EXERCISE MISCELLANEOUS

Q No 1 Evaluate $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

Sol: $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3 = \left(i^{18}+\frac{1}{i^{25}}\right)^3 = \left[i^{18}+\frac{1}{i^{24}\cdot i}\right]^3$

$= \left[(i^2)^9+\frac{1}{(i^2)^{12}\cdot i}\right]^3 = \left[(-1)^9+\frac{1}{(-1)^{12}\cdot i}\right]^3$

$= \left(-1+\frac{1}{i}\right)^3 = \left(-1+\frac{i}{i^2}\right)^3 = \left(-1+\frac{i}{-1}\right)^3$

$= (-1-i)^3 = -(1+i)^3$

$= -\left[(1)^3+(i)^3+3\times1\times i\,(1+i)\right]$

$= -\left[1+i^2\cdot i+3i+3i^2\right]$

$= -\left[1-i+3i-3\right] = -\left[-2+2i\right] = 2-2i$

Q No 2: For any two complex Nos z_1 and z_2 prove that $Re(z_1z_2) = Re\,z_1\,Re\,z_2 - Im\,z_1\,Im\,z_2$.

Sol. Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$

Then $Re\,z_1 = x_1$ $\quad Re\,z_2 = x_2$

$Im\,z_1 = y_1$ $\quad Im\,z_2 = y_2$

Now $z_1z_2 = (x_1+iy_1)(x_2+iy_2)$

$= x_1x_2 + x_1y_2 i + x_2y_1 i + i^2 y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2+x_2y_1)$ $\quad [\because i^2=-1]$

$\therefore Re(z_1z_2) = x_1x_2 - y_1y_2$

$= Re\,z_1\,Re\,z_2 - Im\,z_1\,Im\,z_2$

Hence the proof

Q No 3 Reduce $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to standard form.

Sol :

$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)=\left(\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right)\left(\frac{3-4i}{5+i}\right)$$

$$=\left(\frac{1+i-2+8i}{1+i-4i-4i^2}\right)\left(\frac{3-4i}{5+i}\right)=\left(\frac{-1+9i}{1-3i+4}\right)\left(\frac{3-4i}{5+i}\right)$$

$$=\left(\frac{-1+9i}{5-3i}\right)\left(\frac{3-4i}{5+i}\right)=\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}$$

$$=\frac{-3+4i+27i+36}{25+5i-15i+3}=\frac{33+31i}{28-10i}$$

$$=\frac{33+31i}{28-10i}\times\frac{28+10i}{28+10i}=\frac{924+330i+868i-310}{784+100}$$

$$=\frac{614+1198i}{884}=\frac{307+599i}{442}=\frac{307}{442}+i\frac{599}{442}.$$

Q No 4 : If $x-iy=\sqrt{\frac{a-ib}{c-id}}$ prove that $(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$

Sol

Here $x-iy=\sqrt{\frac{a-ib}{c-id}}$.

$\therefore\ x+iy=\sqrt{\frac{a+ib}{c+id}}$

$$\Rightarrow (x-iy)(x+iy)=\sqrt{\frac{a-ib}{c-id}}\times\sqrt{\frac{a+ib}{c+id}}.$$

$$\Rightarrow x^2-i^2y^2=\sqrt{\frac{(a-ib)(a+ib)}{(c-id)(c+id)}}$$

$$\Rightarrow x^2+y^2=\sqrt{\frac{a^2-i^2b^2}{c^2-i^2d^2}}=\sqrt{\frac{a^2+b^2}{c^2+d^2}}\quad [\because i^2=-1]$$

$$\Rightarrow (x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}.$$

Q No 5: Convert the following in Polar form:

(i) $\frac{1+7i}{(2-i)^2}$ (ii) $\frac{1+3i}{1-2i}$

Sol. (i) $\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{4-1-4i} = \frac{1+7i}{3-4i}$

$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{(3)^2-(4i)^2}$

$= \frac{3+25i-28}{9+16} = \frac{-25+25i}{25} = -1+i$

Let $-1+i = r(\cos\theta + i\sin\theta)$

$\Rightarrow r\cos\theta = -1$

$r\sin\theta = 1$

$\Rightarrow r^2 = (-1)^2+(1)^2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$

and $\therefore \cos\theta = -\frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$

As θ lies in Second quadrant and $-\pi < \theta \le \pi$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

$\therefore -1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(ii) $\frac{1+3i}{1-2i} = \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{1+2i+3i+6i^2}{(1)^2-(2i)^2}$

$= \frac{1+5i-6}{1-4i^2} = \frac{-5+5i}{1+4} = -1+i$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ [Same as in part (i)]

Solve each of the equation in exercises 6 to 9.

Q.No.6 $3x^2 - 4x + \frac{20}{3} = 0$

Sol:

$3x^2 - 4x + \frac{20}{3} = 0$ or $9x^2 - 12x + 20 = 0$

$\therefore$ $D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$

$\therefore$ Roots are $x = \frac{-(-12) \pm \sqrt{-576}}{18} = \frac{12 \pm \sqrt{576}\sqrt{-1}}{18} = \frac{12 \pm 24i}{18}$

$= \frac{2 \pm 4i}{3} = \frac{2}{3} \pm \frac{4}{3}i = \frac{2}{3} + \frac{4}{3}i, \ \frac{2}{3} - \frac{4}{3}i$

Q.No.7 $x^2 - 2x + \frac{3}{2} = 0$

Sol

$x^2 - 2x + \frac{3}{2} = 0$ or $2x^2 - 4x + 3 = 0$

$\therefore$ $x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 2 \times 3}}{2 \times 2} = \frac{4 \pm \sqrt{16 - 24}}{4} = \frac{4 \pm \sqrt{-8}}{4}$

$= \frac{4 \pm 2\sqrt{2}i}{4} = \frac{2 \pm \sqrt{2}i}{2} = 1 \pm \frac{\sqrt{2}}{2}i$

$= 1 + \frac{\sqrt{2}}{2}i, \ 1 - \frac{\sqrt{2}}{2}i$

Q.No.8 $27x^2 - 10x + 1 = 0$

Sol. Roots are $x = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 27 \times 1}}{2 \times 27} = \frac{10 \pm \sqrt{100 - 108}}{54}$

$= \frac{10 \pm \sqrt{-8}}{54} = \frac{10 \pm \sqrt{-4 \times 2}}{54} = \frac{10 \pm 2\sqrt{2}i}{54}$

$= \frac{5 \pm i\sqrt{2}}{27}$

$= \frac{5}{27} \pm \frac{\sqrt{2}}{27}i$

$= \frac{5}{27} + \frac{\sqrt{2}}{27}i, \ \frac{5}{27} - \frac{\sqrt{2}}{27}i$

Q No. 9 $21x^2 - 28x + 10 = 0$

Sol: The given equation is $21x^2 - 28x + 10 = 0$

The Roots are $x = \dfrac{28 \pm \sqrt{(-28)^2 - 4 \times 21 \times 10}}{2 \times 21}$

$$= \frac{28 \pm \sqrt{784 - 840}}{42} = \frac{28 \pm \sqrt{-56}}{42}$$

$$= \frac{20 \pm \sqrt{4 \times 14 \times (-1)}}{42} = \frac{28 \pm i\,2\sqrt{14}}{42}$$

$$= \frac{14 \pm i\sqrt{14}}{21} = \frac{14}{21} \pm i\frac{\sqrt{14}}{21}$$

$$= \frac{2}{3} + i\frac{\sqrt{14}}{21}, \quad \frac{2}{3} - i\frac{\sqrt{14}}{21}$$

Q No 10 If $z_1 = 2 - i$, $z_2 = 1 + i$ find $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$

Sol: Here $z_1 = 2 - i$, $z_2 = 1 + i$

$$\therefore \left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{2 - i + 1 + i + 1}{2 - i - 1 - i + 1}\right| = \left|\frac{4}{2(1-i)}\right| = \left|\frac{2}{1-i}\right| = \frac{|2|}{|1-i|}$$

$$= \frac{|2|}{\sqrt{(1)^2 + (-1)^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Q No 11 If $a + ib = \dfrac{(x+i)^2}{2x^2+1}$, prove that $\dfrac{(x^2+1)^2}{(2x^2+1)^2}$

Sol. $a + ib = \dfrac{(x+i)^2}{2x^2+1} = \dfrac{x^2 + i^2 + 2xi}{2x^2+1} = \dfrac{x^2 - 1 + 2xi}{2x^2+1}$

$$\therefore |a + ib| = \left|\frac{(x^2-1) + 2xi}{2x^2+1}\right|$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{\sqrt{(x^2-1)^2 + (2x)^2}}{2x^2+1}$$

$$\Rightarrow a^2 + b^2 = \frac{(x^2-1)^2 + (2x)^2}{(2x^2+1)^2}$$

$$\Rightarrow \quad a^2+b^2 = \frac{x^4+1-2x^2+4x^2}{(2x^2+1)^2}$$

$$\therefore \quad a^2+b^2 = \frac{x^4+1+2x^2}{(2x^2+1)^2}$$

$$\Rightarrow \quad a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

QNo12 Let $z_1 = 2-i$ $z_2 = -2+i$ find

(i) $Re\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $Im\left(\frac{1}{z_1\bar{z}_1}\right)$

Sol. Here $z_1 = 2-i$ $z_2 = -2+i$

$\therefore \bar{z}_1 = 2+i$

(i) $$\frac{z_1 z_2}{\bar{z}_1} = \frac{(2-i)(-2+i)}{(2+i)} = \frac{-4+2i+2i-i^2}{2+i} = \frac{-4+4i+1}{2+i}$$

$$= \frac{-3+4i}{2+i} = \frac{-3+4i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{-6+3i+8i-4i^2}{4-i^2} = \frac{-6+3i+8i+4}{4+1} = \frac{-2+11i}{5}$$

$$= -\frac{2}{5} + \frac{11}{5}i$$

$$\therefore Re\left(\frac{z_1 z_2}{\bar{z}_1}\right) = -\frac{2}{5}$$

(ii) $$\frac{1}{z_1\bar{z}_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{4-i^2} = \frac{1}{4+1} = \frac{1}{5}$$

$$= \frac{1}{5} + 0i$$

$$\therefore Im\left(\frac{1}{z_1\bar{z}_1}\right) = 0$$

Q.No 13 Find the modulus and argument of complex No. $\frac{1+2i}{1-3i}$

Sol.
$$z = \frac{1+2i}{1-3i} = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{1+3i+2i+6i^2}{(1)^2-(3i)^2} = \frac{1+5i-6}{1-9i^2} = \frac{-5+5i}{1+9}$$

$$= \frac{-5+5i}{10} = -\frac{1}{2}+\frac{1}{2}i$$

Let $-\frac{1}{2}+\frac{1}{2}i = r(\cos\theta + i\sin\theta)$

$\therefore$ $-\frac{1}{2} = r\cos\theta$

$\frac{1}{2} = r\sin\theta$

$\Rightarrow r^2 = \left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2$

$\Rightarrow r^2 = \frac{1}{4}+\frac{1}{4} = \frac{1}{2} \Rightarrow r = \frac{1}{\sqrt{2}}$

$\therefore$ $\cos\theta = -\frac{1}{\sqrt{2}}$; $\sin\theta = \frac{1}{\sqrt{2}}$

Since θ lies in second quadrant and $-\pi < \theta \leq \pi$

$\therefore$ $\theta = \pi - \alpha$ where $\tan\alpha = 1$

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore$ Modulus of $z = \frac{1+2i}{1-3i}$ is $\frac{1}{\sqrt{2}}$

and argument is $\theta = \frac{3\pi}{4}$

Q.No.14: Find the real numbers x and y if $(x-iy)(3+5i)$ is conjugate of $-6-24i$

Sol: Since $(x-iy)(3+5i)$ is conjugate of $-6+24i$

$\therefore$ $(x-iy)(3+5i) = -6-24i$

$\Rightarrow x-iy = \frac{-6-24i}{3+5i}$

$$= \frac{-6-24i}{3+5i} \times \frac{3-5i}{3-5i}$$

$$\Rightarrow \quad x-iy = \frac{-18+30i+72i-120i^2}{9-25i^2}$$

ie $$x-iy = \frac{-18+102i+120}{9+25}$$

ie $$x-iy = \frac{102+102i}{34}$$

ie $$x-iy = 3+3i$$

Equating Real and imaginary parts

$$x=3 \quad y=3.$$

Q No 15 Find modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

Sol. Let $z = \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$

$$= \frac{(1+i^2+2i)-(1+i^2-2i)}{(1)^2-(i)^2}$$

$$= \frac{(1-1+2i)-(1-1-2i)}{1-(-1)} = \frac{2i+2i}{2} = \frac{4i}{2} = 2i$$

$$= 0+2i$$

$\therefore$ Modulus of given complex No. $z = \sqrt{(0)^2+(2)^2} = 2.$

Q No 16 If $(x+iy)^3 = u+iv$, then show that $\frac{u}{x}+\frac{v}{y} = 4(x^2-y^2)$

Sol.
$$(x+iy)^3 = u+iv$$

$$\Rightarrow \quad x^3+i^3y^3+3xyi(x+iy) = u+iv$$

$$\Rightarrow \quad x^3-iy^3+3x^2yi-3xy^2 = u+iv$$

$$\Rightarrow \quad (x^3-3xy^2)+i(3x^2y-y^3) = u+iv$$

Equating real and imaginary parts, we get

$$x^3-3xy^2 = u$$

or $$x(x^2-3y^2) = u$$

$$x^2-3y^2=\frac{u}{x}$$

lly $\quad 3x^2y-y^3=v \Rightarrow 3x^2-y^2=\frac{v}{y}$

On adding

$$x^2-3y^2+3x^2-y^2=\frac{u}{x}+\frac{v}{y}$$

$$\Rightarrow \quad 4x^2-4y^2=\frac{u}{x}+\frac{v}{y}$$

$$\Rightarrow \quad \frac{u}{x}+\frac{v}{y}=4(x^2-y^2)$$

QNo 17 If α and β are different complex Numbers with $|\beta|=1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$

Sol: We know that for any complex No z.

$$|z|^2=z\bar{z}$$

$$\therefore \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2=\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\overline{\frac{\beta-\alpha}{1-\bar{\alpha}\beta}}\right)$$

$$=\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right)$$

$$=\frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}=\frac{\beta\bar{\beta}-\beta\bar{\alpha}-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\bar{\alpha}\beta\alpha\bar{\beta}}$$

$$=\frac{|\beta|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+(\alpha\bar{\alpha})(\beta\bar{\beta})}=\frac{|\alpha|^2-\alpha\bar{\beta}-\bar{\alpha}\beta+1}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}=1.$$

$$\therefore \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=1.$$

QNo18. Find the number of non-zero Integral solutions of Equation $|1-i|^x=2^x$.

Sol: $\quad |1-i|^x=2^x$

$$\Rightarrow \left(\sqrt{(1)^2+(-1)^2}\right)^x = 2^x$$

$$\Rightarrow (\sqrt{2})^x = 2^x$$

$$\Rightarrow (2)^{\frac{x}{2}} = 2^x \Rightarrow 1 = \frac{2^x}{2^{x/2}}$$

$$\Rightarrow 1 = 2^{x-\frac{x}{2}} \Rightarrow 1 = 2^{\frac{x}{2}}$$

$$\Rightarrow 2^0 = 2^{\frac{x}{2}}$$

$$\Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0$$

$\Rightarrow$ The given equation has no non-zero integral solution.

Q.No. 19: If $(a+ib)(c+id)(e+if)(g+ih) = A+iB$, then show that $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

Sol: We have $(a+ib)(c+id)(e+if)(g+ih) = A+iB$

$$\Rightarrow |(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$$

$$\Rightarrow |a+ib||c+id||e+if||g+ih| = |A+iB|$$

$$\Rightarrow \sqrt{a^2+b^2}\sqrt{c^2+d^2}\sqrt{e^2+f^2}\sqrt{g^2+h^2} = \sqrt{A^2+B^2}$$

$$\Rightarrow (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$$

Q.No. 20 If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

Sol.
$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2-i^2} = \frac{1+i^2+2i}{1-(-1)} = \frac{1-1+2i}{1+1} = \frac{2i}{2} = i$$

Now $\left(\frac{1+i}{1-i}\right)^1 = (i)^1 = i$

$$\left(\frac{1+i}{1-i}\right)^2 = i^2 = -1, \quad \left(\frac{1+i}{1-i}\right)^3 = i^3 = i^2 \cdot i = -i$$

$$\left(\frac{1+i}{1-i}\right)^4 = i^4 = i^2 \times i^2 = (-1)\times(-1) = 1$$

$\therefore$ Least value of m for which $\left(\frac{1+i}{1-i}\right)^m = 1$ is 4.

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