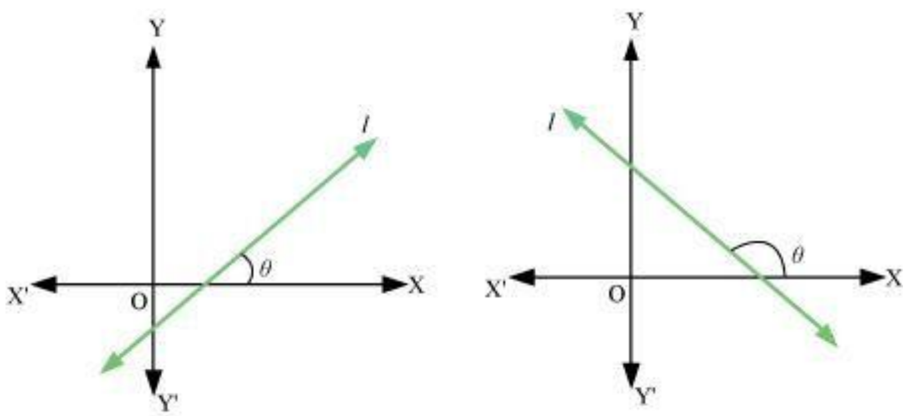
Straight Lines

- **Slope of a line:** If θ is the inclination of a line *l* (the angle between positive *x*-axis and line *l*), then $m = \tan\theta$ is called the slope or gradient of line *l*.



  - The slope of a line whose inclination is 90º is not defined. Hence, the slope of the vertical line, *y*-axis is undefined.
  - The slope of the horizontal line, *x*-axis is zero.

  For example, the slope of a line making an angle of 135° with the positive direction of *x*-axis is $m = \tan 135^\circ = \tan(180^\circ - 45^\circ) = -\tan 45^\circ = -1$

- **Slope of line passing through two given points:**

  The slope (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2$.

  For example, the slope of the line joining the points (–1, 3) and (4, –2) is given by, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 3}{4 - (-1)} = -\frac{5}{5} = -1$

- **Conditions for parallelism and perpendicularity of lines:**

  Suppose l_1 and l_2 are non-vertical lines having slopes m_1 and m_2 respectively.

  - l_1 is parallel to l_2 if and only if $m_1 = m_2$ i.e., their slopes are equal.
  - l_1 is perpendicular to l_2 if and only if $m_1 m_2 = -1$ i.e., the product of their slopes is –1.

  Example: Find the slope of the line which makes an angle of 45° with a line of slope 3.

Solution: Let m be the slope of the required line.

$$\therefore \tan 45^\circ = \left|\frac{m-3}{1+3m}\right|$$

$$\Rightarrow \left|\frac{m-3}{1+3m}\right| = 1$$

$$\Rightarrow \left|\frac{m-3}{1+3m}\right| = \pm 1$$

$$\Rightarrow \frac{m-3}{1+3m} = 1 \quad \text{or} \quad \frac{m-3}{1+3m} = -1$$

$$\Rightarrow m-3 = 1+3m \quad \text{or} \quad m-3 = -1-3m$$

$$\Rightarrow -2m = 4 \quad \text{or} \quad 4m = 2$$

$$\Rightarrow m = -2 \quad \text{or} \quad m = \frac{1}{2}$$

- **Collinearity of three points:** Three points A, B and C are collinear if and only if slope of AB = slope of BC

- **Angle between two lines:** An acute angle, θ, between line l_1 and l_2 with slopes m_1 and m_2 respectively is given by

  $$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|, \ 1+m_1m_2 \neq 0$$.

  Example 1: Two lines AB and CB, intersect at point B. The coordinates of end points are A(−4, −3), B(0, 5), and C(10, 5). Find the measures of angles between AB and CB.

  Solution: Let the angle between the lines AB and BC be θ.

  Slope of line $AB = \frac{5-(-3)}{0-(-4)} = \frac{8}{4} = 2$

  Slope of line $BC = \frac{5-5}{10-0} = 0$

  We know that the angle between two lines with slopes m_1 and m_2 is given by $\tan\theta = \left|\frac{m_2-m_1}{1+m_2m_1}\right|$.

  Therefore, $\tan\theta = \left|\frac{2-0}{1+2\times 0}\right| = 2$

  ⇒ $\theta = \tan^{-1}(2)$.

- The equation of a horizontal line at distance a from the x-axis is either $y = a$ (above x-axis) or $y = -a$ (below x-axis).
- The equation of a vertical line at distance b from the y-axis is either $x = b$ (right of y-axis) or $x = -b$ (left of y-axis).

- **Point-slope form of the equation of a line**

  The point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) if and only if its coordinates satisfy the equation. This means $y - y_0 = m(x - x_0)$.

  Example : Find the equation of the line passing through (4, 5) and making an angle of 120° with the positive direction of x-axis?

  Solution : Slope of the line, $m = \tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$

  Equation of the required line is,

  $$y - 5 = -\sqrt{3}(x - 4)$$
  $$\Rightarrow y - 5 = -\sqrt{3}x + 4\sqrt{3}$$
  $$\Rightarrow \sqrt{3}x + y - (5 + 4\sqrt{3}) = 0$$

- **Two-point form of the equation of a line**

  The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by
  $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

  Example: Find the equation of the line passing through the points (–5, 2) and (1, 6).

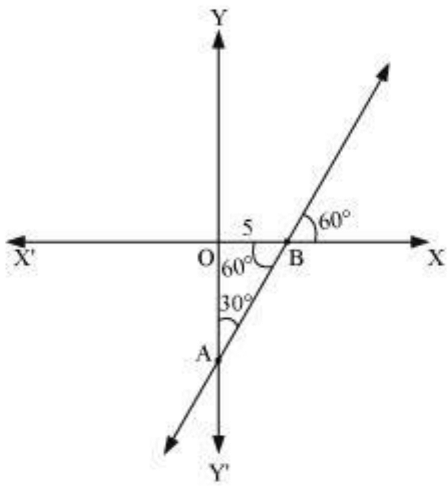
  Solution: Equation of the line passing through points (–5, 2) and (1, 6) is

  $$y - 2 = \frac{6-2}{1-(-5)}(x - (-5))$$
  $$\Rightarrow y - 2 = \frac{4}{6}(x + 5)$$
  $$\Rightarrow y - 2 = \frac{2}{3}(x + 5)$$
  $$\Rightarrow 3y - 6 = 2x + 10$$
  $$\Rightarrow 2x - 3y + 16 = 0$$

- **Slope-intercept form of a line**
  - The equation of the line, with slope m, which makes y-intercept c is given by $y = mx + c$.
  - The equation of the line, with slope m, which makes x-intercept d is given by $y = m(x - d)$.

  Example: Find the equation of the line which cuts off an intercept 5 on the x-axis and makes an angle of 30° with the y-axis.

  Solution:



Slope of the line, m = tan 60° $= \sqrt{3}$
OB = 5
Intercept on the x-axis, c = –OB = –5 and tan 60° = $-5\sqrt{3}$
Equation of the required line is $y = \sqrt{3}x + \left(-5\sqrt{3}\right)$.

- **General equation of line**

  Any equation of the form $Ax + By + C = 0$, where A and B are not zero simultaneously is called the general linear equation or general equation of line.
  Slope of the line = $-\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\dfrac{A}{B}$
  y- intercept = $-\dfrac{C}{B}$

  Example: Find the slope and the y-intercept of the line $2x - 3y = -16$.

  Solution: The equation of the given line can be rewritten as $2x - 3y + 16 = 0$.
  Here, $A = 2$, $B = -3$ and $C = 16$.
  Slope of the line $= -\dfrac{A}{B} = -\dfrac{2}{(-3)} = \dfrac{2}{3}$
  Intercept on the y-axis $= -\dfrac{C}{B} = -\dfrac{16}{(-3)} = \dfrac{16}{3}$

- **Intercept form**

  The equation of the line making intercepts a and b on x-axis and y-axis respectively is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

  Example: If a line passes through (3, 2) and cuts off intercepts on the axes in such a way that the product of the intercepts is 24, then find the equation of the line.

  Solution: The equation of a line in intercept form is

$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots(1)$

Where, *a* and *b* are the intercepts on *x* and *y* axes respectively.

Since the line passes through (3, 2), we obtain

$\frac{3}{a} + \frac{2}{b} = 1$

$\Rightarrow 3b + 2a = ab$

$\Rightarrow 2a + 3b = 24 \quad \ldots(2)$ (Since product of intercepts is given as 24)

Now,

$(2a - 3b)^2 = 24ab$

$= (24)^2 - 24(24) \quad$ [From equation (2)]

$= 0$

$\therefore 2a - 3b = 0 \quad \ldots(3)$

On adding equations (2) and (3), we obtain

$4a = 24 \Rightarrow a = 6$

$\therefore 3b = 2a = 2 \times 6 = 12$

$\Rightarrow b = 4$

Hence, from (1), the required equation of line is

$\frac{x}{6} + \frac{y}{4} = 1$

$\Rightarrow 4x + 6y = 24$

$\Rightarrow 2x + 3y = 12$

- **Normal form of the equation of a line**

  The equation of the line at normal distance *p* from the origin and angle *ω*, which the normal makes with the positive direction of the *x*-axis is given by $x\cos\omega + y\sin\omega = p$

  Example: Reduce the equation $x - \sqrt{3y} - 6 = 0$ to normal form and hence find the length of perpendicular to the line from the origin. Also find angle between the normal and positive direction of the *x*-axis.

  Solution: The given equation is $x - \sqrt{3y} - 6 = 0$.

  $\Rightarrow x - \sqrt{3y} = 6 \quad \ldots(1)$

  On dividing (1) by $\sqrt{\left(\sqrt{1}\right)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$, we obtain

  $\frac{1}{2}x - \frac{\sqrt{2}}{2}y = 3$

  $\Rightarrow x \cos 300^\circ + y \sin 300^\circ = 3 \quad \ldots(2)$

  On comparing equation (2) with $x\cos\omega + y\sin\omega = p$, we obtain $\omega = 300^\circ$ and $p = 3$

  Therefore, the length of perpendicular to the line from the origin is 3 units and the angle between the normal and the positive *x*-axis is 300°.

- **Distance of a Point From a Line**
  The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is
  $$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

  Example: Find the distance of point (1, –2) from the line $8x - 6y - 12 = 0$.

  Solution: On comparing the equation of the given line i.e., $8x - 6y - 12 = 0$ with $Ax + By + C = 0$, we obtain
  $A = 8$, $B = -6$, $C = -12$
  The distance (d) of point (1, –2) from line $8x - 6y - 12 = 0$ is

  $$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|8 \times 1 + (-6)(-2) + (-12)|}{\sqrt{8^2 + 6^2}} = \frac{|8 + 12 - 12|}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$

- **Distance between parallel lines**

  The distance (d) between two parallel lines i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by, $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$

  Example: Find the distance between the lines $4x + 3y = 11$ and $4x + 3y = 8$.

  Solution: The given lines are $4x + 3y - 11 = 0$ and $4x + 3y - 8 = 0$
  Slope of the line $4x + 3y - 11 = 0$ is $-\frac{4}{3}$.
  Slope of the line $4x + 3y - 8 = 0$ is $-\frac{4}{3}$.
  Since the slopes of the given lines are equal, the lines are parallel.
  Here, $A = 4$, $B = 3$, $C_1 = -11$ and $C2 = -8$
  Distance between the lines $= \left|\frac{-11 - (-8)}{\sqrt{4^2 + 3^2}}\right| = \left|\frac{-11 + 8}{\sqrt{16 + 9}}\right| = \left|\frac{-3}{\sqrt{25}}\right| = \frac{3}{5}$