 2x}{2}dx+b\int_0^{\pi/2}\frac{1-\cos 2x}{2}dx$

$$\Rightarrow I=\frac{a}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}+\frac{b}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$\Rightarrow I=\frac{a}{2}\left[\left\{\frac{\pi}{2}+\frac{\sin 2.\frac{\pi}{2}}{2}\right\}-0\right]+\frac{b}{2}\left[\left(\frac{\pi}{2}-\frac{\sin 2.\frac{\pi}{2}}{2}\right)-0\right]$$

$$\Rightarrow I=\frac{a}{2}\left[\frac{\pi}{2}+0\right]+\frac{b}{2}\left[\frac{\pi}{2}-0\right] \quad \Rightarrow I=\frac{\pi}{2}\left(\frac{a}{2}+\frac{b}{2}\right) \quad \therefore\ I=\frac{\pi}{4}(a+b)$$

42. $\int_{\pi/3}^{\pi/4}(\tan x+\cot x)^2\,dx$

Sol. Let $I=\int_{\pi/3}^{\pi/4}(\tan x+\cot x)^2\,dx \quad \Rightarrow I=\int_{\pi/3}^{\pi/4}\left(\tan^2 x+2\tan x.\cot x+\cot^2 x\right)dx$

$$\Rightarrow I=\int_{\pi/3}^{\pi/4}\left(\tan^2 x+2+\cot^2 x\right)dx \quad \Rightarrow I=\int_{\pi/3}^{\pi/4}\left(\sec^2 x-1+2+\operatorname{cosec}^2 x-1\right)dx$$

$$\Rightarrow I=\int_{\pi/3}^{\pi/4}\left(\sec^2 x+\operatorname{cosec}^2 x\right)dx \quad \Rightarrow [\tan x-\cot x]_{\pi/3}^{\pi/4}$$

$$\Rightarrow I=\left(\tan\frac{\pi}{4}-\cot\frac{\pi}{4}\right)-\left(\tan\frac{\pi}{3}-\cot\frac{\pi}{3}\right) \quad \Rightarrow I=(1-1)-\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)=\frac{-2}{\sqrt{3}} \quad \therefore I=\frac{-2}{\sqrt{3}}$$

43. $\int_0^{\pi/2}\cos^4 x\,dx$

Sol. Let $I=\int_0^{\pi/2}\cos^4 x\,dx \quad \Rightarrow I=\int_0^{\pi/2}\left(\cos^2 x\right)^2 dx \quad \Rightarrow I=\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)^2 dx$

$$\Rightarrow I=\frac{1}{4}\int_0^{\pi/2}\left(1+2\cos 2x+\cos^2 2x\right)dx \quad \Rightarrow I=\frac{1}{4}\int_0^{\pi/2}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$$

$$\Rightarrow I=\frac{1}{4}\int_0^{\pi/2}\left(\frac{3}{2}+2\cos 2x+\frac{\cos 4x}{2}\right)dx \quad \Rightarrow I=\frac{1}{4}\left[\frac{3}{2}x+\frac{2\sin 2x}{2}+\frac{\sin 4x}{2\times 4}\right]_0^{\pi/2}$$

$$\Rightarrow I=\frac{1}{4}\left[\frac{3}{2}.\frac{\pi}{2}+\sin\pi+\frac{\sin 2\pi}{8}-0\right]=\frac{1}{4}.\frac{3\pi}{4}=\frac{3\pi}{16}$$

44. $\int_0^a \frac{dx}{\left(ax+a^2-x^2\right)}$

Sol. Let $I=\int_0^a\frac{dx}{ax+a^2-x^2} \quad \Rightarrow I=\int_0^a\frac{dx}{-\left(x^2-ax-a^2\right)}$

$$\Rightarrow I=-\int_0^a \frac{dx}{x^2-2.\frac{a}{2}.x+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2-a^2} \quad \Rightarrow I=-\int_0^a \frac{dx}{\left(x-\frac{a}{2}\right)^2-\left(\frac{\sqrt{5}a}{2}\right)^2}$$

$$\Rightarrow I=\int_0^a \frac{dx}{\left(\frac{\sqrt{5}a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2} \quad \Rightarrow I=\frac{1}{2.\frac{\sqrt{5}}{2}.a}\left[\log\frac{\frac{\sqrt{5}a}{2}+\left(x-\frac{a}{2}\right)}{\frac{\sqrt{5}a}{2}-\left(x-\frac{a}{2}\right)}\right]_0^a$$

$$\Rightarrow I=\frac{1}{\sqrt{5}a}\left[\log\frac{\sqrt{5}a+2x-a}{\sqrt{5}a-2x+a}\right]_0^a \quad I=\frac{1}{\sqrt{5}a}\left[\log\frac{\sqrt{5}a+a}{\sqrt{5}a-a}-\log\frac{\sqrt{5}a-a}{\sqrt{5}a+a}\right]$$

$$\Rightarrow I=\frac{1}{\sqrt{5}a}\left[\log\frac{\sqrt{5}+1}{\sqrt{5}-1}-\log\frac{\sqrt{5}-1}{\sqrt{5}+1}\right] \quad I=\frac{1}{\sqrt{5}a}\log\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)$$

$$\Rightarrow I=\frac{1}{\sqrt{5}a}\log\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)^2 \quad \Rightarrow \frac{2}{\sqrt{5}a}\log\frac{\sqrt{5}+1}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}+1}$$

$$\Rightarrow I=\frac{2}{\sqrt{5}a}\log\frac{5+1+2\sqrt{5}}{5-1} \quad \Rightarrow I=\frac{2}{\sqrt{5}a}\log\frac{2\left(3+\sqrt{5}\right)}{4} \quad \Rightarrow I=\frac{2}{\sqrt{5}a}\log\left(\frac{3+\sqrt{5}}{2}\right)$$

45. $\displaystyle\int_{1/4}^{1/2}\frac{dx}{\sqrt{x-x^2}}$

Sol. Let $I=\displaystyle\int_{1/4}^{1/2}\frac{dx}{\sqrt{x-x^2}} \quad \Rightarrow I=\int_{1/4}^{1/2}\frac{dx}{\sqrt{-\left[x^2-x\right]}} \quad \Rightarrow I=\int_{1/4}^{1/2}\frac{dx}{\sqrt{-\left[(x)^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}}$

$$\Rightarrow I=\int_{1/4}^{1/2}\frac{dx}{\sqrt{-\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}} \quad \Rightarrow I=\int_{1/4}^{1/2}\frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} \quad \Rightarrow I=\left[\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right]_{1/4}^{1/2}$$

$$\Rightarrow I=\left[\sin^{-1}\left(2x-1\right)\right]_{1/4}^{1/2} \quad \Rightarrow I=\left[\sin^{-1}\left(2.\frac{1}{2}-1\right)-\sin^{-1}\left(2.\frac{1}{4}-1\right)\right] \quad \Rightarrow I=-\sin^{-1}\left(\frac{-1}{2}\right)$$

$$\Rightarrow I=\sin^{-1}\left(\frac{1}{2}\right) \quad \therefore\ I=\frac{\pi}{6}$$

46. $\displaystyle\int_0^1\sqrt{x\left(1-x\right)}\,dx$

Sol. Let $I=\displaystyle\int_0^1\sqrt{x\left(1-x\right)}\,dx \quad \Rightarrow I=\int_0^1\sqrt{x-x^2}\,dx \quad \Rightarrow I=\int_0^1\sqrt{-\left[x^2-x\right]}\,dx$

$$\Rightarrow I=\int_0^1\sqrt{-\left[(x)^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}\,dx \quad \Rightarrow I=\int_0^1\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$\Rightarrow I=\left[\frac{x-\frac{1}{2}}{2}\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}+\frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$\Rightarrow I=\left[\frac{2x-1}{4}\sqrt{x-x^2}+\frac{1}{8}\sin^{-1}(2x-1)\right]_0^1 \quad \Rightarrow I=\left[\left\{\frac{2\times1-1}{4}\sqrt{1-1}+\frac{1}{8}\sin^{-1}(2\times1-1)\right\}-\frac{1}{8}\sin^{-1}(-1)\right]$$

$$\Rightarrow I=\frac{1}{8}\sin^{-1}(1)+\frac{1}{8}\sin^{-1}(1) \quad \Rightarrow I=\frac{1}{8}.\frac{\pi}{2}+\frac{1}{8}.\frac{\pi}{2}=\frac{2\pi}{16} \quad \therefore I=\frac{\pi}{8}$$

47. $\int_1^3\frac{dx}{x^2(x+1)}$

Sol. Let $I=\int_1^3\frac{dx}{x^2(x+1)} \quad \Rightarrow I=[I_1]_1^3$

$$\therefore\ I_1=\int\frac{1}{x^2(x+1)}dx \quad \Rightarrow \frac{1}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$\Rightarrow \frac{1}{x^2(x+1)}=\frac{Ax(x+1)+B(x+1)+C(x^2)}{x^2(x+1)} \quad \Rightarrow 1=Ax^2+Ax+Bx+B+Cx^2$$

$$\Rightarrow 1=x^2(A+C)+(A+B)x+B$$

Now, Equating co-efficient both side we get

$A+C=0 \qquad ...(1)$

$A+B=0 \qquad ...(2)$

$B=1 \qquad ...(3)$

By, Solving equation (1), (2) & (3) we get $A=-1,\ B=1,\ C=1$

$$\Rightarrow I_1=\int\left(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}\right)dx \quad \Rightarrow I_1=-\int\frac{1}{x}dx+\int\frac{1}{x^2}dx+\int\frac{1}{x+1}dx$$

$$\Rightarrow I_1=-\log|x|-\frac{1}{x}+\log|x+1| \quad \Rightarrow I_1=\log\left(\frac{x+1}{x}\right)-\frac{1}{x}$$

$$\therefore\ I=\left[\log\left(\frac{x+1}{x}\right)-\frac{1}{x}\right]_1^3 \quad \Rightarrow I=\log4-\log3-\frac{1}{3}-\log2+1 \quad \Rightarrow I=2\log2-\log3-\log2+\frac{2}{3}$$

$$\therefore\ I=\log2-\log3+\frac{2}{3}$$

48. $\int_1^2\frac{dx}{x(1+2x)^2}$

Sol. Let $I=\int_1^2\frac{dx}{x(1+2x)^2} \quad \Rightarrow I=[I_1]_1^2$

$$I_1=\int\frac{1}{x(1+2x)^2}dx \quad \Rightarrow \frac{1}{x(1+2x)^2}=\frac{A}{x}+\frac{B}{1+2x}+\frac{C}{(1+2x)^2}$$

$$\Rightarrow \frac{1}{x(1+2x)^2}=\frac{A(1+2x)^2+B(1+2x).x+Cx}{x(1+2x)^2}$$

$\Rightarrow 1 = A\left(1+4x+4x^2\right)+B\left(x+2x^2\right)+Cx \quad \Rightarrow 1 = A+4Ax+4Ax^2+Bx+2Bx^2+Cx$

$\Rightarrow 1 = x^2\left(4A+2B\right)+x\left(4A+B+C\right)+A$

Now, Equating co-efficient we get

$4A+2B=0 \quad \Rightarrow 2A+B=0 \quad \quad ...(1)$

$4A+B+C=0 \quad \quad ...(2)$

$A=1 \quad \quad ...(3)$

By, Solving equation (1), (2) & (3) then we get $A=1,\ B=-2,\ C=-2$

$I_1 = \int\left\{\frac{A}{x}+\frac{B}{1+2x}+\frac{C}{(1+2x)^2}\right\}dx \quad \Rightarrow I_1 = A\int\frac{1}{x}dx+B\int\frac{1}{1+2x}dx+C\int\frac{1}{(1+2x)^2}dx$

$\Rightarrow I_1 = \log|x|-2\frac{\log(1+2x)}{2}-2\left(-\frac{1}{1+2x}\right)\times\frac{1}{2}$

$\Rightarrow I_1 = \log|x|-\log(1+2x)+\frac{1}{1+2x} \quad \therefore\ I = \left[\log|x|-\log(1+2x)+\frac{1}{1+2x}\right]_1^2$

$\Rightarrow I = \left(\log 2-\log(5)+\frac{1}{5}\right)-\left(\log 1-\log 3+\frac{1}{3}\right)$

$\Rightarrow I = \log 2-\log 5+\frac{1}{5}+\log 3-\frac{1}{3} \quad \therefore\ I = \log 6-\log 5-\frac{2}{15}$

49. $\int_0^1 x e^x\, dx$

Sol. Let $I = \int_0^1 x e^x\, dx \quad \Rightarrow I = [I_1]_0^1$

$I_1 = \int x e^x\, dx \quad \Rightarrow I_1 = x\int e^x dx-\int\left[\frac{d(x)}{dx}\int e^x dx\right]dx \quad \Rightarrow I_1 = x e^x-\int 1.e^x\, dx$

$\Rightarrow I_1 = x e^x - e^x \quad \Rightarrow I_1 = e^x(x-1)$

$I = \left[e^x(x-1)\right]_0^1 \quad \Rightarrow I = \left[e^1(1-1)-e^0(0-1)\right] \quad \Rightarrow I = -1(-1) \quad \therefore\ I = 1$

50. $\int_0^{\pi/2} x^2 \cos x\, dx$

Sol. Let $I = \int_0^{\pi/2} x^2\cos x\, dx \quad \Rightarrow I = [I_1]_0^{\pi/2}$

$\therefore\ I_1 = \int x^2\cos x\, dx \quad \Rightarrow I_1 = x^2\int\cos x\, dx-\int\left[\frac{d(x^2)}{dx}\int\cos x\, dx\right]dx$

$\Rightarrow I_1 = x^2(\sin x)-\int 2x\sin x\, dx \quad \Rightarrow I_1 = x^2\sin x-2\int x\sin x\, dx \quad \Rightarrow I_1 = x^2\sin x-2\int x\sin x\, dx$

$\Rightarrow I_1 = x^2\sin x-2\left[x\int\sin x\, dx-\int\left[\frac{d(x)}{dx}\int\sin x\, dx\right]dx\right]$

$\Rightarrow I_1 = x^2\sin x-2\left[-x\cos x+\int\cos x\, dx\right]$

$$\Rightarrow I=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2} \qquad \Rightarrow I=\left\{\left(\frac{\pi}{2}\right)^2\sin\frac{\pi}{2}+2\frac{\pi}{2}\cos\frac{\pi}{2}-2\sin\frac{\pi}{2}\right\}-\{0\}$$

$$\therefore \ I=\frac{\pi^2}{4}-2$$

51. $\int_0^{\pi/4} x^2 \sin x\, dx$

Sol. Let $I=\int_0^{\pi/4} x^2\sin x\, dx \quad \Rightarrow I=[I_1]_0^{\pi/4}$

$$\therefore \ I_1=\int x^2\sin x\, dx \quad \Rightarrow I_1=x^2\int\sin x\,dx-\int\left\{\frac{d(x^2)}{dx}\int\sin x\,dx\right\}dx$$

$$\Rightarrow I_1=-x^2\cos x+\int 2x\cos x\,dx \quad \Rightarrow I_1=-x^2\cos x+2\int x\cos x\,dx$$

$$\Rightarrow I_1=-x^2\cos x+2\left[x\int\cos x\,dx-\int\left[\frac{d(x)}{dx}\int\cos x\,dx\right]dx\right]$$

$$\Rightarrow I_1=-x^2\cos x+2\left[x\sin x-\int 1.\sin x\,dx\right] \quad \Rightarrow I_1=-x^2\cos x+2\left[x\sin x+\cos x\right]$$

$$\Rightarrow I_1=-x^2\cos x+2x\sin x+2\cos x$$

$$\Rightarrow I=\left[-x^2\cos x+2x\sin x+2\cos x\right]_0^{\pi/4} \qquad \Rightarrow I=\left\{-\left(\frac{\pi}{4}\right)^2\cos\frac{\pi}{4}+2.\frac{\pi}{4}\sin\frac{\pi}{4}+2\cos\frac{\pi}{4}\right\}-\{0+2\cos 0\}$$

$$\Rightarrow I=-\frac{\pi^2}{16}.\frac{1}{\sqrt{2}}+\frac{\pi}{2}.\frac{1}{\sqrt{2}}+2.\frac{1}{\sqrt{2}}-2 \qquad \therefore \ I=-\frac{\pi^2}{16\sqrt{2}}+\frac{\pi}{2\sqrt{2}}+\sqrt{2}-2$$

52. $\int_0^{\pi/2} x^2\cos 2x\, dx$

Sol. Let $I=\int_0^{\pi/2} x^2\cos 2x\, dx \quad \Rightarrow I=[I_1]_0^{\pi/2}$

$$I_1=\int x^2\cos 2x\,dx \quad \Rightarrow I_1=x^2\int\cos 2x\,dx-\int\left[\frac{d(x^2)}{dx}\int\cos 2x\,dx\right]dx$$

$$\Rightarrow I_1=x^2\frac{\sin 2x}{2}-\int 2x.\frac{\sin 2x}{2}dx \quad \Rightarrow I_1=\frac{x^2\sin 2x}{2}-\int x\sin 2x\,dx$$

$$\Rightarrow I_1=\frac{x^2\sin 2x}{2}-\left[x\int\sin 2x\,dx-\int\left[\frac{d(x)}{dx}\int\sin 2x\,dx\right]dx\right]$$

$$\Rightarrow I_1=\frac{x^2\sin 2x}{2}-\left[-x\frac{\cos 2x}{2}+\int 1.\frac{\cos 2x}{2}dx\right] \quad \Rightarrow I_1=\frac{x^2\sin 2x}{2}+\frac{x\cos 2x}{2}-\frac{1}{2}.\frac{\sin 2x}{2}$$

$$\Rightarrow I_1=\frac{x^2\sin 2x}{2}+\frac{x\cos 2x}{2}-\frac{1}{4}\sin 2x$$

$$\Rightarrow I=\left[\frac{x^2\sin 2x}{2}+\frac{x\cos 2x}{2}-\frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$\Rightarrow I=\left\{\frac{\left(\frac{\pi}{2}\right)^2\sin\left(2.\frac{\pi}{2}\right)}{2}+\frac{\pi}{2}\frac{\cos\left(2.\frac{\pi}{2}\right)}{2}-\frac{1}{4}\sin\left(2.\frac{\pi}{2}\right)\right\}-\{0\} \quad \Rightarrow I=\frac{\frac{\pi}{2}(-1)}{2} \quad \therefore\ I=-\frac{\pi}{4}$$

53. $\int_0^{\pi/2} x^3 \sin 3x\, dx$

Sol. Let $I=\int_0^{\pi/2} x^3\sin 3x\, dx \quad \Rightarrow I=[I_1]_0^{\pi/2}$

$$\therefore\ I_1=\int x^3\sin 3x\, dx \quad \Rightarrow I_1=x^3\int\sin 3x\, dx-\int\left[\frac{d(x^3)}{dx}\int\sin 3x\, dx\right]dx$$

$$\Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\int 3x^2.\frac{\cos 3x}{3}dx \quad \Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\int x^2\cos 3x\, dx$$

$$\Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\left[x^2\int\cos 3x\, dx-\int\left[\frac{d(x^2)}{dx}\int\cos 3x\, dx\right]dx\right]$$

$$\Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\left[x^2\frac{\sin 3x}{3}-\int 2x.\frac{\sin 3x}{3}dx\right] \quad \Rightarrow I_1=\frac{-x^3\cos 3x}{3}+x^2\frac{\sin 3x}{3}-\frac{2}{3}\int x\sin 3x\, dx$$

$$\Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\frac{x^2}{3}\sin 3x-\frac{2}{3}\left[x\int\sin 3x\, dx-\int\left[\frac{d(x)}{dx}\int\sin 3x\, dx\right]dx\right]$$

$$\Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\frac{x^2\sin 3x}{3}-\frac{2}{3}\left[-x\frac{\cos 3x}{3}+\int\frac{\cos 3x}{3}dx\right]$$

$$\Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\frac{x^2\sin 3x}{3}+\frac{2x}{9}\cos 3x-\frac{2}{9}.\frac{\sin 3x}{3}$$

$$\Rightarrow I_1=\frac{-x^3\cos 3x}{3}+\frac{x^2\sin 3x}{3}+\frac{2x}{9}\cos 3x-\frac{2}{27}.\sin 3x$$

$$\Rightarrow I=\left[\frac{-x^3\cos 3x}{3}+\frac{x^2\sin 3x}{3}+\frac{2x}{9}\cos 3x-\frac{2}{27}\sin 3x\right]_0^{\pi/2}$$

$$\Rightarrow I=\left\{\frac{-\left(\frac{\pi}{2}\right)^3\cos\left(3.\frac{\pi}{2}\right)}{3}+\frac{\left(\frac{\pi}{2}\right)^2\sin\left(3.\frac{\pi}{2}\right)}{3}+\frac{2}{9}.\frac{\pi}{2}\cos\left(3.\frac{\pi}{2}\right)-\frac{2}{27}\sin\left(3.\frac{\pi}{2}\right)\right\}-0$$

$$\Rightarrow I=\left[\frac{-\pi^2}{12}+\frac{2}{27}\right] \quad \therefore\ I=\frac{2}{27}-\frac{\pi^2}{12}$$

54. $\int_0^{\pi/2} x^2\cos^2 x\, dx$

Sol. Let $I=\int_0^{\pi/2} x^2\cos^2 x\, dx \quad \Rightarrow I=[I_1]_0^{\pi/2}$

$$I_1=\int x^2\cos^2 x\, dx \quad \Rightarrow I_1=\int x^2\left(\frac{1+\cos 2x}{2}\right)dx \quad \Rightarrow I_1=\frac{1}{2}\left[\int x^2dx+\int x^2\cos 2xdx\right]$$

$$\Rightarrow I_1 = \frac{1}{2}\left[\frac{x^3}{3} + \left\{x^2\int \cos 2x\, dx - \int \frac{d(x^2)}{dx}\int \cos 2x\, dx\right\}\right]$$

$$\Rightarrow I_1 = \frac{x^3}{6} + \frac{1}{2}\left[x^2\frac{\sin 2x}{2} - \int 2x\frac{\sin 2x}{2}dx\right] \quad \Rightarrow I_1 = \frac{x^3}{6} + \frac{1}{2}\left[\frac{x^2\sin 2x}{2} - \int x\sin 2x\, dx\right]$$

$$\Rightarrow I_1 = \frac{x^3}{6} + \frac{x^2\sin 2x}{4} - \frac{1}{2}\left[x\int \sin 2x\, dx - \int\left[\frac{d(x)}{dx}\int \sin 2x\, dx\right]dx\right]$$

$$\Rightarrow I_1 = \frac{x^3}{6} + \frac{x^2\sin 2x}{4} - \frac{1}{2}\left[\frac{-x\cos 2x}{2} + \int\frac{\cos 2x}{2}dx\right]$$

$$\Rightarrow I_1 = \frac{x^3}{6} + \frac{x^2\sin 2x}{4} + \frac{x}{4}\cos 2x - \frac{1}{4}\frac{\sin 2x}{2} \qquad \Rightarrow I_1 = \frac{x^3}{6} + \frac{x^2\sin 2x}{4} + \frac{x}{4}\cos 2x - \frac{\sin 2x}{8}$$

$$\Rightarrow I = \left[\frac{x^3}{6} + \frac{x^2\sin 2x}{4} + \frac{x}{4}\cos 2x - \frac{\sin 2x}{8}\right]_0^{\pi/2}$$

$$\Rightarrow I = \left\{\frac{\left(\frac{\pi}{2}\right)^3}{6} + \frac{\left(\frac{\pi}{2}\right)^2\sin\left(2.\frac{\pi}{2}\right)}{4} + \frac{\pi}{2}.\frac{1}{4}\cos\left(2.\frac{\pi}{2}\right) - \frac{\sin\left(2.\frac{\pi}{2}\right)}{8}\right\} - \{0\} \quad \therefore\ I = \frac{\pi^3}{48} - \frac{\pi}{8}$$

55. $\int_1^2 \log x\, dx$

Sol. Let $I = \int_1^2 \log x\, dx \quad \Rightarrow I = [I_1]_1^2$

$$I_1 = \int \log x\, dx \quad \Rightarrow I_1 = \log x\int 1\, dx - \int\left[\frac{d(\log x)}{dx}\int dx\right]dx \quad \Rightarrow I_1 = \log x.x - \int\frac{1}{x}.x\, dx$$

$$\Rightarrow I_1 = x\log x - \int dx \quad \Rightarrow I_1 = x\log x - x \quad \Rightarrow I = [x\log x - x]_1^2$$

$$\Rightarrow I = (2\log 2 - 2) - (1\log 1 - 1) \quad \Rightarrow I = 2\log 2 - 2 + 1 \quad \therefore\ I = 2\log 2 - 1$$

56. $\int_1^3 \frac{\log x}{(1+x)^2}dx$

Sol. Let $I = \int_1^3 \frac{\log x}{(1+x)^2}dx \quad \Rightarrow I = [I_1]_1^3$

$$\therefore\ I_1 = \int\frac{\log x}{(1+x)^2}dx \quad \Rightarrow I_1 = \int(\log x)(1+x)^{-2}dx$$

$$\Rightarrow I_1 = \log x\int(1+x)^{-2}dx - \int\left[\frac{d(\log x)}{dx}\int(1+x)^{-2}dx\right]dx$$

$$\Rightarrow I_1 = \log x\frac{(1+x)^{-1}}{-1} - \int\frac{1}{x}.\frac{(1+x)^{-1}}{-1}dx \quad \Rightarrow I_1 = -\frac{1}{(1+x)}\log x + \int\frac{1}{x(1+x)}dx$$

$$\Rightarrow I_1 = -\frac{1}{(1+x)}\log x + \int\frac{(1+x) - x}{x(1+x)}dx \quad \Rightarrow I_1 = -\frac{\log x}{(1+x)} + \int\frac{1}{x}dx - \int\frac{1}{(x+1)}dx$$

$$\Rightarrow I_1 = -\frac{\log x}{(1+x)} + \log|x| - \log|x+1| \quad \Rightarrow I_1 = -\frac{\log x}{(1+x)} + \log\left|\frac{x}{x+1}\right|$$

$$I = \left[-\frac{\log x}{(1+x)} + \log\left|\frac{x}{x+1}\right|\right]_1^3 \quad \Rightarrow I = \left[\left\{-\frac{\log 3}{(1+3)} + \log\left(\frac{3}{3+1}\right)\right\} - \left\{-\frac{\log 1}{(1+1)} + \log\left(\frac{1}{1+1}\right)\right\}\right]$$

$$\Rightarrow I = \left[-\frac{\log 3}{4} + \log\left(\frac{3}{4}\right) - \log\left(\frac{1}{2}\right)\right] \quad \Rightarrow I = \left[-\frac{\log 3}{4} + \log 3 - \log 4 - \log 1 + \log 2\right]$$

$$\Rightarrow I = \left[\frac{3}{4}\log 3 - 2\log 2 + \log 2\right] \quad \therefore\ I = \frac{3}{4}\log 3 - \log 2$$

57. $\int_0^{e^2}\left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\}dx$

Sol. Let $I = \int_0^{e^2}\left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\}dx$, Put $t = \log x \quad \Rightarrow e^t = x \quad \Rightarrow \frac{dt}{dx} = \frac{1}{x} \quad \Rightarrow x\,dt = dx \quad \Rightarrow e^t dt = dx$

When $x = 0$, $t = 0$ and when $x = e^2$, $t = 2$

$$I = \int_0^2\left(\frac{1}{t} - \frac{1}{t^2}\right)e^t dt, \text{ Here } f(t) = \frac{1}{t},\quad f'(t) = -\frac{1}{t^2}dt$$

$$\Rightarrow I = \int_0^2 e^t\{f(t) + f'(t)\}dt = \left[e^t.f(t)\right]_0^2 = \left[\frac{e^t}{t}\right]_0^2 = \frac{e^2}{2} - \frac{e^0}{0} = \frac{e^2}{2}$$

58. $\int_1^e e^x\left(\frac{1+x\log x}{x}\right)dx$

Sol. Let $I = \int_1^e e^x\left(\frac{1+x\log x}{x}\right)dx \quad \Rightarrow I = [I_1]_1^e$

$$\Rightarrow I_1 = \int e^x\left(\frac{1+x\log x}{x}\right)dx \quad \Rightarrow I_1 = \int e^x\left(\frac{1}{x} + \log x\right)dx$$

Where $f(x) = \log x$, $f'(x) = \frac{1}{x}$; $\quad \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c$

$$\therefore\ I_1 = e^x.\log x \quad \Rightarrow I = \left[e^x\log x\right]_1^e \quad \Rightarrow I = \left(e^e\log e - e^1\log 1\right) \quad \therefore\ I = e^e$$

59. $\int_0^1 \frac{x\,e^x}{(1+x)^2}dx$

Sol. Let $I = \int_0^1 \frac{x\,e^x}{(1+x)^2}dx \quad \Rightarrow I = [I_1]_0^1$

$$\Rightarrow I_1 = \int e^x\left\{\frac{(x+1)-1}{(1+x)^2}\right\}dx \quad \Rightarrow I_1 = \int e^x\left\{\frac{1}{1+x} - \frac{1}{(1+x)^2}\right\}dx$$

Where $f(x) = \frac{1}{1+x}$, $f'(x) = -\frac{1}{(1+x)^2}$; $\quad \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c$

$$I_1 = e^x\left(\frac{1}{1+x}\right) \quad \Rightarrow I = \left[\frac{e^x}{1+x}\right]_0^1 \quad \Rightarrow I = \left(\frac{e^1}{1+1} - \frac{e^0}{1+0}\right) \quad \therefore\ I = \left(\frac{e}{2} - 1\right)$$

60. $\int_0^{\pi/2} 2\tan^3 x\, dx$

Sol. Let $I = \int_0^{\pi/2} 2\tan^3 x\, dx \quad \Rightarrow I = 2\int_0^{\pi/2} \tan^2 x \tan x\, dx = 2\int_0^{\pi/2} \left(\sec^2 x - 1\right)\tan x\, dx$

$\Rightarrow I = 2\int_0^{\pi/2} \left(\sec^2 x \tan x - \tan x\right)dx = 2\int_0^{\pi/2} \sec^2 x \tan x\, dx - 2\int_0^{\pi/2} \tan x\, dx$

Put $t = \tan x \quad \Rightarrow dt = \sec^2 x\, dx$

When $x = 0,\ t = 0$ and when $= \dfrac{\pi}{2},\ t = 1;\ \ I = 2\int_0^1 t\, dt - 2\left[\log\sec x\right]_0^{\pi/4}$

$\Rightarrow I = 2\left[\dfrac{t^2}{2}\right]_0^1 - 2\left[\log\sec\dfrac{\pi}{4} - \log\sec 0\right] \quad \Rightarrow I = 2\left[\dfrac{1}{2} - 0\right] - 2\left[\log\sqrt{2} - \log 1\right]$

$\Rightarrow I = 1 - 2\left[\dfrac{1}{2}\log 2 - 0\right] = 1 - \log 2$

61. $\int_1^2 \dfrac{5x^2}{x^2 + 4x + 3}dx$

Sol. Let $I = \int_1^2 \dfrac{5x^2}{x^2 + 4x + 3}dx \quad \Rightarrow I = \left[I_1\right]_1^2$

$\therefore\ I_1 = \int \dfrac{5x^2}{x^2 + 4x + 3}dx \quad \Rightarrow I_1 = \int\left(5 + \dfrac{-20x - 15}{x^2 + 4x + 3}\right)dx$

$\Rightarrow I_1 = \int 5dx - 5\int \dfrac{4x + 3}{(x+3)(x+1)}dx \quad \Rightarrow I_1 = 5x - 5\int \dfrac{4x+3}{(x+3)(x+1)}dx$

$\because\ \dfrac{4x+3}{(x+3)(x+1)} = \dfrac{A}{x+3} + \dfrac{B}{x+1} \qquad \Rightarrow \dfrac{4x+3}{(x+3)(x+1)} = \dfrac{A(x+1) + B(x+3)}{(x+3)(x+1)}$

$\Rightarrow 4x + 3 = Ax + A + Bx + 3B \quad \Rightarrow 4x + 3 = x(A+B) + A + 3B$

By, Equating co-efficient both side we get

$A + B = 4 \quad ...(1) \qquad\qquad A + 3B = 3 \quad ...(2)$

Now, Solving equation (1) & (2) then we get, $A = \dfrac{9}{2},\ B = \dfrac{-1}{2}$

$I_1 = 5x - 5\int\left(\dfrac{A}{x+3} + \dfrac{B}{x+1}\right)dx \quad \Rightarrow I_1 = 5x - 5.\dfrac{9}{2}\log|x+3| - 5.\left(\dfrac{-1}{2}\right)\log(x+1)$

$\Rightarrow I_1 = 5x - \dfrac{45}{2}\log|x+3| + \dfrac{5}{2}\log|x+1|$

$I = \left[5x - \dfrac{45}{2}\log|x+3| + \dfrac{5}{2}\log|x+1|\right]_1^2$

$\Rightarrow I = \left[5.2 - \dfrac{45}{2}\log|2+3| + \dfrac{5}{2}\log(2+1)\right] - \left[5.1 - \dfrac{45}{2}\log(1+3) + \dfrac{5}{2}\log(1+1)\right]$

$\Rightarrow I = \left[10 - \dfrac{45}{2}\log 5 + \dfrac{5}{2}\log 3 - 5 + \dfrac{45}{2}\log 4 - \dfrac{5}{2}\log 2\right]$

$\Rightarrow I = 5 - \dfrac{45}{2}\log 5 + \dfrac{5}{2}\log 3 + 45\log 2 - \dfrac{5}{2}\log 2 \qquad \therefore\ I = 5 - \dfrac{5}{2}\left(9\log\dfrac{5}{4} - \log\dfrac{3}{2}\right)$

EXERCISE 16B (Pg.No.: 811)

Very-Short-Answer Questions

Evaluate the following integrals:

1. $\int_0^1 \frac{dx}{2x-3}$

Sol. Let $I=\int_0^1 \frac{dx}{2x-3} \Rightarrow I=\left[\frac{\log|2x-3|}{2}\right]_0^1 \Rightarrow I=\log\frac{|2-3|}{2}-\log\frac{|2.0-3|}{2}$

$\Rightarrow I=\frac{1}{2}(\log 1-\log 3) \quad \therefore I=\frac{-1}{2}\log 3$

2. $\int_0^1 \frac{2x}{(1+x^2)}dx$

Sol. Let $I=\int_0^1 \frac{2x}{(1+x^2)}dx$; Put $t=1+x^2 \Rightarrow dt=2x\,dx$

When $x=0,\ t=1$ and when $x=1,\ t=2$

$I=\int_1^2 \frac{dt}{t}=[\log t]_1^2=\log 2-\log 1=\log 2-0=\log 2$

3. $\int_1^2 \frac{3x}{(9x^2-1)}dx$

Sol. Let $I=\int_1^2 \frac{3x}{(9x^2-1)}dx$, Put, $9x^2-1=t \Rightarrow 18x\,dx=dt \Rightarrow 3x\,dx=\frac{dt}{6}$

$I_1=\int \frac{3x}{(9x^2-1)}dx \Rightarrow I_1=\frac{1}{6}\int\frac{dt}{t} \Rightarrow I_1=\frac{1}{6}\left[\log\left|9x^2-1\right|\right]_1^2$

$\Rightarrow I_1=\frac{1}{6}\left[\log\left|9(2)^2-1\right|-\log\left|9(1)^2-1\right|\right] \qquad \therefore I_1=\frac{1}{6}[\log 35-\log 8]$

4. $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

Sol. Let $I=\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$, Put, $\tan^{-1}x=t \Rightarrow \frac{1}{1+x^2}dx=dt$

When $x=0,\ t=0$ and when $x=1,\ t=\frac{\pi}{4}$

$I=\int_0^{\pi/4} t\,dt \Rightarrow I=\left[\frac{t^2}{2}\right]_0^{\pi/4} \Rightarrow I=\frac{\left(\frac{\pi}{4}\right)^2}{2}-0 \quad \therefore I=\frac{\pi^2}{32}$

5. $\int_0^1 \frac{e^x}{(1+e^{2x})}dx$

Sol. Let $I=\int_0^1 \frac{e^x}{(1+e^{2x})}dx$, Put $t=e^x \Rightarrow dt=e^x dx$

When $x=0,\ t=1$ and when $x=1,\ t=e$

$$\Rightarrow I=\int_1^e \frac{dt}{1+t^2}=\left[\tan^{-1} t\right]_1^e=\tan^{-1} e-\tan^{-1} 1=\tan^{-1} e-\frac{\pi}{4}$$

6. $\int_0^1 \frac{2x}{\left(1+x^4\right)} dx$

Sol. Let $I=\int_0^1 \frac{2x}{\left(1+x^4\right)} dx$, Put $t=x^2 \Rightarrow dt=2x\,dx$

When $x=0, t=0$ and when $x=1, t=1$

$$\Rightarrow I=\int_0^1 \frac{dt}{1+t^2}=\left[\tan^{-1} t\right]_0^1=\tan^{-1} 1-\tan^{-1} 0=\frac{\pi}{4}-0=\frac{\pi}{4}$$

7. $\int_0^1 x\, e^{x^2} dx$

Sol. $\int_0^1 x\, e^{x^2} dx$, Put $t=x^2 \Rightarrow dx=2x\,dx \Rightarrow \frac{dt}{2}=xdx$

When $x=0,\ t=0$ and $x=1\ t=1$, $\int_0^1 e^t \frac{dt}{2}=\frac{1}{2}\int_0^1 e^t dt=\frac{1}{2}\left[e^t\right]_0^1=\frac{1}{2}(e-1)$

8. $\int_1^2 \frac{e^{1/x}}{\left(x^2\right)} dx$

Sol. Let $I=\int_1^2 \frac{e^{1/x}}{x^2} dx$, Put $t=\frac{1}{x} \Rightarrow \frac{dt}{dx}=-\frac{1}{x^2} \Rightarrow -dt=\frac{1}{x^2} dx$

When $x=1,\ t=1$ and $x=2$, $t=\frac{1}{2}$

$$\Rightarrow I=\int_1^2 \frac{e^{1/x}}{x^2} dx=\int_1^{1/2} e^t(-dt)=-\left[e^t\right]_1^{1/2}=-\left[e^{1/2}-e\right]=\left[e-e^{1/2}\right]=e-\sqrt{e}$$

9. $\int_0^{\pi/6} \frac{\cos x}{(3+4\sin x)} dx$

Sol. Let $I=\int_0^{\pi/6} \frac{\cos x}{(3+4\sin x)} dx$, Put $t=3+4\sin x \Rightarrow dt=4\cos x\,dx \Rightarrow \frac{dt}{4}=\cos x\,dx$

When $x=0,\ t=3$ and $x=\frac{\pi}{6},\ t=3+4.\frac{1}{2}=5$

$$\Rightarrow I=\int_3^5 \frac{dt/4}{t}=\frac{1}{4}[\log t]_3^5=\frac{1}{4}[\log 5-\log 3]=\frac{1}{4}\log\frac{5}{3}$$

10. $\int_0^{\pi/2} \frac{\sin x}{\left(1+\cos^2 x\right)} dx$

Sol. Let $I=\int_0^{\pi/2} \frac{\sin x}{\left(1+\cos^2 x\right)} dx$, Put $t=\cos x \Rightarrow dt=-\sin x\,dx$

When $x=0,\ t=1$ and $x=\frac{\pi}{2},\ t=0$

$$\therefore I = \int_{1}^{0} -\frac{dt}{1+t^2} = -\left[\tan^{-1} t\right]_1^0 = -\left[\tan^{-1} 0 - \tan^{-1} 1\right] = -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

11. $\int_{0}^{1} \frac{dx}{\left(e^x + e^{-x}\right)}$

Sol. Let $I = \int_{0}^{1} \frac{e^x}{e^{2x}+1} dx$, Put $t = e^x \;\Rightarrow dt = e^x dx$

When $x = 0,\; t = 1$ and $x = 1,\; t = e$

$$\Rightarrow I = \int_{1}^{e} \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_1^e = \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}$$

12. $\int_{1/e}^{e} \frac{dx}{x\left(\log x\right)^{1/3}}$

Sol. Let $I = \int_{1/e}^{e} \frac{dx}{x\left(\log x\right)^{1/2}}$, Put $t = \log x \;\Rightarrow dt = \frac{1}{x} dx$

When $x = \frac{1}{e},\; t = -1$ and $x = e,\; t = 1$

$$\Rightarrow \int_{-1}^{1} \frac{dt}{t^{1/2}} \quad {}^{1/3} dt = \left[\frac{t^{2/3}}{2/3}\right]_{-1}^{1} = \frac{3}{2}\left[t^{2/3}\right]_{-1}^{1} = \frac{3}{2}\left[(1)^{2/3} - (-1)^{2/3}\right] = \frac{3}{2}(1-1) = 0$$

Short-Answer Question
Evaluate the following integrals:

13. $\int_{0}^{1} \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx$

Sol. Let $I = \int_{0}^{1} \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx$, Put $\tan^{-1} x = t \;\Rightarrow \frac{1}{1+x^2} dx = dt$

When $x = 0,\, t = 0$ and when $x = 1,\, t = \frac{\pi}{4}$

$$\Rightarrow I = \int_{0}^{\pi/4} \sqrt{t}\; dt \;\Rightarrow I = \left[\frac{t^{3/2}}{3/2}\right]_0^{\pi/4} \;\Rightarrow I = \frac{2}{3}\left[t^{3/2}\right]_0^{\pi/4} \;\Rightarrow I = \frac{2}{3}\left[\left(\frac{\pi}{4}\right)^{3/2}\right] \quad \therefore I = \frac{1}{12}\pi^{3/2}$$

14. $\int_{0}^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}} dx$

Sol. Let $I = \int_{0}^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}} dx$, Put $t = 1 + \cos x \;\Rightarrow dt = -\sin x\, dx$

When $x = 0,\; t = 2$ and $x = \frac{\pi}{2},\; t = 1$

$$\Rightarrow I = \int_{2}^{1} \frac{-dt}{\sqrt{t}} = -\int_{2}^{1} e^{-1/2} dt = -\left[2t^{1/2}\right]_2^1 = -2\left[1 - \sqrt{2}\right] = 2\left(\sqrt{2} - 1\right)$$

15. $\int_{0}^{\pi/2} \sqrt{\sin x} \cos^5 x \; dx$

Sol. Let $I = \int_0^{\pi/2} \sqrt{\sin x} \cos^5 x \, dx$

$\Rightarrow I = \int_0^{\pi/2} \sqrt{\sin x} \cos^4 x \cos x \, dx = \int_0^{\pi/2} \sqrt{\sin x} \left(1-\sin^2 x\right)^2 \cos x \, dx$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$, When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$\therefore \; I_1 = \int \sqrt{t}\left(1-t^2\right)\left(1-t^2\right) dt \;\; \Rightarrow I_1 = \int \sqrt{t}\left(1-2t^2+t^4\right) dt \;\; \Rightarrow \int\left(\sqrt{t}-2t^{5/2}+t^{9/2}\right) dt$

$\Rightarrow I_1 = \frac{t^{3/2}}{3/2} - 2\frac{t^{7/2}}{7/2} + \frac{t^{11/2}}{11/2} \;\; \Rightarrow I_1 = \left[\frac{2}{3}t^{3/2} - \frac{4}{7}t^{7/2} + \frac{2}{11}t^{11/2}\right]_0^1$

$\Rightarrow I = \frac{2}{3} - \frac{4}{7} + \frac{2}{11} \;\; \Rightarrow I = \frac{154-132+42}{231} \quad \therefore \; I = \frac{64}{231}$

16. $\int_0^{\pi/2} \frac{\sin x \cos x}{\left(1+\sin^4 x\right)} dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sin x \cos x}{\left(1+\sin^4 x\right)} dx \quad \Rightarrow I = [I_1]_0^{\pi/2}$

$\Rightarrow I_1 = \int \frac{\sin x \cos x}{\left(1+\sin^4 x\right)} dx \;\; \Rightarrow I_1 = \int \frac{\sin x . \cos x}{1+\left(\sin^2 x\right)^2} dx$

Put $\sin^2 x = t \;\Rightarrow 2\sin x \cos x \, dx = dt \;\Rightarrow \sin x \cos x \, dx = \frac{dt}{2}$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$\therefore \; I_1 = \frac{1}{2}\int \frac{dt}{\left(1+t^2\right)} \;\; \Rightarrow I_1 = \frac{1}{2}\tan^{-1}(t)$

$\Rightarrow I = [I_1]_0^1 \;\; \Rightarrow I = \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1} 0\right] = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$

17. $\int_0^a \sqrt{a^2 - x^2} \, dx$

Sol. Let $I = \int_0^a \sqrt{a^2 - x^2} \, dx$

$\Rightarrow I_1 = \int \sqrt{a^2 - x^2} \, dx \;\; \Rightarrow I_1 = \frac{x}{a}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$\Rightarrow I = \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin\left(\frac{x}{a}\right)\right]_0^a \quad \Rightarrow I = \left\{\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right\} - \left\{\frac{a}{2}\sqrt{a^2 - 0} + \frac{a^2}{2}\sin^{-1}(0)\right\}$

$\Rightarrow I = \frac{a^2}{2}\sin^{-1}(1) \;\; \Rightarrow I = \frac{a^2}{2}\sin^{-1}\left(\sin\frac{\pi}{2}\right) \;\; \Rightarrow I = \frac{a^2}{2}.\frac{\pi}{2} \quad \therefore \; I = \frac{\pi a^2}{4}$

18. $\int_0^{\sqrt{2}} \sqrt{2 - x^2} \, dx$

Sol. Let $I=\int_0^{\sqrt{2}}\sqrt{2-x^2}\,dx$

$$\Rightarrow I_1=\int\sqrt{\left(\sqrt{2}\right)^2-\left(x\right)^2}\;dx\quad\Rightarrow I_1=\frac{x}{2}\sqrt{\left(\sqrt{2}\right)^2-\left(x\right)^2}+\frac{2}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$\Rightarrow I=\left[I_1\right]_0^{\sqrt{2}}\quad\Rightarrow I=\left[\frac{x}{2}\sqrt{2-x^2}+\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$$

$$\Rightarrow I=\left[\frac{\sqrt{2}}{2}\sqrt{2-2}+\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\right]-\left[\frac{0}{2}\sqrt{2-0}+\sin^{-1}(0)\right]\quad\Rightarrow I=\left[0+\sin^{-1}(1)\right]\quad\therefore\ I=\frac{\pi}{2}$$

19. $\int_0^a\frac{x^4}{\sqrt{a^2-x^2}}dx$

Sol. Let $I=\int_0^a\frac{x^4}{\sqrt{a^2-x^2}}dx$

Let, $x=a\sin x\ \Rightarrow dx=a\cos\theta\,d\theta$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\left(a\sin\theta\right)^4}{\sqrt{a^2-a^2\sin^2\theta}}.a\cos\theta d\theta$$

When $x=0\ \Rightarrow\ \theta=0\quad\Rightarrow\ x=a\quad\therefore\ \theta=\frac{\pi}{2}$

$$\therefore\ \ I=\int_0^{\pi/2}\frac{a^4\sin^4\theta}{\sqrt{a^2\left(1-\sin^2\theta\right)}}.a\cos\theta\ d\theta\quad\Rightarrow I=a^4\int_0^{\pi/2}\sin^4\theta\ d\theta\quad\Rightarrow I=a^4\int\sin^2\theta\sin^2\theta\ d\theta$$

$$\Rightarrow I=a^4\int\left(\frac{1-\cos2\theta}{2}\right)\left(\frac{1-\cos2\theta}{2}\right)d\theta\ \Rightarrow I=\frac{a^4}{4}\int\left(1-2\cos2\theta+\cos^22\theta\right)d\theta$$

$$\Rightarrow I=\frac{a^4}{4}\left[\int d\theta-2\int\cos2\theta\ d\theta+\int\left(\frac{1-\cos4\theta}{2}\right)d\theta\right]$$

$$\Rightarrow I=\frac{a^4}{4}\left[\theta-\frac{2\sin2\theta}{2}+\frac{1}{2}\left(\theta+\frac{\sin4\theta}{4}\right)\right]_0^{\pi/2}\quad\Rightarrow I=\frac{a^4}{4}\left[\frac{3\theta}{2}-\sin2\theta+\frac{\sin4\theta}{8}\right]_0^{\pi/2}$$

$$\Rightarrow I=\frac{a^4}{4}\left[\left(3\frac{\pi}{4}\right)-\sin^2\left(\frac{\pi}{2}\right)+\frac{\sin\pi\left(\frac{\pi}{2}\right)}{8}\right]-\left[\frac{3(0)}{2}-\sin(0)+\frac{\sin(4)(0)}{8}\right]$$

$$\Rightarrow I=\frac{a^4}{4}\left[\frac{3\pi}{4}-0\right]\quad\therefore\ I=\frac{3\pi a^4}{16}$$

20. $\int_0^a\frac{x}{\sqrt{a^2+x^2}}dx$

Sol. Let $I=\int_0^a\frac{x}{\sqrt{a^2+x^2}}dx$, Put $a^2+x^2\Rightarrow t\quad\Rightarrow 2x\ dx=dt\quad\Rightarrow x\ dx=\frac{dt}{2}$

When $x=0,\ t=a^2$, & $x=a, t=2a^2$

$$\Rightarrow I=\int_{a^2}^{2a^2}\frac{1}{\sqrt{t}}.\frac{dt}{2} \Rightarrow I=\frac{1}{2}\left[\frac{t^{1/2}}{1/2}\right]_{a^2}^{2a^2} \Rightarrow I=\left[\sqrt{t}\right]_{a^2}^{2a^2} \Rightarrow I=\sqrt{2}a-a \quad \therefore I=a\left(\sqrt{2}-1\right)$$

21. $\int_0^2 x\sqrt{2-x}\,dx$

Sol. Let $I=\int_0^2 x\sqrt{2-x}\,dx$, Put $2-x=t \Rightarrow -dx=dt \Rightarrow dx=-dt$

$\therefore\ x=2-t$, when $x=0,\ t=2$ & $x=2,\ t=0$

$$\Rightarrow I=\int_2^0(2-t)\sqrt{t}(-dt) \Rightarrow I=\int_0^2(2-t)\sqrt{t}\,dt \Rightarrow I=2\int_0^2 t^{1/2}\,dt-\int_0^2 t^{3/2}\,dt$$

$$\Rightarrow I=2\left[\frac{t^{3/2}}{3/2}\right]_0^2-\left[\frac{t^{5/2}}{5/2}\right]_0^2 \Rightarrow I=\frac{4}{3}\left(2^{3/2}\right)-\frac{2}{5}(2)^{5/2}$$

$$\Rightarrow I=\frac{4}{3}\left\{\left(\sqrt{2}\right)^2\right\}^{3/2}-\frac{2}{5}\left\{\left(\sqrt{2}\right)^2\right\}^{5/2} \Rightarrow I=\frac{4}{3}\left(\sqrt{2}\right)^3-\frac{2}{5}\left(\sqrt{2}\right)^5$$

$$\Rightarrow I=\frac{8\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}=8\sqrt{2}\left(\frac{1}{3}-\frac{1}{5}\right)=8\sqrt{2}\left(\frac{5-3}{15}\right)=\frac{8\sqrt{2}\times 2}{15}=\frac{16\sqrt{2}}{15}$$

22. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Sol. Let $I=\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx \Rightarrow I=\left[I_1\right]_0^1$

$$\Rightarrow I_1=\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx,$$

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta\,d\theta$. When $x=0,\ \theta=0$ and when $x=1,\ \theta=\frac{\pi}{4}$

$\therefore\ \theta=\tan^{-1}x$

$$\therefore\ I_1=\int \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\,d\theta \Rightarrow I_1=\int \sin^{-1}(\sin 2\theta)\sec^2\theta\,d\theta$$

$$\Rightarrow I_1=\int 2\theta\sec^2\theta\,d\theta \Rightarrow I_1=2\left[\theta\int\sec^2\theta\,d\theta-\int\left[\frac{d\theta}{d\theta}\right]\int\sec^2\theta\,d\theta\right]d\theta$$

$$\Rightarrow I_1=2\left[\theta\tan\theta-\int 1.\tan\theta\,d\theta\right] \Rightarrow I_1=2\left[\theta\tan\theta+\log\left|\cos\theta\right|\right]$$

$$\Rightarrow I_1=2\left[\theta\tan\theta+\log\left|\cos\theta\right|\right]_0^{\pi/4} \Rightarrow I_1=2\left[\frac{\pi}{4}.1+\log\left(\frac{1}{\sqrt{2}}\right)\right] \Rightarrow I_1=2\left[\frac{\pi}{4}-\frac{1}{2}\log 2\right]=\frac{\pi}{2}-\log 2$$

23. $\int_0^{\pi/2}\sqrt{1+\cos x}\,dx$

Sol. Let $I=\int_0^{\pi/2}\sqrt{1+\cos x}\,dx \Rightarrow I=\int_0^{\pi/2}\sqrt{2\cos^2\left(\frac{x}{2}\right)}\,dx \Rightarrow I=\sqrt{2}\int_0^{\pi/2}\cos\frac{x}{2}dx$

$$\Rightarrow I=\sqrt{2}.\left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} \Rightarrow I=2\sqrt{2}\left[\sin\frac{x}{2}\right]_0^{\pi/2} \Rightarrow I=2\sqrt{2}\left\{\sin\frac{\pi}{4}-0\right\} \Rightarrow I=2\sqrt{2}.\frac{1}{\sqrt{2}}$$

$\therefore\ I = 2$

24. $\int_0^{\pi/2} \sqrt{1+\sin x}\,dx$

Sol. Let $I = \int_0^{\pi/2} \sqrt{1+\sin x}\,dx \Rightarrow I = \int_0^{\pi/2} \sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\,dx \Rightarrow I = \int_0^{\pi/2} \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx$

$$\Rightarrow I = \left[\frac{\sin\frac{x}{2}}{\frac{1}{2}} - \frac{\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} \Rightarrow I = 2\left[\sin\frac{x}{2} - \cos\frac{x}{2}\right]_0^{\pi/2} \Rightarrow I = 2\left[\left\{\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right\} - \{\sin 0 - \cos 0\}\right]$$

$\therefore\ I = 2$

25. $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \Rightarrow I = [I_1]_0^{\pi/2}$

$$\Rightarrow I_1 = \int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Dividing numerator and denominator by $\cos^2 x$

$$\therefore\ I_1 = \int \frac{\frac{1}{\cos^2 x}}{\frac{a^2\cos^2 x}{\cos^2 x} + \frac{b^2\sin^2 x}{\cos^2 x}}\,dx \Rightarrow I_1 = \int \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$, When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$$\therefore\ I_1 = \int_0^{\infty} \frac{dt}{a^2 + b^2t^2} \Rightarrow I_1 = \frac{1}{b^2}\int_0^{\infty} \frac{1}{(t)^2 + \left(\frac{a}{b}\right)^2}dt \Rightarrow I_1 = \frac{1}{ab}\left[\tan^{-1}\left(\frac{b\tan x}{a}\right)\right]_0^{\infty}$$

$$\Rightarrow I = \frac{1}{ab}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right] \Rightarrow I = \frac{1}{ab}\left[\frac{\pi}{2} - 0\right] \quad \therefore\ I = \frac{\pi}{2ab}$$

26. $\int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{1+\cos^2 x} \Rightarrow I = [I_1]_0^{\pi/2}$

$$\Rightarrow I_1 = \int \frac{1}{1+\cos^2 x}dx$$

By, Dividing numerator and denominator by $\cos^2 x$, $I_1 = \int \frac{\frac{1}{\cos^2 x}}{\frac{1+\cos^2 x}{\cos^2 x}}dx \Rightarrow I_1 = \int \frac{\sec^2 x}{\sec^2 x + 1}\,dx$

$$\Rightarrow I_1 = \int \frac{\sec^2 x}{1+\tan^2 x + 1}dx = \int \frac{\sec^2 x}{2+\tan^2 x}dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$, When $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \infty$

$$\therefore\ I_1=\int_0^{\infty}\frac{dt}{2+t^2}\quad\Rightarrow I_1=\int_0^{\infty}\frac{dt}{(t)^2+\left(\sqrt{2}\right)^2}\quad\Rightarrow I_1=\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^{\infty}$$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]\quad\Rightarrow I=\frac{1}{\sqrt{2}}.\frac{\pi}{2}\quad\therefore\ I=\frac{\pi}{2\sqrt{2}}$$

27. $\displaystyle\int_0^{\pi/2}\frac{dx}{4+9\cos^2 x}$

Sol. Let $I=\displaystyle\int_0^{\pi/2}\frac{dx}{4+9\cos^2 x}$

By dividing numerator and denominator by $\cos^2 x$, $I=\displaystyle\int_0^{\pi/2}\frac{\frac{1}{\cos^2 x}}{\frac{4+9\cos^2 x}{\cos^2 x}}dx$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sec^2 x}{4\sec^2 x+9}dx\quad\Rightarrow I=\int_0^{\pi/2}\frac{\sec^2 x}{4\left(1+\tan^2 x\right)+9}dx\quad\Rightarrow I=\int_0^{\pi/2}\frac{\sec^2 x}{13+4\tan^2 x}dx$$

Put $\tan x=t\quad\Rightarrow\sec^2 x\,dx=dt$, When $x=0,\,t=0$ and when $x=\frac{\pi}{2},\ t=\infty$

$$\Rightarrow I=\int_0^{\infty}\frac{dt}{13+4t^2}\quad\Rightarrow I=\frac{1}{4}\int_0^{\infty}\frac{1}{\frac{13}{4}+t^2}dt\quad\Rightarrow I=\frac{1}{4}\int\frac{1}{\left(\frac{\sqrt{13}}{2}\right)^2+(t)^2}dt$$

$$\Rightarrow I=\frac{1}{4}.\frac{1}{\frac{\sqrt{13}}{2}}\left[\tan^{-1}\left(\frac{t}{\sqrt{13}/2}\right)\right]_0^{\infty}\quad\Rightarrow I=\frac{1}{2\sqrt{13}}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]$$

$$\Rightarrow I=\frac{1}{2\sqrt{13}}.\frac{\pi}{2}\quad\therefore\ I=\frac{\pi}{4\sqrt{13}}$$

28. $\displaystyle\int_0^{\pi/2}\frac{dx}{5+4\sin x}$

Sol. Let $I=\displaystyle\int_0^{\pi/2}\frac{dx}{5+4\sin x}\quad\Rightarrow I=\int_0^{\pi/2}\frac{dx}{5+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\left(1+\tan^2\frac{x}{2}\right)}{5\left(1+\tan^2\frac{x}{2}\right)+8\tan\frac{x}{2}}dx\quad\Rightarrow I=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{5\tan^2\frac{x}{2}+8\tan\frac{x}{2}+5}dx$$

Put, $\tan\frac{x}{2}=t\quad\Rightarrow\sec^2\frac{x}{2}.\frac{1}{2}dx=dt\quad\Rightarrow\sec^2\frac{x}{2}dx=2dt$

When $x=0,\,t=0$ and when $x=\frac{\pi}{2},\,t=1$

$$\Rightarrow I = 2\int_0^{\pi/2} \frac{dt}{5t^2+8t+5} \quad \Rightarrow I = 2\int_0^{\pi/2} \frac{dt}{5\left(t^2+\frac{8}{5}t+1\right)}$$

$$\Rightarrow I = \frac{2}{5}\int_0^1 \frac{dt}{(t)^2+2.t.\frac{4}{5}+\left(\frac{4}{5}\right)^2-\left(\frac{4}{5}\right)^2+1} \quad \Rightarrow I = \frac{2}{5}\int_0^1 \frac{dt}{\left(t+\frac{4}{5}\right)^2+\frac{9}{25}}$$

$$\Rightarrow I = \frac{2}{5}\int_0^1 \frac{dt}{\left(t+\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2} \quad \Rightarrow I = \frac{2}{5}.\frac{1}{3/5}\left[\tan^{-1}\left(\frac{t+\frac{4}{5}}{\frac{3}{5}}\right)\right]_0^1 \quad \Rightarrow I = \frac{2}{3}\left[\tan^{-1}\left(\frac{5t+4}{3}\right)\right]_0^1$$

$$\Rightarrow I = \frac{2}{3}\left[\tan^{-1}(3)-\tan^{-1}\left(\frac{4}{3}\right)\right] \quad \Rightarrow I = \frac{2}{3}\tan^{-1}\left(\frac{3-4/3}{1+3\times 4/3}\right) \quad \Rightarrow I = \frac{2}{3}\tan^{-1}\left(\frac{9-4/3}{1+4}\right) \Rightarrow$$

$$I = \frac{2}{3}\left[\tan^{-1}\left(\frac{5}{3}\times\frac{1}{5}\right)\right] \quad \therefore\ I = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$$

29. $\int_0^{\pi} \frac{dx}{6-\cos x}$

Sol. Let $I = \int_0^{\pi} \frac{dx}{6-\cos x} \Rightarrow I = [I_0]_0^{\pi}$

$$\Rightarrow I_1 = \int \frac{1}{6-\cos x}dx \quad \Rightarrow I_1 = \int \frac{1}{6-\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx \quad \Rightarrow I_1 = \int \frac{1+\tan^2\frac{x}{2}}{6\left(1+\tan^2\frac{x}{2}\right)-\left(1-\tan^2\frac{x}{2}\right)}dx$$

$$\Rightarrow I_1 = \int \frac{\sec^2\frac{x}{2}}{6+6\tan^2\frac{x}{2}-1+\tan^2\frac{x}{2}}dx \quad \Rightarrow I_1 = \int \frac{\sec^2\frac{x}{2}}{7\tan^2\frac{x}{2}+5}dx$$

Put $\tan\frac{x}{2} = t \quad \Rightarrow \sec^2\frac{x}{2}.\frac{1}{2}dx = dt \quad \Rightarrow \sec^2\frac{x}{2}dx = 2dt$

When $x = 0,\ t = 0$ and when $x = \pi,\ t = \infty$

$$\therefore\ I_1 = 2\int \frac{dt}{7t^2+5} \quad \Rightarrow I_1 = 2\int \frac{dt}{7\left(t^2+\frac{5}{7}\right)} \quad \Rightarrow I_1 = \frac{2}{7}\int \frac{dt}{(t)^2+\left(\frac{\sqrt{5}}{\sqrt{7}}\right)^2}$$

$$\Rightarrow I_1 = \frac{2}{7}.\frac{1}{\sqrt{5}/\sqrt{7}}\tan^{-1}\left(\frac{t}{\sqrt{5}/\sqrt{7}}\right) \quad \Rightarrow I_1 = \frac{2}{\sqrt{35}}\left[\tan^{-1}\left(\frac{\sqrt{7}t}{\sqrt{5}}\right)\right]_0^{\infty}$$

$$\Rightarrow I = \frac{2}{\sqrt{35}}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right] = \frac{2}{\sqrt{35}}\left[\frac{\pi}{2}-0\right] \quad \therefore I = \frac{\pi}{\sqrt{35}}$$

30. $\int_0^{\pi} \frac{dx}{5+4\cos x}$

Sol. Let $I=\int_0^{\pi}\frac{dx}{5+4\cos x} \Rightarrow I=[I_1]_0^{\pi}$

$$\Rightarrow I_1=\int\frac{dx}{5+4\cos x} \Rightarrow I_1=\int\frac{dx}{5+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} \Rightarrow I_1=\int\frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)}dx$$

$$\Rightarrow I_1=\int\frac{\sec^2\frac{x}{2}}{5\tan^2\frac{x}{2}+5+4-4\tan^2\frac{x}{2}}dx \Rightarrow I_1=\int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9}dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \sec^2\frac{x}{2}.\frac{1}{2}dx=dt \Rightarrow \sec^2\frac{x}{2}dx=2dt$

When $x=0,\ t=0$ and when $x=\pi,\ t=\infty$

$$\therefore\ I_1=2\int\frac{dt}{t^2+(3)^2} \Rightarrow I_1=2.\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)=\frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\infty}$$

$$\Rightarrow I=\frac{2}{3}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right] \Rightarrow I=\frac{2}{3}.\frac{\pi}{2}\ \therefore\ I=\frac{\pi}{3}$$

31. $\int_0^{\pi/2}\frac{dx}{\cos x+2\sin x}$

Sol. Let $I=\int_0^{\pi/2}\frac{dx}{\cos x+2\sin x} \Rightarrow I=[I_1]_0^{\pi/2}$

$$\Rightarrow I_1=\int\frac{dx}{\cos x+2\sin x} \Rightarrow I_1=\int\frac{dx}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+2.\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} \Rightarrow I_1=\int\frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}+4\tan\frac{x}{2}}dx$$

$$\Rightarrow I_1=\int\frac{\sec^2\frac{x}{2}}{1-\tan^2\frac{x}{2}+4\tan\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \sec^2\frac{x}{2}.\frac{1}{2}dx=dt \Rightarrow \sec^2\frac{x}{2}dx=2dt$

When $x=0,\ t=0$ and when $x=\frac{\pi}{2},\ t=1$

$$I_1=\int\frac{2\,dt}{-t^2+4t+1} \Rightarrow I_1=-2\int\frac{dt}{t^2-4t-1} \Rightarrow I_1=-2\int\frac{1}{(t)^2-2.t.2+(2)^2-(2)^2-1}$$

$$\Rightarrow I_1=-2\int\frac{1}{(t-2)^2-\left(\sqrt{5}\right)^2}dt \Rightarrow I_1=2\int\frac{1}{\left(\sqrt{5}\right)^2-(t-2)^2}dt \Rightarrow I=2.\frac{1}{2.\sqrt{5}}\log\left|\frac{\sqrt{5}+t-2}{\sqrt{5}-t+2}\right|$$

$$\Rightarrow I=\frac{1}{\sqrt5}\log\left(\frac{\sqrt5+t-2}{\sqrt5-t+2}\right)_0^1 \Rightarrow I=\frac{1}{\sqrt5}\left[\log\left|\frac{\sqrt5-1}{\sqrt5+1}\right|-0\right] \Rightarrow I=\frac{1}{\sqrt5}\left[\log\left|\frac{(\sqrt5-1)\times(\sqrt5-1)}{(\sqrt5+1)(\sqrt5-1)}\right|\right]$$

$$\Rightarrow I=\frac{1}{\sqrt5}\left[\log\left|\frac{5-2\sqrt5+1}{5-1}\right|\right] \Rightarrow I=\frac{1}{\sqrt5}\log\left|\frac{6-2\sqrt5}{4}\right| \quad \therefore I=\frac{1}{\sqrt5}\log\left|\frac{3-\sqrt5}{2}\right|$$

32. $\int_0^{\pi}\frac{dx}{3+2\sin x+\cos x}$

Sol. Let $I=\int_0^{\pi}\frac{dx}{3+2\sin x+\cos x} \Rightarrow I=[I_1]_0^{\pi}$

$$\Rightarrow I_1=\int\frac{dx}{3+2\sin x+\cos x} \Rightarrow I_1=\int\frac{dx}{3+2.\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$\Rightarrow I_1=\int\frac{1+\tan^2\frac{x}{2}}{3\left(1+\tan^2\frac{x}{2}\right)+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}} \Rightarrow I_1=\int\frac{\sec^2\frac{x}{2}}{3+3\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

$$\Rightarrow I=\int\frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}dx \Rightarrow I_1=\frac{1}{2}\int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+2}dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \sec^2\frac{x}{2}.\frac{1}{2}dx=dt \Rightarrow \sec^2\frac{x}{2}dx=2dt$

When $x=0,\ t=0$ and when $x=\pi,\ t=\infty$

$$\therefore I_1=\frac{1}{2}\int\frac{2\,dt}{t^2+2t+2} \Rightarrow I_1=\int\frac{dt}{(t)^2+2.t.1+(1)^2-(1)^2+2} \Rightarrow I_1=\int\frac{dt}{(t+1)^2+(1)^2}dt$$

$$\Rightarrow I_1=\tan^{-1}(t+1) \Rightarrow I=\left[\tan^{-1}(t+1)\right]_0^{\infty}=\left[\tan^{-1}\infty-\tan^{-1}(0+1)\right]=\frac{\pi}{2}-\frac{\pi}{4} \qquad \therefore I=\frac{\pi}{4}$$

33. $\int_0^{\pi/4}\frac{\tan^3 x}{1+\cos 2x}dx$

Sol. Let $I=\int_0^{\pi/4}\frac{\tan^3 x}{1+\cos 2x}dx \Rightarrow I=[I_1]_0^{\pi/4}$

$$\therefore I_1=\int\frac{\tan^3 x}{1+\cos 2x}dx \Rightarrow I_1=\int\frac{\tan^3 x}{2\cos^2 x}dx \Rightarrow I_1=\frac{1}{2}\int\tan^3 x\sec^2 x\,dx$$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$, when $x=0,\ t=0$ and when $x=\frac{\pi}{4},\ t=1$

$$\Rightarrow I_1=\frac{1}{2}\int_0^1 t^3\,dt \Rightarrow I_1=\frac{1}{2}.\left[\frac{t^4}{4}\right]_0^1 \quad \therefore I=\frac{1}{8}$$

34. $\int_0^{\pi/2}\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx$

Sol. Let $I=\int_0^{\pi/2}\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx \Rightarrow I=[I_1]_0^{\pi/2}$; $I_1=\int\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx$

Put $\cos x=t \Rightarrow -\sin x\,dx=dt \Rightarrow \sin x\,dx=-dt$

When $x=0,\ t=1$ and when $x=\frac{\pi}{2},\ t=0$

$\Rightarrow I_1=-\int\frac{t}{t^2+3t+2}dt \Rightarrow I_1=-\int\frac{t}{(t+2)(t+1)}dt$

$\frac{t}{(t+2)(t+1)}=\frac{A}{t+2}+\frac{B}{t+1} \Rightarrow \frac{t}{(t+2)(t+1)}=\frac{A(t+1)+B(t+2)}{(t+2)(t+1)}$

$\Rightarrow t=At+A+Bt+2B \Rightarrow t=(A+B)t+A+2B$

By, Equating co-efficient both side we get

$A+B=1$ ...(1)

$A+2B=0$ ...(2)

Now, Solving equation (1) & (2) then we get $A=2,\ B=-1$

$I_1=-\left[\int\left(\frac{A}{t+2}+\frac{B}{t+1}\right)dt\right] \Rightarrow I_1=-\left[A\int\frac{1}{t+2}dt+B\int\frac{1}{t+1}dt\right]$

$\Rightarrow I_1=-\left[2\log(t+2)-\log(t+1)\right] \Rightarrow I=\left[\log(t+1)-2\log(t+2)\right]_1^0$

$\Rightarrow I=(\log 1-2\log 2)-(\log 2-2\log 3)$

$\Rightarrow I=-3\log 2+2\log 3$

$\therefore\ I=\log 9-\log 8$

35. $\int_0^{\pi/2}\frac{\sin 2x}{\cos^4 x+\sin^4 x}dx$

Sol. Let $I=\int_0^{\pi/2}\frac{\sin 2x}{\cos^4 x+\sin^4 x}dx \Rightarrow I=[I_1]_0^{\pi/2}$

$I_1=\int\frac{\sin 2x}{\cos^4 x+\sin^4 x}dx$

By, Dividing numerator and denominator by $\cos^4 x$, $I_1=\int\frac{\frac{\sin 2x}{\cos^4 x}}{\frac{\cos^4 x}{\cos^4 x}+\frac{\sin^4 x}{\cos^4 x}}dx$

$\Rightarrow I_1=\int\frac{\sin 2x.\sec^4 x}{1+\tan^4 x}dx \Rightarrow I_1=\int\frac{\frac{2\tan x}{1+\tan^2 x}.\sec^2 x.\sec^2 x}{1+\tan^4 x}dx \Rightarrow I_1=\int\frac{2\tan x\sec^2 x}{1+\left(\tan^2 x\right)^2}dx$

Put $\tan^2 x=t \Rightarrow 2\tan x\sec^2 x\,dx=dt$. When $x=0,\ t=0$ and when $x=\frac{\pi}{2},\ t=\infty$

$\therefore\ I_1=\int\frac{dt}{1+t^2} \Rightarrow I_1=\tan^{-1}(t) \Rightarrow I=\left[\tan^{-1}t\right]_0^{\infty} \Rightarrow I=\tan^{-1}(\infty)-\tan^{-1}(0)\ \therefore I=\frac{\pi}{2}$

36. $\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx$

Sol. Let $I=\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx \Rightarrow I=[I_1]_{\pi/3}^{\pi/2}$; $\quad I_1=\int\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx$

$$\Rightarrow I_1=\int\frac{\sqrt{2\cos^2\frac{x}{2}}}{\left(2\sin^2\frac{x}{2}\right)^{5/2}}dx \Rightarrow I_1=\int\frac{\sqrt{2}\cos\frac{x}{2}}{2^{5/2}\sin^5\frac{x}{2}}dx \Rightarrow I_1=\frac{1}{4}\int\frac{\cos\frac{x}{2}}{\sin^5\frac{x}{2}}dx$$

Put $\sin\frac{x}{2}=t \Rightarrow \cos\frac{x}{2}.\frac{1}{2}dx=dt \Rightarrow \cos\frac{x}{2}dx=2dt$

When $x=\frac{\pi}{3}, t=\frac{1}{2}$ and when $x=\frac{\pi}{2}, t=\frac{1}{\sqrt{2}}$

$$I_1=\frac{1}{4}\int\frac{2dt}{t^5} \Rightarrow I_1=\frac{1}{2}\int t^{-5}dt \Rightarrow I_1=\frac{1}{2}.\frac{t^{-4}}{-4} \Rightarrow I_1=-\frac{1}{8}.\frac{1}{t^4} \Rightarrow I=\frac{-1}{8}\left[\frac{1}{t^4}\right]_{1/2}^{1/\sqrt{2}}$$

$$\Rightarrow I=-\frac{1}{8}\left[\left(\sqrt{2}\right)^4-(2)^4\right] \Rightarrow I=-\frac{1}{8}[4-16] \Rightarrow I=\frac{12}{8} \quad \therefore I=\frac{3}{2}$$

37. $\int_0^1\left(\cos^{-1}x\right)^2dx$

Sol. Let $I=\int_0^1\left(\cos^{-1}x\right)^2dx \Rightarrow I=[I_1]_0^1$; $\quad I_1=\int\left(\cos^{-1}x\right)^2dx$

Put $\cos^{-1}x=t \quad \therefore x=\cos t$. When $x=0, t=\frac{\pi}{2}$ and when $x=1, t=0$

$$-\frac{1}{\sqrt{1-x^2}}dx=dt \Rightarrow -dx=\sqrt{1-x^2}dt \Rightarrow dx=-\sqrt{1-\cos^2 t}\,dt \Rightarrow dx=-\sin t\,dt$$

$$I_1=\int t^2.(-\sin t)\,dt \Rightarrow I_1=-\int t^2\sin t\,dt \Rightarrow I_1=-\left[t^2\int\sin t\,dt-\int\left[\frac{d(t^2)}{dt}\int\sin t\,dt\right]dt\right]$$

$$\Rightarrow I_1=-\left[-t^2\cos t+\int 2t\cos t\,dt\right] \Rightarrow I_1=t^2\cos t-2\int t\cos t\,dt$$

$$\Rightarrow I_1=t^2\cos t-2\left[t\int\cos t\,dt-\int\left[\frac{d(t)}{dt}\int\cos t\,dt\right]dt\right]$$

$$\Rightarrow I_1=t^2\cos t-2\left[t\sin t-\int 1.\sin t\,dt\right] \Rightarrow I_1=t^2\cos t-2[t\sin t+\cos t]$$

$$\Rightarrow I_1=t^2\cos t-2t\sin t-2\cos t \Rightarrow I=\left[t^2\cos t-2t\sin t-2\cos t\right]_{\pi/2}^0$$

$$\Rightarrow I=(0-2)-(-\pi) \quad \therefore I=\pi-2$$

38. $\int_0^1 x\left(\tan^{-1}x\right)^2dx$

Sol. Let $I=\int_0^1 x\left(\tan^{-1}x\right)^2dx \Rightarrow I=[I_1]_0^1$; $\quad I_1=\int x\left(\tan^{-1}x\right)^2dx$

Put $\tan^{-1}x=t \quad \therefore x=\tan t$. When $x=0, t=0$ and when $x=1, t=\frac{\pi}{4}$

$$\frac{1}{1+x^2}dx=dt \Rightarrow dx=\left(1+x^2\right)dt \Rightarrow dx=\left(1+\tan^2 t\right)dt \Rightarrow dx=\sec^2 t\,dt$$

$\therefore\ I_1 = \int \tan t.\, t^2 \sec^2 t\, dt \quad \Rightarrow I_1 = \int t^2 \tan t \sec^2 t\, dt$

$\Rightarrow I_1 = t^2 \int \tan t \sec^2 t\, dt - \int \left[\frac{d(t^2)}{dt} \int \tan t \sec t\, dt \right] dt$

Again put $\tan t = y \ \Rightarrow\ \sec^2 t\, dt = dy$

$I_1 = t^2 \int y\, dy - \int \left[2t. \int y\, dy \right] dt \quad \Rightarrow I_1 = t^2.\frac{y^2}{2} - \int 2t.\frac{y^2}{2}\, dt \quad \Rightarrow I_1 = t^2.\frac{\tan^2 t}{2} - \int t.\tan^2 t\, dt$

$\Rightarrow I_1 = \frac{t^2}{2}\tan^2 t - \left[t\int \tan^2 t\, dt. - \int \left[\frac{d(t)}{dt} \int \tan^2 t\, dt \right] dt \right]$

$\Rightarrow I_1 = \frac{t^2}{2}\tan^2 t - \left[t\int (\sec^2 t - 1)\, dt - \int \left[1. \int (\sec^2 t - 1)\, dt \right] dt \right]$

$\Rightarrow I_1 = \frac{t^2}{2}\tan^2 t - \left[t(\tan t - t) - \int (\tan t - t)\, dt \right] \ \Rightarrow I_1 = \frac{t^2}{2}\tan^2 t - \left[t\tan t - t^2 + \log|\cos t| + \frac{t^2}{2} \right]$

$\Rightarrow I_1 = \frac{t^2}{2}\tan^2 t - \left[t\tan t + \log|\cos t| - \frac{t^2}{2} \right] \ \Rightarrow I = \left[\frac{t^2}{2}\tan^2 t - t\tan t - \log|\cos t| + \frac{t^2}{2} \right]_0^{\frac{\pi}{4}}$

$\Rightarrow I = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \frac{\pi}{4} + \frac{1}{2}\log 2 + \frac{\pi^2}{32} \quad \therefore\ I = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2$

39. $\int_0^1 \sin^{-1}\left(\sqrt{x}\right) dx$

Sol. Let $I = \int_0^1 \sin^{-1}\left(\sqrt{x}\right) dx,$

Put $t = \sin^{-1}\sqrt{x} \ \Rightarrow \sqrt{x} = \sin t \ \Rightarrow x = \sin^2 t$, when $x = 0,\ t = 0$ and $x = 1,\ t = \frac{\pi}{2}$

$\frac{dt}{dx} = \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}} \ \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-\sin^2 t}}.\frac{1}{2\sin t} \ \Rightarrow dt = \frac{dx}{2\cos t \sin t} \ \Rightarrow \sin 2t\, dt = dx$

$\therefore\ I = \int_0^{\pi/2} t.\sin 2t\, dt,$ Let $I = [I_1]_0^{\pi/2}$

$I_1 = t\int \sin 2t\, dt - \int \left(\frac{dt}{dt} \int \sin 2t\, dt \right) dt = t.\left(\frac{-\cos 2t}{2} \right) + \int 1.\frac{\cos 2t}{2} dt \ \Rightarrow I_1 = -\frac{t}{2}\cos 2t + \frac{1}{4}\sin 2t$

$\therefore\ I = \left[\frac{-t}{2}\cos 2t + \frac{1}{4}\sin 2t \right]_0^{\pi/2} = \left[-\frac{\pi}{4}\cos \pi + \frac{1}{4}(0) + 0 \right] = -\frac{\pi}{4}(-1) = \frac{\pi}{4}$

40. $\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

Sol. Let $I = \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx \quad \Rightarrow I = [I_1]_0^a; \quad I_1 = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

Put $x = a\tan^2\theta \quad \Rightarrow \frac{x}{a} = \tan^2\theta \quad \Rightarrow \tan\theta = \sqrt{\frac{x}{a}} \quad \Rightarrow \theta = \tan^{-1}\left(\sqrt{\frac{x}{a}}\right)$

When $x = 0,\ \theta = 0$ and when $x = a,\ \theta = \frac{\pi}{4}$

$\therefore\ I_1=\int \sin^{-1}\sqrt{\dfrac{a\tan^2\theta}{a+a\tan^2\theta}}.2a\tan\theta\sec^2\theta\ d\theta \ \Rightarrow I_1=\int \sin^{-1}\sqrt{\dfrac{\tan^2\theta}{1+\tan^2\theta}}.2a\ \tan\theta\ \sec^2\theta\ d\theta$

$\Rightarrow I_1=\int \sin^{-1}\sqrt{\dfrac{\tan^2\theta}{\sec^2\theta}}.2a\ \tan\theta\sec^2\theta\ d\theta \ \Rightarrow I_1=\int \sin^{-1}(\sin\theta)2a\tan\theta\sec^2\theta\ d\theta$

$\Rightarrow I_1=\int \theta\ 2a\ \tan\theta\ \sec^2\theta\ d\theta \quad \Rightarrow I_1=2a\int\theta\ \tan\theta\sec^2\theta\ d\theta$

$\Rightarrow I=2a\left[\theta\int\tan\theta\sec^2\theta d\theta-\int\left[\dfrac{d\theta}{d\theta}\int\tan\theta\sec^2\theta\ d\theta\right]d\theta\right]$

Put $\tan\theta=t \quad \Rightarrow \sec^2\theta\ d\theta=dt$

$\therefore\ I_1=2a\left[\theta\int t\ dt-\int\left[1.\int t\ dt\right]dt\right] \ \Rightarrow I_1=2a\left[\theta\dfrac{t^2}{2}-\int\dfrac{t^2}{2}dt\right]$

$\Rightarrow I_1=2a\left[\theta\dfrac{\tan^2\theta}{2}-\dfrac{1}{2}\int\tan^2\theta\ d\theta\right] \ \Rightarrow I_1=a\theta\ \tan^2\theta-a\int\left(\sec^2\theta-1\right)d\theta$

$\Rightarrow I_1=a\ \theta\ \tan^2\theta-a\left(\tan\theta-\theta\right) \ \Rightarrow I=\left[a\theta\tan^2\theta-a\left(\tan\theta-\theta\right)\right]_0^{\pi/4}=a.\dfrac{\pi}{4}.1-a\left(1-\dfrac{\pi}{4}\right)$

$\Rightarrow I=2a.\dfrac{\pi}{4}-a \quad \therefore\ I=a\left(\dfrac{\pi}{2}-1\right)$

41. $\displaystyle\int_0^9\frac{dx}{1+\sqrt{x}}$

Sol. Let $I=\displaystyle\int_0^9\frac{dx}{1+\sqrt{x}} \ \Rightarrow I=\left[I_1\right]_0^9;\quad I_1=\int\frac{1}{1+\sqrt{x}}dx \ \Rightarrow I_1=\int\frac{1}{1+\sqrt{x}}\times\frac{1-\sqrt{x}}{1-\sqrt{x}}dx$

$\Rightarrow I_1=\displaystyle\int\frac{1-\sqrt{x}}{1-x}dx \ \Rightarrow I_1=\int\frac{1}{1-x}dx-\int\frac{\sqrt{x}}{1-\left(\sqrt{x}\right)^2}dx$

Put $\sqrt{x}=t \quad \Rightarrow \dfrac{1}{2\sqrt{x}}dx=dt \quad \Rightarrow dx=2t\ dt$

$\therefore\ I_1=-\log\left|1-x\right|-\displaystyle\int\frac{t}{1-t^2}.2t\ dt \ \Rightarrow I_1=-\log\left|1-t^2\right|+2\int\frac{t^2}{t^2-1}dt$

$\Rightarrow I_1=-\log\left|1-t^2\right|+2\displaystyle\int\frac{\left(t^2-1\right)+1}{t^2-1}dt \ \Rightarrow I_1=-\log\left|1-t^2\right|+2\int\frac{t^2-1}{t^2-1}dt+2\int\frac{1}{t^2-1}dt$

$\Rightarrow I_1=-\log\left|1-t^2\right|+2\displaystyle\int dt+2.\frac{1}{2.1}\log\left|\frac{t-1}{t+1}\right| \ \Rightarrow I_1=-\log\left|1-t^2\right|+2t+\log\left|\frac{t-1}{t+1}\right|$

$\Rightarrow I_1=-\log\left|1-x\right|+2\sqrt{x}+\log\left|\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right|$

$\therefore\ I=\left[\log\left|\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right|+2\sqrt{x}-\log\left|1-x\right|\right]_0^9$

$\Rightarrow I=\left\{\log\left(\dfrac{\sqrt{9}-1}{\sqrt{9}+1}\right)+2\sqrt{9}-\log\left|1-9\right|\right\}-\{0\} \ \Rightarrow I=\log\left(\dfrac{2}{4}\right)+2.3-\log 8$

$$\Rightarrow I = \log\left|\frac{1}{2}\right| + 6 - \log 8 \quad \Rightarrow I = \log 1 - \log 2 + 6 - 3\log 2 \quad \therefore I = 6 - 4\log 2$$

42. $\int_0^1 x^3\sqrt{1+3x^4}\, dx$

Sol. Let $I = \int_0^1 x^3\sqrt{1+3x^4}\, dx \Rightarrow I = [I_1]_0^1; \quad I_1 = \int x^3\sqrt{1+3x^4}\, dx$

Put $1+3x^4 = t \Rightarrow 12x^3 dx = dt \Rightarrow x^3\, dx = \frac{dt}{12}$. When $x = 0,\ t = 1$ and $x = 1,\ t = 4$

$$\Rightarrow I_1 = \int \sqrt{t}.\frac{dt}{12} \quad \Rightarrow I_1 = \frac{1}{12}.\frac{t^{3/2}}{3/2} \Rightarrow I = \frac{1}{18}\left[t^{3/2}\right]_1^4 \quad \Rightarrow I = \frac{1}{18}\left[\left(2^2\right)^{3/2} - 1\right]$$

$$\Rightarrow I = \frac{1}{18}[8-1] \quad \therefore I = \frac{7}{18}$$

43. $\int_0^1 \frac{1-x^2}{\left(1+x^2\right)^2} dx$

Sol. Let $I = \int_0^1 \frac{1-x^2}{\left(1+x^2\right)^2} dx \Rightarrow I = [I_1]_0^1; \quad I_1 = \int \frac{1-x^2}{\left(1+x^2\right)^2} dx \quad \Rightarrow I_1 = \int \frac{1-x^2}{\left(1+x^2\right)^2} dx$

$$\Rightarrow I_1 = \int \frac{1-x^2}{1+2x^2+x^4} dx \quad \Rightarrow I_1 = \int \frac{x^2\left(\frac{1}{x^2} - 1\right)}{x^2\left(\frac{1}{x^2} + 2 + x^2\right)} dx \quad \Rightarrow I_1 = \int \frac{\left(\frac{1}{x^2} - 1\right)}{\left(\frac{1}{x^2} + x^2 + 2\right)} dx$$

$$\Rightarrow I_1 = -\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + 2 + \frac{1}{x^2}\right)} dx \quad \Rightarrow I_1 = -\int \frac{1 - \frac{1}{x^2}}{(x)^2 + 2.x.\frac{1}{x} + \left(\frac{1}{x}\right)^2} dx \quad \Rightarrow I_1 = -\int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2} dx$$

Put $x + \frac{1}{x} = t \quad \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$, when $x = 0,\ t = \infty$ and when $x = 1,\ t = 2$

$$\Rightarrow I_1 = -\int \frac{dt}{t^2} \quad \Rightarrow I_1 = -\left(-\frac{1}{t}\right) \quad \Rightarrow I_1 = \frac{1}{t} \quad \Rightarrow I = \left[\frac{1}{t}\right]_\infty^2 \quad \Rightarrow I = \left(\frac{1}{2} - 0\right) \quad \therefore I = \frac{1}{2}$$

44. $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$

Sol. Let $I = \int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}} \quad \Rightarrow I = [I_1]_1^2; \quad I_1 = \int \frac{1}{(x+1)\sqrt{x^2-1}} dx$

Put $x + 1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$, when $x = 0,\ t = \frac{1}{2}$ and when $x = 2,\ t = \frac{1}{3}$

$$\therefore I_1 = \int \frac{\left(-\frac{1}{t^2}\right) dt}{\frac{1}{t}\sqrt{\frac{1}{t}\left(\frac{1}{t} - 2\right)}} \quad \therefore x^2 - 1 = (x+1)(x-1) = \frac{1}{t}\left(\frac{1}{t} - 1 - 1\right) = \frac{1}{t}\left(\frac{1}{t} - 2\right)$$

$$\Rightarrow I_1 = \int \frac{-dt}{\sqrt{1-2t}} \quad \Rightarrow I_1 = \frac{-(1-2t)^{1/2}}{\frac{-1}{2}} \quad \Rightarrow I_1 = 2(1-2t)^{1/2}$$

$$\Rightarrow I = 2\left[\sqrt{1-2t}\right]_{1/2}^{1/3} = 2\left(\sqrt{1-\frac{2}{3}} - \sqrt{1-2.\frac{1}{2}}\right) = 2\sqrt{\frac{1}{3}} - 0 \quad \therefore\ I = \frac{2}{\sqrt{3}}$$

45. $\int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

Sol. Let $I = \int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx \quad \Rightarrow I = \int_0^{\pi/2} \left\{\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right\} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx \quad \Rightarrow I = \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx$$

Put $(\sin x - \cos x) = t$ and $(\cos x + \sin x) dx = dt$

Squaring both sides, $t^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x \quad \Rightarrow t^2 = 1 - 2\sin x \cos x \quad \Rightarrow 2\sin x \cos x = 1 - t^2$

Also $x = 0,\ t = -1$ and $x = \frac{\pi}{2},\ t = 1$

$$\therefore\ I = \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1} \quad \Rightarrow I = \sqrt{2}\left\{\sin^{-1}(1) - \sin^{-1}(-1)\right\}$$

$$\Rightarrow I = \sqrt{2}\left\{2\sin^{-1}(1)\right\} \quad \Rightarrow I = \sqrt{2} \times 2 \times \frac{\pi}{2} \quad \therefore\ I = \sqrt{2}\,\pi$$

46. $\int_2^3 \frac{2-x}{\sqrt{5x-6-x^2}} dx$

Sol. Let $I = \int_2^3 \frac{2-x}{\sqrt{5x-6-x^2}} dx \quad \Rightarrow I = [I_1]_2^3; \quad I_1 = \int \frac{2-x}{\sqrt{5x-6-x^2}} dx$

$$\Rightarrow 2 - x = A.\frac{d}{dx}\left(5x - 6 - x^2\right) + B \quad \Rightarrow 2 - x = A(5-2x) + B$$

$$\Rightarrow 2 - x = 5A - 2Ax + B \quad \Rightarrow 2 - x = (5A + B) - 2Ax$$

Equating co-efficient both side we get

$5A + B = 2$ ...(1)

$-2A = -1 \Rightarrow A = \frac{1}{2}$ ...(2)

Putting the value of A in equation (1) we get, $5A + B = 2$

$$\Rightarrow B = 2 - 5A \quad \Rightarrow B = 2 - 5.\frac{1}{2} \quad \Rightarrow B = 2 - \frac{5}{2} \quad \therefore\ B = \frac{-1}{2}$$

$$I_1 = \int \frac{A(5-2x) + B}{\sqrt{5x-6-x^2}} dx$$

$$\Rightarrow I_1 = A\int \frac{5-2x}{\sqrt{5x-6-x^2}} + B\int \frac{1}{\sqrt{5x-6-x^2}} dx \quad \Rightarrow I_1 = \frac{1}{2} I_1 - \frac{1}{2} I_3 \qquad \ldots(1)$$

$$I_2 = \int \frac{5-2x}{\sqrt{5x-6-x^2}} dx$$

Put $5x-6-x^2=t \Rightarrow (5-2x)dx=dt$

$$I_2=\int\frac{dt}{\sqrt{t}} \Rightarrow I_2=\frac{t^{1/2}}{1/2} \Rightarrow I_2=2\sqrt{5x-6-x^2}$$

Again $I_3=\int\frac{1}{\sqrt{5x-6-x^2}}dx \Rightarrow I_3=\int\frac{1}{\sqrt{-\left[x^2-5x+6\right]}}dx$

$$\Rightarrow I_3=\int\frac{1}{\sqrt{-\left[(x)^2-2.x.\frac{5}{2}+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6\right]}} \Rightarrow I_3=\int\frac{1}{\sqrt{-\left[\left(x-\frac{5}{2}\right)^2-\frac{1}{4}\right]}}$$

$$\Rightarrow I_3=\int\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}}dx \Rightarrow I_3=\sin^{-1}\left(\frac{x-\frac{5}{2}}{\frac{1}{2}}\right) \Rightarrow I_3=\sin^{-1}(2x-5)$$

Putting the value of I_2 & I_3 in equation (1)

$$I_1=\frac{1}{2}.2\sqrt{5x-6-x^2}-\frac{1}{2}\sin^{-1}(2x-5) \Rightarrow I_1=\sqrt{5x-6-x^2}-\frac{1}{2}\sin^{-1}(2x-5)$$

$$I=\left[\sqrt{5x-6-x^2}-\frac{1}{2}\sin^{-1}(2x-5)\right]_2^3$$

$$\Rightarrow I=\left[\left\{\sqrt{15-6-9}-\frac{1}{2}\sin^{-1}(6-5)\right\}-\left\{\sqrt{10-6-4}-\frac{1}{2}\sin^{-1}(4-5)\right\}\right]$$

$$\Rightarrow I=-\frac{1}{2}\sin^{-1}(1)+\frac{1}{2}\sin^{-1}(-1) \Rightarrow I=-\frac{1}{2}.\frac{\pi}{2}-\frac{1}{2}.\frac{\pi}{2} \Rightarrow I=-\frac{\pi}{4}-\frac{\pi}{4} \quad \therefore I=-\frac{\pi}{2}$$

47. $\displaystyle\int_{\pi/4}^{\pi/2}\frac{\cos\theta}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^3}d\theta$

Sol. Let $I=\displaystyle\int_{\pi/4}^{\pi/2}\frac{\cos\theta}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^3}d\theta$

$$\Rightarrow I=\int_{\pi/4}^{\pi/2}\frac{\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^3}d\theta=\int_{\pi/4}^{\pi/2}\frac{\left\{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right\}\left\{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right\}}{\left\{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right\}^3}d\theta$$

$$=\int\frac{(\cos\theta/2-\sin\theta/2)}{(\cos\theta/2+\sin\theta/2)}dx$$

Put $t=\cos\frac{\theta}{2}+\sin\frac{\theta}{2} \Rightarrow \frac{dt}{d\theta}=-\frac{1}{2}\sin\frac{\theta}{2}+\frac{1}{2}\cos\frac{\theta}{2}$

$$\Rightarrow dt=\frac{1}{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\theta \Rightarrow 2dt=\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\theta$$

When $x=\frac{\pi}{4}$, $t=\cos\frac{\pi}{8}+\sin\frac{\pi}{8}$; When $x=\frac{\pi}{2}$, $t=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$

$$I=\int_{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}^{\sqrt{2}}\frac{2dt}{t^2}=-2\left[\frac{1}{t}\right]_{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}^{\sqrt{2}}=-2\left[\frac{1}{\sqrt{2}}-\frac{1}{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}\right]$$

$$\therefore I=\frac{-2}{\sqrt{2}}+\frac{2}{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}=\frac{2}{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}-\sqrt{2}$$

48. $\int_0^{(\pi/2)^{1/3}} x^2\sin x^3 dx$

Sol. Let $I=\int_0^{(\pi/2)^{1/3}} x^2\sin x^3 dx$, Put $t=x^3 \Rightarrow dt=3x^2dx$

When $x=0,\ t=0$; $x=\left(\frac{\pi}{2}\right)^{1/3}$, $t=\left(\frac{\pi}{2}\right)^{\frac{1}{3}\times 3} \Rightarrow t=\frac{\pi}{2}$

$$\therefore\ I=\int_0^{\pi/2}\sin t\,dt/3=\frac{1}{3}\left[-\cos t\right]_0^{\pi/2}=-\frac{1}{3}\left[\cos\frac{\pi}{2}-\cos 0\right]=-\frac{1}{3}\left[0-1\right]=\frac{1}{3}$$

49. $\int_1^2\frac{dx}{x(1+\log x)^2}$

Sol. Let $I=\int_1^2\frac{dx}{x(1+\log x)^2}$, Put $t=1+\log x \Rightarrow dt=\frac{1}{x}dx$

When $x=1,\ t=1$; $x=2,\ t=1+\log 2$

$$\therefore\ I=\int_1^{1+\log 2}\frac{dt}{t^2}=\left[-\frac{1}{t}\right]_1^{1+\log 2}=-\frac{1}{1+\log 2}+1=1-\frac{1}{1+\log 2}=\frac{1+\log 2-1}{1+\log 2}=\frac{\log 2}{1+\log 2}$$

50. $\int_{\pi/6}^{\pi/2}\frac{\operatorname{cosec} x\cot x}{1+\operatorname{cosec}^2 x}dx$

Sol. Let $I=\int_{\pi/6}^{\pi/2}\frac{\operatorname{cosec} x\cot x}{1+\operatorname{cosec}^2 x}dx$, Put $t=\operatorname{cosec} x \Rightarrow dt=-\operatorname{cosec} x\cot x\,dx$

When $x=\frac{\pi}{2},\ t=1$; $x=\frac{\pi}{6},\ t=2$

$$\therefore\ I=\int_2^1\frac{-dt}{1+t^2}=-\left[\tan^{-1}t\right]_2^1=-\left[\tan^{-1}1-\tan^{-1}2\right]=-\left[\frac{\pi}{4}-\tan^{-1}2\right]=\tan^{-1}2-\frac{\pi}{4}$$

EXERCISE 16C (Pg.No.: 834)

Prove that

1. $\int_0^{\pi/2} \frac{\cos x}{(\sin x+\cos x)}dx=\frac{\pi}{4}$

Sol. L.H.S., $I=\int_0^{\pi/2} \frac{\cos x}{(\sin x+\cos x)}dx$ ...(1)

$\Rightarrow I=\int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \quad \Rightarrow I=\int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}$ ...(2)

By, Adding equation (1) and (2), then

$\Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\cos x}{\sin x+\cos x}+\frac{\sin x}{\cos x+\sin x}\right)dx \quad \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\cos x+\sin x}{\sin x+\cos x}\right)dx \quad \Rightarrow 2I=\int_0^{\pi/2} dx$

$\Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \quad \Rightarrow 2I=\frac{\pi}{2} \quad \therefore\ I=\frac{\pi}{4}=$ R.H.S.

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x}+\sqrt{\cos x})}dx=\frac{\pi}{4}$

Sol. L.H.S, $I=\int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x}+\sqrt{\cos x})}dx$ ...(1)

$\Rightarrow I=\int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\times\left(\frac{\pi}{2}-x\right)}}dx \quad \Rightarrow I=\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ ...(2)

By, Adding equation (1) and (2), then

$\Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx \quad \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx$

$\Rightarrow 2I=\int_0^{\pi/2} dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \quad \Rightarrow 2I=\frac{\pi}{2} \quad \therefore\ I=\frac{\pi}{4}=$ R.H.S

3. (i) $\int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x+\cos^3 x)}dx=\frac{\pi}{4}$ (ii) $\int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x+\cos^3 x}dx=\frac{\pi}{4}$

Sol. **(i)** L.H.S, $I=\int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x+\cos^3 x)}dx$ ...(1)

$\Rightarrow I=\int_0^{\pi/2} \frac{\sin^3 x\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right)+\cos^3\left(\frac{\pi}{2}-x\right)}dx \quad \Rightarrow I=\int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x+\sin^3 x}dx$ ...(2)

By, Adding equation (1) and (2), then, $I_1 = \int_0^{\pi/2} \left(\frac{\sin^3 x}{\sin^3 x + \cos^3 x} + \frac{\cos^3 x}{\cos^3 x + \sin^3 x} \right) dx$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} dx \quad \Rightarrow 2I = [x]_0^{\pi/2}$$

$$\Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \quad \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

(ii) L.H.S, $I = \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos^3 x} dx \qquad ...(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{\cos^3 x + \sin^3 x} dx \qquad ...(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x}{\sin^3 x + \cos^3 x} + \frac{\sin^3 x}{\cos^3 x + \sin^3 x} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x + \sin^3 x}{\sin^3 x + \cos^3 x} \right) dx$$

$$\Rightarrow I = \int_0^{\pi/2} dx \quad \Rightarrow 2I = [x]_0^{\pi/2} \quad \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \quad \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

4. (i) $\int_0^{\pi/2} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx = \frac{\pi}{4}$ (ii) $\int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx = \frac{\pi}{4}$

Sol. **(i)** L.H.S, $I = \int_0^{\pi/2} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx \qquad ...(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^7\left(\frac{\pi}{2} - x\right)}{\sin^7\left(\frac{\pi}{2} - x\right) + \cos^7\left(\frac{\pi}{2} - x\right)} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{\cos^7 x}{\cos^7 x + \sin^7 x} dx \qquad ...(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^7 x}{\sin^7 x + \cos^7 x} + \frac{\cos^7 x}{\cos^7 x + \sin^7 x} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^7 x + \cos^7 x}{\sin^7 x + \cos^7 x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \quad \Rightarrow 2I = [x]_0^{\pi/2} \quad \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \quad \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

(ii) L.H.S, $I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx \qquad ...(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} dx \qquad ...(2)$$

By, Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^5 x}{\sin^5 x + \cos^5 x} + \frac{\cos^5 x}{\cos^5 x + \sin^5 x} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^5 x + \cos^5 x}{\sin^2 x + \cos^5 x} \right) dx$$

$$\Rightarrow 2I=\int_0^{\pi/2} dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \quad \Rightarrow 2I=\frac{\pi}{2} \quad \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}= \text{R.H.S}$$

5. $\displaystyle\int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x+\cos^4 x)}dx=\frac{\pi}{4}$

Sol. L.H.S, $\displaystyle I=\int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x+\cos^4 x)}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\pi/2} \frac{\cos^4\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx \quad \Rightarrow I=\int_0^{\pi/2} \frac{\sin^4 x}{\cos^4 x+\sin^4 x}dx \qquad ...(2)$$

Adding equation (1) and (2), then

$$2I=\int_0^{\pi/2}\left(\frac{\cos^4 x}{\sin^4 x+\cos^4 x}+\frac{\sin^4 x}{\cos^4 x+\sin^4 x}\right)dx \quad \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\cos^4 x+\sin^4 x}{\sin^4 x+\cos^4 x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2} dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \quad \Rightarrow 2I=\frac{\pi}{2} \quad \Rightarrow I=\frac{\pi}{4}=\text{R.H.S}$$

6. $\displaystyle\int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x+\cos^{1/4} x)}dx=\frac{\pi}{4}$

Sol. L.H.S, $\displaystyle I=\int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x+\cos^{1/4} x)}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\pi/2} \frac{\cos^{1/4}\left(\frac{\pi}{2}-x\right)}{\sin^{1/4}\left(\frac{\pi}{2}-x\right)+\cos^{1/4}\left(\frac{\pi}{2}-x\right)}dx \quad \Rightarrow I=\int_0^{\pi/2} \frac{\sin^{1/4} x}{(\cos^{1/4} x+\sin^{1/4} x)}dx \qquad ...(2)$$

Adding equation (1) and (2), then

$$2I=\int_0^{\pi/2}\left(\frac{\cos^{1/4} x}{\sin^{1/4} x+\cos^{1/4} x}+\frac{\sin^{1/4} x}{\cos^{1/4} x+\sin^{1/4} x}\right)dx \quad \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\cos^{1/4} x+\sin^{1/4} x}{\sin^{1/4} x+\cos^{1/4} x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2} dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \quad \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}=\text{R.H.S}$$

7. $\displaystyle\int_0^{\pi/2} \frac{\sin^{3/2} x}{(\sin^{3/2} x+\cos^{3/2} x)}dx=\frac{\pi}{4}$

Sol. L.H.S, $\displaystyle I=\int_0^{\pi/2} \frac{\sin^{3/2} x}{(\sin^{3/2} x+\cos^{3/2} x)}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2}-x\right)}{\sin^{3/2}\left(\frac{\pi}{2}-x\right)+\cos^{3/2}\left(\frac{\pi}{2}-x\right)}dx \quad \Rightarrow I=\int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x+\sin^{3/2} x}dx \qquad ...(2)$$

By, Adding equation (1) and (2), then

$$2I=\int_0^{\pi/2}\left(\frac{\sin^{3/2}x}{\sin^{3/2}x+\cos^{3/2}x}+\frac{\cos^{3/2}x}{\cos^{3/2}x+\sin^{3/2}x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sin^{3/2}x+\cos^{3/2}x}{\sin^{3/2}x+\cos^{3/2}x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \quad \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}=\text{R.H.S}$$

8. $\displaystyle\int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}dx=\frac{\pi}{4}$

Sol. L.H.S, $\displaystyle I=\int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}dx$ ...(1)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sin^n\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)+\cos^n\left(\frac{\pi}{2}-x\right)}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\cos^n x}{\cos^n x+\sin^n x}dx \quad ...(2)$$

By, Adding equation (1) and (2), then

$$2I=\int_0^{\pi/2}\left(\frac{\sin^n x}{\sin^n x+\cos^n x}+\frac{\cos^n x}{\cos^n x+\sin^n x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sin^n x+\cos^n x}{\sin^n x+\cos^n x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \quad \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}=\text{R.H.S}$$

9. $\displaystyle\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx=\frac{\pi}{4}$

Sol. Let $\displaystyle I=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx$ ...(1)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}{\sqrt{\tan\left(\frac{\pi}{2}-x\right)}+\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx \quad ...(2)$$

By, Adding (1) and (2), we get, $\displaystyle 2I=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx+\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx$

$$\Rightarrow 2I=\int_0^{\pi/2}\frac{\sqrt{\tan x}+\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx \quad \Rightarrow 2I=\int_0^{\pi/2}dx=[x]_0^{\pi/2}=\frac{\pi}{2}-0 \quad \Rightarrow I=\frac{\pi}{4}$$

10. $\displaystyle\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx=\frac{\pi}{4}$

Sol. Let $\displaystyle I=\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx$ ...(1)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}{\sqrt{\tan\left(\frac{\pi}{2}-x\right)}+\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx \quad ...(2)$$

By, Adding (1) and (2), we get,

$$2I=\int_0^{\pi/2}\frac{\sqrt{\tan x}+\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx \Rightarrow 2I=\int_0^{\pi/2}dx=[x]_0^{\pi/2}=\frac{\pi}{2} \Rightarrow I=\frac{\pi}{4}$$

11. $\int_0^{\pi/2}\frac{1}{(1+\tan x)}dx=\frac{\pi}{4}$

Sol. L.H.S, $I=\int_0^{\pi/2}\frac{1}{(1+\tan x)}dx \Rightarrow I=\int_0^{\pi/2}\frac{1}{\left(1+\frac{\sin x}{\cos x}\right)}dx \Rightarrow I=\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx$ ...(1)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}dx \Rightarrow I=\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx \quad ...(2)$$

By, Adding equation (1) and (2), then

$$2I=\int_0^{\pi/2}\left(\frac{\cos x}{\cos x+\sin x}+\frac{\sin x}{\sin x+\cos x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\cos x+\sin x}{\cos x+\sin x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}dx$$

$$\Rightarrow 2I=[x]_0^{\pi/2} \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}=\text{R.H.S}$$

12. $\int_0^{\pi/2}\frac{1}{(1+\cot x)}dx=\frac{\pi}{4}$

Sol. L.H.S, $I=\int_0^{\pi/2}\frac{1}{(1+\cot x)}dx \Rightarrow I=\int_0^{\pi/2}\frac{1}{\left(1+\frac{\cos x}{\sin x}\right)}dx \Rightarrow I=\int_0^{\pi/2}\frac{\sin x}{(\sin x+\cos x)}dx$ ...(1)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \Rightarrow I=\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx \quad ...(2)$$

By, Adding equation (1) and (2), then

$$2I=\int_0^{\pi/2}\left(\frac{\sin x}{\sin x+\cos x}+\frac{\cos x}{\cos x+\sin x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sin x+\cos x}{\sin x+\cos x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}dx$$

$$\Rightarrow 2I=[x]_0^{\pi/2} \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}=\text{R.H.S}$$

13. $\int_0^{\pi/2}\frac{1}{(1+\tan^3 x)}dx=\frac{\pi}{4}$

Sol. L.H.S, $I=\int_0^{\pi/2}\frac{1}{(1+\tan^3 x)}dx \Rightarrow I=\int_0^{\pi/2}\frac{1}{\left(1+\frac{\sin^3 x}{\cos^3 x}\right)}dx \Rightarrow I=\int_0^{\pi/2}\frac{\cos^3 x}{(\cos^3 x+\sin^3 x)}dx$ ...(1)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\cos^3\left(\frac{\pi}{2}-x\right)}{\cos^3\left(\frac{\pi}{2}-x\right)+\sin^3\left(\frac{\pi}{2}-x\right)}dx \Rightarrow I=\int_0^{\pi/2}\frac{\sin^3 x}{\sin^3 x+\cos^3 x}dx \quad ...(2)$$

Adding equation (1) and (2), then

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x}{\cos^3 x + \sin^3 x} + \frac{\sin^3 x}{\sin^3 x + \cos^3 x} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos^3 x + \sin^3 x}{\cos^3 x + \sin^3 x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \quad \Rightarrow 2I = [x]_0^{\pi/2} \quad \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \quad \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

14. $\int_0^{\pi/2} \frac{1}{(1 + \cot^3 x)} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{1}{(1 + \cot^3 x)} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{1}{\left(1 + \frac{\cos^3 x}{\sin^3 x}\right)} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)} dx \qquad ...(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \qquad ...(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sin^3 x}{\sin^3 x + \cos^3 x} + \frac{\cos^3 x}{\cos^3 x + \sin^3 x} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} dx$$

$$\Rightarrow 2I = [x]_0^{\pi/2} \quad \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \quad \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

15. $\int_0^{\pi/2} \frac{dx}{(1 + \sqrt{\tan x})} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{dx}{(1 + \sqrt{\tan x})} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{1}{\left(1 + \frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad ...(1)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \qquad ...(2)$$

By, Adding equation (1) and (2), then

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx \quad \Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx \quad \Rightarrow 2I = [x]_0^{\pi/2} \quad \Rightarrow 2I \left(\frac{\pi}{2} - 0 \right) \quad \Rightarrow 2I = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} = \text{R.H.S}$$

16. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}} dx = \frac{\pi}{4}$

Sol. L.H.S, $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}} dx \quad \Rightarrow I = \int_0^{\pi/2} \frac{\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)}{\left(1 + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)} dx$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)}{\left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}}\right)}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad ...(2)$$

By,Adding equation (1) and (2), then

$$2I=\int_0^{\pi/2}\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}+\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}=\text{R.H.S}$$

17. $\int_0^{\pi/2}\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx=\frac{\pi}{4}$

Sol. L.H.S, $I=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)}{\left(1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)}dx$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)}{\left(\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}}\right)}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx \quad \Rightarrow I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(2)$$

By, Adding equation (1) and (2), then

$$\therefore \quad 2I=\int_0^{\pi/2}\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx \Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}dx \quad \Rightarrow 2I=[x]_0^{\pi/2} \quad \Rightarrow 2I=\left(\frac{\pi}{2}-0\right) \Rightarrow 2I=\frac{\pi}{2} \quad \therefore I=\frac{\pi}{4}=\text{R.H.S}$$

18. $\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx=0$

Sol. L.H.S, $I=\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx \Rightarrow I=\int_0^{\pi/2}\frac{\cos x-\sin x}{1+\cos x\sin x}dx \quad ...(2)$$

Adding equation (1) and (2), then $2I=\int_0^{\pi/2}\left(\frac{\sin x-\cos x}{1+\sin x\cos x}+\frac{\cos x-\sin x}{1+\cos x\sin x}\right)dx$

$$\Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sin x-\cos x+\cos x-\sin x}{1+\sin x\cos x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}0 \Rightarrow 2I=0 \quad \therefore I=0=\text{R.H.S}$$

19. $\int_0^1 x(1-x)^5\,dx=\frac{1}{42}$

Sol. Let $I=\int_0^1 x(1-x)^5\,dx=\int_0^1(1-x)\{1-(1-x)\}^5\,dx=\int_0^1(1-x)(x)^5\,dx=\int_0^1\left(x^5-x^6\right)dx$

$$=\left[\frac{x^6}{6}-\frac{x^7}{7}\right]_0^1=\frac{1}{6}-\frac{1}{7}-0=\frac{7-6}{42}=\frac{1}{42}$$

20. $\int_0^2 x\sqrt{2-x}\,dx=\frac{16\sqrt{2}}{15}$

Sol. Let $I=\int_0^2 x\sqrt{2-x}\,dx=\int_0^2(2-x)\sqrt{2-(2-x)}\,dx=\int_0^2(2-x)x^{1/2}dx=\int_0^2\left(2x^{1/2}-x^{3/2}\right)dx$

$$=\left[2\frac{x^{3/2}}{3/2}-\frac{x^{5/2}}{5/2}\right]_0^2=\left[\frac{4}{3}x^{3/2}-\frac{2}{5}x^{5/2}\right]_0^2=\frac{4}{3}2^{3/2}-\frac{2}{5}2^{5/2}=\frac{4}{3}2\sqrt{2}-\frac{2}{5}4\sqrt{2}$$

$$=8\sqrt{2}\left(\frac{1}{3}-\frac{1}{5}\right)=8\sqrt{2}\,\frac{2}{15}=\frac{16\sqrt{2}}{15}$$

21. $\int_0^\pi x\cos^2 x\,dx=\frac{\pi^2}{4}$

Sol. Let $I=\int_0^\pi x\cos^2 x\,dx \quad ...(1)$

Then, $I=\int_0^\pi(\pi-x)\cos^2(\pi-x)dx$ or $I=\int_0^\pi(\pi-x)\cos^2 x\,dx \quad ...(2)$

By, Adding (1) and (2) we get, $2I=\int_0^\pi\left\{x\cos^2 x+(\pi-x)\cos^2 x\right\}dx \Rightarrow 2I=\int_0^\pi \pi\cos^2 x\,dx$

$$\Rightarrow 2I=\pi\int_0^\pi\cos^2 x\,dx \Rightarrow 2I=\pi\int_0^\pi\left(\frac{1+\cos 2x}{2}\right)dx \Rightarrow 2I=\frac{\pi}{2}\left[x+\frac{\sin 2x}{2}\right]_0^\pi$$

$$\Rightarrow 2I=\frac{\pi}{2}\left[\left\{\pi+\frac{\sin 2\pi}{2}\right\}-(0)\right] \Rightarrow 2I=\frac{\pi}{2}[\pi+0] \Rightarrow 2I=\frac{\pi^2}{2} \quad \therefore I=\frac{\pi^2}{4}$$

22. $\int_0^\pi\frac{x\tan x}{\sec x\operatorname{cosec} x}dx=\frac{\pi^2}{4}$

Sol. Let $I=\int_0^\pi\frac{x\tan x}{\sec x\operatorname{cosec} x}dx \quad ...(1)$

Then, $I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\operatorname{cosec}(\pi-x)}dx$ or $I=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x\operatorname{cosec} x}dx$ ...(2)

Adding (1) and (2) we get

$$2I=\int_0^{\pi}\left[\frac{x\tan x}{\sec x\operatorname{cosec} x}+\frac{(\pi-x)\tan x}{\sec x\operatorname{cosec} x}\right]dx \Rightarrow 2I=\pi\int_0^{\pi}\sin^2 x\,dx \Rightarrow 2I=\pi\int_0^{\pi}\frac{1-\cos 2x}{2}dx$$

$$\Rightarrow 2I=\frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi} \Rightarrow 2I=\frac{\pi}{2}\left[\left\{\pi-\frac{\sin 2\pi}{2}\right\}-\{0\}\right] \Rightarrow 2I=\frac{\pi}{2}[\pi-0] \quad \therefore I=\frac{\pi^2}{4}$$

23. $\int_0^{\pi/2}\frac{\cos^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$

Sol. Let $I=\int_0^{\pi/2}\frac{\cos^2 x}{\sin x+\cos x}dx$ 1)

Then, $I=\int_0^{\pi/2}\frac{\cos^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} \Rightarrow I=\int_0^{\pi/2}\frac{\sin^2 x}{\cos x+\sin x}dx$ ...(2)

By, Adding (1) & (2) we get, $2I=\int_0^{\pi/2}\left(\frac{\cos^2 x}{\sin x+\cos x}+\frac{\sin^2 x}{\cos x+\sin x}\right)dx$

$$\Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\cos^2 x+\sin^2 x}{\sin x+\cos x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\frac{1}{\sin x+\cos x}dx \Rightarrow 2I=\frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{1}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x}dx$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\cos\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}\cos x}=\frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\sin\left(x+\frac{\pi}{4}\right)}$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\left[\log\left|\operatorname{cosec}\left(\frac{\pi}{2}+\frac{\pi}{4}\right)-\cot\left(\frac{\pi}{2}+\frac{\pi}{4}\right)\right|-\log\left|\operatorname{cosec}\left(0+\frac{\pi}{4}\right)-\cot\left(0+\frac{\pi}{4}\right)\right|\right]$$

$$\Rightarrow 2I-\frac{1}{\sqrt{2}}\left[\log\left|\operatorname{cosec}\frac{3\pi}{4}-\cot\frac{3\pi}{4}\right|-\log\left|\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}\right|\right]$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\left[\log\left(\sqrt{2}+1\right)-\log\left(\sqrt{2}-1\right)\right] \Rightarrow 2I=\frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right] \Rightarrow 2I=\frac{1}{\sqrt{2}}\log\left[\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right]$$

$$\Rightarrow 2I=\frac{2}{\sqrt{2}}\log\left(\sqrt{2}+1\right) \quad \therefore I=\frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

24. $\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx=\frac{\pi^2}{4}$

Sol. Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx$ ...(1)

Then, $I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\cos(\pi-x)}dx$ or $I=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\cos x}dx$ ...(2)

Adding (1) and (2) we get, $2I=\int_0^{\pi}\left[\frac{x\tan x}{\sec x+\cos x}+\frac{(\pi-x)\tan x}{\sec x+\cos x}\right]dx$

$\Rightarrow 2I=\int_0^{\pi}\left[\frac{x\tan x+\pi\tan x-x\tan x}{\sec x+\cos x}\right]dx \Rightarrow 2I=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\cos x}dx \Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

Put $\cos x=t \Rightarrow -\sin x\,dx=dt \Rightarrow \sin x\,dx=-dt$

Clearly $x=0, t=1$ and $x=\pi, t=-1$

$2I=\pi\int_1^{-1}\frac{-dt}{1+t^2} \Rightarrow 2I=\pi\int_{-1}^{1}\frac{1}{1+t^2}dt \Rightarrow 2I=\pi\left[\tan^{-1}(t)\right]_{-1}^{1} \Rightarrow 2I=\pi\left[\tan^{-1}(1)-\tan^{-1}(-1)\right]$

$\Rightarrow 2I=\pi\left[\tan^{-1}(1)+\tan^{-1}(1)\right] \Rightarrow 2I=\pi\left[2\tan^{-1}(1)\right] \therefore I=\frac{\pi^2}{4}$

25. $\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx=\pi\left(\frac{\pi}{2}-1\right)$

Sol. Let $I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$ ...(1)

Then $I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx$ or $I=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}dx$ ...(2)

Adding (1) and (2) we get

$2I=\int_0^{\pi}\left[\frac{x\sin x}{1+\sin x}+\frac{(\pi-x)\sin x}{1+\sin x}\right]dx \Rightarrow 2I=\int_0^{\pi}\frac{x\sin x+\pi\sin x-x\sin x}{1+\sin x}dx$

$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx \Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx \Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x(1-\sin x)}{1-\sin^2 x}dx$

$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x(1-\sin x)}{\cos^2 x}dx \Rightarrow 2I=\pi\left[\int_0^{\pi}\tan x\sec x\,dx-\int_0^{\pi}\tan^2 x\,dx\right]$

$\Rightarrow 2I=\pi\left[[\sec x]_0^{\pi}-\int_0^{\pi}(\sec^2 x-1)dx\right] \Rightarrow 2I=\pi\left[\{\sec\pi-\tan\pi+\pi\}-\{\sec 0-\tan 0+0\}\right]$

$\Rightarrow 2I=\pi[-1+\pi-1] \Rightarrow 2I=\pi(\pi-2) \Rightarrow I=\frac{\pi}{2}(\pi-2) \quad \therefore I=\pi\left(\frac{\pi}{2}-1\right)$

26. $\int_0^{\pi}\frac{x}{1+\sin^2 x}dx=\frac{\pi^2}{2\sqrt{2}}$

Sol. Let $I=\int_0^{\pi}\frac{x}{1+\sin^2 x}dx$ ...(1)

Then $I=\int_0^{\pi}\frac{\pi-x}{1+\sin^2(\pi-x)}dx$ or $I=\int_0^{\pi}\frac{\pi-x}{1+\sin^2(\pi-x)}dx$ ...(2)

Adding (1) and (2) we get,

$$2I=\int_0^{\pi}\left\{\frac{x}{1+\sin^2 x}+\frac{\pi-x}{1+\sin^2 x}\right\}dx \Rightarrow 2I=\int_0^{\pi}\left(\frac{x+\pi-x}{1+\sin^2 x}\right)dx \Rightarrow 2I=\pi\int_0^{\pi}\frac{1}{1+\sin^2 x}dx$$

$$\Rightarrow 2I=2\pi\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2 x}\cdot dx \Rightarrow I=\pi\int_0^{\frac{\pi}{2}}\frac{1}{a+\sin^2 x}\cdot dx=\pi\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{\sec^2 x+\tan^2 x}dx$$

$$\Rightarrow I=\pi\int_0^{\frac{\pi}{2}}\frac{\sec^2 xdx}{1+2\tan^2 x}\ \textit{put}\ \tan x=t\Rightarrow \sec^2 x\cdot dx=dt$$

$$\Rightarrow I=\pi\int_0^{\infty}\frac{dt}{1+\sqrt{2}t^2}=\pi\frac{\left[\tan^{-1}\left(\sqrt{2}t\right)\right]_0^{\infty}}{\sqrt{2}}=\frac{\pi\left[\tan^{-1}\infty-\tan^{-1}0\right]}{\sqrt{2}}=\frac{\pi}{\sqrt{2}}\left(\frac{\pi}{2}-0\right)=\frac{\pi^2}{2\sqrt{2}}$$

27. $\int_0^{\pi/2}\left(2\log\cos x-\log\sin 2x\right)dx=-\frac{\pi}{2}(\log 2)$

Sol. Let $I=\int_0^{\pi/2}\left\{2\log(\cos x)-\log(\sin 2x)\right\}dx \Rightarrow I=\int_0^{\pi/2}\left\{2\log\cos x-\log(2\sin x\cos x)\right\}dx$

$$\Rightarrow I=\int_0^{\pi/2}\left\{2\log\cos x-\log 2-\log\sin x-\log\cos x\right\}dx$$

$$\Rightarrow I=\int_0^{\pi/2}\left(\log\cos x-\log\sin x-\log 2\right)dx \qquad \ldots(1)$$

$$\Rightarrow I=\int_0^{2}\left(\log\sin x-\log\cos x-\log 2\right)dx \qquad \ldots(2)$$

Adding (1) and (2), we get, $2I=\int_0^{\pi/2}-2\log 2\,dx=-2\log 2\left[x\right]_0^{\pi/2}=-2.\frac{\pi}{2}\log 2$

$$\Rightarrow 2I=-\pi\log 2 \quad \Rightarrow I=-\frac{\pi}{2}\log 2$$

28. $\int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx=\frac{\pi}{4}$

Sol. Let $I=\int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx$

Put $x=\tan\theta \quad \Rightarrow dx=\sec^2\theta\,d\theta$. Clearly $x=0,\theta=0$ and $x=\infty,\theta=\frac{\pi}{2}$.

$$I=\int_0^{\pi/2}\frac{\tan\theta\sec^2\theta\,d\theta}{(1+\tan\theta)(1+\tan^2\theta)} \Rightarrow I=\int_0^{\pi/2}\frac{\tan\theta}{1+\tan\theta}d\theta \Rightarrow I=\int_0^{\pi/2}\frac{\sin\theta}{\sin\theta+\cos\theta}d\theta \qquad \ldots(1)$$

Then, $I=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}d\theta$ or $I=\int_0^{\pi/2}\frac{\cos\theta}{\cos\theta+\sin\theta}d\theta \qquad \ldots(2)$

By, Adding (1) and (2) we get, $2I = \int_0^{\pi/2} \left(\frac{\sin\theta}{\sin\theta + \cos\theta} + \frac{\cos\theta}{\sin\theta + \cos\theta} \right) d\theta$

$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta} \right) d\theta \Rightarrow 2I = \int_0^{\pi/2} d\theta \Rightarrow 2I = [\theta]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - \theta \right) \quad \therefore I = \frac{\pi}{4}$

29. $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} = \frac{\pi}{4}$

Sol. Let $I = \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} dx$

Let $x = a \sin\theta$, $dx = a\cos\theta \, d\theta$. Clearly $x = 0$, $\theta = 0$, and $x = a, \theta = \frac{\pi}{2}$.

$I = \int_0^{\pi/2} \frac{a\cos\theta \, d\theta}{a\sin\theta + \sqrt{a^2 - a^2 \sin^2\theta}} \Rightarrow I = \int_0^{\pi/2} \frac{a\cos\theta}{a\sin\theta + \sqrt{a^2(1 - \sin^2\theta)}} d\theta$

$\Rightarrow I = \int_0^{\pi/2} \frac{a\cos\theta}{a\sin\theta + a\cos\theta} d\theta \Rightarrow I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta \quad ...(1)$

Then $I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)} d\theta$ or $I = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta \quad ...(2)$

By, Adding (1) and (2) we get, $2I = \int_0^{\pi/2} \left(\frac{\cos\theta}{\sin\theta + \cos\theta} + \frac{\sin\theta}{\cos\theta + \sin\theta} \right) d\theta$

$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\cos\theta + \sin\theta}{\sin\theta + \cos\theta} \right) d\theta \Rightarrow 2I = \int_0^{\pi/2} d\theta \Rightarrow 2I = [\theta]_0^{\pi/2} \Rightarrow 2I = \left(\frac{\pi}{2} - 0 \right) \quad \therefore I = \frac{\pi}{4}$

30. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx = \frac{a}{2}$

Sol. Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx \quad ...(1)$

$\Rightarrow I = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{a - (a - x)}} dx \Rightarrow I = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx \quad ...(2)$

By, Adding (1) and (2) we get, $2I = \int_0^a \left(\frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} + \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} \right) dx$

$\Rightarrow 2I = \int_0^a \left[\frac{\sqrt{x} + \sqrt{a - x}}{\sqrt{x} + \sqrt{a - x}} \right] dx \Rightarrow 2I = \int_0^a dx \Rightarrow 2I = [x]_0^a \Rightarrow 2I = a \quad \therefore I = \frac{a}{2}$

31. $\int_0^{\pi} \sin^2 x \cos^3 x \, dx = 0$

Sol. Let $I = \int_0^{\pi} \sin^2 x \cos^3 x \, dx \quad ...(1)$

Then $I=\int_0^{\pi}\sin^2(\pi-x)\cos^3(\pi-x)dx \Rightarrow I=-\int_0^{\pi}\sin^2 x\cos^3 x\,dx$ ...(2)

By, Adding (1) and (2) we get, $2I=\int_0^{\pi}\left(\sin^2 x\cos^3 x-\sin^2 x\cos^3 x\right)dx \Rightarrow 2I=0 \quad \therefore I=0$

32. $\int_0^{\pi}\sin^{2m} x\cos^{2m+1} x\,dx=0$, where m is a positive integer

Sol. Let $I=\int_0^{\pi}\sin^{2m} x\cos^{2m+1} x\,dx$ ...(1)

$\Rightarrow I=\int_0^{\pi}\sin^{2m} x(\pi-x)\cos^{2m+1}(\pi-x)dx$

When m is positive integer then, $I=-\int_0^{\pi}\sin^{2m} x\cos^{2m+1} x\,dx$ ...(2)

By, Adding (1) and (2) we get, $2I=\int_0^{\pi}\left(\sin^{2m} x\cos^{2m+1} x-\sin^{2m} x\cos^{2m+1} x\right)dx \Rightarrow 2I=0$

$\therefore I=0$

33. $\int(\sin x-\cos x)\log(\sin x+\cos x)dx=0$

Sol. Let $I=\int_0^{\pi/2}(\sin x-\cos x)\log(\sin x+\cos x)dx$ ...(1)

Then $I=\int\left\{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)\right\}\log\left\{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)\right\}$

or $I=\int_0^{\pi/2}(\cos x-\sin x)\log(\cos x+\sin x)dx$ ...(2)

By, Adding (1) and (2) we get

$$2I=\int_0^{\pi/2}\{(\sin x-\cos x)\log(\sin x+\cos x)+(\cos x-\sin x)\log(\cos x+\sin x)\}\,dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\log(\sin x+\cos x)\{\sin x-\cos x+\cos x-\sin x\}dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\log(\sin x+\cos x).0\,dx \Rightarrow 2I=0 \quad \therefore I=0$$

34. $\int_0^{\pi/2}\log(\sin 2x)dx$

Sol. Let $I=\int_0^{\pi/2}\log(\sin 2x)dx$

Put $2x=t \Rightarrow 2\,dx=dt \Rightarrow dx=\frac{dt}{2}$. Clearly $x=0, t=0$ and $x=\frac{\pi}{2}, t=\pi$.

$$I=\int_0^{\pi}\log(\sin t).\frac{dt}{2} \Rightarrow I=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt$$

$$\Rightarrow I=\frac{1}{2}\times 2.\int_0^{\pi/2}\log(\sin t)dt \quad \Rightarrow I=\int_0^{\pi/2}\log(\sin t)dt \qquad \ldots(1)$$

Then $I=\int_0^{\pi/2}\log\left\{\sin\left(\frac{\pi}{2}-t\right)\right\}dt$ or $I=\int_0^{\pi/2}\log(\cos t)dt \qquad \ldots(2)$

Adding (1) and (2) we get,

$$2I=\int_0^{\pi/2}(\log\sin t+\log\cos t)dt=\int_0^{\pi/2}\log(\sin t\cos t)dt=\int_0^{\pi/2}\log\left(\frac{\sin 2t}{2}\right)dt$$

$$\Rightarrow 2I=\int_0^{\pi/2}\log(\sin 2t)dt-\int_0^{\pi/2}(\log 2)dt$$

$$\Rightarrow 2I=\frac{1}{2}\int_0^{\pi}\log\sin t\ dt-\log 2\int_0^{\pi/2}dt \qquad \text{[Putting } 2x=t \text{ in the 1st integral]}$$

$$\Rightarrow 2I=\frac{1}{2}\int_0^{\pi}\log\sin t\ dt-\log 2[t]_0^{\pi/2} \quad \Rightarrow 2I=\int_0^{\pi/2}\log(\sin t)dt-\frac{\pi}{2}\log 2$$

$$\Rightarrow 2I=I-\frac{\pi}{2}\log 2 \quad \therefore I=-\frac{\pi}{2}\log 2$$

35. $\int_0^{\pi}x\log(\sin x)dx=-\frac{\pi^2}{2}\log 2$

Sol. Let $I=\int_0^{\pi}x\log(\sin x)dx \qquad \ldots(1)$

$$\Rightarrow I=\int_0^{\pi}(\pi-x)\log\sin(\pi-x)dx \quad \Rightarrow I=\int_0^{\pi}(\pi-x)\log\sin x\,dx \qquad \ldots(2)$$

Adding (1) and (2), we get,

$$2I=\int_0^{\pi}x\log\sin x\,dx+\int_0^{\pi}(\pi-x)\log\sin x\,dx=\int_0^{\pi}(x+\pi-x)\log\sin x\,dx=\int_0^{\pi}\pi\log\sin x\,dx$$

$$\Rightarrow 2I=-2\pi\int_0^{\pi/2}\log\sin x\,dx \quad \Rightarrow I=\pi\int_0^{\pi/2}\log\sin x\,dx \qquad \ldots(3)$$

$$\Rightarrow I=\pi\int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-x\right)dx \quad \Rightarrow I=\pi\int_0^{\pi/2}\log\cos x\,dx \qquad \ldots(4)$$

By, Adding equation (3) and (4), we have

$$2I=\pi\int_0^{\pi/2}(\log\sin x+\log\cos x)dx=\pi\int_0^{\pi/2}\log\sin x.\cos x\,dx=\pi\int_0^{\pi/2}\log\frac{2\sin x.\cos x}{2}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi/2}\log\frac{\sin 2x}{2}dx=\pi\int_0^{\pi/2}(\log\sin 2x-\log 2)dx=\pi\int_0^{\pi/2}\log\sin 2x\,dx-\pi\int_0^{\pi/2}\log 2\,dx$$

Put $t=2x \Rightarrow \frac{dt}{2}=dx$. When $x=0,\ t=0$ and when $x=\frac{\pi}{2},\ t=\pi$

$$\Rightarrow 2I=\pi\int_0^{\pi}\log\sin t\,dt/2-\pi\log 2[x]_0^{\pi/2}$$

$$\Rightarrow 2I=\frac{\pi}{2}\times 2\int_0^{\pi/2}\log\sin t\,dt-\pi\log 2\left[\frac{\pi}{2}-0\right]=\pi\int_0^{\pi/2}\log\sin x\,dx-\frac{\pi^2}{2}\log 2\quad\left[\because \int_a^b f(x)dx=\int_a^b f(t)dt\right]$$

$$\Rightarrow 2I=I-\frac{\pi^2}{2}\log 2\quad\text{[From (3)]}\quad\Rightarrow I=-\frac{\pi^2}{2}\log 2$$

36. $\int_0^{\pi}\log(1+\cos x)\,dx=-\pi(\log 2)$

Sol. Let $I=\int_0^{\pi}\log(1+\cos x)dx$ ...(1)

$$\Rightarrow I=\int_0^{\pi}\log\{1+\cos(\pi-x)\}dx\quad\Rightarrow I=\int_0^{\pi}\log(1-\cos x)dx \quad ...(2)$$

By, Adding (1) and (2) we get, $2I=\int_0^{\pi}\{\log(1+\cos x)+\log(1-\cos x)\}dx$

$$\Rightarrow 2I=\int_0^{\pi}\log\{(1+\cos x)(1-\cos x)\}dx\quad\Rightarrow 2I=\int_0^{\pi}\log(1-\cos^2 x)dx$$

$$\Rightarrow 2I=\int_0^{\pi}\log(\sin^2 x)dx\quad\Rightarrow 2I=2\int_0^{\pi}\log(\sin x)dx\quad\Rightarrow I=\int_0^{\pi}\log(\sin x)dx$$

$$\Rightarrow I=2\int_0^{\pi/2}\log(\sin x)dx\quad\Rightarrow I=2\int_0^{\pi/2}\log(\sin x)dx\quad\Rightarrow I=2I_1 \quad ...(3)$$

$$\Rightarrow I_1=\int_0^{\pi/2}\log(\sin x)dx \quad ...(4)$$

Then $I_1=\int_0^{\pi/2}\log\left\{\sin\left(\frac{\pi}{2}-x\right)\right\}dx$ or $I_1=\int_0^{\pi/2}\log(\cos x)dx$ ...(5)

By, Adding (4) and (5) we get, $2I_1=\int_0^{\pi/2}\{\log(\sin x)+\log(\cos x)\}dx$

$$\Rightarrow 2I_1=\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx\quad\Rightarrow 2I_1=\int_0^{\pi/2}\{\log(\sin 2x)-\log 2\}dx$$

$$\Rightarrow 2I_1=\frac{1}{2}\int_0^{\pi}\log\sin t\,dt-(\log 2)\int_0^{\pi/2}dx \quad \text{[Putting } 2x=t \text{ in the 1st integral]}$$

$$\Rightarrow 2I_1=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt-\log 2[x]_0^{\pi/2}\quad\Rightarrow 2I_1=\left(\frac{1}{2}\times 2\right)\int_0^{\pi/2}\log(\sin t)dt-\log 2\left(\frac{\pi}{2}-0\right)$$

$$\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin t)dt-\frac{\pi}{2}\log 2\quad\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin x)dx-\frac{\pi}{2}\log 2\quad\left[\because \int_a^b f(x)dx=\int_a^b f(t)dt\right]$$

$$\Rightarrow 2I_1=I_1-\frac{\pi}{2}\log 2\quad\therefore I_1=-\frac{\pi}{2}\log 2$$

Putting the value of I_1 in equation (3), $I=2\left(-\frac{\pi}{2}\log 2\right)\quad\therefore I=-\pi\log 2$

37. $\int\log(\tan x+\cot x)dx=\pi(\log 2)$

Sol. Let $I=\int_0^{\pi/2}\log(\tan x+\cot x)dx \Rightarrow I=\int_0^{\pi/2}\log\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)dx \Rightarrow I=\int_0^{\pi/2}\log\left(\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right)dx$

$\Rightarrow I=\int_0^{\pi/2}\log\left(\frac{1}{\sin x\cos x}\right)dx \Rightarrow I=\int_0^{\pi/2}\log\left(\frac{1}{\sin x\cos x}\right)dx \Rightarrow I=\int_0^{\pi/2}\log\left(\frac{2}{2\sin x\cos x}\right)dx$

$\Rightarrow I=\int_0^{\pi/2}\log\left(\frac{2}{\sin 2x}\right)dx \Rightarrow I=\int_0^{\pi/2}\log 2dx-\int_0^{\pi/2}\log(\sin 2x)dx \Rightarrow I=\log 2\int_0^{\pi/2}dx-I_1$

$\Rightarrow I=\log 2[x]_0^{\pi/2}-I_1 \Rightarrow I=\log 2\left(\frac{\pi}{2}-0\right)-I_1 \Rightarrow I=\log 2\left(\frac{\pi}{2}-0\right)-I_1$

$\Rightarrow I=\frac{\pi}{2}\log 2-I_1$ ...(1)

Now, $I_1=\int_0^{\pi/2}\log(\sin 2x)dx$

Put $2x=t \Rightarrow 2dx=dt \Rightarrow dx=\frac{dt}{2}$. Clearly $x=0, t=0$ and $x=\frac{\pi}{2}, t=\pi$.

$I_1=\int_0^{\pi}\log(\sin t).\frac{dt}{2} \Rightarrow I_1=\frac{1}{2}\times 2\int_0^{\pi/2}\log(\sin t)dt \Rightarrow I=\int_0^{\pi/2}\log(\sin t)dt$ ...(2)

Then $I_1=\int_0^{\pi/2}\log\left\{\sin\left(\frac{\pi}{2}-t\right)\right\}dt$ or $I_1=\int_0^{\pi/2}\log(\cos t)dt$ ...(3)

By, Adding (2) and (3) we get, $2I_1=\int_0^{\pi/2}(\log\sin t+\log\cos t)dt \Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin t.\cos t)dt$

$\Rightarrow 2I_1=\int_0^{\pi/2}\log\left(\frac{\sin 2t}{2}\right)dt \Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin 2t)dt-\int_0^{\pi/2}(\log 2)dt$

$\Rightarrow 2I_1=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt-(\log 2)\int_0^{\pi/2}dt$ [Putting $2x=t$ in the 1st integral]

$\Rightarrow 2I_1=\frac{1}{2}\int_0^{\pi}\log\sin t\, dt-\log 2[t]_0^{\pi/2} \Rightarrow 2I_1=\frac{1}{2}\times 2\int_0^{\pi/2}\log(\sin t)dt-\log 2\left(\frac{\pi}{2}-0\right)$

$\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin t)dt-\frac{\pi}{2}\log 2 \Rightarrow 2I_1=I_1-\frac{\pi}{2}\log 2 \Rightarrow I_1=-\frac{\pi}{2}\log 2$

Putting the value of I_1 in equation (1), $I=\frac{\pi}{2}\log 2+\frac{\pi}{2}\log 2 \quad \therefore I=\pi\log 2$

38. $\int_{\pi/8}^{3\pi/8}\frac{\cos x}{\cos x+\sin x}dx=\frac{\pi}{4}$

Sol. Let $I=\int_{\pi/8}^{3\pi/8}\frac{\cos x}{\cos x+\sin x}dx$ ...(1)

$$\Rightarrow I=\int_{\pi/8}^{3\pi/8}\frac{\cos\left\{\left(\frac{\pi}{8}+\frac{3\pi}{8}\right)-x\right\}}{\cos\left\{\left(\frac{\pi}{8}+\frac{3\pi}{8}\right)-x\right\}+\sin\left\{\left(\frac{\pi}{8}+\frac{3\pi}{8}\right)-x\right\}}dx$$

$$\Rightarrow I=\int_{\pi/8}^{3\pi/8}\frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}dx \quad \Rightarrow I=\int_{\pi/8}^{3\pi/8}\frac{\sin x}{\sin x+\cos x}dx \qquad \ldots(2)$$

By, Adding (1) and (2) we get, $2I=\int_{\pi/8}^{3\pi/8}\left(\frac{\cos x}{\cos x+\sin x}+\frac{\sin x}{\sin x+\cos x}\right)dx \Rightarrow 2I=\int_{\pi/8}^{3\pi/8}\frac{\cos x+\sin x}{\cos x+\sin x}dx$

$$\Rightarrow 2I=\int_{\pi/8}^{3\pi/8}dx \quad \Rightarrow 2I=[x]_{\pi/8}^{3\pi/8} \quad \Rightarrow 2I=\left(\frac{3\pi}{8}-\frac{\pi}{8}\right) \quad \Rightarrow 2I=\frac{2\pi}{8} \quad \therefore\ I=\frac{\pi}{8}$$

39. $\int_{\pi/6}^{\pi/3}\frac{1}{1+\sqrt{\tan x}}dx=\frac{\pi}{12}$

Sol. Let $I=\int_{\pi/6}^{\pi/3}\frac{1}{1+\sqrt{\tan x}}dx \quad \Rightarrow I=\int_{\pi/6}^{\pi/3}\frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}dx \quad \Rightarrow I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad \ldots(1)$

Then, $I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left\{\left(\frac{\pi}{6}+\frac{\pi}{3}\right)-x\right\}}}{\sqrt{\cos\left\{\left(\frac{\pi}{6}+\frac{\pi}{3}\right)-x\right\}}+\sqrt{\sin\left\{\left(\frac{\pi}{6}+\frac{\pi}{3}\right)-x\right\}}}dx$

$$\Rightarrow I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx \quad \Rightarrow I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad \ldots(2)$$

By, Adding (1) and (2) we get, $2I=\int_{\pi/6}^{\pi/3}\left(\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}+\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx$

$$\Rightarrow 2I=\int_{\pi/6}^{\pi/3}\left(\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx \quad \Rightarrow 2I=\int_{\pi/6}^{\pi/3}dx \quad \Rightarrow 2I=[x]_{\pi/6}^{\pi/3}$$

$$\Rightarrow 2I=\left(\frac{\pi}{3}-\frac{\pi}{6}\right) \quad \Rightarrow 2I=\frac{2\pi-\pi}{6} \quad \Rightarrow 2I=\frac{\pi}{6} \quad \therefore I=\frac{\pi}{12}$$

40. $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}=2$

Sol. Let $I=\int_{\pi/4}^{3\pi/4}\frac{1}{1+\cos x}dx \quad \Rightarrow I=\int_{\pi/4}^{3\pi/4}\frac{1}{1+\cos x}\times\frac{1-\cos x}{1-\cos x}dx \quad \Rightarrow I=\int_{\pi/4}^{3\pi/4}\frac{1-\cos x}{1-\cos^2 x}dx$

$$\Rightarrow I=\int_{\pi/4}^{3\pi/4}\frac{1-\cos x}{\sin^2 x}dx \quad \Rightarrow I=\int_{\pi/4}^{3\pi/4}\frac{1-\cos x}{\sin^2 x}dx \quad \Rightarrow I=\int_{\pi/4}^{3\pi/4}\left(\operatorname{cosec}^2 x-\cot x\operatorname{cosec} x\right)dx$$

$$\Rightarrow I=\left[-\cot x+\operatorname{cosec}\right]_{\pi/4}^{3\pi/4} \quad \Rightarrow I=\left[\operatorname{cosec} x-\cot x\right]_{\pi/4}^{3\pi/4}$$

$$\Rightarrow I=\left[\left\{\operatorname{cosec}\left(\frac{3\pi}{4}\right)-\cot\left(\frac{3\pi}{4}\right)\right\}-\left\{\operatorname{cosec}\left(\frac{\pi}{4}\right)-\cot\left(\frac{\pi}{4}\right)\right\}\right]$$

$$\Rightarrow I=\left\{\sqrt{2}-(-1)\right\}-\left\{\sqrt{2}-1\right\} \quad \Rightarrow I=\sqrt{2}+1-\sqrt{2}+1 \quad \therefore\ I=2$$

41. $\int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x}dx = \pi\left(\sqrt{2}-1\right)$

Sol. Let $I = \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x}dx \qquad ...(1)$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)-x}{1+\sin\left\{\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)-x\right\}}dx \quad \Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1+\sin(\pi - x)}dx$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\pi - x}{1+\sin x}dx \qquad ...(2)$$

Adding (1) and (2) we get, $2I = \int_{\pi/4}^{3\pi/4}\left(\frac{x}{1+\sin x}+\frac{\pi - x}{1+\sin x}\right)dx \quad \Rightarrow 2I = \int_{\pi/4}^{3\pi/4}\left(\frac{x+\pi-x}{1+\sin x}\right)dx$

$$\Rightarrow 2I = \pi\int_{\pi/4}^{3\pi/4}\frac{1}{1+\sin x}dx \quad \Rightarrow 2I = \pi\int_{\pi/4}^{3\pi/4}\frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx \quad \Rightarrow 2I = \pi\int_{\pi/4}^{3\pi/4}\frac{1-\sin x}{1-\sin^2 x}dx$$

$$\Rightarrow 2I = \pi\int_{\pi/4}^{3\pi/4}\frac{1-\sin x}{\cos^2 x}dx \quad \Rightarrow 2I = \pi\int_{\pi/4}^{3\pi/4}\left(\sec^2 x - \tan x\sec x\right)dx \quad \Rightarrow 2I = \pi\left[\tan x - \sec x\right]_{\pi/4}^{3\pi/4}$$

$$\Rightarrow 2I = \pi\left[\left\{\tan\left(\frac{3\pi}{4}\right)-\sec\left(\frac{3\pi}{4}\right)\right\}-\left\{\tan\frac{\pi}{4}-\sec\frac{\pi}{4}\right\}\right]$$

$$\Rightarrow 2I = \pi\left[-1+\sqrt{2}-1+\sqrt{2}\right] \quad \Rightarrow 2I = \pi\left[2\sqrt{2}-2\right] \quad \therefore\ I = \pi\left(\sqrt{2}-1\right)$$

42. $\int_{a/4}^{3a/4} \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}dx = \frac{a}{4}$

Sol. Let $I = \int_{a/4}^{3a/4} \frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(1)$

Then, $I = \int_{a/4}^{3a/4} \frac{\sqrt{\left(\frac{a}{4}+\frac{3a}{4}\right)-x}}{\sqrt{a-\left\{\left(\frac{a}{4}+\frac{3a}{4}\right)-x\right\}}+\sqrt{\left\{\left(\frac{a}{4}+\frac{3a}{4}\right)-x\right\}}}dx$

$$\Rightarrow I = \int_{a/4}^{3a/4} \frac{\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad ...(2)$$

Adding (1) and (2) we get, $2I = \int_{a/4}^{3a/4}\left(\frac{\sqrt{x}}{\sqrt{a-x}+\sqrt{x}}+\frac{\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\right)dx$

$$\Rightarrow 2I = \int_{a/4}^{3a/4}\left(\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\right)dx \quad \Rightarrow 2I = \int_{a/4}^{3a/4}dx \quad \Rightarrow 2I = [x]_{a/4}^{3a/4}$$

$$\Rightarrow 2I = \left(\frac{3a}{4}-\frac{a}{4}\right) \quad \Rightarrow 2I = \frac{2a}{4} \quad \therefore I = \frac{a}{4}$$

43. $\int_{1}^{4} \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx = \frac{3}{2}$

Sol. Let $I=\int_1^4 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ ...(1)

Then $I=\int_1^4 \frac{\sqrt{(1+4)-x}}{\sqrt{5-\{(1+4)-x\}}+\sqrt{(1+4)-x}}dx$ or $I=\int_1^4 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)}+\sqrt{5-x}}dx$

$\Rightarrow I=\int_1^4 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$ ...(2)

By, Adding (1) and (2) we get, $2I=\int_1^4\left(\frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}+\frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\right)dx$

$\Rightarrow 2I=\int_1^4\left(\frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}\right)dx \Rightarrow 2I=\int_1^4 dx \Rightarrow 2I=[x]_1^4 \Rightarrow 2I=(4-1) \therefore I=\frac{3}{2}$

44. $\int_0^{\pi/2} x\cot x\,dx=\frac{\pi}{2}\log 2$

Sol. Let $I=\int_0^{\pi/2} x\cot x\,dx \Rightarrow I=\int_0^{\pi/2} x\cot x\,dx \Rightarrow I=x\int_0^{\pi/2}\cot x\,dx-\int_0^{\pi/2}\left[\frac{d(x)}{dx}\int\cot x\,dx\right]dx$

$\Rightarrow I=x\left[\log(\sin x)\right]_0^{\pi/2}-\int_0^{\pi/2}1.\log(\sin x)dx$

$\Rightarrow I=\left\{\frac{\pi}{2}\log\left(\sin\frac{\pi}{2}\right)-0\right\}-\int_0^{\pi/2}\log(\sin x)dx \Rightarrow I=-\int_0^{\pi/2}\log(\sin x)dx$

$\Rightarrow I=-I_1$ ...(1)

$I_1=\int_0^{\pi/2}\log(\sin x)dx$ ...(2)

Then $I_1=\int_0^{\pi/2}\log\left[\sin\left(\frac{\pi}{2}-x\right)\right]dx$ or $I_1=\int_0^{\pi/2}\log(\cos x)dx$ ...(3)

By, Adding (2) and (3) we get, $2I_1=\int_0^{\pi/2}\left[\log(\sin x)+\log(\cos x)\right]dx$

$\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin x.\cos x)dx \Rightarrow 2I_1=\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx$

$\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin 2x)dx-\int_0^{\pi/2}\log 2\,dx \Rightarrow 2I_1=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt-(\log 2)\int_0^{\pi/2}dx$

[Putting $2x=t$ in the 1st integral]

$2I_1=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt-(\log 2)[x]_0^{\pi/2} \Rightarrow 2I_1=\left(\frac{1}{2}\times 2\right).\int_0^{\pi/2}\log\sin t\,dt-(\log 2)\left(\frac{\pi}{2}-0\right)$

$\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin x)dx-\frac{\pi}{2}\log 2 \Rightarrow 2I_1=I_1-\frac{\pi}{2}\log 2 \Rightarrow I_1=\frac{-\pi}{2}\log 2$

Putting the value of I_1 in equation (1), $I=\frac{\pi}{2}\log 2$

45. $\int_0^1 \left(\frac{\sin^{-1} x}{x}\right) dx = \frac{\pi}{2}(\log 2)$

Sol. Let $I = \int_0^1 \left[\frac{\sin^{-1}(x)}{x}\right] dx$

Put $x = \sin\theta \quad \Rightarrow dx = \cos\theta\ d\theta$. Clearly $x = 0,\ \theta = 0,$ and $x = 1,\ \theta = \frac{\pi}{2}$.

Now $I = \int_0^{\pi/2} \frac{\sin^{-1}(\sin\theta)}{\sin\theta}.\cos\theta\ d\theta \quad \Rightarrow I = \int_0^{\pi/2} \theta.\ \cot\theta\ d\theta$

$\Rightarrow I = \left[\theta \log(\sin\theta)\right]_0^{\pi/2} - \int_0^{\pi/2} 1.\log(\sin\theta) d\theta$

$\Rightarrow I = \theta\left\{\frac{\pi}{2}\log\left(\sin\frac{\pi}{2}\right) - 0^*\right\} - \int_0^{\pi/2} \log(\sin\theta) d\theta$

$$\left(\begin{aligned} * \lim_{\theta\to 0^+} \theta \log\sin\theta &= \lim_{\theta\to 0^+} \frac{\log\sin\theta}{\frac{1}{\theta}} = \lim_{\theta\to 0^+} \frac{\frac{\cos\theta}{\sin\theta}}{-\frac{1}{\theta^2}} \\ &= \lim_{\theta\to 0^+} -\frac{\theta}{\sin\theta} \times \theta\cos\theta = -1\times 0 \times 1 = 0 \end{aligned}\right)$$

$\Rightarrow I = 0 - \int_0^{\frac{\pi}{2}} \log(\sin\theta) d\theta$

$\Rightarrow I = -\int_0^{\pi/2} \log(\sin\theta) d\theta \quad \Rightarrow I = -I_1 \qquad \ldots(1)$

$I_1 = \int_0^{\pi/2} \log(\sin\theta) d\theta \qquad \ldots(2)$

Then $I_1 = \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - \theta\right)\right] d\theta$ or $I_1 = \int_0^{\pi/2} \log(\cos\theta) d\theta$

By, Adding (2) and (3) we get, $2I_1 = \int_0^{\pi/2} \left[\log(\sin\theta) + \log(\cos\theta)\right] d\theta$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin\theta.\cos\theta) d\theta \quad \Rightarrow 2I_1 = \int_0^{\pi/2} \log\left(\frac{\sin 2\theta}{2}\right) d\theta$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log(\sin 2\theta) d\theta - \int_0^{\pi/2} \log 2 d\theta \quad \Rightarrow 2I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t) dt - (\log 2)\int_0^{\pi/2} d\theta$

[Putting $2\theta = t$ in the 1st integral]

$2I_1 = \left(\frac{1}{2}\times 2\right).\int_0^{\pi/2} \log(\sin t) dt - \log 2\left(\frac{\pi}{2} - 0\right) \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$\Rightarrow 2I_1 = \int_0^{\pi/2} \log\ \sin\theta\ d\theta - \frac{\pi}{2}\log 2 \quad \Rightarrow 2I_1 = I_1 - \frac{\pi}{2}\log 2 \quad \Rightarrow I_1 = -\frac{\pi}{2}\log 2$

Putting value of I_1 in equation (1), $I = \frac{\pi}{2}\log 2$

46. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx = -\frac{\pi}{2}\log 2$

Sol. Let $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

Put $x=\sin\theta \Rightarrow dx=\cos\theta\, d\theta$. Clearly $x=0, \theta=0,$ and $x=1, \theta=\frac{\pi}{2}$.

$$I=\int_0^{\pi/2}\frac{\log(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta \Rightarrow I=\int_0^{\pi/2}\log(\sin\theta)d\theta$$

$$I_1=\int_0^{\pi/2}\log(\sin\theta)d\theta \qquad ...(2)$$

Then $I_1=\int_0^{\pi/2}\log\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]d\theta$ or $I_1=\int_0^{\pi/2}\log(\cos\theta)d\theta$

By, Adding (2) and (3) we get, $2I_1=\int_0^{\pi/2}\left[\log(\sin\theta)+\log(\cos\theta)\right]d\theta$

$$\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin\theta.\cos\theta)d\theta \Rightarrow 2I_1=\int_0^{\pi/2}\log\left(\frac{\sin 2\theta}{2}\right)d\theta$$

$$\Rightarrow 2I_1=\int_0^{\pi/2}\log(\sin 2\theta)d\theta-\int_0^{\pi/2}\log 2d\theta \Rightarrow 2I_1=\frac{1}{2}\int_0^{\pi}\log(\sin t)dt-(\log 2)\int_0^{\pi/2}d\theta$$

[Putting $2\theta=t$ in the 1st integral]

$$2I_1=\left(\frac{1}{2}\times 2\right).\int_0^{\pi/2}\log(\sin t)dt-\log 2\left(\frac{\pi}{2}-0\right) \Rightarrow 2I_1=\int_0^{\pi/2}\log\sin\theta\, d\theta-\frac{\pi}{2}\log 2$$

$$\Rightarrow 2I_1=I_1-\frac{\pi}{2}\log 2 \Rightarrow I_1=-\frac{\pi}{2}\log 2$$

Now, Putting value of I_1 in equation (1), $I=\frac{\pi}{2}\log 2$

47. $\int_0^1\frac{\log(1+x)}{1+x^2}dx=\frac{\pi}{8}\log 2$

Sol. Let $I=\int_0^1\frac{\log(1+x)}{1+x^2}dx$

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta\, d\theta$. Clearly $x=0, \theta=0$ and $x=1, \theta=\frac{\pi}{4}$.

$$\therefore\ I=\int_0^{\pi/4}\frac{\log(1+\tan\theta)}{1+\tan^2\theta}.\sec^2\theta\, d\theta \Rightarrow I=\int_0^{\pi/4}\log(1+\tan\theta)d\theta \qquad ...(1)$$

Then $I=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$ or $I=\int_0^{\pi/4}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]d\theta$

$$\Rightarrow I=\int_0^{\pi/4}\log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta \Rightarrow I=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$\Rightarrow I=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta \Rightarrow I=\int_0^{\pi/4}\left[\log 2-\log(1+\tan\theta)\right]d\theta \qquad ...(2)$$

By, Adding (1) and (2) we get

$$2I=\int_0^{\pi/4}\left[\log(1+\tan 0)+\log 2-\log(1+\tan 0)\right]d\theta \Rightarrow 2I=\int_0^{\pi/4}\log 2\, d\theta$$

$$\Rightarrow 2I=\log 2\int_0^{\pi/4}d\theta \Rightarrow 2I=\log 2[\theta]_0^{\pi/4} \Rightarrow 2I=\log 2\left(\frac{\pi}{4}-0\right) \quad \therefore I=\frac{\pi}{8}\log 2$$

48. $\int_{-a}^{a} x^3\sqrt{a^2-x^2}\,dx=0$

Sol. Let $f(x)=x^3\sqrt{a^2-x^2}$

Then, $f(-x)=(-x)^3\sqrt{a^2-(-x)^2}=-x^3\sqrt{a^2-x^2}=-f(x)$

$\therefore$ $f(x)$ is an odd function of x.

But $\int_{-a}^{a} f(x)dx=0$, when $f(x)$ is odd. $\therefore \int_{-a}^{a} x^3\sqrt{a^2-x^2}\,dx=0$

49. $\int_{-\pi}^{\pi}\left(\sin^{75}x+x^{125}\right)dx=0$

Sol. Let $f(x)=\sin^{75}x+x^{125}$

Then, $f(-x)=\sin^{75}(-x)+(-x)^{125}=-\sin^{75}x-x^{125}=-\left(\sin^{75}x+x^{125}\right)=-f(x)$

$\therefore$ $f(x)$ is an odd function of x.

But $\int_{-a}^{a} f(x)dx=0$, where $f(x)$ is odd. $\therefore \int_{-\pi}^{\pi}\left(\sin^{75}x+x^{125}\right)dx=0$

50. $\int_{-\pi}^{\pi} x^{12}\sin^9 x\,dx=0$

Sol. Let $f(x)=x^{12}\sin^9 x$

Then, $f(-x)=(-x)^{12}\sin^9(-x)=-x^{12}\sin^9 x=-f(x)$.

$\therefore$ $f(x)$ is an odd function of x.

But $\int_{-a}^{a} f(x)dx=0$, when $f(x)$ is odd. $\therefore \int_{-\pi}^{\pi} x^{12}\sin^9 x\,dx=0$

51. $\int_{-1}^{1} e^{|x|}\,dx=2(e-1)$

Sol. Clearly, $|x|=\begin{cases}-x & \text{when } -1\le x\le 0\\ x & \text{when } \ 0\le x\le 1\end{cases}$

$$\therefore \int_{-1}^{1}e^{|x|}dx=\int_{-1}^{0}e^{|x|}dx+\int_{0}^{1}e^{|x|}dx=\int_{-1}^{0}e^{-x}dx+\int_{0}^{1}e^{x}dx=-\left[e^{-x}\right]_{-1}^{0}+\left[e^{x}\right]_0^1$$

$$=-\left[e^0-e^1\right]+e^1-e^0=-[1-e]+e^1=-1+e+e-1=2e-2=2(e-1)$$

52. $\int_{-2}^{2}|x+1|dx=5$

Sol. Clearly $|x+1| = \begin{cases} -(x+1) & \text{when } -2 \le x \le -1 \\ x+1 & \text{when } -1 \le x \le 2 \end{cases}$

$$\therefore \int_{-2}^{2} |x+1|\,dx = \int_{-2}^{1} |x+1|\,dx + \int_{-1}^{2} |x+1|\,dx = -\int_{-2}^{-1} (x+1)\,dx + \int_{-1}^{2} (x+1)\,dx$$

$$= -\left[\frac{x^2}{2}+x\right]_{-2}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{2} = -\left[\left(\frac{1}{2}-1\right)-\left(\frac{4}{2}-2\right)\right] + \left[\frac{4}{2}+2-\left(\frac{1}{2}-1\right)\right]$$

$$= -\left[-\frac{1}{2}-0\right] + \left[4+\frac{1}{2}\right] = \frac{1}{2}+\frac{9}{2} = 5$$

53. $\int_{0}^{8} |x-5|\,dx = 17$

Sol. Clearly $|x-5| = \begin{cases} -(x-5) & \text{when } 0 \le x \le 5 \\ (x-5) & \text{when } 5 \le x \le 8 \end{cases}$

$$\therefore \int_{0}^{8} (x-5)\,dx = \int_{0}^{5} |x-5|\,dx + \int_{5}^{8} |x-5|\,dx = \int_{0}^{5} -(x-5)\,dx + \int_{5}^{8} (x-5)\,dx = -\left[\frac{x^2}{2}-5x\right]_{0}^{5} + \left[\frac{x^2}{2}-5x\right]_{5}^{8}$$

$$= -\left[\left\{\frac{(5)^2}{2}-5\times 5\right\}-(0)\right] + \left[\left\{\frac{(8)^2}{2}-5\times 8\right\}-\left\{\frac{(5)^2}{2}-(5\times 5)\right\}\right]$$

$$= -\left[\left(\frac{25}{2}-25\right)\right] + \left[(32-40)-\left(\frac{25}{2}-25\right)\right]$$

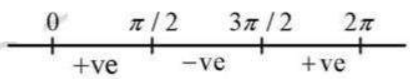
$$= -\left(\frac{-25}{2}\right) + \left[-8-\left(-\frac{25}{2}\right)\right] = \frac{25}{2}-8+\frac{25}{2} = \frac{25}{2}+\frac{9}{2} = \frac{34}{2} = 17$$

54. $\int_{0}^{2\pi} (\cos x)\,dx = 4$

Sol. Given $\int_{0}^{2\pi} (\cos x)\,dx = 4$; $\quad |\cos x| = \begin{cases} \cos x, & \text{when } 0 \le x \le \frac{\pi}{2} \\ -\cos x, & \text{when } \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ \cos x, & \text{when } \frac{3\pi}{2} \le x \le 2\pi \end{cases}$

$$\int_{0}^{2\pi} \cos x\,dx = \int_{0}^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{3\pi/2} |\cos x|\,dx + \int_{3\pi/2}^{2\pi} |\cos x|\,dx$$

$$= \int_{0}^{\pi/2} \cos x\,dx - \int_{\pi/2}^{3\pi/2} \cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$$



$$= [\sin x]_{0}^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$

$$= \left[\sin\frac{\pi}{2}-\sin 0\right] - \left[\sin\frac{3\pi}{2}-\sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

$$= [1-0]-[-1-1]+[0+1] = 1+2+1 = 4$$

55. $\int_{-\pi/4}^{\pi/4} |\sin x|\,dx = 2-2\sqrt{2}$

Sol. Clearly $\left|\sin x\right| = \begin{cases} -\sin x & \text{when } -\dfrac{\pi}{4} \le x \le 0 \\ \sin x & \text{when } 0 \le x \le \dfrac{\pi}{4} \end{cases}$

$$\therefore \int_{-\pi/4}^{\pi/4} \left|\sin x\right| dx = \int_{-\pi/4}^{0} \left|\sin x\right| dx + \int_{0}^{\pi/4} (\sin x)\, dx = \int_{-\pi/4}^{0} -\sin x\, dx + \int_{0}^{\pi/4} \sin x\, dx$$

$$= \int_{0}^{-\pi/4} \sin x\, dx + \int_{0}^{\pi/4} \sin x\, dx = \left[-\cos x\right]_0^{-\pi/4} + \left[-\cos x\right]_0^{\pi/4}$$

$$= \left[-\cos\left(-\frac{\pi}{4}\right) - (-\cos 0)\right] + \left[-\cos\frac{\pi}{4} - (-\cos 0)\right]$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(-\frac{1}{\sqrt{2}} + 1\right) = 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2}$$

56. $f(x) = \begin{cases} 2x+1 & \text{when } 1 \le x \le 2 \\ x^2+1 & \text{when } 2 \le x \le 3 \end{cases}$, show that $\int_1^3 f(x)\, dx = \dfrac{34}{3}$.

Sol. $$\int_1^3 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx = \int_1^2 (2x+1)dx + \int_2^3 \left(x^2+1\right)dx = \left[2\frac{x^2}{2} + x\right]_1^2 + \left[\frac{x^3}{3} + x\right]_2^3$$

$$= \left[x^2 + x\right]_1^2 + \left[\frac{x^3}{3} + x\right]_2^3 = \left[\left\{(2)^2 + 2\right\} - \left\{(1)^2 + 1\right\}\right] + \left[\left\{\frac{(3)^3}{3} + 3\right\} - \left\{\frac{(2)^3}{3} + 2\right\}\right]$$

$$= \left[(4+2) - (2)\right] + \left[(9+3) - \left(\frac{8}{3} + 2\right)\right] = 6 - 2 + 12 - \frac{14}{3} = 16 - \frac{14}{3} = \frac{34}{3}$$

57. Let $f(x) = \begin{cases} 3x^2 + 4 & \text{when } 0 \le x \le 2 \\ 9x - 2 & \text{when } 2 \le x \le 4 \end{cases}$ show that $\int_0^4 f(x)dx = 66$

Sol. $$\int_0^4 f(x)dx = \int_0^2 \left(3x^2+4\right)dx + \int_2^4 (9x-2)dx = \left[3\frac{x^3}{3} + 4x\right]_0^2 + \left[9\frac{x^2}{2} - 2x\right]_2^4 = \left[x^3 + 4x\right]_0^2 + \left[\frac{9x^2}{2} - 2x\right]_2^4$$

$$= \left[\left\{(2)^3 + 4(2)\right\} - \left\{(0)^3 + 4(0)\right\}\right] + \left[\left\{\frac{9(4)^2}{2} - 2(4)\right\} - \left\{\frac{9(2)^2}{2} - 2(2)\right\}\right]$$

$$= \left[(8+8) - 0\right] + \left[(72-8) - (18-4)\right] = 16 + \left[64 - 14\right] = 16 + 50 = 66$$

58. Prove that $\int_0^4 \left\{\left|x\right| + \left|x-2\right| + \left|x-4\right| dx\right\} = 20$

Sol. $$\text{L.H.S} = \int_0^4 \left|x\right| dx + \int_0^4 \left|x-2\right| dx + \int_0^4 \left|x-4\right| dx = \int_0^4 x\, dx + \int_0^2 (2-x)dx + \int_2^4 (x-2)dx + \int_0^4 (4-x)dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 + \left[4x - \frac{x^2}{2}\right]_0^4$$

$$= 8 + (4-2) + (8-8) - (2-4) + (16-8) = 20$$

EXERCISE 16D (Pg. No.: 843)

Evaluate each of the following integrals as the limits of sums.

1. $\int_0^2 (x+4)\,dx$

Sol. Let $f(x)=x+4$ i.e., $a=0$ $b=2,\ nh=2-0=2$

$$\therefore \int_0^2 (x+4)\,dx=\lim_{h\to 0} h\left[f(0)+f(0+h)+....+f(0+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[4+(h+4)+(2h+4)+.....+((n-1)h+4)\right]$$

$$=\lim_{h\to 0} h\left[4n+h(1+2+3+.....+(n-1))\right]=\lim_{h\to 0} h\left[4n+h.\frac{n(n-1)}{2}\right]$$

$$=\lim_{h\to 0}\left[4nh+\frac{h^2(n^2-n)}{2}\right]=\lim_{h\to 0}\left[4nh+\frac{n^2h^2}{2}-\frac{nh^2}{2}\right]=4(2)+\frac{(2)^2}{2}-\frac{2.0}{2}=8+2=10$$

2. $\int_1^2 (3x-2)\,dx$

Sol. Let $f(x)=3x-2,\ a=1,\ b=2,\ nh=2-1=1$

$$\therefore \int_1^2 (3x-2)\,dx=\lim_{h\to 0} h\left[f(1)+f(1+h)+.....+f(1+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[\{3(1)-2\}+\{3(1+h)-2\}+\{3(1+2h)-2\}+.....+\{3(1+(n-1)h)-2\}\right]$$

$$=\lim_{h\to 0} h\left[1+(1+3h)+(1+6h)+.....+\{1+3(n-1)h\}\right]$$

$$=\lim_{h\to 0} h\left[n+3h(1+2+3+.....+(n-1))\right]=\lim_{h\to 0} h\left[n+3h.\frac{n(n-1)}{2}\right]$$

$$=\lim_{h\to 0}\left[nh+\frac{3h^2(n^2-n)}{2}\right]=\lim_{h\to 0}\left[nh+\frac{3.h^2n^2}{2}-\frac{3.h^2n}{2}\right]=\lim_{h\to 0}\left[nh+\frac{3(nh)^2}{2}-\frac{3(nh).h}{2}\right]$$

$$=1+\frac{3(1)^2}{2}-\frac{3.1.0}{2}=1+\frac{3}{2}=\frac{5}{2}$$

3. $\int_1^3 x^2\,dx$

Sol. Let $f(x)=x^2$ i.e. 0 $a=1,\ b=3,\ nh=3-1=2$

$$\therefore \int_1^3 x^2\,dx=\lim_{h\to 0} h\left[f(1)+f(1+h)+.....+f(1+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[(1)^2+(1+h)^2+(1+2h)^2+.....+(1+(n-1)h)^2\right]$$

$$=\lim_{h\to 0} h\left[1+(1+2h+h^2)+(1+4h+4h^2)+.....+\{1+2(n-1)h+(n-1)^2h^2\}\right]$$

$$=\lim_{h\to 0} h\left[n+2h(1+2+3+....+(n-1))+h^2\left(1^2+2^2+3^2+.....+(n-1)^2\right)\right]$$

$$= \lim_{h\to 0} h\left[n + 2h.\frac{n(n-1)}{2} + h^2.\frac{n(n-1)(2n-1)}{6}\right] = \lim_{h\to 0}\left[nh + h^2\left(n^2 - n\right) + h^3\frac{\left(2n^3 - 3n^2 + n\right)}{6}\right]$$

$$= \lim_{h\to 0}\left[nh + (nh)^2 - (nh).h + \frac{2(nh)^3}{6} - \frac{3(nh)^2.h}{6} + \frac{(nh).h^2}{6}\right]$$

$$= 2 + (2)^2 - 2.0 + \frac{(2)^3}{3} - \frac{3(2)^2.0}{6} + \frac{2.0}{6} = 6 + \frac{8}{3} = \frac{26}{3}$$

4. $\int_0^3 \left(x^2 + 1\right) dx$

Sol. Let $f(x) = x^2 + 1$ i.e. $a = 0,\ b = 3,\ nh = 3 - 0 = 3$

$$\therefore \int_0^3 \left(x^2 + 1\right) dx = \lim_{h\to 0} h\left[f(0) + f(0+h) + f(0+2h) + ... + f(0 + (n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[1 + \left(h^2 + 1\right) + \left(4h^2 + 1\right) + \left(9h^2 + 1\right) + ... + \left\{(n-1)^2 h^2 + 1\right\}\right]$$

$$= \lim_{h\to 0} h\left[n + h^2\left(1 + 4 + 9 + ... + (n-1)^2\right)\right] = \lim_{h\to 0}\left[h + h^2.\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[nh + h^3.\frac{\left(2n^3 - 3n^2 + n\right)}{6}\right] = \lim_{h\to 0}\left[nh + 2.\frac{(nh)^3}{6} - 3.\frac{(nh)^2.h}{6} + \frac{(nh).h^2}{6}\right]$$

$$= 3 + \frac{(3)^3}{3} - \frac{3.(3)^2.0}{6} + \frac{3.0}{6} = 3 + 9 = 12$$

5. $\int_2^5 \left(3x^2 - 5\right) dx$

Sol. Let $f(x) = 3x^2 - 5$ i.e. $a = 2,\ b = 5,\ nh = 5 - 2 = 3$

$$\therefore \int_2^5 \left(3x^2 - 5\right) dx = \lim_{h\to 0} h\left[f(2) + f(2+h) + f(2+2h) + ... + f(2 + (n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[7 + \left(7 + 12h + 3h^2\right) + \left(7 + 24h + 12h^2\right) + ... + \left\{7 + 12(n-1)h + 3(n-1)^2 h^2\right\}\right]$$

$$= \lim_{h\to 0} h\left[7n + 12h\left(1 + 2 + 3 + ... + (n-1)\right) + 3h^2\left(1 + 2^2 + ... + (n-1)^2\right)\right]$$

$$= \lim_{h\to 0} h\left[7n + 12h.\frac{n(n-1)}{2} + 3h^2.\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[7nh + 6h^2\left(n^2 - n\right) + \frac{h^3\left(2n^3 - 3n^2 + n\right)}{2}\right]$$

$$= \lim_{h\to 0}\left[7nh + 6(nh)^2 - 6(nh).h + (nh)^3 - \frac{3(nh)^2.h}{2} + \frac{(nh)h^2}{2}\right]$$

$$= \left[7.3 + 6(3)^2 - 6.3.0 + (3)^3 - \frac{3(3)^2.0}{2} + \frac{3.0}{2}\right] = 21 + 54 + 27 = 102$$

6. $\int_0^3 (x^2+2x)dx$

Sol. Let $f(x)=x^2+2x$ i.e. $a=0,\ b=3,\ nh=3-0=3$

$$\therefore \int_0^3 (x^2+2x)dx = \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+.....+f(0+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[0+(h^2+2h)+(4h^2+4h)+(9h^2+6h)+.....+\left\{(n-1)^2h^2+2(n-1)h\right\}\right]$$

$$=\lim_{h\to 0} h\left[2h(1+2+3+.....+(n-1))+h^2\left(1^2+2^2+3^2+.....+(n-1)^2\right)\right]$$

$$=\lim_{h\to 0} h\left[2h.\frac{n(n-1)}{2}+h^2.\frac{n(n-1)(2n-1)}{6}\right]=\lim_{h\to 0}\left[h^2(n^2-n)+h^3.\frac{(2n^3-3n^2+n)}{6}\right]$$

$$=\lim_{h\to 0}\left[(nh)^2-(nh).h+\frac{2(nh)^3}{6}-\frac{3(nh)^2.h}{6}+\frac{(nh).h^2}{6}\right]$$

$$=\left[(3)^2-3.0+\frac{(3)^3}{3}-\frac{3.(3)^2.0}{6}+\frac{3.0}{6}\right]=9+\frac{27}{3}=9+9=18$$

7. $\int_1^4 (3x^2+2x)dx$

Sol. Let $f(x)=3x^2+2x$ i.e. $a=1,\ b=4,\ nh=4-1=3$

$$\therefore \int_1^3 (3x^2+2x)dx = \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+.....+f(1+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[5+(5+8h+3h^2)+(5+16h+12h^2)+.....+\left\{5+8(n-1)h+3(n-1)^2h^2\right\}\right]$$

$$=\lim_{h\to 0} h\left[5n+8h(1+2+3+.....+(n-1))+3h^2\left(1^2+2^2+3^2+.....+(n-1)^2\right)\right]$$

$$=\lim_{h\to 0} h\left[5n+8h.\frac{n(n-1)}{2}+3h^2.\frac{n(n-1)(2n-1)}{6}\right]$$

$$=\lim_{h\to 0}\left[5nh+4h^2(n^2-n)+\frac{h^3(2n^3-3n^2+n)}{2}\right]$$

$$=\lim_{h\to 0}\left[5(nh)+4(nh)^2-4(nh).h+(nh)^3-\frac{3(nh)^2.h}{2}+\frac{(nh).h^2}{2}\right]$$

$$=5(3)+4(3)^2-4.3.0+(3)^3-\frac{3(3)^2.0}{2}+\frac{3.0}{2}=15+36+27=78$$

8. $\int_1^3 (x^2+5x)dx$

Sol. Let $f(x)=x^2+5x$ i.e. $a=1,\ b=3,\ nh=3-1=2$

$$\therefore \int_1^3 (x^2+5x)dx = \lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h)+......+f(1+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[6+\left(6+7h+h^2\right)+\left(6+14h+4h^2\right)+.....+\left\{6+7(n-1)h+(n-1)^2h^2\right\}\right]$$

$$= \lim_{h\to 0} h\left[6n+7h\left(1+2+3+.....+(n-1)\right)+h^2\left(1^2+2^2+3^2+.....+(n-1)^2\right)\right]$$

$$= \lim_{h\to 0} h\left[6n+7h.\frac{n(n-1)}{2}+h^2.\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[6nh+7h^2.\frac{(n^2-n)}{2}+h^3.\frac{(2n^3-3n^2+n)}{6}\right]$$

$$= \lim_{h\to 0}\left[6nh+\frac{7(nh)^2}{2}-\frac{7(nh).h}{2}+\frac{(nh)^3}{3}-\frac{(nh)^2.h}{2}+\frac{(nh)h^2}{6}\right]$$

$$= 6(2)+\frac{7(2)^2}{2}-\frac{7.2.0}{2}+\frac{(2)^3}{3}-\frac{(2)^2.0}{2}+\frac{2.0}{6}=12+14+\frac{8}{3}=26+\frac{8}{3}=\frac{86}{3}$$

9. $\int_1^3 \left(2x^2+3x\right)dx$

Sol. Let $I=\int_1^3\left(2x^2+3x\right)dx$, i.e., $a=1, b=3, nh=b-a \Rightarrow nh=3-1=2$

$f(x)=2x^2+3x$

$f(1)=2+3=5$

$f(1+h)=2(1+h)^2+3(1+h)$

$f(1+2h)=2(1+2h)^2+3(1+2h)$

$f(1+3h)=2(1+3h)^2+3(1+3h)$

....

$f\{1+(n-1)h\}=2\{1+(n-1)h\}^2+3\{1+(n-1)h\}$

Now, $\int_1^3\left(2x^2+3x\right)dx=\lim_{h\to 0} h\left[f(1)+f(1+h)+f(1+2h).....+f\{1+(n-1)h\}\right]$

$$= \lim_{h\to 0} h\left[5+\left\{2(1+h)^2+3(1+h)\right\}+\left\{2(1+2h)+3(1+2h)\right\}^2\right.$$
$$\left.+\left\{2(1+3h)^2+3(1+3h)\right\}+.....+2\{1+(n-1)h\}^2\right]$$

$$= \lim_{h\to 0}\left[5n+2\left\{h^2+(2h)^2+(3h)^2+.....+(n-1)^2h^2\right\}+2.\{2.1.h+2.1.2h+2.1.3h+.....+2.1.(n-1)h\}\right.$$
$$\left.+\{3h+3.2.h+3.3h+3.4h+......+3.(n-1)h\}\right]$$

$$= \lim_{h\to 0} h\left[5n+2h^2\left\{1^2+2^2+3^2+.....+(n-1)^2\right\}+2.2h\{1+2+3+......+(n-1)\}\right.$$
$$\left.+3h\{1+2+3+.....+(n-1)\}\right]$$

$$= \lim_{h\to 0} h\left[5n + 2.h^2.\frac{n(n-1)(2n-1)}{6} + 7h.\frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[5nh + \frac{nh(nh-h)(2nh-h)}{3} + \frac{7}{2}nh(nh-h)\right]$$

$$= \left[5.2 + \frac{2(2-0)(2.2-0)}{3} + \frac{7}{2}.2(2-0)\right] = 10 + \frac{16}{3} + 14 = 24 + \frac{16}{3} = \frac{72+16}{3} = \frac{88}{3}$$

10. $\int_0^2 x^3 dx$

Sol. Let $f(x) = x^3$ i.e. $a = 0,\ b = 2,\ nh = 2 - 0 = 2$

$$\therefore\ \int_0^2 x^3 dx = \lim_{h\to 0} h\left[f(0) + f(0+h) + f(0+2h) + ... + f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[0 + h^3 + 8h^3 + 27h^3 + ... + (n-1)^3 h^3\right]$$

$$= \lim_{h\to 0}\left[h.h^3\left(1^3 + 2^3 + 3^3 + ...(n-1)^3\right)\right] = \lim_{h\to 0} h^4\left(\frac{n(n-1)}{2}\right)^2 = \lim_{h\to 0} h^4.\frac{n^2(n-1)^2}{4}$$

$$= \lim_{h\to 0}\left[h^4.\frac{n^2(n^2-2n+1)}{4}\right] = \lim_{h\to 0}\left[\frac{(nh)^4}{4} - 2.\frac{(nh)^3.h}{4} + \frac{(nh)^2.h^2}{4}\right]$$

$$= \frac{(2)^4}{4} - \frac{2.(2)^3.0}{4} + \frac{(2)^2.0}{4} = \frac{16}{4} - 0 + 0 = 4$$

11. $\int_2^4 (x^2 - 3x + 2)dx$

Sol. Let $f(x) = x^2 - 3x + 2$ i.e. $a = 2,\ b = 4,\ nh = 4 - 2 = 2$

$$\therefore\ \int_2^4 (x^2 - 3x + 2)dx = \lim_{h\to 0} h\left[f(2) + f(2+h) + f(2+2h) + ... + f(2+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[0 + (h + h^2) + (2h + 4h^2) + (2h + 9h^2) + ... + \left\{(n-1)h + (n-1)^2 h^2\right\}\right]$$

$$= \lim_{h\to 0} h\left[h(1 + 2 + 3 + ... + (n-1)) + h^2\left(1^2 + 2^2 + 3^3 + ... + (n-1)^2\right)\right]$$

$$= \lim_{h\to 0} h\left[h.\frac{n(n-1)}{2} + h^2.\frac{n(n-1)(2n-1)}{6}\right] = \lim_{h\to 0}\left[\frac{h^2(n^2-n)}{2} + \frac{h^3(2n^3 - 3n^2 + n)}{6}\right]$$

$$= \lim_{h\to 0}\left[\frac{(nh)^2}{2} - \frac{(nh).h}{2} + \frac{(nh)^3}{3} - \frac{(nh)^2.h}{2} + \frac{(nh).h^2}{6}\right]$$

$$= \frac{(2)^2}{2} - \frac{2.0}{2} + \frac{(2)^3}{3} - \frac{(2)^2.0}{2} + \frac{2.0}{6} = 2 + \frac{8}{3} = \frac{14}{3}$$

12. $\int_0^2 (x^2 + x)dx$

Sol. Let $f(x) = x^2 + x$ i.e. $a = 0,\ b = 2,\ nh = 2 - 0 = 2$

$$\therefore \int_0^2 \left(x^2+x\right)dx = \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+.....+f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[0+\left(h^2+h\right)+\left(4h^2+h\right)+\left(9h^2+3h\right)+....+\left\{(n-1)^2h^2+(n-1)h\right\}\right]$$

$$= \lim_{h\to 0} h\left[h^2\left(1^2+2^2+3^3+.....+(n-1)^2\right)+h\left(1+2+3+.....+(n-1)\right)\right]$$

$$= \lim_{h\to 0} h\left[h^2.\frac{n(n-1)(2n-1)}{6}+h.\frac{n(n-1)}{2}\right] = \lim_{h\to 0}\left[h^3\frac{\left(2n^3-3n^2+n\right)}{6}+\frac{h^2\left(n^2-n\right)}{2}\right]$$

$$= \lim_{h\to 0}\left[\frac{(nh)^3}{3}-\frac{3.(nh)^2.h}{6}+\frac{(nh).h^2}{6}+\frac{(nh)^2}{2}-\frac{(nh).h}{2}\right]$$

$$= \frac{(2)^3}{3}-\frac{3(2)^2.0}{6}+\frac{2.0}{6}+\frac{(2)^2}{2}-\frac{2.0}{2}=\frac{8}{3}+2=\frac{14}{3}$$

13. $\int_0^3 \left(2x^2+3x+5\right)dx$

Sol. Let $f(x)=2x^2+3x+5$ i.e. $a=0,\ b=3,\ nh=3-0=3$

$$\therefore \int_0^3 \left(2x^2+3x+5\right)dx = \lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+.....+f(0+(n-1)h)\right]$$

$$= \lim_{h\to 0} h\left[5+\left(2h^2+3h+5\right)+\left(8h^2+6h+5\right)+.....+\left\{2(n-1)^2h^2+3(n-1)h+5\right\}\right]$$

$$= \lim_{h\to 0} h\left[5n+\left\{3h\left(1+2+3+.....+(n-1)\right)\right\}+2h^2\left(1^2+2^2+3^2+.....+(n-1)^2h^2\right)\right]$$

$$= \lim_{h\to 0} h\left[5n+3h.\frac{n(n-1)}{2}+2h^2.\frac{n(n-1)(2n-1)}{6}\right]$$

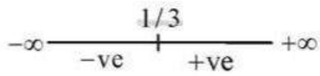
$$= \lim_{h\to 0}\left[5nh+\frac{3h^2\left(n^2-n\right)}{2}+\frac{h^3\left(2n^3-3n^2+n\right)}{3}\right]$$

$$= \lim_{h\to 0}\left[5nh+\frac{3(nh)^2}{2}-\frac{3(nh).h}{2}+\frac{2(nh)^3}{3}-\frac{3(nh)^2.h}{3}+\frac{(nh).h^2}{3}\right]$$

$$= 5(3)+\frac{3(3)^2}{2}-\frac{3.3.0}{2}+\frac{2(3)^3}{3}-\frac{3(3)^2.0}{3}+\frac{3.0}{3}=15+\frac{27}{2}+18=33+\frac{27}{2}=\frac{93}{2}$$

14. $\int_0^1 \left|3x-1\right|dx$

Sol. Let $I=\int_0^1 \left|3x-1\right|dx$



$$3x-1=0 \Rightarrow x=\frac{1}{3}$$

$$I=\int_0^1 \left|3x-1\right|dx \quad \Rightarrow I=\int_0^{1/3}\left|3x-1\right|dx+\int_{1/3}^1\left|3x-1\right|dx \quad \Rightarrow I=\int_0^{1/3}-(3x-1)dx+\int_{1/3}^1(3x-1)dx$$

$$\Rightarrow I=\int_0^{1/3}(1-3x)dx+\int_{1/3}^1(3x-1)dx$$

$$\Rightarrow I = I_1 + I_2,\ \text{where } I_1 = \int_0^{1/3} (1-3x)\,dx,\quad I_2 = \int_{1/3}^{1} (3x-1)\,dx$$

$$I_1 = \int_0^{1/3} (1-3x)\,dx,\quad nh = \frac{1}{3} - 0 = \frac{1}{3}$$

$$f(x) = 1-3x$$

$$f(0) = 1$$

$$f(0+h) = 1-3(0+h)$$

$$f(0+2h) = 1-3(0+2h)$$

$$f(0+3h) = 1-3(0+3h)$$

....

$$f\{1+(n-1)h\} = 1-3(n-1)h$$

$$\int_0^{1/3} (1-3x)\,dx = \lim_{h\to 0} h\left[f(0) + f(0+h) + f(0+2h) + ... + f\left(0+(n-1)h\right)\right]$$

$$= \lim_{h\to 0} h\left[1 + (1-3h) + 1 - 3.2h + (1-3.3h) + ... + 1 - 3(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[(1+1+1+.....) - 3h\{1+2+3+.....+(n-1)\}\right]$$

$$= \lim_{h\to 0} h\left[n - 3h.\frac{n(n+1)}{2}\right] = \lim_{h\to 0}\left[nh - 3nh\frac{(nh+h)}{2}\right] = \frac{1}{3} - \frac{3.\frac{1}{3}\left(\frac{1}{3}+0\right)}{2} = \frac{1}{3} - \frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}$$

$$I_2 = \int_{1/3}^{1} (3x-1)\,dx,\quad nh = 1 - \frac{1}{3} = \frac{2}{3}$$

$$f(x) = 3x-1$$

$$f\left(\frac{1}{3}\right) = 3.\frac{1}{3} - 1 = 0$$

$$f\left(\frac{1}{3}+h\right) = 3\left(\frac{1}{3}+h\right) - 1 = 3h$$

$$f\left(\frac{1}{3}+2h\right) = 3\left(\frac{1}{3}+2h\right) - 1 = 3.2h$$

....

$$f\left\{\frac{1}{3}+(n-1)h\right\} = 3\left(\frac{1}{3}+(n-1)h\right) - 1 = 3(n-1)h$$

$$\int_{1/3}^{1} (3x-1)\,dx = \lim_{h\to 0} h\left[f\left(\frac{1}{3}\right) + f\left(\frac{1}{3}+h\right) + f\left(\frac{1}{3}+2h\right) + ... + f\left\{\frac{1}{3}+(n-1)h\right\}\right]$$

$$= \lim_{h\to 0} h\left[0 + 3h + 3.2h + 3.3h + ... + 3(n-1)h\right] = \lim_{h\to 0} h.3h\left[1+2+3+....+(n-1)\right]$$

$$= \lim_{h\to 0} 3h.h.\frac{n(n+1)}{2} = \lim_{h\to 0}\frac{3nh(nh+h)}{2} = \frac{3.\frac{2}{3}\left(\frac{2}{3}-0\right)}{2} = \frac{2}{3}$$

So, $I = \frac{1}{6} + \frac{2}{3} = \frac{1+4}{6} = \frac{5}{6}$

15. $\int_0^2 e^x dx$

Sol. Let $f(x)=e^x$ i.e. $a=0,\ b=2,\ nh=2-0=2$

$$\therefore \int_0^2 e^x dx=\lim_{h\to 0} h\left[f(0)+f(0+h)+f(0+2h)+.....+f(0+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[e^0+e^{0+h}+e^{0+2h}+.....+e^{0+(n-1)h}\right]=\lim_{h\to 0} h\left[e^0+e^0.e^h+e^0.e^{2h}+.....+e^0.e^{(n-1)h}\right]$$

$$=\lim_{h\to 0} h\left[1+e^h+\left(e^h\right)^2+.....+\left(e^h\right)^{n-1}\right]=\lim_{h\to 0} h\left(1.\frac{\left(e^n\right)^h-1}{e^n-1}\right)=\lim_{h\to 0} h\left(\frac{e^{nh}-1}{e^h-1}\right)=\underset{h\to 0}{lt}\frac{e^2-1}{\frac{e^h-1}{h}}=e^2-1$$

16. $\int_1^3 e^{-x} dx$

Sol. Let $f(x)=e^{-x}$ i.e. $a=1,\ b=3,\ nh=3-1=2$

$$\therefore \int_1^3 e^{-x} dx=\lim_{h\to 0} h\left[f(1)+f(1+h)+.....+f(1+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[e^{-1}+e^{-(1+h)}+e^{-(1+2h)}+.....+e^{-\{1+(n-1)h\}}\right]=\lim_{h\to 0} h\left[e^{-1}+e^{-1}.e^{-h}+e^{-1}.e^{-2h}+.....+e^{-1}.e^{-(h-1)}\right]$$

$$=\lim_{h\to 0} e^{-1}h\left[1+\left(e^{-h}\right)+\left(e^{-h}\right)^2+.....+\left(e^{-h}\right)^{n-1}\right]=\lim_{h\to 0} h.e^{-1}\left\{\frac{\left(e^{-h}\right)^n-1}{e^{-h}-1}\right\}$$

$$=e^{-1}\left(\frac{e^{-nh}-1}{-1}\right)=-\left(\frac{e^{-2}-1}{e}\right)=-\left(\frac{e^{\frac{1}{2}}-1}{e}\right)=\frac{e^2-1}{e^3}$$

17. $\int_a^b \cos x\, dx$

Sol. Let $f(x)=\cos x$ i.e. $a=a,\ b=b,\ nh=b-a$

$$\therefore \int_a^b \cos x\, dx=\lim_{h\to 0} h\left[f(a)+f(a+h)+.....+f(a+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\left[\cos a+\cos(a+h)+\cos(a+2h)+.....+\cos(a+(n-1)h)\right]$$

$$=\lim_{h\to 0} h\frac{\sin\frac{nh}{2}\cos\left\{a+\left(\frac{n-1}{2}\right)h\right\}}{\sin\left(\frac{h}{2}\right)}=\lim_{h\to 0}\frac{h\sin\frac{nh}{2}\cos\left(\frac{2a+nh-h}{2}\right)}{\frac{\sin(h/2)}{h/2}.\frac{h}{2}}$$

$$=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{2a+b-a-0}{2}\right)=2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$=\sin\left(\frac{b-a}{2}+\frac{a+b}{2}\right)+\sin\left(\frac{b-a}{2}-\frac{a+b}{2}\right)=\sin\left(\frac{b-a+a+b}{2}\right)+\sin\left(\frac{b-a-a-b}{2}\right)$$

$$=\sin(b)+\sin(-a)=\sin b-\sin a$$