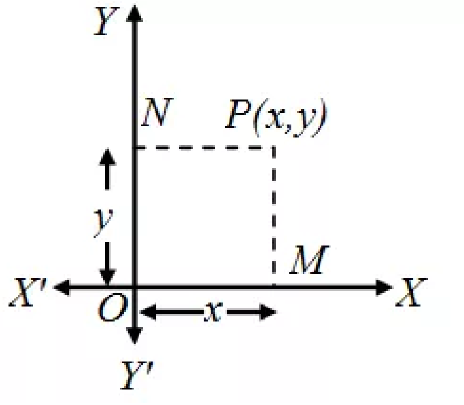
6. Graphs

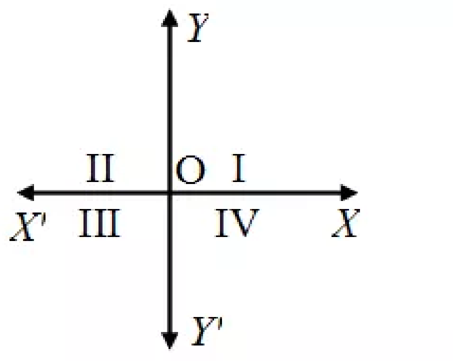
What Is The Cartesian Coordinate System

In Cartesian co-ordinates the position of a point P is determined by knowing the distances from two perpendicular lines passing through the fixed point. Let O be the fixed point called the origin and XOX' and YOY', the two perpendicular lines through O, called Cartesian or Rectangular co-ordinates axes.



Draw PM and PN perpendiculars on OX and OY respectively. OM (or NP) and ON (or MP) are called the x-coordinate (or abscissa) and y-coordinate (or ordinate) of the point P respectively.

1. **Axes of Co-ordinates**
   In the figure OX and OY are called as x-axis and y-axis respectively and both together are known as axes of co-ordinates.
2. **Origin**
   It is point O of intersection of the axes of co-ordinates.
3. **Co-ordinates of the Origin**
   It has zero distance from both the axes so that its abscissa and ordinate are both zero. Therefore, the coordinates of origin are (0, 0).
4. **Abscissa**
   The distance of the point P from y-axis is called its abscissa. In the figure OM = PN is the Abscissa.
5. **Ordinate**
   The distance of the point P from x-axis is called its ordinate. ON = PM is the ordinate in the figure.
6. **Quadrant**
   The axes divide the plane into four parts. These four parts are called quadrants. So, the plane consists of axes and quadrants. The plane is called the cartesian plane or the coordinate plane or the xy-plane. These axes are called the co-ordinate axes.
   A quadrant is 1/4 part of a plane divided by co-ordinate axes.

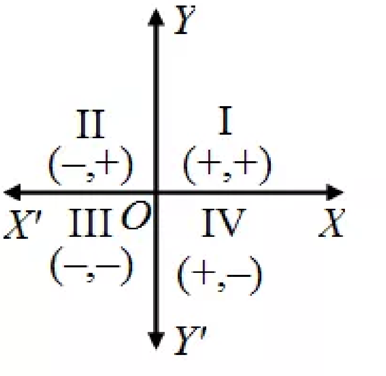


   (i) XOY is called the first quadrant
   (ii) YOX' the second.
   (iii) X'OY' the third.
   (iv) Y'OX the fourth
   as marked in the figure.

RULES OF SIGNS OF CO-ORDINATES

1. In the first quadrant, both co-ordiantes i.e., abscissa and ordinate of a point are positive.

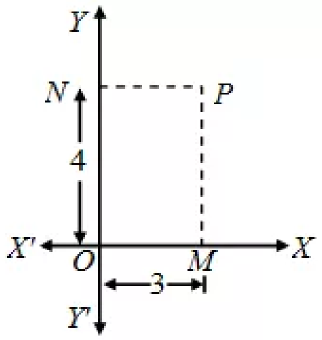
2. In the second quadrant, for a point, abscissa is negative and ordinate is positive.
3. In the third quadrant, for a point, both abscissa and ordinate are negative.
4. In the fourth quadrant, for a point, the abscissa is positive and the ordinate is negative.



Quadrant	x-co-ordinate	y-cordinate	Point
First quadrant	+	+	(+,+)
Second quadrant	–	+	(–,+)
Third quadrant	–	–	(–,–)
Fourth quadrant	+	–	(+,–)

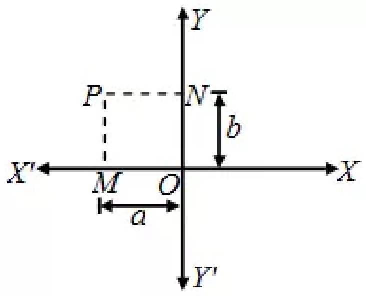
Cartesian Coordinate System Example Problems With Solutions

Example 1: From the adjoining figure find



(i) Abscissa
(ii) Ordinate
(iii) Co-ordinates of a point P
Solution: (i) Abscissa = PN = OM = 3 units
(ii) Ordinate = PM = ON = 4 units
(iii) Co-ordinates of the point P = (Abscissa, ordinate) = (3, 4)

Example 2: Determine

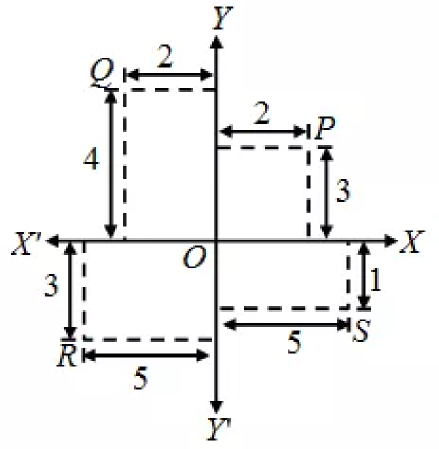


(i) Abscissa (ii) ordinate (iii) Co-ordinates of point P given in the following figure.

Solution: (i) Abscissa of the point P = – NP = –OM = – a
(ii) Ordinate of the point P = MP = ON = b

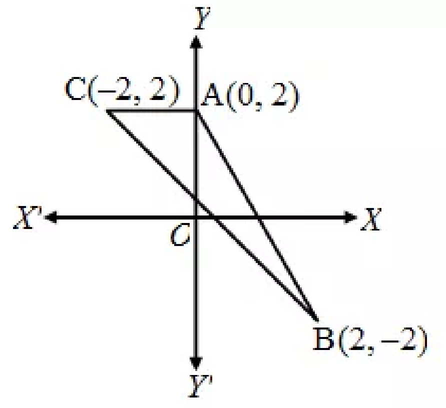
(iii) Co-ordinates of point P = (abscissa, ordinate)
= (–a, b)

Example 3: Write down the (i) abscissa (ii) ordinate (iii) Co-ordinates of P, Q, R and S as given in the figure.



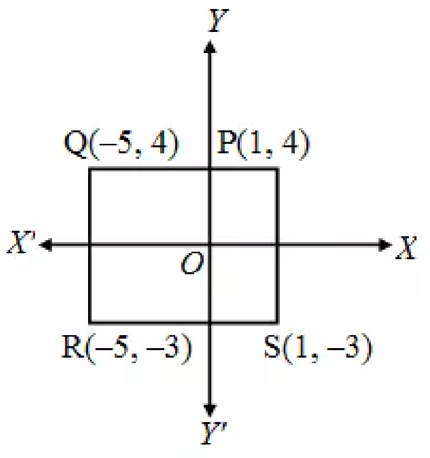
Solution: Point P
Abscissa of P = 2; Ordinate of P = 3
Co-ordinates of P = (2, 3)
Point Q
Abscissa of Q = – 2; Ordinate of Q = 4
Co-ordinate of Q = (–2, 4)
Point R
Abscissa of R = – 5; Ordinate of R = – 3
Co-ordinates of R = (–5, –3)
Point S
Abscissa of S = 5; Ordinate of S = – 1
Co-ordinates of S = (5, – 1)

Example 4: Draw a triangle ABC where vertices A, B and C are (0, 2), (2, – 2), and (–2, 2) respectively.
Solution: Plot the point A by taking its abscissa O and ordinate = 2.
Similarly, plot points B and C taking abscissa 2 and –2 and ordinates – 2 and 2 respectively. Join A, B and C. This is the required triangle.

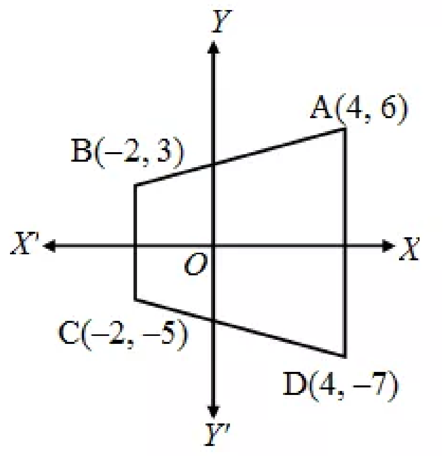


Example 5: Draw a rectangle PQRS in which vertices P, Q, R and S are (1, 4), (–5, 4), (–5, –3) and (1, – 3) respectively.
Solution: Plot the point P by taking its abscissa 1 and ordinate – 4.
Similarly, plot the points Q, R and S taking abscissa as –5, –5 and 1 and ordinates as 4, – 3 and –3 respectively.

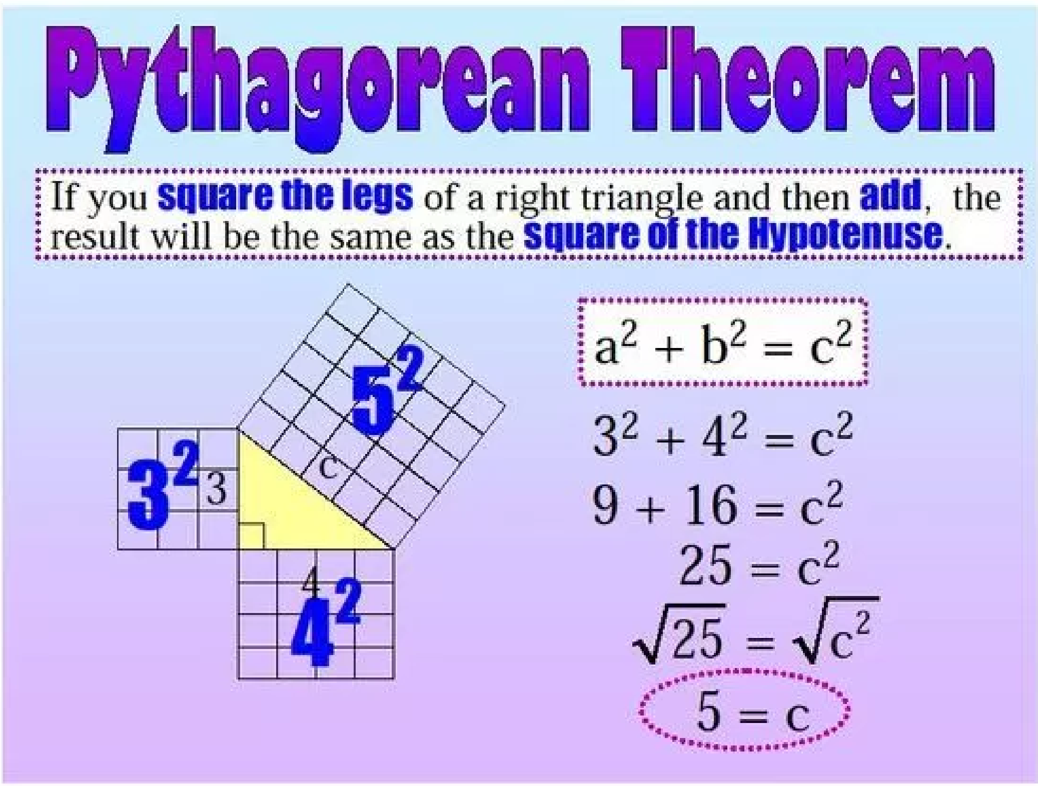
Join the points PQR and S. PQRS is the required rectangle.



Example 6: Draw a trapezium ABCD in which vertices A, B, C and D are (4, 6), (–2, 3), (–2, –5) and (4, –7) respectively.
Solution: Plot the point A taking its abscissa as 4 and ordinate as 6.
Similarly plot the point B, C and D taking abscissa as – 2, –2 and 4 and ordinates as 3, – 5, and –7 respectively. Join A, B, C and D ABCD is the required trapezium.

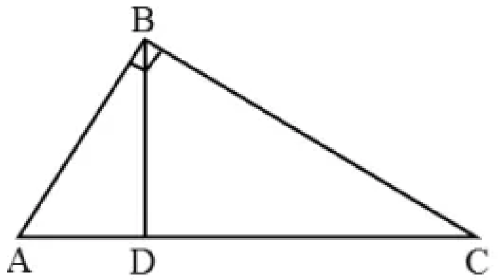


Pythagoras Theorem



Theorem 1: In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
Given: A right-angled triangle ABC in which B = ∠90º.
To Prove: $(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{Perpendicular})^2$.
i.e., $AC^2 = AB^2 + BC^2$

Construction: From B draw BD ⊥ AC.

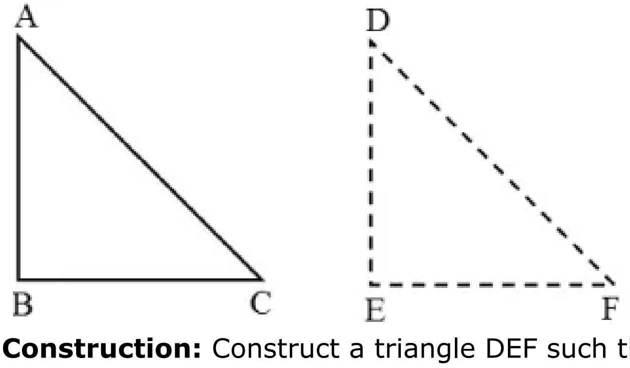


Proof: In triangle ADB and ABC, we have
∠ADB = ∠ABC [Each equal to 90º]
and, ∠A = ∠A [Common]
So, by AA-similarity criterion, we have
ΔADB ~ ΔABC
$\Rightarrow \frac{AD}{AB} = \frac{AB}{AC}$ [∵ In similar triangles corresponding sides are proportional]
$\Rightarrow AB^2 = AD \times AC$ ....(i)
In triangles BDC and ABC, we have
∠CDB = ∠ABC [Each equal to 90º]
and, ∠C = ∠C [Common]
So, by AA-similarity criterion, we have
ΔBDC ~ ΔABC
$\Rightarrow \frac{DC}{BC} = \frac{BC}{AC}$ [∵ In similar triangles corresponding sides are proportional]
$\Rightarrow BC^2 = AC \times DC$ ....(ii)
Adding equation (i) and (ii), we get
$AB^2 + BC^2 = AD \times AC + AC \times DC$
$\Rightarrow AB^2 + BC^2 = AC\,(AD + DC)$
$\Rightarrow AB^2 + BC^2 = AC \times AC$

$\Rightarrow AC^2 = AB^2 + BC^2$
Hence, $AC^2 = AB^2 + BC^2$
The converse of the above theorem is also true as proved below.

Theorem 2: (Converse of Pythagoras Theorem).
In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the side is a right angle.

Given: A triangle ABC such that $AC^2 = AB^2 + BC^2$



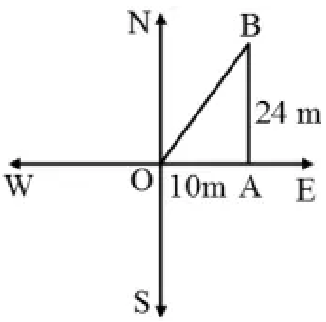
Construction: Construct a triangle DEF such that DE = AB, EF = BC and ∠E = 90º,

Proof: In order to prove that B = ∠90º, it is sufficient to show that ΔABC ~ ΔDEF.
For this we proceed as follows :
Since ΔDEF is a right angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have
$DF^2 = DE^2 + EF^2$
⇒ $DF^2 = AB^2 + BC^2$ [∵ DE = AB and EF = BC (By construction)]
⇒ $DF^2 = AC^2$ [∵ $AB^2 + BC^2 = AC^2$ (Given)]
⇒ DF = AC ....(i)
Thus, in ΔABC and ΔDEF, we have
AB = DE, BC = EF [By construction]
and, AC = DF [From equation (i)]
∴ ΔABC ~ ΔDEF
⇒ ∠B = ∠E = 90º
Hence, ΔABC is a right triangle right angled at B.

Pythagoras Theorem With Examples

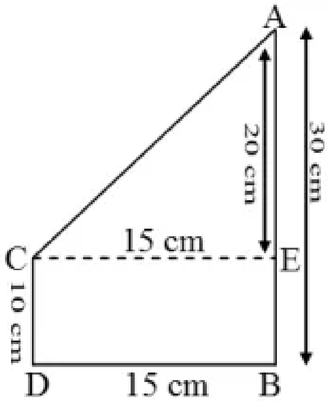
Example 1: Side of a triangle is given, determine it is a right triangle.
(2a – 1) cm, $2\sqrt{2a}$ cm, and (2a + 1) cm
Sol. Let p = (2a – 1) cm, q = $2\sqrt{2a}$ cm and r = (2a + 1) cm.
Then, $(p^2 + q^2) = (2a - 1)^2 \text{ cm}^2 + (2\ )^2 \text{ cm}^2$
$= \{(4a^2 + 1 - 4a) + 8a\}\text{cm}^2$
$= (4a^2 + 4a + 1)\text{cm}^2$
$= (2a + 1)^2 \text{ cm}^2 = r^2$.
$(p^2 + q^2) = r^2$.
Hence, the given triangle is right angled.

Example 2: A man goes 10 m due east and then 24 m due north. Find the distance from the starting point.
Sol. Let the initial position of the man be O and his final position be B. Since the man goes 10 m due east and then 24 m due north. Therefore, ΔAOB is a right triangle right-angled at A such that OA = 10 m and AB = 24 m.



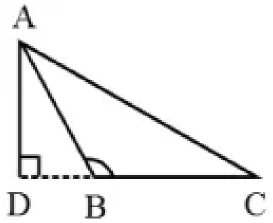
By Phythagoras theorem, we have
$OB^2 = OA^2 + AB^2$
⇒ $OB^2 = 10^2 + 24^2 = 100 + 576 = 676$
⇒ OB = $\sqrt{676}$ = 26 m
Hence, the man is at a distance of 26 m from the starting point.

Example 3: Two towers of heights 10 m and 30 m stand on a plane ground. If the distance between their feet is 15 m, find the distance between their tops.
Sol.



By Phythagoras theorem, we have
$AC^2 = CE^2 + AE^2$
$\Rightarrow AC^2 = 15^2 + 20^2 = 225 + 400 = 625$
$\Rightarrow AC = \sqrt{625} = 25$ m.

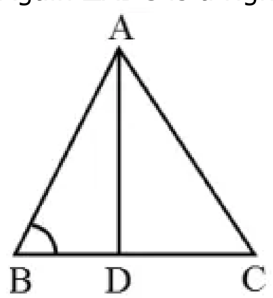
Example 4: In Fig., ΔABC is an obtuse triangle, obtuse angled at B. If AD ⊥ CB, prove that
$AC^2 = AB^2 + BC^2 + 2BC \times BD$
Sol. **Given:** An obtuse triangle ABC, obtuse-angled at B and AD is perpendicular to CB produced.
To Prove: $AC^2 = AB^2 + BC^2 + 2BC \times BD$
Proof: Since ΔADB is a right triangle right angled at D. Therefore, by Pythagoras theorem, we have $AB^2 = AD^2 + DB^2$ ....(i)



Again ΔADC is a right triangle right angled at D.
Therefore, by Phythagoras theorem, we have
$AC^2 = AD^2 + DC^2$
$\Rightarrow AC^2 = AD^2 + (DB + BC)^2$
$\Rightarrow AC^2 = AD^2 + DB^2 + BC^2 + 2BC \bullet BD$
$\Rightarrow AC^2 = AB^2 + BC^2 + 2BC \bullet BD$ [Using (i)]
Hence, $AC^2 = AB^2 + BC^2 + 2BC \bullet BD$

Example 5: In figure, ∠B of ΔABC is an acute angle and AD ⊥ BC, prove that
$AC^2 = AB^2 + BC^2 - 2BC \times BD$
Sol. **Given:** A ΔABC in which ∠B is an acute angle and AD ⊥ BC.

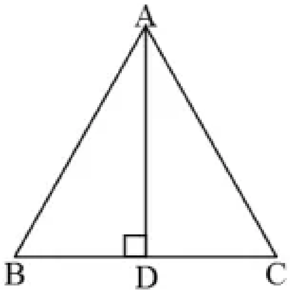
To Prove: $AC^2 = AB^2 + BC^2 - 2BC \times BD$.
Proof: Since ΔADB is a right triangle right-angled at D. So, by Pythagoras theorem, we have
$AB^2 = AD^2 + BD^2$ ....(i)
Again ΔADC is a right triangle right angled at D.



So, by Pythagoras theorem, we have
$AC^2 = AD^2 + DC^2$
$\Rightarrow AC^2 = AD^2 + (BC - BD)^2$
$\Rightarrow AC^2 = AD^2 + (BC^2 + BD^2 - 2BC \bullet BD)$
$\Rightarrow AC^2 = (AD^2 + BD^2) + BC^2 - 2BC \bullet BD$
$\Rightarrow AC^2 = AB^2 + BC^2 - 2BC \bullet BD$ [Using (i)]
Hence, $AC^2 = AB^2 + BC^2 - 2BC \bullet BD$

Example 6: If ABC is an equilateral triangle of side a, prove that its altitude = $\frac{\sqrt{3}}{2}a$.

Sol. ΔABD is an equilateral triangle.
We are given that AB = BC = CA = a.
AD is the altitude, i.e., AD ⊥ BC.
Now, in right angled triangles ABD and ACD, we have
AB = AC (Given)
and AD = AD (Common side)
ΔABD ≅ ΔACD (By RHS congruence)
⇒ BD = CD ⇒ BD = DC = $\frac{1}{2}BC$ = $\frac{a}{2}$



From right triangle ABD.
$AB^2 = AD^2 + BD^2$

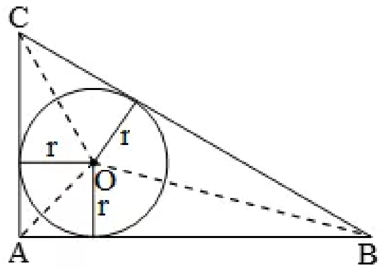
$$\Rightarrow a^2 = AD^2 + \left(\frac{a}{2}\right)^2$$

$$\Rightarrow AD^2 = a^2 - \frac{a^2}{4} = \frac{3}{4}a^2$$

$$\Rightarrow AD = \frac{\sqrt{3}}{2}a$$

Example 7: ABC is a right-angled triangle, right-angled at A. A circle is inscribed in it. The lengths of the two sides containing the right angle are 5 cm and 12 cm. Find the radius of the circle.

Sol. Given that ΔABC is right angled at A.
AC = 5 cm and AB = 12 cm
$BC^2 = AC^2 + AB^2 = 25 + 144 = 169$
BC = 13 cm
Join OA, OB, OC



Let the radius of the inscribed circle be r

Area of ΔABC = Area of ΔOAB + Area of ΔOBC + Area of ΔOCA

⇒ 1/2 × AB × AC

$= \frac{1}{2}(12 \times r) + \frac{1}{2}(13 \times r) + \frac{1}{2}(5 \times r)$

⇒ 12 × 5 = r × {12 + 13 + 5}

⇒ 60 = r × 30 ⇒ r = 2 cm

Example 7: ABCD is a rhombus. Prove that

$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

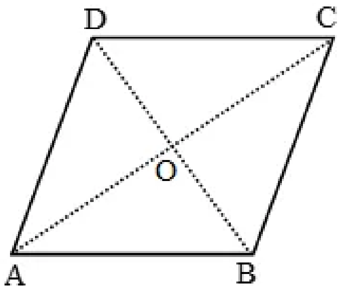
Sol. Let the diagonals AC and BD of rhombus ABCD intersect at O.

Since the diagonals of a rhombus bisect each other at right angles.

∴ ∠AOB = ∠BOC = ∠COD = ∠DOA = 90º

and AO = CO, BO = OD.

Since ΔAOB is a right triangle right-angle at O.



∴ $AB^2 = OA^2 + OB^2$

$\Rightarrow AB^2 = \left(\frac{1}{2}AC\right)^2 + \left(\frac{1}{2}BD\right)^2$ [∵ OA = OC and OB = OD]

⇒ $4AB^2 = AC^2 + BD^2$ ….(i)

Similarly, we have

$4BC^2 = AC^2 + BD^2$ ….(ii)

$4CD^2 = AC^2 + BD^2$ ….(iii)

and, $4AD^2 = AC^2 + BD^2$ ….(iv)

Adding all these results, we get

$4(AB^2 + BC^2 + AD^2) = 4(AC^2 + BD^2)$

⇒ $AB^2 + BC^2 + AD^2 + DA^2 = AC^2 + BD^2$

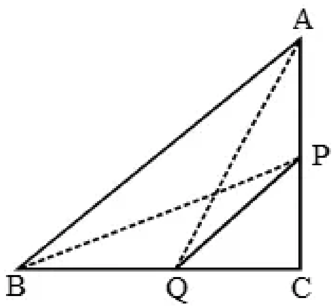
Example 8: P and Q are the mid-points of the sides CA and CB respectively of a ΔABC, right angled at C. Prove that:

(i) $4AQ^2 = 4AC^2 + BC^2$

(ii) $4BP^2 = 4BC^2 + AC^2$

(iii) $(4AQ^2 + BP^2) = 5AB^2$

Sol.



(i) Since ΔAQC is a right triangle right-angled at C.

$\therefore AQ^2 = AC^2 + QC^2$

$\Rightarrow 4AQ^2 = 4AC^2 + 4QC^2$ [Multiplying both sides by 4]

$\Rightarrow 4AQ^2 = 4AC^2 + (2QC)^2$

$\Rightarrow 4AQ^2 = 4AC^2 + BC^2$ [$\because$ BC = 2QC]

(ii) Since ΔBPC is a right triangle right-angled at C.

$\therefore BP^2 = BC^2 + CP^2$

$\Rightarrow 4BP^2 = 4BC^2 + 4CP^2$ [Multiplying both sides by 4]

$\Rightarrow 4BP^2 = 4BC^2 + (2CP)^2$

$\Rightarrow 4BP^2 = 4BC^2 + AC^2$ [$\because$ AC = 2CP]

(iii) From (i) and (ii), we have

$4AQ^2 = 4AC^2 + BC^2$ and, $4BC^2 = 4BC^2 + AC^2$

$\therefore 4AQ^2 + 4BP^2 = (4AC^2 + BC^2) + (4BC^2 + AC^2)$

$\Rightarrow 4(AQ^2 + BP^2) = 5\ (AC^2 + BC^2)$

$\Rightarrow 4(AQ^2 + BP^2) = 5\ AB^2$

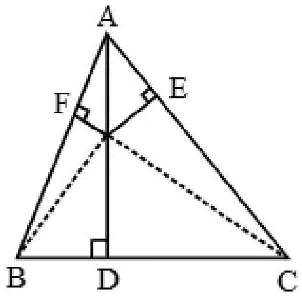
[In ΔABC, we have $AB^2 = AC^2 + BC^2$]

Example 9: From a point O in the interior of a ΔABC, perpendicular OD, OE and OF are drawn to the sides BC, CA and AB respectively. Prove that :

(i) $AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2$

(ii) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$

Sol.



Let O be a point in the interior of ΔABC and let $OD \perp BC$, $OE \perp CA$ and $OF \perp AB$.

(i) In right triangles ΔOFA, ΔODB and ΔOEC, we have

$OA^2 = AF^2 + OF^2$

$OB^2 = BD^2 + OD^2$

and, $OC^2 = CE^2 + OE^2$

Adding all these results, we get

$OA^2 + OB^2 + OC^2 = AF^2 + BD^2 + CE^2 + OF^2 + OD^2 + OE^2$

$\Rightarrow AF^2 + BD^2 + CE^2 = OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2$

(ii) In right triangles ΔODB and ΔODC, we have

$OB^2 = OD^2 + BD^2$

and, $OC^2 = OD^2 + CD^2$

$OB^2 - OC^2 = (OD^2 + BD^2) - (OD^2 + CD^2)$

$\Rightarrow OB^2 - OC^2 = BD^2 - CD^2$ ....(i)

Similarity, we have

$OC^2 - OA^2 = CE^2 - AE^2$ ....(ii)

and, $OA^2 - OB^2 = AF^2 - BF^2$ ....(iii)

Adding (i), (ii) and (iii), we get

$(OB^2 - OC^2) + (OC^2 - OA^2) + (OA^2 - OB^2)$

$= (BD^2 - CD^2) + (CE^2 - AE^2) + (AF^2 - BF^2)$

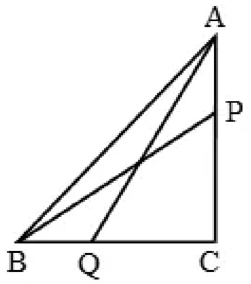
$\Rightarrow (BD^2 + CE^2 + AF^2) - (AE^2 + CD^2 + BF^2) = 0$

$\Rightarrow AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$

Example 10: In a right triangle ABC right-angled at C, P and Q are the points on the sides CA and CB respectively, which divide these sides in the ratio 2 : 1. Prove that

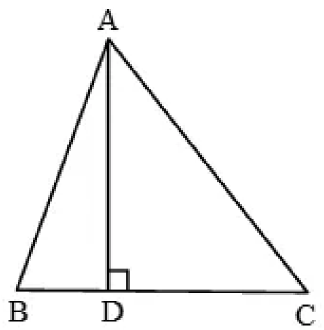
(i) $9\ AQ^2 = 9\ AC^2 + 4\ BC^2$

(ii) $9\ BP^2 = 9\ BC^2 + 4\ AC^2$
(iii) $9\ (AQ^2 + BP^2) = 13\ AB^2$
Sol. It is given that P divides CA in the ratio 2 : 1. Therefore,
$CP = \frac{2}{3}AC$ ....(i)
Also, Q divides CB in the ratio 2 : 1.
$\therefore QC = \frac{2}{3}BC$ ....(ii)



(i) Applying pythagoras theorem in right-angled triangle ACQ, we have
$AQ^2 = QC^2 + AC^2$
$\Rightarrow AQ^2 = \frac{4}{9} BC^2 + AC^2$ [Using (ii)]
$\Rightarrow 9\ AQ^2 = 4\ BC^2 + 9\ AC^2$ ....(iii)
(ii) Applying pythagoras theorem in right triangle BCP, we have
$BP^2 = BC^2 + CP^2$
$\Rightarrow BP^2 = BC^2 + AC^2$ [Using (i)]
$\Rightarrow 9\ BP^2 = 9\ BC^2 + 4\ AC^2$ ....(iv)
(iii) Adding (iii) and (iv), we get
$9\ (AQ^2 + BP^2) = 13\ (BC^2 + AC^2)$
$\Rightarrow 9\ (AQ^2 + BP^2) = 13\ AB^2$ [$\because$ $BC^2 = AC^2 + AB^2$]

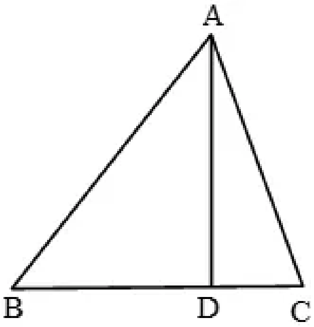
Example 11: In a ΔABC, AD ⊥ BC and $AD^2 = BC \times CD$. Prove ΔABC is a right triangle.
Sol.



In right triangles ADB and ADC, we have
$AB^2 = AD^2 + BD^2$ ....(i)
and, $AC^2 = AD^2 + DC^2$ ....(ii)
Adding (i) and (ii), we get
$AB^2 + AC^2 = 2\ AD^2 \times BD^2 + DC^2$
$\Rightarrow AB^2 + AC^2 = 2BD \times CD + BD^2 + DC^2$ [$\because$ AD2 = BD × CD (Given)]
$\Rightarrow AB^2 + AC^2 = (BD + CD)^2 = BC^2$
Thus, in ΔABC, we have
$AB^2 = AC^2 + BC^2$
Hence, ΔABC, is a right triangle right-angled at A.

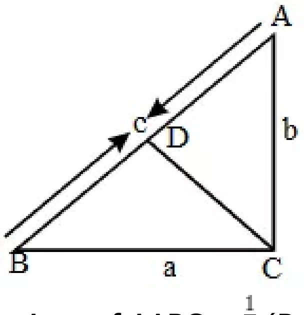
Example 12: The perpendicular AD on the base BC of a ΔABC intersects BC at D so that DB = 3 CD. Prove that $2AB^2 = 2AC^2 + BC^2$.

Sol. We have,



DB = 3CD
BC = BD + DC
The perpendicular AD on the base BC of a ΔABC intersects BC at D so that DB = 3 CD. Prove that 2AC2 + BC2.
We have,
DB = 3CD
∴ BC = BD + DC
⇒ BC = 3 CD + CD
⇒ BD = 4 CD ⇒ CD = $\frac{1}{4}$ BC
∴ CD = $\frac{1}{4}$ BC and BD = 3CD = $\frac{1}{4}$ BC ....(i)
Since ΔABD is a right triangle right-angled at D.
∴ $AB^2 = AD^2 + BD^2$ ....(ii)
Similarly, ΔACD is a right triangle right angled at D.
∴ $AC^2 = AD^2 + CD^2$ ....(iii)
Subtracting equation (iii) from equation (ii) we get
$AB^2 - AC^2 = BD^2 - CD^2$
⇒ $AB^2 - AC^2 = \left(\frac{3}{4}BC\right)^2 - \left(\frac{1}{4}BC\right)^2$ $\left[From\ (i)\ CD = \frac{1}{4}BC,\ BD = \frac{3}{4}BC\right]$
⇒ $AB^2 - AC^2 = \frac{9}{16}BC^2 - \frac{1}{16}BC^2$
⇒ $AB^2 - AC^2 = \frac{1}{2}BC^2$
⇒ $2(AB^2 - AC^2) = BC^2$
⇒ $2AB^2 = 2AC^2 + BC^2$.

Example 13: ABC is a right triangle right-angled at C. Let BC = a, CA = b, AB = c and let p be the length of perpendicular from C on AB, prove that
(i) cp = ab
(ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$
Sol. **(i)** Let CD ⊥ AB. Then, CD = p.

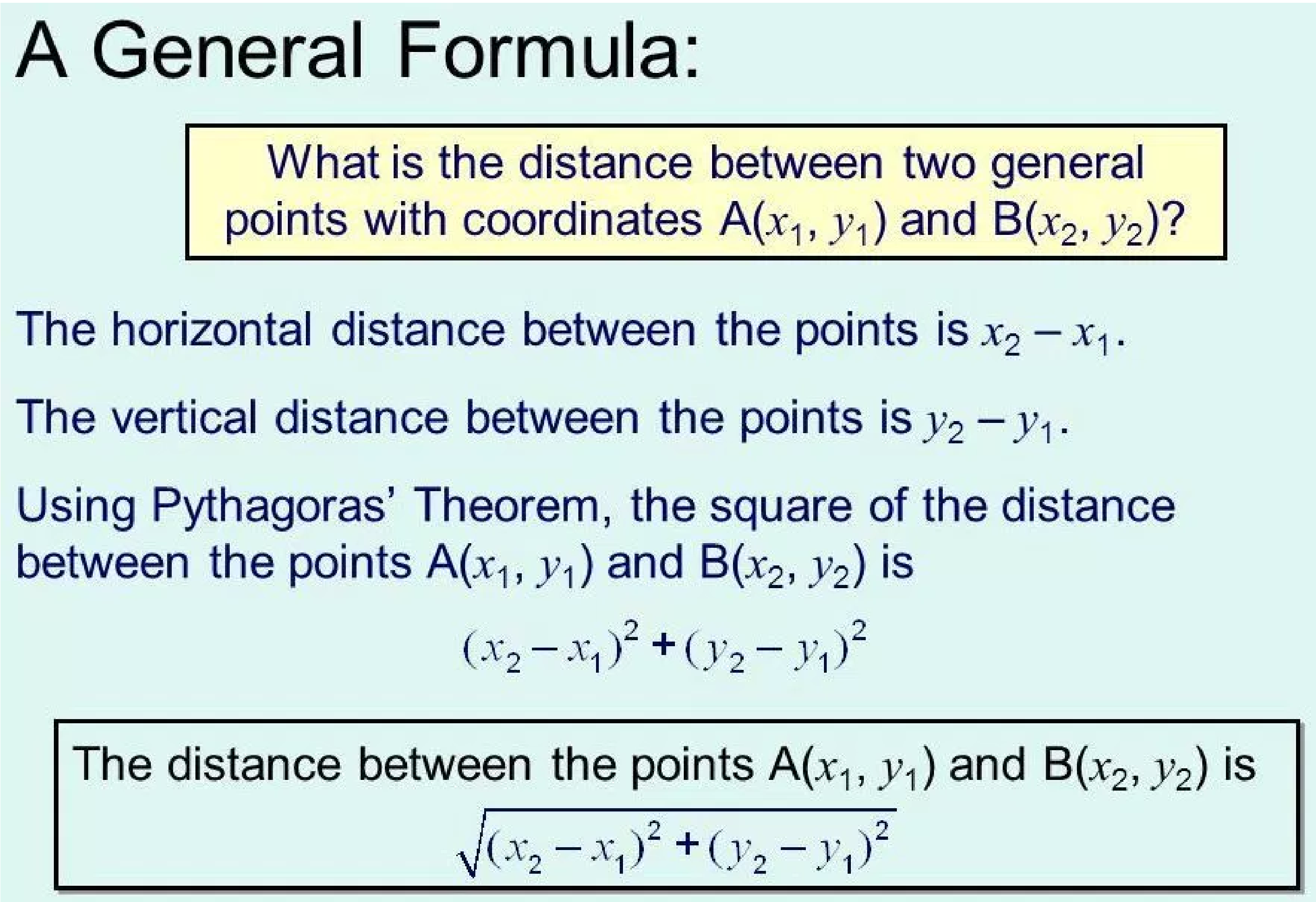


∴ Area of ΔABC = $\frac{1}{2}$ (Base × Height)
⇒ Area of ΔABC = $\frac{1}{2}$ (AB × CD) = $\frac{1}{2}$ cp
Also,
Area of ΔABC = $\frac{1}{2}$ (BC × AC) = $\frac{1}{2}$ ab
∴ $\frac{1}{2}$ cp = $\frac{1}{2}$ ab
⇒ cp = ab
(ii) Since ΔABC is right triangle right-angled at C.
∴ $AB^2 = BC^2 + AC^2$
⇒ $c^2 = a^2 + b^2$

$$\Rightarrow \left(\frac{ab}{p}\right)^2 = a^2 + b^2 \qquad \left[\because\ cp = ab \quad \therefore c = \frac{ab}{p}\right]$$
$$\Rightarrow \frac{a^2b^2}{p^2} = a^2 + b^2$$
$$\Rightarrow \frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2} \Rightarrow \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2}$$
$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

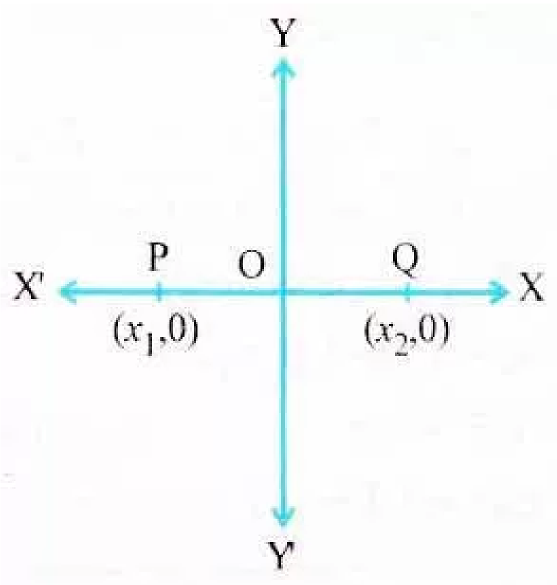
Distance Between Two Points

Distance Formula

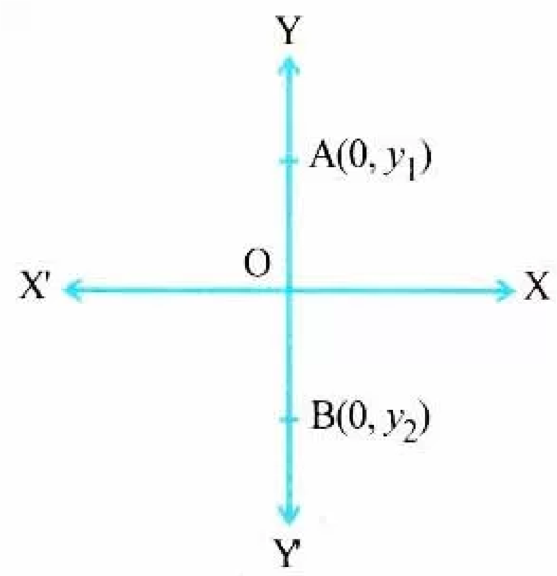


(1) Distance between two points on X-axis :

The coordinate axes in the coordinate plane can be treated as number lines.
If P(x_1, 0) and Q(x_2, 0) are two points on X-axis, the distance between them is taken as
PQ = $|x_1 - x_2|$ ……….(i)

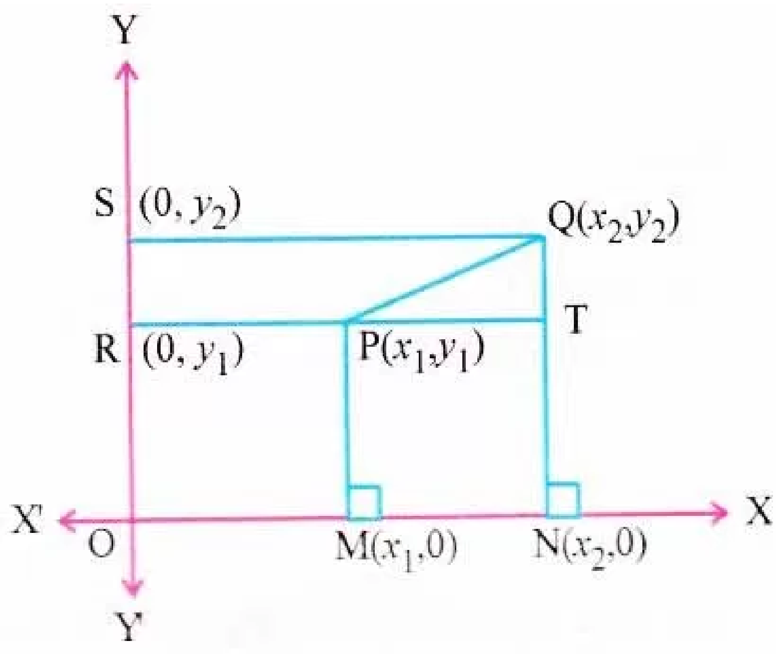


(2) Distance between two points on Y-axis:
If the points $A(0, y_1)$ and $B(0, y_2)$ are two points on Y-axis, the distance between them is taken as
$AB = |y_1-y_2|$ ....(ii)



(3) Distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$:

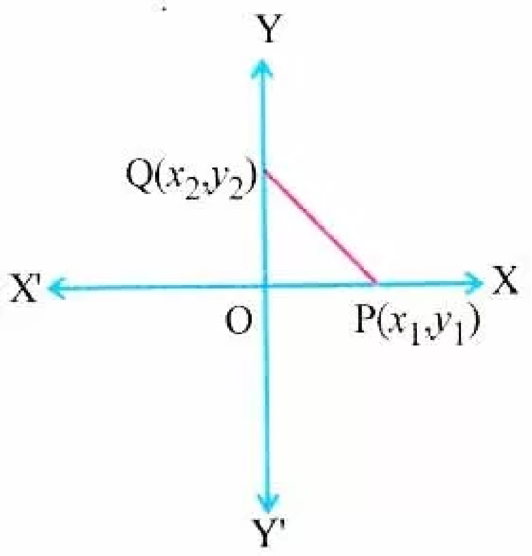
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points in the coordinate plane.
Let M and N be the feet of perpendiculars from $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively to X-axis.



∴ M and N are respectively (x_1, 0) and (x_2, 0).
MN = $|x_1-x_2|$ ……….(i)
Let R and S be the feet of perpendiculars from P(x_1, y_1) and Q(x_2, y_2) to Y-axis.
R and S are respectively (0, y_1) and (0, y_2).
RS = $|y_1-y_2|$ ……….(ii)
Let PR and QN intersect in T.
Clearly in ΔPQT, ∠PTQ is a right angle.
Using Pythagoras' theorem we have,
$PQ^2 = PT^2 + QT^2 = MN^2 + RS^2$
because QTRS and PTNM are rectangles.
Now, by (i) and (ii), we have
$PQ^2 = MN^2 + RS^2$
$=|x_1-x_2|^2 + |y_1-y_2|^2$
$= (x_1-x_2)^2 + (y_1-y_2)^2$

∴ $$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Formula (iii) gives the distance between two points whose coordinates are (x_1, y_1) and (x_2, y_2). The distance between the points P and Q is also denoted by d(P, Q).

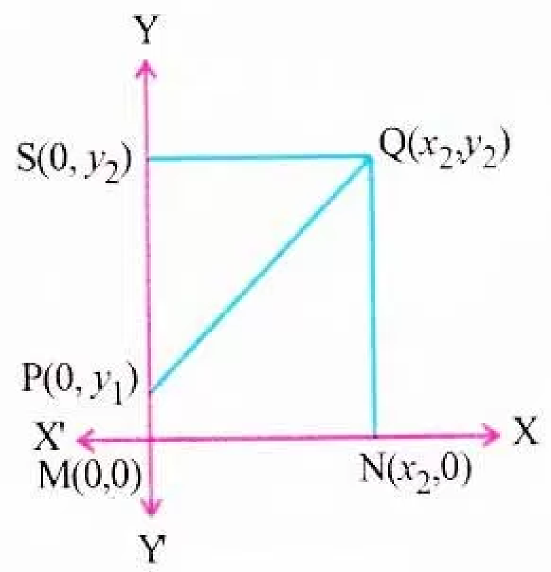


Thus, d(P, Q) = PQ

= d(P(x_1, y_1), Q(x_2, y_2)) = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
If P and Q lie on X-axis then also formula remains same.

Here, M = P and N = O.
R = O and S = Q.
MN = OP = $|x_1-0| = |x_1| = |x_1-y_1|$ ($y_1 = 0$)
RS = OQ = $|0-y_2| = |x_2-y_2|$ ($x_2 = 0$)

$$\therefore PQ = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



[Note : P may lie on Y-axis
Here $x_1 = 0$.
Here also PS = $|y_1-y_2|$ (P = R)
MN = = $|0-x_2| = |x_2-y_2|$
Similarly if P lies on X-axis, the formula remains same.]
If PQ is parallel to any axis then $x_1 = x_2$ or $y_1 = y_2$ and formula remains same.

'Distance' is also known as a 'Metric'. Metric plays an important role in all kinds of geometry, including Euclidean geometry. In fact the nature of Metric defines the type of geometry. The following properties of Metric are not only interesting but also very useful.
(1) d(A, B) = AB ≥ 0 i.e. the distance between two points is a non-negative real number.
(2) d(A, B) = AB = 0, if and only if A = B.
(3) d(A, B) = d(B, A) i.e. AB = BA.
(4) If A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) are three points in coordinate plane, then
d(A, B) + d(B, C) ≥ d(A, C) i.e. AB + BC ≥ AC.
If, A, B, C are collinear points and A—B—C then, AB + BC = AC.
If, A, B, C are non-collinear points or collinear but B—A—C or A—C—B, then AB + BC > AC
The formula for PQ can also be written as $PQ^2 = (x_1-x_2)^2 + (y_1-y_2)^2$. While solving the examples, this form of distance formula is advantageous. We have to be careful that at the end when we find the distance 'PQ' we have to take the positive square root of the expression for PQ^2.

Distance Between Two Points With Examples

Example 1: Find the distance between two points
(i) P(−6, 7) and Q(−1, −5)
(ii) R(a + b, a − b) and S(a − b, −a − b)
(iii) $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$

Sol. **(i)** Here, $x_1 = -6$, $y_1 = 7$ and $x_2 = -1$, $y_2 = -5$

$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow PQ = \sqrt{(-1 + 6)^2 + (-5 - 7)^2}$$

$$\Rightarrow PQ = \sqrt{25 + 144} = \sqrt{169} = 13$$

(ii) We have,

$$RS = \sqrt{(a-b-a-b)^2 + (-a-b-a+b)^2}$$

$$\Rightarrow RS = \sqrt{4b^2 + 4a^2} = 2\sqrt{a^2+b^2}$$

(iii) We have,

$$AB = \sqrt{(at_2{}^2 - at_1{}^2)^2 + (2at_2 - 2at_1)^2}$$

$$\Rightarrow AB = \sqrt{a^2(t_2-t_1)^2(t_2+t_1)^2 + 4a^2(t_2-t_1)^2}$$

$$\Rightarrow AB = a(t_2-t_1)\sqrt{(t_2+t_1)^2 + 4}$$

Example 2: If the point (x, y) is equidistant from the points (a + b, b − a) and (a − b, a + b), prove that bx = ay.

Sol. Let P(x, y), Q(a + b, b − a) and R (a − b, a + b) be the given points. Then
PQ = PR (Given)

$$\Rightarrow \sqrt{\{x-(a+b)\}^2 + \{y-(b-a)\}^2} = \sqrt{\{x-(a-b)\}^2 + \{y-(a+b)\}^2}$$

$\Rightarrow \{x - (a + b)\}^2 + \{y - (b - a)\}^2 = \{x - (a - b)\}^2 + \{y - (a + b)\}^2$
$\Rightarrow x^2 - 2x(a + b) + (a + b)^2 + y^2 - 2y(b - a) + (b - a)^2 = x^2 + (a - b)^2 - 2x(a - b) + y^2 - 2(a + b) + (a + b)^2$
$\Rightarrow -2x(a + b) - 2y(b - a) = -2x(a - b) - 2y(a + b)$
$\Rightarrow ax + bx + by - ay = ax - bx + ay + by$
$\Rightarrow 2bx = 2ay \Rightarrow bx = ay$

Example 3: Find the value of x, if the distance between the points (x, − 1) and (3, 2) is 5.

Sol. Let P(x, − 1) and Q(3, 2) be the given points, Then,
PQ = 5 (Given)

$$\Rightarrow \sqrt{(x-3)^2 + (-1-2)^2} = 5$$

$\Rightarrow (x - 3)^2 + 9 = 5^2$
$\Rightarrow x^2 - 6x + 18 = 25 \Rightarrow x^2 - 6x - 7 = 0$
$\Rightarrow (x - 7)(x + 1) = 0 \Rightarrow x = 7$ or $x = -1$

Example 4: Show that the points (a, a), (−a, −a) and (− √3 a, √3 a) are the vertices of an equilateral triangle. Also find its area.

Sol. Let A (a, a), B(−a, −a) and C(− √3 a, √3 a) be the given points. Then, we have

$$AB = \sqrt{(-a-a)^2 + (-a-a)^2} = \sqrt{4a^2 + 4a^2} = 2\sqrt{2}a$$

$$BC = \sqrt{(-\sqrt{3}a + a)^2 + (\sqrt{3}a + a)^2}$$

$$\Rightarrow BC = \sqrt{a^2(1-\sqrt{3})^2 + a^2(\sqrt{3}+1)^2}$$

$$\Rightarrow BC = a\sqrt{1+3-2\sqrt{3}+1+3+2\sqrt{3}} = a\sqrt{8} = 2\sqrt{2}a$$

$$and,\ AC = \sqrt{(-\sqrt{3}a - a^2) + (\sqrt{3}a - a)^2}$$

$$\Rightarrow AC = \sqrt{a^2(\sqrt{3}+1) + a^2(\sqrt{3}-1)^2}$$

$$\Rightarrow AC = \sqrt{3+1+2\sqrt{3}+3+1-2\sqrt{3}} = a\sqrt{8} = 2\sqrt{2}a$$

Clearly, we have
AB = BC = AC
Hence, the triangle ABC formed by the given points is an equilateral triangle.
Now,
Area of ΔABC = $\frac{\sqrt{3}}{4}$ (side)^2
⇒ Area of ΔABC = $\frac{\sqrt{3}}{4} \times AB^2$
⇒ Area of ΔABC = $\frac{\sqrt{3}}{4} \times (2\sqrt{2}a)^2$ sq. units = $2\sqrt{3}a^2$ sq. units

Example 5: Show that the points (1, – 1), (5, 2) and (9, 5) are collinear.

Sol. Let A (1, –1), B (5, 2) and C (9, 5) be the given points. Then, we have Clearly, AC = AB + BC

$$AB = \sqrt{(5-1)^2 + (2+1)^2} = \sqrt{16+9} = 5$$

$$BC = \sqrt{(5-9)^2 + (2-5)^2} = \sqrt{16+9} = 5$$

$$and, \quad AC = \sqrt{(1-9)^2 + (-1-5)^2} = \sqrt{64+36} = 10$$

Hence, A, B, C are collinear points.

Example 6: Show that four points (0, – 1), (6, 7), (–2, 3) and (8, 3) are the vertices of a rectangle. Also, find its area.

Sol. Let A (0, –1), B(6, 7), C(–2, 3) and D (8, 3) be the given points. Then,

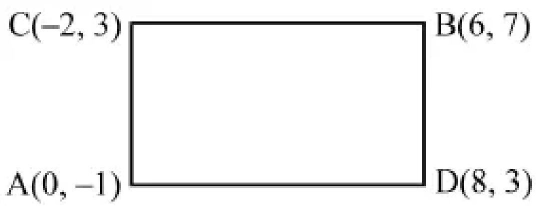
$$AD = \sqrt{(8-0)^2 + (3+1)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$BC = \sqrt{(6+2)^2 + (7-3)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$AC = \sqrt{(-2-0)^2 + (3+1)^2} = \sqrt{4+16} = 2\sqrt{5}$$

$$and, \ BD = \sqrt{(8-6)^2 + (3-7)^2} = \sqrt{4+16} = 2\sqrt{5}$$

∴ AD = BC and AC = BD.
So, ADBC is a parallelogram,



$$Now, \ AB = \sqrt{(6-0)^2 + (7+1)^2} = \sqrt{36+64} = 10$$

$$and \quad CD = \sqrt{(8+2)^2 + (3-3)^2} = 10$$

Clearly, $AB^2 = AD^2 + DB^2$ and $CD^2 = CB^2 + BD^2$
Hence, ADBC is a rectangle.
Now, Area of rectangle ADBC = AD × DB
= $(4\sqrt{5} \times 2\sqrt{5})$ sq. units = 40 sq. units

Example 7: If P and Q are two points whose coordinates are $(at^2, 2at)$ and $(a/t^2, 2a/t)$ respectively and S is the point (a, 0). Show that $\frac{1}{SP} + \frac{1}{SQ}$ is independent of t.

Sol. We have,

$$SP = \sqrt{(at^2 - a)^2 + (2at - 0)^2}$$

$$= \sqrt{(t^2-1)^2 + 4t^2} = a(t^2 + 1)$$

$$and \quad SQ = \sqrt{\left(\frac{a}{t^2} - a\right)^2 + \left(\frac{2a}{t} - 0\right)^2}$$

$$\Rightarrow SQ = \sqrt{\frac{a^2(1-t^2)^2}{t^4} + \frac{4a^2}{t^2}}$$

$$\Rightarrow SQ = \frac{a}{t^2}\sqrt{(1-t^2)^2 + 4t^2} = \frac{a}{t^2}\sqrt{(1+t^2)^2}$$

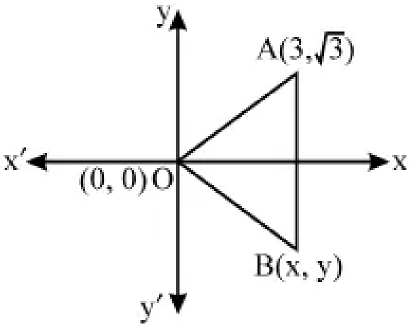
which is independent of t.

$$= \frac{a}{t^2}(1 + t^2)$$

$$\therefore \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t^2+1)} + \frac{t^2}{a(t^2+1)}$$
$$\Rightarrow \frac{1}{SP} + \frac{1}{SQ} = \frac{1+t^2}{a(t^2+1)} = \frac{1}{a}$$

Example 8: If two vertices of an equilateral triangle be (0, 0), (3, √3), find the third vertex.

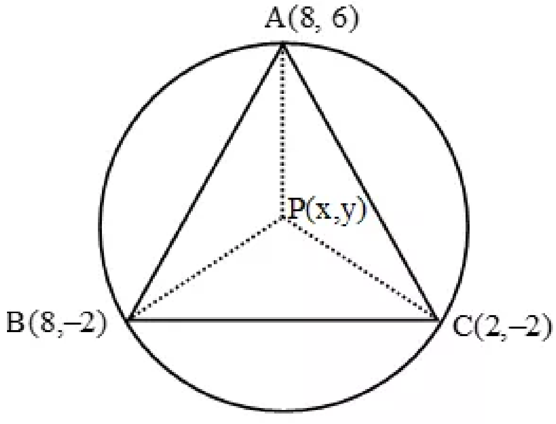
Sol. O(0, 0) and A(3, √3) be the given points and let B(x, y) be the third vertex of equilateral ΔOAB.
Then, OA = OB = AB
⇒ $OA^2 = OB^2 = AB^2$



We have, $OA^2 = (3 - 0)^2 + (\sqrt{3} - 0)^2 = 12$,
$OB^2 = x^2 + y^2$
and, $AB^2 = (x - 3)^2 + (y - \sqrt{3})^2$
⇒ $AB^2 = x^2 + y^2 - 6x - 2\ y + 12$
∴ $OA^2 = OB^2 = AB^2$
⇒ $OA^2 = OB^2$ and $OB^2 = AB^2$
⇒ $x^2 + y^2 = 12$
and, $x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}y + 12$
⇒ $x^2 + y^2 = 12$ and $6x + 2\sqrt{3}y = 12$
⇒ $x^2 + y^2 = 12$ and $3x + \sqrt{3}y = 6$
$$\Rightarrow \quad x^2 + \left(\frac{6-3x}{\sqrt{3}}\right)^2 = \ 12 \ \left[\because \ 3x + \sqrt{3}y = 6 \quad \therefore \quad y = \frac{6-3x}{\sqrt{3}}\right]$$
⇒ $3x^2 + (6 - 3x)^2 = 36$
⇒ $12x^2 - 36x = 0$ ⇒ $x = 0, 3$
∴ $x = 0$ ⇒ $\sqrt{3}y = 6$
$$\Rightarrow \ y = \frac{6}{\sqrt{3}} = 2\sqrt{3} \ \left[\text{Putting x} = 0 \text{ in } 3\text{x} + \sqrt{3}\text{y} = 6\right]$$
and, x = 3 ⇒ $9 + \sqrt{3}\ y = 6$
$$\Rightarrow \ y = \frac{6-9}{\sqrt{3}} = -\sqrt{3} \ \left[\text{Putting x} = 0 \text{ in } 3\text{x} + \sqrt{3}\text{y} = 6\right]$$
Hence, the coordinates of the third vertex B are (0, 2 √3) or (3, – √3).

Example 9: Find the coordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, – 2) and (2, – 2). Also, find its circum radius.

Sol. Recall that the circumcentre of a triangle is equidistant from the vertices of a triangle. Let A (8, 6), B(8, –2) and C(2, – 2) be the vertices of the given triangle and let P (x, y) be the circumcentre of this triangle. Then,
PA = PB = PC ⇒ $PA^2 = PB^2 = PC^2$



Now, $PA^2 = PB^2$

⇒ $(x - 8)^2 + (y - 6)^2 = (x - 8)2 + (y + 2)^2$

⇒ $x^2 + y^2 - 16x - 12y + 100 = x^2 + y^2 - 16x + 4y + 68$

⇒ $16y = 32$ ⇒ $y = 2$

and, $PB^2 = PC^2$

⇒ $(x - 8)^2 + (y + 2)^2 = (x - 2)^2 + (y + 2)^2$

⇒ $x^2 + y^2 - 16x + 4y + 68 = x^2 + y^2 - 4x + 4y + 8$

⇒ $12x = 60$ ⇒ $x = 5$

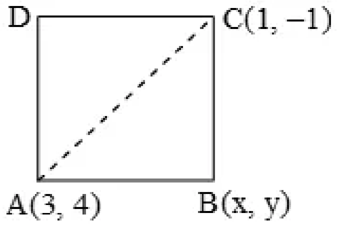
So, the coordinates of the circumcentre P are (5, 2).

Also, Circum-radius = PA = PB = PC

$= \sqrt{(5-8)^2 + (2-6)^2} = \sqrt{9+16} = 5$

Example 10: If the opposite vertices of a square are (1, – 1) and (3, 4), find the coordinates of the remaining angular points.

Sol. Let A(1, – 1) and C(3, 4) be the two opposite vertices of a square ABCD and let B(x, y) be the third vertex.



Then, AB = BC

⇒ $AB^2 = BC^2$

⇒ $(x - 1)^2 + (y + 1)^2 = (3 - x)^2 + (4 - y)^2$

⇒ $x^2 - 2x + 1 + y^2 + 2y + 1 = 9 - 6x + x^2 + 16 - 8y + y2$

⇒ $x^2 + y^2 - 2x + 2y + 2 = x^2 + y^2 - 6x - 8y + 25$

⇒ $4x + 10y = 23$

⇒ $x = \frac{23-10y}{4}$ ....(1)

In right-angled triangle ABC, we have

$AB^2 + BC^2 = AC^2$

⇒ $(x - 3)^2 + (y - 4)^2 + (x - 1)^2 + (y + 1)^2 = (3 -1)^2 + (4 + 1)^2$

⇒ $x^2 + y^2 - 4x - 3y - 1 = 0$ .... (2)

Substituting the value of x from (1) and (2),

we get

$\left(\frac{23-10y}{4}\right)^2 + y^2 - (23 - 10y) - 3y - 1 = 0$

⇒ $4y^2 - 12y + 5 = 0$ ⇒ $(2y - 1)(2y - 5) = 0$

⇒ $y = \frac{1}{2}$ or $\frac{5}{2}$

Putting $y = \frac{1}{2}$ and $y = \frac{5}{2}$ respectively in (1) we get

x = $\frac{9}{2}$ and x = $\frac{-1}{2}$ respectively.
Hence, the required vertices of the square are $\left(\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$.

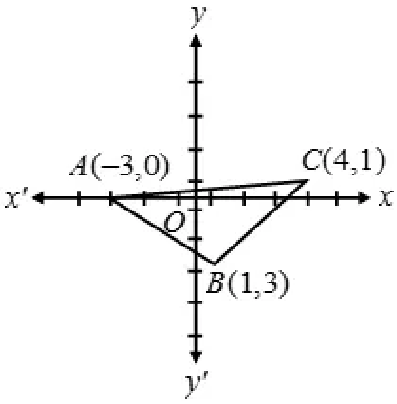
Example 11: Prove that the points (–3, 0), (1, –3) and (4, 1) are the vertices of an isosceles right angled triangle. Find the area of this triangle.

Sol. Let A (–3, 0), B (1, –3) and C (4, 1) be the given points. Then,

$$AB = \sqrt{\{1-(-3)\}^2 + (-3-0)^2} = \sqrt{16+9} = 5\ units.$$

$$BC = \sqrt{(4-1)^2 + (1+3)^2} = \sqrt{9+16} = 5\ units.$$

$$CA = \sqrt{(4+3)^2 + (1-0)^2} = \sqrt{49+1} = 5\sqrt{2}\ units.$$



Clearly, AB = BC. Therefore, ΔABC is isosceles.
Also, $AB^2 + BC^2 = 25 + 25 = (5)^2 = CA^2$
⇒ ΔABC is right-angled at B.
Thus, ΔABC is a right-angled isosceles triangle.
Now, Area of ΔABC = $\frac{1}{2}$ (Base × Height)
= $\frac{1}{2}$ (AB × BC)
⇒ Area of ΔABC = $\frac{1}{2}$ × 5 × 5 sq. units
= $\frac{25}{2}$ sq. units.

Example 12: If P (2, – 1), Q(3, 4), R(–2, 3) and S(–3, –2) be four points in a plane, show that PQRS is a rhombus but not a square. Find the area of the rhombus.

Sol. The given points are P(2, –1), Q(3, 4), R(–2, 3) and S(–3, –2).
We have,

$$PQ = \sqrt{(3-2)^2 + (4+1)^2} = \sqrt{1^2+5^2} = \sqrt{26}\ units$$

$$QR = \sqrt{(-2-3)^2 + (3-4)^2} = \sqrt{25+1} = \sqrt{26}\ units$$

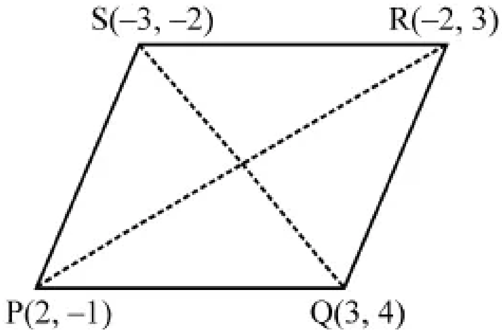
$$RS = \sqrt{(-3+2)^2 + (-2-3)^2} = \sqrt{1+25} = \sqrt{26}\ units$$

$$SP = \sqrt{(-3-2)^2 + (-2+1)^2} = \sqrt{25+1} = \sqrt{26}\ units$$

$$PR = \sqrt{(-2-2)^2 + (3+1)^2} = \sqrt{16+16} = 4\sqrt{2}\ units$$

$$QS = \sqrt{(-3-3)^2 + (-2-4)^2} = \sqrt{36+36} = 6\sqrt{2}\ units$$

∴ PQ = QR = RS = SP = units
and, PR ≠ QS



This means that PQRS is a quadrilateral whose sides are equal but diagonals are not equal.
Thus, PQRS is a rhombus but not a square.

Now, Area of rhombus PQRS = $\frac{1}{2}$ × (Product of lengths of diagonals)

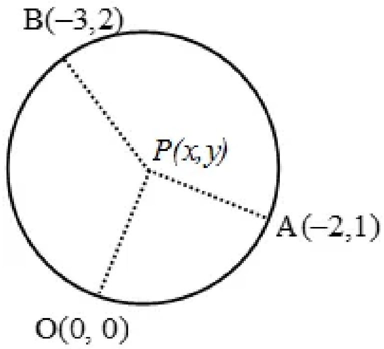
⇒ Area of rhombus PQRS = $\frac{1}{2}$ × (PR × QS)

⇒ Area of rhombus PQRS

$= \left(\frac{1}{2} \times 4\sqrt{2} \times 6\sqrt{2}\right) sq.\ units\ =\ 24\ sq.\ units$

Example 13: Find the coordinates of the centre of the circle passing through the points (0, 0), (–2, 1) and (–3, 2). Also, find its radius.

Sol. Let P (x, y) be the centre of the circle passing through the points O(0, 0), A(–2,1) and B(–3,2). Then, OP = AP = BP



Now, OP = AP ⇒ $OP^2 = AP^2$

⇒ $x^2 + y^2 = (x + 2)^2 + (y - 1)^2$

⇒ $x^2 + y^2 = x^2 + y^2 + 4x - 2y + 5$

⇒ $4x - 2y + 5 = 0$ ....(1)

and, OP = BP ⇒ $OP^2 = BP^2$

⇒ $x^2 + y^2 = (x + 3)^2 + (y - 2)^2$

⇒ $x^2 + y^2 = x^2 + y^2 + 6x - 4y + 13$

⇒ $6x - 4y + 13 = 0$ ....(2)

On solving equations (1) and (2), we get

x = $\frac{3}{2}$ and y = $\frac{11}{2}$

Thus, the coordinates of the centre are $\left(\frac{3}{2},\ \frac{11}{2}\right)$

$$Now,\ Radius = OP = \sqrt{x^2 + y^2} = \sqrt{\frac{9}{4} + \frac{121}{4}}$$

$$= \frac{1}{2}\sqrt{130}\ units.$$