Dynamics Of A Solid Body (Part - 1)

Q. 234. A thin uniform rod AB of mass m = 1.0 kg moves translationally with acceleration w = 2.0 m/s^2 due to two antiparallel forces F_1 a nd F_2 (Fig. 1.52). The distance between the points at which these forces are applied is equal to a = 20 cm. Besides, it is known that F_2 = 5.0 N. Find the length of the rod.

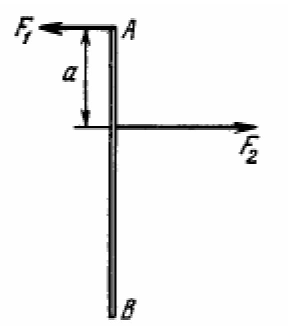


Fig. 1.52.

Ans. Since, motion of the rod is purely translational, net torque about the C.M. of the rod should be equal to zero.

Thus $F_1 \frac{l}{2} = F_2\left(\frac{l}{2} - a\right)$ or, $\frac{F_1}{F_2} = 1 - \frac{a}{l/2}$ (1)

For the translational motion of rod.

$F_2 - F_1 = mw_c$ or $1 - \frac{F_1}{F_2} = \frac{mw_c}{F_2}$ (2)

From (1) and (2)

$\frac{a}{l/2} = \frac{mw_c}{F_2}$ or, $l = \frac{2aF_2}{mw_c} = 1$ m

Q. 235. A force F = Ai + Bj is applied to a point whose radius vector relative to the origin of coordinates O is equal to r = ai + bj, where a, b, A, B are constants, and i, j are the unit vectors of the x and y axes. Find the moment N and the arm l of the force F relative to the point O.

Ans. Sought moment

$$\vec{N} = \vec{r} \times \vec{F} = (a\vec{i} + b\vec{j}) \times (A\vec{i} + B\vec{j})$$
$$= aB\vec{k} + Ab(-\vec{k}) = (aB - Ab)\vec{k}$$

and arm of the force $l = \frac{N}{F} = \frac{aB - Ab}{\sqrt{A^2 + B^2}}$

Q. 236. A force $F_1 = Aj$ is applied to a point whose radius vector $r_1 = ai$, while a force $F_2 = Bi$ is applied to the point whose radius vector $r_2 = bj$. Both radius vectors are determined relative to the origin of coordinates O, i and j are the unit vectors of the x and y axes, a, b, A, B are constants. Find the arm l of the resultant force relative to the point O.

Ans. Relative to point O, the net moment of force :

$$\vec{N} = \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} = (a\vec{i} \times A\vec{j}) + (B\vec{j} \times B\vec{i})$$

$$= ab\vec{k} + AB(-\vec{k}) = (ab - AB)\vec{k} \qquad (1)$$

Resultant of the external force

$$\vec{F} = \vec{F_1} + \vec{F_2} = A\vec{j} + B\vec{i} \qquad (2)$$

As $\vec{N} \cdot \vec{F} = 0$ (as $\vec{N} \perp \vec{F}$) so the sought arm l o f the force $\vec{F}$

$$l = N/F = \frac{ab - AB}{\sqrt{A^2 + B^2}}$$

Q. 237. Three forces are applied to a square plate as shown in Fig. 1.53. Find the modulus, direction, and the point of application of the resultant force, if this point is taken on the side BC.

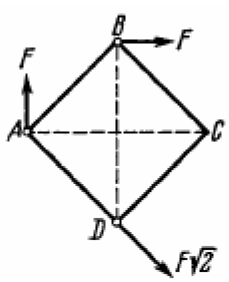


Fig. 1.53.

Ans. For coplanar forces, about any point in the same plane $\sum \vec{r_i} \times \vec{F_i} = \vec{r} \times \vec{F}_{net}$

(where $\vec{F}_{net} = \sum \vec{F}_i$ = resultant force) or, $\vec{N}_{net} = \vec{r} \times \vec{F}_{net}$

Thus length of the arm, $l = \frac{N_{net}}{F_{net}}$

Here obviously $|\vec{F}_{net}| = 2F$ and it is directed toward right along AC. Take the origin at C. Then about C,

$\vec{N} = \left(\sqrt{2}\, a F + \frac{a}{\sqrt{2}} F - \sqrt{2}\, a F\right)$ directed normally into the plane of figure.

(Here a = side of the square.)

Thus $\vec{N} = F \frac{a}{\sqrt{2}}$ directed into the plane of the figure.

Hence $l = \frac{F(a/\sqrt{2})}{2F} = \frac{a}{2\sqrt{2}} = \frac{a}{2} \sin 45°$

Thus the point of application of force is at the mid point of the side BC.

Q. 238. Find the moment of inertia

(a) of a thin uniform rod relative to the axis which is perpendicular to the rod and passes through its end, if the mass of the rod is m and its length l;
(b) of a thin uniform rectangular plate relative to the axis passing perpendicular to the plane of the plate through one of its vertices, if the sides of the plate are equal to a and b, and its mass is m.

Ans. (a) Consider a strip of length dx at a perpendicular distance x from the axis about which we have to find the moment of inertia of the rod. The elemental mass of the rod equals

$dm = \frac{m}{l} dx$

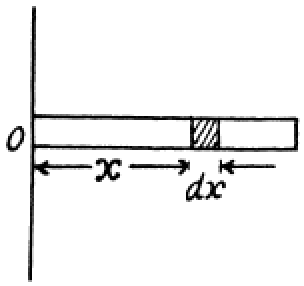
Moment of inertia of this element about the axis

$dI = dm\, x^2 = \frac{m}{l} dx \cdot x^2$

Thus, moment of inertia of the rod, as a whole about the given axis

$$I = \int_0^l \frac{m}{l} x^2 dx = \frac{m l^2}{3}$$

(b) Let us imagine the plane of plate as xy plane taking the origin at the intersection point of the sides of the plate (Fig.).



Obviously $I_x = \int dm\, y^2$

$$= \int_0^a \left(\frac{m}{ab} b\, dy\right) y^2$$

$$= \frac{m a^2}{3}$$

Similarly $I_y = \frac{m b^2}{3}$

Hence from perpendicular axis theorem

$$I_z = I_x + I_y = \frac{m}{3}\left(a^2 + b^2\right),$$

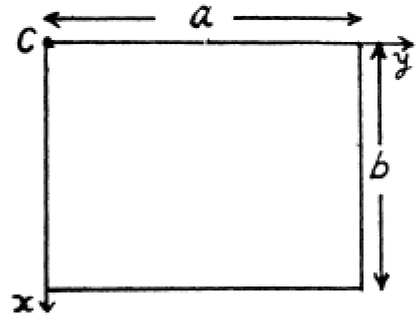
which is the sought moment of inertia.

Q. 239. Calculate the moment of inertia

(a) of a copper uniform disc relative to the symmetry axis perpendicular to the plane of the disc, if its thickness is equal to b = 2.0 mm and its radius to R = 100 mm;
(b) of a uniform solid cone relative to its symmetry axis, if the mass of the cone is equal to m and the radius of its base to R.

Ans. (a) Consider an elementry disc of thickness dx. Moment of inertia of this element about the 2 -axis, passing through its C.M.

$$dI_z = \frac{(dm)R^2}{2} = \rho S\, dx\, \frac{R^2}{2}$$



where p = density of the material of the plate and S = area of cross section of the plate.

Thus the sought moment of inertia

$$I_z = \frac{\rho S R^2}{2} \int_0^b dx = \frac{R^2}{2} \rho S b$$

$$= \frac{\pi}{2} \rho b R^4 \left(\text{as } S = \pi R^2\right)$$

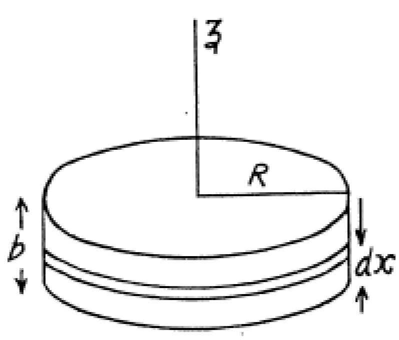
putting all the vallues we get,l_z = 2.gm.m^2

(b) Consider an element disc of radius r and thickness dx at a distance x from the point O. Then r = x tana and volume of the disc

$$= \pi x^2 \tan^2 \alpha\, dx$$

Hence, its mass dm = πx^2 tan α dx.p (where p = density of the cone $= m/\frac{1}{3}\pi R^2 h)$

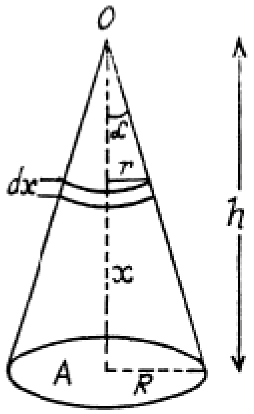
Moment of inertia of this element, about the axis OA,



$$dI = dm\,\frac{r^2}{2}$$

$$= (\pi x^2 \tan^2\alpha\, dx)\,\frac{x^2 \tan^2\alpha}{2}$$

$$= \frac{\pi\rho}{2}\, x^4 \tan^4\alpha\, dx$$



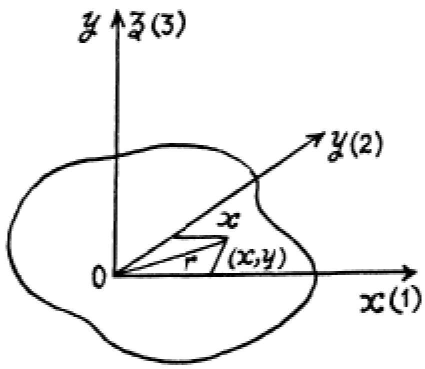
Thus the sought moment of inertia $I = \frac{\pi\rho}{2}\tan^4\alpha \int_0^h x^4\,dx$

$$= \frac{\pi\rho R^4 \cdot h^5}{10 h^4}\left(\text{as } \tan\alpha = \frac{R}{h}\right)$$

Hence $I = \frac{3mR^2}{10}\left(\text{putting } \rho = \frac{3m}{\pi R^2 h}\right)$

Q. 240. Demonstrate that in the case of a thin plate of arbitrary shape there is the following relationship between the moments of inertia: $l_1 + I_2 = l_.$, where subindices 1, 2, and 3 define three mutually perpendicular axes passing through one point, with axes 1 and 2 lying in the plane of the plate. Using this relationship, find the moment of inertia of a thin uniform round disc of radius R and mass m relative to the axis coinciding with one of its diameters.

Ans. (a) Let us consider a lamina of an arbitrary shape and indicate by 1,2 and 3, three axes coinciding with x, y and z - axes and the plane o f lamina as x - y plane.



Now, moment of inertia of a point mass about

x - axis, $dI_x = dm\,y^2$

Thus moment of inertia of the lamina about

this axis, $I_x = \int dmy^2$

Similarly, $I_y = \int dmx^2$

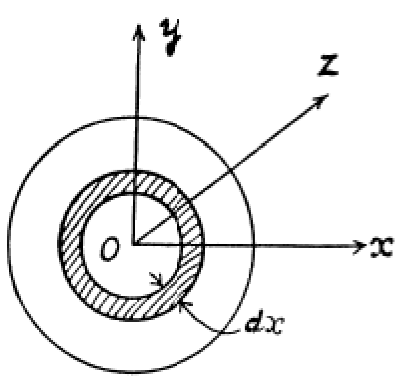
and $I_z = \int dmr^2$

$= \int dm\,(x^2 + y^2)$ as $r = \sqrt{x^2 + y^2}$

Thus, $I_z = I_x + I_y$ or, $I_3 = I_1 + I_2$

(b) Let us take the plane of the disc as x - y plane and origin to the centre of the disc (Fig.) From the symmetry $I_x = I_y$. Let us consider a ring element of radius r and thickness dr, then the moment of inertia of the ring element about the y - axis. \

$$dI_z = dmr^2 = \frac{m}{\pi R^2}(2\pi rdr)\,r^2$$



Thus the moment of inertia of the disc about

z − axis

$$I_z = \frac{2m}{R^2}\int_0^R r^3\,dr = \frac{mR^2}{2}$$

But we have $I_z = I_x + I_y = 2I_x$

Thus $I_x = \frac{I_z}{2} = \frac{mR^2}{4}$

Q. 241. A uniform disc of radius R = 20 cm has a round cut as shown in Fig. 1.54. The mass of the remaining (shaded) portion of the disc equals m = 7.3 kg. Find the moment of inertia of such a disc relative to the axis passing through its centre of inertia and perpendicular to the plane of the disc.

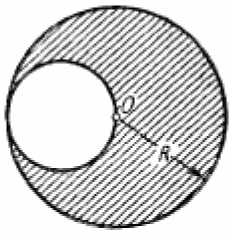
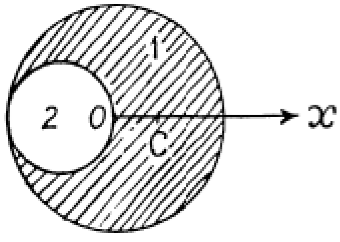


Fig. 1.54.

Ans. For simplicity let us use a mathematical trick. We consider the portion of the given disc as the superposition of two- complete discs (without holes), one of positive density and radius R and other of negative density but of same magnitude and radius R/2.
As (area) α (mass), the respective masses of the considered discs are (4m/3) and (-m/3) respectively, and these masses can be imagined to be situated at their respective centers (C.M). Let us take point O as origin and point x - axis towards right Obviously the C.M. of the shaded position of given shape lies on the x - axis. Hence the C.M. (C) of the shaded portion is given by

$$x_c = \frac{(-m/3)(-R/2) + (4m/3)0}{(-m/3) + 4m/3} = \frac{R}{6}$$



Thus C.M. of the shape is at a distance R/6 from point O toward x - axis
Using parallel axis theorem and bearing in mind that the moment of inertia of a complete homogeneous disc of radius m_0 and radius r_0 equals $\frac{1}{2} m_0 r_0^2.$

The moment of inetia of the small disc of mass (- m / 3) and radius R / 2 about the axis passing through point C and perpendicular to the plane of the disc

$$I_{2C} = \frac{1}{2}\left(-\frac{m}{3}\right)\left(\frac{R}{2}\right)^2 + \left(-\frac{m}{3}\right)\left(\frac{R}{2}+\frac{R}{6}\right)^2$$
$$= -\frac{mR^2}{24} - \frac{4}{27}mR^2$$

Simflarly

$$I_{1C} = \frac{1}{2}\left(\frac{4m}{3}\right)R^2 + \left(\frac{4m}{3}\right)\left(\frac{R}{6}\right)^2$$
$$= \frac{2}{3}mR^2 + \frac{mR^2}{27}$$

Thus the sought moment of inertia,

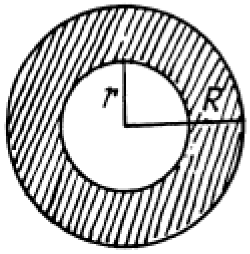
$$I_C = I_{1C} + I_{2C} = \frac{15}{24}mR^2 - \frac{3}{27}mR^2 = \frac{37}{72}mR^2$$

Q. 242. Using the formula for the moment of inertia of a uniform sphere, find the moment of inertia of a thin spherical layer of mass m and radius R relative to the axis passing through its centre.

Ans. Moment of inertia of the shaded portion, about the axis passing through it 's certre,

$$I = \frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2 - \frac{2}{5}\left(\frac{4}{3}\pi r^3\rho\right)r^3$$
$$= \frac{2}{5}\frac{4}{3}\pi\rho\left(R^5 - r^5\right)$$

Now, if R = r + dr, the shaded portion becomes a shell, which is the required shape to calculate the moment of inertia.



Now, $I = \frac{2}{5} - \frac{4}{3}\pi\rho\left\{(r+dr)^5 - r^5\right\}$

$= \frac{2}{5}\frac{4}{3}\pi\rho\left(r^5 + 5r^4\,dr + \ldots\ldots - r^5\right)$

Neglecting higher terms,

$= \frac{2}{3}\left(4\pi r^2\,dr\rho\right)r^2 = \frac{2}{3}mr^2$

Q. 243. A light thread with a body of mass m tied to its end is wound on a uniform solid cylinder of mass M and radius R (Fig. 1.55). At a moment t = 0 the system is set in motion. Assuming the friction in the axle of the cylinder to be negligible, find the time dependence of

(a) the angular velocity of the cylinder;
(b) the kinetic energy of the whole system.

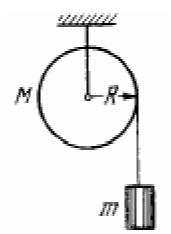


Fig. 1.55.

Ans. (a) Net force which is effective on the system (cylinder M + body m) is the weight of the body m in a uniform gravitational field, which is a constant. Thus the initial acceleration of the body m is also constant.

From the conservation of mechanical energy of the said system in the uniform field of gravity at time

$t = \Delta t : \Delta T + \Delta U = 0$

or $\quad \frac{1}{2}mv^2 + \frac{1}{2}\frac{MR^2}{2}\omega^2 - mg\,\Delta h = 0$

or, $\quad \frac{1}{4}(2m + M)v^2 - mg\,\Delta h = 0$ [as $v = \omega R$ at all times]

But $\quad v^2 = 2w\,\Delta h$

Hence using it in Eq. (1), we get

$$\frac{1}{4}(2m + M)2w\,\Delta h - mg\,\Delta h = 0 \text{ or } w = \frac{2mg}{(2m + M)}$$

From the kinematical relationship, $\beta = \frac{w}{R} = \frac{2mg}{(2m + M)R}$

$$\omega(t) = \beta t = \frac{2mg}{(2m + M)R}t = \frac{gt}{(1 + M/2m)R}$$

(b) Sought kinetic energy.

$$T(t) = \frac{1}{2}mv^2 + \frac{1}{2}\frac{Ml^2}{2}\omega^2 = \frac{1}{4}(2m + M)R^2\omega^2$$

Q. 244. The ends of thin threads tightly wound on the axle of radius r of the Maxwell disc are attached to a horizontal bar. When the disc unwinds, the bar is raised to keep the disc at the same height. The mass bf the disc with the axle is equal to m, the moment of inertia of the arrangement relative to its axis is I. Find the tension of each thread and the acceleration of the bar.

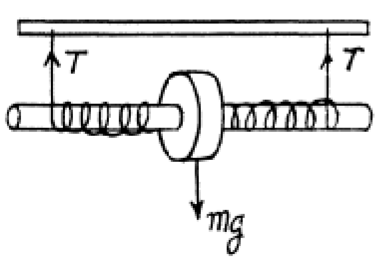
Ans. For equilibrium of the disc and axle

$$2T = mg \text{ or } T = mg/2$$

As the disc unwinds, it has an angular acceleration β given by

$$I\beta = 2Tr \text{ or } \beta = \frac{2Tr}{I} = \frac{mgr}{I}$$

The corresponding linear acceleration is $r\beta = w = \frac{mgr^2}{I}$



Since the disc remains stationary under the combined action of this acceleration and the acceleration (-w) of the bar which is transmitted to the axle, we must have $w = \frac{mgr^2}{I}$

Q. 245. A thin horizontal uniform rod AB of mass m and length l can rotate freely about a vertical axis passing through its end A. At a certain moment the end B starts experiencing a constant force F which is always perpendicular to the original position of the stationary rod and directed in a horizontal plane. Find the angular velocity of the rod as a function of its rotation angle op counted relative to the initial position.

Ans. Let the rod be deviated through an angle φ'from its initial position at an arbitrary instant of time, measured relative to the initial position in the positive direction. From the equation of the increment of the mechanical energy of the system.

$$\Delta T = A_{ext}$$

or, $$\frac{1}{2} I\omega^2 = \int N_z \, d\varphi$$

or, $$\frac{1}{2}\frac{Ml^2}{3}\omega^2 = \int_0^{\varphi} Fl\cos\varphi \, d\varphi = Fl\sin\varphi$$

Thus, $$\omega = \sqrt{\frac{6F\sin\varphi}{Ml}}$$

Q. 246. In the arrangement shown in Fig. 1.56 the mass of the uniform solid cylinder of radius R is equal to m and the masses of two bodies are equal to m_1 and m_2. The thread slipping and the friction in the axle of the cylinder are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions T_1/T_2 of the vertical sections of the thread in the process of motion.

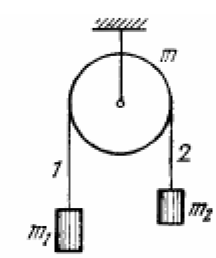
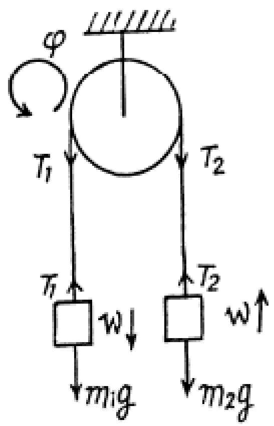


Fig. 1.56.

Ans. First of all, let us sketch free body diagram of each body. Since the cylinder is rotating and massive, the tension will be different in both the sections of threads. From Newton's law in projection form for the bodies m_1 and m_2 and noting that $w_1 = w_2 = \beta R$, (as no thread slipping), we have ($m_1 > m_2$)



$$m_1 g - T_1 = m_1 w = m_1 \beta R$$

$$\text{and } T_2 - m_2 g = m_2 w \qquad (1)$$

Now from the equation of rotational dynamics of a solid about stationary axis of rotation, i.e.
$N_z = I \beta_z$, for the cylinder.

$$\text{or, } (T_1 - T_2) R = I\beta = mR^2 \beta/2 \qquad (2)$$

Similtaneous solution of the above equations yields :

$$\beta = \frac{(m_1 - m_2) g}{R\left(m_1 + m_2 + \frac{m}{2}\right)} \text{ and } \frac{T_1}{T_2} = \frac{m_1 (m + 4m_2)}{m_2 (m + 4m_1)}$$

Q. 247. In the system shown in Fig. 1.57 the masses of the bodies are known to be m_1 and m_2, the coefficient of friction between the body mi and the horizontal plane is equal to k, and a pulley of mass m is assumed to be a uniform disc. The thread does not slip over the pulley. At the moment t = 0 the body m_2 starts descending. Assuming the mass of the thread and the friction in the axle of the pulley to be negligible, find the work performed by the friction forces acting on the body m_1 over the first t seconds after the beginning of motion.

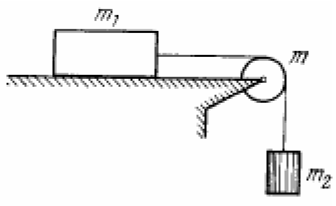


Fig. 1.57.

Ans. As the systenr ($m + m_1 + m_2$) is under constant forces, the acceleration of body m_1 an m_2 is constant In addition to it the velocities and accelerations of bodies m_1 and m_2 equal in magnitude (say v and tv) because the length of the thread is constant From the equation of increament of mechanical energy i.e. $\Delta T + \Delta U = A_{fr}$, at time t whe block m_1 is distance h below from initial position corresponding to t = 0,

$$\frac{1}{2}(m_1 + m_2) v^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\frac{v^2}{R^2} - m_2 gh = -km_1 gh \qquad (1$$

(as angular velocity ω - v/R for no slipping of thread.)

But $v^2 = 2wh$

So using it in (1), we get $w = \frac{2(m_2 - km_1)g}{m + 2(m_1 + m_2)}$. (2

Thus the work done by the friction force on m_1

$$A_{fr} = -km_1 gh = -km_1 g\left(\frac{1}{2}wt^2\right)$$

$$= -\frac{km_1(m_1 - km_1)g^2 t^2}{m + 2(m_1 + m_2)} \qquad \text{(using 2).}$$

Q. 248. A uniform cylinder of radius R is spinned about its axis to the angular velocity ω_0 and then placed into a corner (Fig. 1.58). The coefficient of friction between the corner walls and the cylinder is equal to k. How many turns will the cylinder accomplish before it stops?

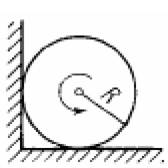


Fig. 1.58.

Ans. In the problem, the rigid body is in translation equlibrium but there is an angular retardation. We first sketch the free body diagram of the cylinder. Obviously the friction forces, acting on the cylinder, are kinetic. From the condition of translational equlibrium for the cylinder,

$$mg = N_1 + kN_2; \quad N_2 = kN_1$$

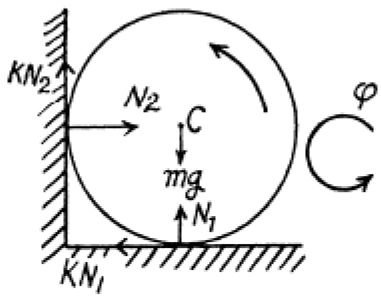
Hence, $N_1 = \frac{mg}{1+k^2}; \; N_2 = k\frac{mg}{1+k^2}$

For pure rotation of the cylinder about its rotation axis, Nz - $I\beta_z$

$$\text{or,} \quad -kN_1R - kN_2R = \frac{mR^2}{2}\beta_z$$

$$\text{or,} \quad -\frac{kmgR(1+k)}{1+k^2} = \frac{mR^2}{2}\beta_z$$

$$\text{or,} \quad \beta_z = -\frac{2k(1+k)g}{(1+k^2)R}$$



Now, from the kinematical equation,

$$\omega^2 = \omega_0^2 + 2\beta_z\Delta\varphi \text{ we have,}$$

$$\Delta\varphi = \frac{\omega_0^2(1+k^2)R}{4k(1+k)g}, \qquad \text{because } \omega = 0$$

Hence, the sought number of turns,

$$n = \frac{\Delta\varphi}{2\pi} = \frac{\omega_0^2(1+k^2)R}{8\pi k(1+k)g}$$

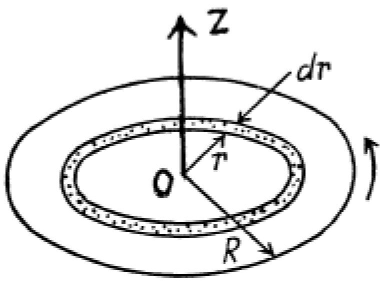
Q. 249. A uniform disc of radius R is spinned to the angular velocity ω and then carefully placed on a horizontal surface. How long will the disc be rotating on the surface if the friction coefficient is equal to k? The pressure exerted by the disc on the surface can be regarded as uniform.

Ans. It is the moment of friction force which brings the disc to rest The force of friction is applied to each section of the disc, and since these sections lie at different distances from the axis, the moments of the forces of friction differ from section to section.
To find N_z, where z is the axis of rotation of the disc let us partition the disc into thin rings (Fig.). The force of friction acting on the considered element $dfr = k(2\pi r\, dr\, \sigma)g,$ (where σ is the density of the disc)

$$dN_z = -r\, dfr = -2\pi k \sigma g r^2 dr$$

Integrating with respect to r from zero to R, we get

$$N_z = -2\pi k \sigma g \int_0^R r^2 dr = -\frac{2}{3}\pi k \sigma g R^3.$$



For the rotation of the disc about the stationary axis z, from the equation $N_z = I\beta z$

$$-\frac{2}{3}\pi k \sigma g R^3 = \frac{(\pi R^2 \sigma)R^2}{2}\beta_z \quad \text{or} \quad \beta_z = -\frac{4kg}{3R}$$

Thus from the angular kinematical equation

$$\omega_z = \omega_{0z} + \beta_z t$$

$$0 = \omega_0 + \left(-\frac{4kg}{3R}\right)t \quad \text{or} \quad t = \frac{3R\,\omega_0}{4kg}$$

Q. 250. A flywheel with the initial angular velocity ω_0 decelerates due to the forces whose moment relative to the axis is proportional to the square root of its angular velocity. Find the mean angular velocity of the flywheel averaged over the total deceleration time.

Ans. According to the question,

$$I\frac{d\omega}{dt} = -k\sqrt{\omega} \quad \text{or,} \quad I = \frac{d\omega}{\sqrt{\omega}} = -k\,dt$$

Integrating, $\sqrt{\omega} = -\frac{kt}{2I} + \sqrt{\omega_0}$

or, $\omega = \frac{k^2 t^2}{4I^2} - \frac{\sqrt{\omega_0}\,kt}{I} + \omega_0$, (Noting that at $t = 0$, $\omega = \omega_0$.)

Let the flywheel stops at $t = t_O$ then from Eq. (1), $t_0 = \frac{2I\sqrt{\omega_0}}{k}$

Hence sought average angular velocity

$$< \omega > = \frac{\int_0^{\frac{2I\sqrt{\omega_0}}{k}} \left(\frac{k^2 t^2}{4I^2} - \frac{\sqrt{\omega_0}\,kt}{I} + \omega_0 \right) dt}{\int_0^{\frac{2I\sqrt{\omega_0}}{k}} dt} = \frac{\omega_0}{3}$$

Dynamics Of A Solid Body (Part - 2)

Q. 251. A uniform cylinder of radius R and mass M can rotate freely about a stationary horizontal axis O (Fig. 1.59). A thin cord of length l and mass m is wound on the cylinder in a single layer. Find the angular acceleration of the cylinder as a function of the length x of the hanging part of the cord. The wound part of the cord is supposed to have its centre of gravity on the cylinder axis.

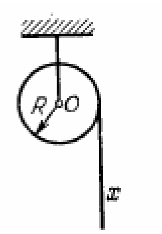


Fig. 1.59.

Ans. Let us use the equation $\frac{dM_z}{dt} = N_z$ relative to the axis through O (1)

For this purpose, let us find the angular momentum of the system M_z about the given rotation axis and the corresponding torque N_z. The angular momentum is

$$M_z = I\omega + mvR = \left(\frac{m_0}{2} + m\right)R^2\omega$$

[where $I = \frac{m_0}{2}R^2$ and $v = \omega R$ (no cord slipping)]

So, $\frac{dM_z}{dt} = \left(\frac{MR^2}{2} + mR^2\right)\beta_z$ (2)

The downward pull of gravity on the overhanging part is the only external force, which exerts a torque about the z -axis, passing through O and is given by,

$$N_z = \left(\frac{m}{l}\right)xgR$$

Hence from the equation $\frac{dM_z}{dt} = N_z$

Thus, $$\left(\frac{MR^2}{2} + mR^2\right)\beta_z = \frac{m}{l}xgR$$

$$\beta_z = \frac{2mgx}{lR(M+2m)} > 0$$

Note : We may solve this problem using conservation of mechanical energy of the system (cylinder + thread) in the uniform field of gravity.

Q. 252. A uniform sphere of mass m and radius R rolls without slipping down an inclined plane set at an angle α to the horizontal. Find:
(a) the magnitudes of the friction coefficient at which slipping is absent;
(b) the kinetic energy of the sphere t seconds after the beginning of motion.

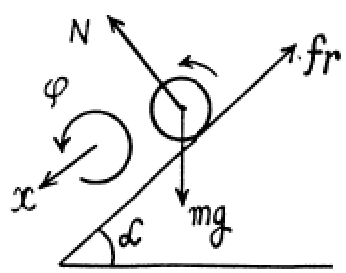
Ans. (a) Let us indicate the forces acting on the sphere and their points of application. Choose positive direction of x and cp (rotation angle) along the incline in downward direction and in the sense of $\vec{\omega}$ (for undirectional rotation) respectively. Now from equations of dynamics o f rigid body i.e.

$F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$ we get :

$mg \sin \alpha - f_r = mw \quad (1)$

and $\quad fr\, R = \frac{2}{5} mR^2 \beta \quad (2)$

But $\quad fr \le kmg \cos\alpha \quad (3)$



In addition, the absence of slipping provides the kinematical realtionship between the accelerations :

w = βR (4)

The simultaneous solution of all the four equations yields :

$k \cos \alpha \ge \frac{2}{7} \sin \alpha$, or $k \ge \frac{2}{7} \tan \alpha$

(b) Solving Eqs. (1) and (2) [of part (a)], we :

$$w_c = \frac{5}{7} g \sin \alpha.$$

As the sphere starts at t = 0 along positive x axis, for pure rolling

$$v_c(t) = w_c t = \frac{5}{7} g \sin \alpha\, t \qquad (5)$$

Hence the sought kinetic energy

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2}\frac{2}{5} m R^2 \omega^2 = \frac{7}{10} m v_c^2 \text{ (as } \omega = v_c/R)$$

$$= \frac{7}{10} m \left(\frac{5}{7} g \sin \alpha\, t\right)^2 = \frac{5}{14} m g^2 \sin^2 \alpha\, t^2$$

Q. 253. A uniform cylinder of mass m = 8.0 kg and radius R = 1.3 cm (Fig. 1.60) starts descending at a moment t = 0 due to gravity. Neglecting the mass of the thread, find:

(a) the tension of each thread and the angular acceleration of the cylinder;
(b) the time dependence of the instantaneous power developed by the gravitational force.

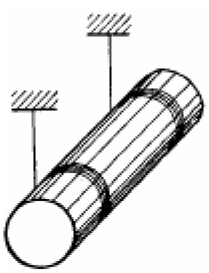
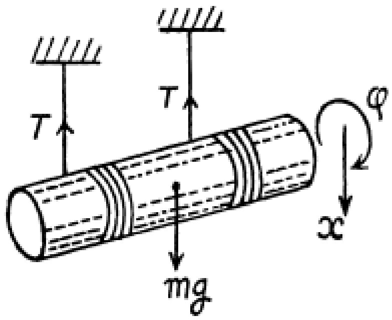


Fig. 1.60.

Ans. (a) Let us indicate the forces and their points of application fox the cylinder. Choosing the positive direction for x and φ as shown in the figure, we write the equation of motion of the cylinder axis and the equation of moments in the C.M. frame relative to that axis i.e. from equation

$F_x = mw_c$ and $N_z = I_c \beta_z$

$$mg - 2T = mw_c; \; 2TR = \frac{mR^2}{2}\beta$$



As there is no slipping of thread on the cylinder

$w_c = \beta R$

From these three equations

$$T = \frac{mg}{6} = 13 \text{ N}, \; \beta = \frac{2}{5}\frac{g}{R} = 5 \times 10^2 \text{ rad/s}^2$$

(b) we have $\beta = \frac{2}{3}\frac{g}{R}$

So, $w_c = \frac{2}{3}g > 0$ or, in vector form $\vec{w}_c = \frac{2}{3}\vec{g}$

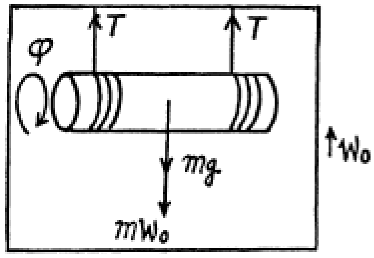
$$P = \vec{F} \cdot \vec{v} = \vec{F} \cdot (\vec{w}_c t)$$

$$= m\vec{g} \cdot \left(\frac{2}{3}\vec{g}t\right) = \frac{2}{3}mg^2 t$$

Q. 254. Thin threads are tightly wound on the ends of a uniform solid cylinder of m ass m. The free ends of the threads are attached to the ceiling of an elevator car. The car starts going up with an accelera- tion w_0. Find the acceleration w' of the cylinder relative to the car and the force F exerted by the cylinder on the ceiling (through the threads).

Ans. Let us depict the forces and their points of application corresponding to the cylinder attached with the elevator. Newton's second law for solid in vector form in the frame of elevator, gives :

$$2\vec{T} + m\vec{g} + m(-\vec{w}_0) = m\vec{w}' \qquad (1)$$



The equation of moment in the C.M. frame relative to the cylinder axis i.e. from

$$N_z = I_c \beta_z \cdot$$

$$2TR = \frac{mR^2}{2}\beta = \frac{mR^2}{2}\frac{w'}{R}$$

[as thread does not slip on the cylinder, W' = PR]

or, $T = \frac{mw'}{4}$

As (1) $\vec{T} \uparrow\downarrow \vec{w}$

so in vector form

$$\vec{T} = -\frac{m\vec{w}}{4} \qquad (2)$$

Solving Eqs. (1) and (2), $\vec{w}' = \frac{2}{3}(\vec{g} - \vec{w}_0)$

$$\vec{F} = 2\vec{T} = \frac{1}{3}m(\vec{g} - \vec{w}_0).$$

Q. 255. A spool with a thread wound on it is placed on an inclined smooth plane set at an angle α = 30° to the horizontal. The free end of the thread is attached to the wall as shown in Fig. 1.61. The mass of the spool is m = 200 g, its moment of inertia relative to its own axis I = 0.45 g.m^2, the radius of the wound thread layer r = 3.0 cm. Find the acceleration of the spool axis.

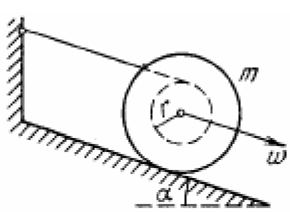
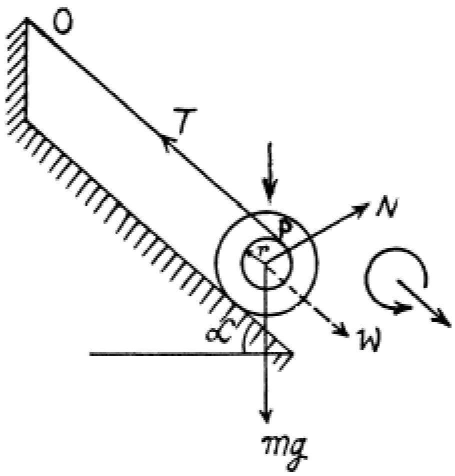


Fig. 1.61.

Ans. Let us depict the forces and their points of application for the spool. Choosing the positive direction for x and φ as shown in the fig., we apply F = mw and N* - I. p. and get $F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$ and get

$mg \sin \alpha - T = mw$; $Tr = I\beta$



"Notice that if a point of a solid in plane motion is connected with a thread, the projection of velocity vector of the solid's point of contact along the length of the thread equals the velocity of the other end of the thread (if it is not slacked)"

Thus in our problem, $v_P = v_0$ but $v_0 = 0$, hence point P is the instantaneous centre of rotation of zero velocity for the spool. Therefore $v_c = \omega r$ and subsequently $w_c = \beta r$

Solving the equations simultaneously, we get

$$w = \frac{g \sin \alpha}{1 + \frac{I}{mr^2}} = 1{\cdot}6 \text{ m/s}^2$$

Q. 256. A uniform solid cylinder of mass m rests on two horizontal planks. A thread is wound on the cylinder. The hanging end of the thread is pulled vertically down with a constant force F (Fig. 1.62). Find the maximum magnitude of the force F which still does not bring about any sliding of the cylinder, if the coefficient of friction between the cylinder and the planks is equal to k. What is the ac- celeration w_{max} of the axis of the cylinder rolling down the inclined plane?

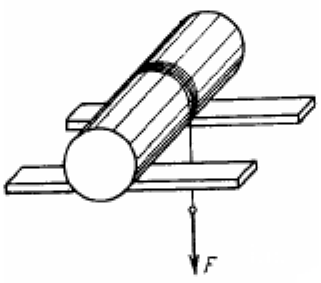
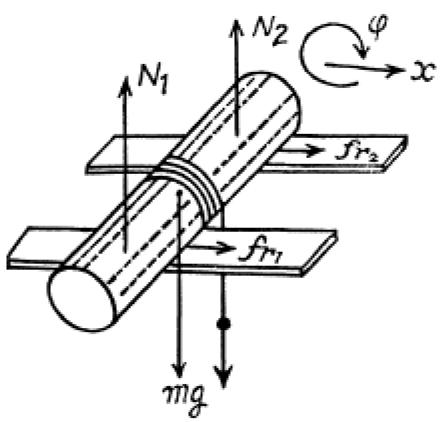


Fig. 1.62.

Ans. Let us sketch the force diagram for solid cylinder and apply Newton's second law in projection form along x and y axes (Fig.) :



$$fr_1 + fr_2 = mw_c \qquad (1)$$

and $$N_1 + N_2 - mg - F = 0$$

or $$N_1 + N_2 = mg + F \qquad (2)$$

Now choosing positive direction of <p as shown in the figure and using Ncz = I_c βz, $N_{cz} = I_c \beta_z$,

we get

$$FR - (fr_1 + fr_2)R = \frac{mR^2}{2}\beta = \frac{mR^2}{2}\frac{w_c}{R} \qquad (3)$$

[as for pure rolling $w_c = \beta R$]. In addition to,

$$fr_1 + fr_2 \le k(N_1 + N_2) \qquad (4)$$

Solving the Eqs., we get

$$F \le \frac{3\,k\,mg}{(2-3k)}, \quad \text{or} \quad F_{\max} = \frac{3\,k\,mg}{2-3\,k}$$

and $$w_{c\,(\max)} = \frac{k\,(N_1+N_2)}{m}$$

$$= \frac{k}{m}\,[mg + F_{\max}] = \frac{k}{m}\left[mg + \frac{3\,kmg}{2-3k}\right] = \frac{2\,kg}{2-3k}$$

Q. 257. A spool with thread wound on it, of mass m, rests on a rough horizontal surface. Its moment of inertia relative to its own axis is equal to I = γmR^2, where γ is a numerical factor, and R is the outside radius of the spool. The radius of the wound thread layer is equal to r. The spool is pulled without sliding by the thread with a constant force F directed at an angle α to the horizontal (Fig. 1.63). Find:

(a) the projection of the acceleration vector of the spool axis on the x-axis;
(b) the work performed by the force F during the first t seconds after the beginning of motion.

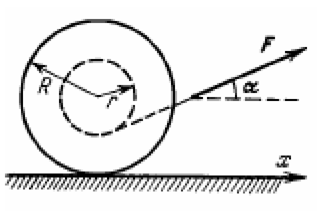
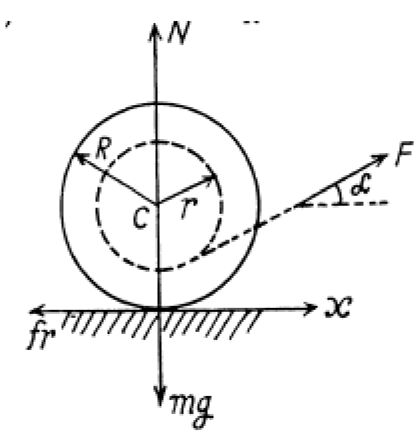


Fig. 1.63.

Ans. (a) Let us choose the positive direction of the rotation angle φ, such that w_{cx} and β_z have identical signs (Fig.). Equation of motion, $F_x = mw_{cx}$ and $N_{cz} = I_c\,\beta_z$ gives :

$$F\cos\alpha - fr = mw_{cx} : fr\,R - Fr = I_c\,\beta_z = \gamma m R^2\,\beta_z$$



In the absence of the slipping of the spool $w_{cx} = \beta_z R$

From the three equations $w_{cx} = w_c = \frac{F[\cos\alpha - (r/R)]}{m(1+\gamma)}$, where $\cos\alpha > \frac{r}{R}$ (1)

(b) As static friction (fr) does not work on the spool, from the equation of the increment of mechanical eneigy $A_{ext} = \Delta T$.

$$A_{ext} = \frac{1}{2} m v_c^2 + \frac{1}{2} \gamma m R^2 \frac{v_c^2}{R^2} = \frac{1}{2} m (1+\gamma) v_c^2$$

$$= \frac{1}{2} m (1+\gamma) 2 w_c x = \frac{1}{2} m (1+\gamma) 2 w_c \left(\frac{1}{2} w_c t^2\right)$$

$$= \frac{F^2 \left(\cos\alpha - \frac{r}{R}\right)^2 t^2}{2\, m (1+\gamma)}$$

Note\that at cos α = r/R, there is no rolling and for $\cos\alpha < r/R$, $w_{cx} < 0$, *i.e.* the spool will move towards negative x-axis and rotate in anticlockwise sense.

Q. 258. The arrangement shown in Fig. 1.64 consists of two identical uniform solid cylinders, each of mass m, on which two light threads are wound symmetrically. Find the tension of each thread in the process of motion. The friction in the axle of the upper cylinder is assumed to be absent.

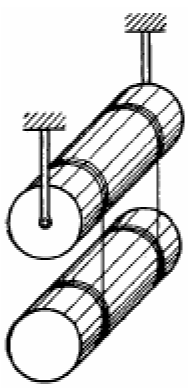
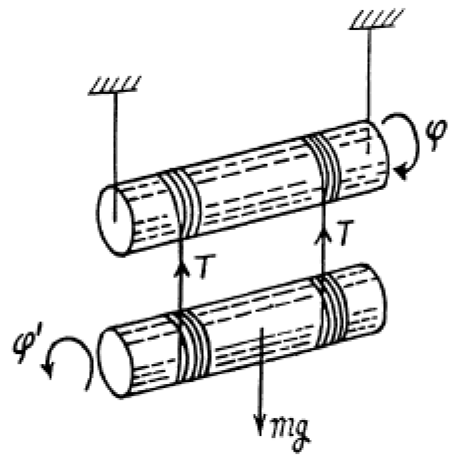


Fig. 1.64.

Ans. For the cylinder from the equation $N_z = I\beta_z$ about its stationary axis of rotation.

$$2\, Tr = \frac{mr^2}{2}\beta \quad \text{or} \quad \beta = \frac{4T}{mr} \qquad (1)$$



For the rotation of the lower cylinder from the equation $N_{cz} = I_c \beta_z$

$$2Tr = \frac{mr^2}{2}\beta' \quad \text{or,} \quad \beta' = \frac{4T}{mr} = \beta$$

Now for the translational motion of lower cylinder from the Eq. $F_x = mw_{cx}$:

$$mg - 2T = mw_c \qquad (2)$$

As there is no slipping of threads on the cylinders :

$$w_c = \beta' r + \beta r = 2\beta r \qquad (3)$$

Simultaneous solu tion of (1), (2) and (3) yields

$$T = \frac{mg}{10}.$$

Q. 259. In the arrangement shown in Fig. 1.65 a weight A possesses mass m, a pulley B possesses mass M. Also known are the moment of inertia I of the pulley relative to its axis and the radii of the pulley R and 2R. The mass of the threads is negligible. Find the acceleration of the weight A after the system is set free.

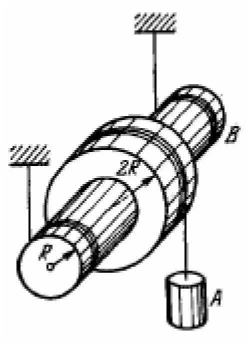
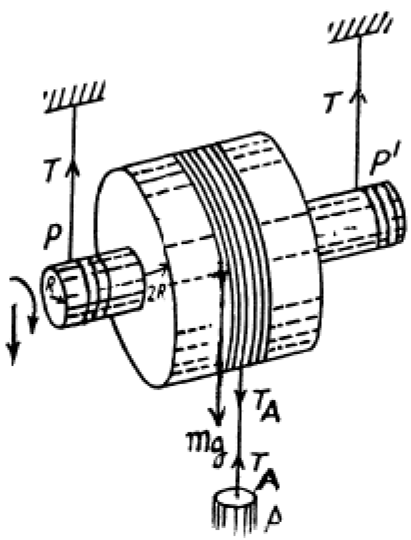


Fig. 1.65.

Ans. Let us depict the forces acting on the pulley and weight A, and indicate positive direction for x and φ as shown in the figure. For the cylinder from the equation $F_x = m\ddot{x}$ and $N_{cz} = I_c \beta_z$, we get

$$Mg + T_A - 2T = M w_c \qquad (1)$$

and $$2TR + T_A (2R) = I\beta = \frac{I w_c}{R} \qquad (2)$$



For the weight A from the equation

$$F_x = m w_x$$

$$mg - T_A = m w_A \qquad (3)$$

As there is no slipping of the threads on the pulleys.

$$w_A = w_c + 2\beta R = w_c + 2w_c = 3w_c \qquad (4)$$

Simultaneous solutions of above four equations gives :

$$w_A = \frac{3(M+3m)g}{\left(M+9m+\frac{I}{R^2}\right)}$$

Q. 260. A uniform solid cylinder A of mass m_1 can freely rotate about a horizontal axis fixed to a mount B of mass m_2 (Fig. 1.66). A constant horizontal force F is applied to the end K of a light thread tightly wound on the cylinder. The friction between the mount and the supporting horizontal plane is assumed to be absent. Find:

(a) the acceleration of the point K;
(b) the kinetic energy of this system t seconds after the beginning of motion.

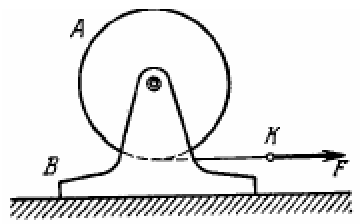


Fig. 1.66.

Ans. (a) For the translational motion of the system $(m_1 + m_2)$, from the equation $F_x = mw_{cx}$

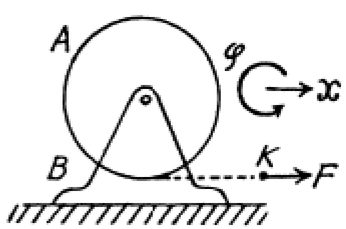
$$F = (m_1 + m_2)\, w_c \quad \text{or,} \quad w_c = F/(m_1+m_2) \qquad (1)$$

Now for the rotational motion of cylinder from the equation : $N_{cx} = I_c \beta_z$

$$Fr = \frac{m_1 r^2}{2}\beta \quad \text{or} \quad \beta r = \frac{2F}{m_1} \qquad (2)$$

But $\quad w_K = w_c + \beta r$, So

$$w_K = \frac{F}{m_1+m_2} + \frac{2F}{m_1} = \frac{F(3m_1+2m_2)}{m_1(m_1+m_2)} \qquad (3)$$



(b) From the equation of increment of m echanical energy : $\Delta T = A_{ext}$

Here $\Delta T = T(t)$, so, $T(t) = A_{ext}$

As force F is constant and is directed along jc-axis the sought work done.

$A_{ext} = Fx$

(where x is the displacement of the point of application of the force F during time interval t)

$$= F\left(\frac{1}{2} w_K t^2\right) = \frac{F^2 t^2 (3m_1 + 2m_2)}{2m_1(m_1 + m_2)} = T(t)$$

(using Eq. (3)

Alternate : $T(t) = T_{translation}(t) + T_{rotation}(t)$

$$= \frac{1}{2}(m_1 + m_2)\left(\frac{Ft}{(m_1 + m_2)}\right)^2 + \frac{1}{2}\frac{m_1 r^2}{2}\left(\frac{2Ft}{m_1 r}\right)^2 = \frac{F^2 t^2 (3m_1 + 2m_2)}{2m_1(m_{1+} m_2)}$$

Q. 261. A plank of mass m_1 with a uniform sphere of mass m_2 placed on it rests on a smooth horizontal plane. A constant horizontal force F is applied to the plank. With what accelerations will the plank and the centre of the sphere move provided there is no sliding between the plank and the sphere?

Ans. Choosing the positive direction for x and <p as shown in Fig, let us we write the equation of motion for the sphere $F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$

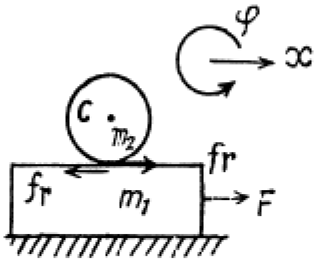
$$fr = m_2 w_2 \, ; \; fr\, r = \frac{2}{5} m_2 r^2 \beta$$

(w_2 is the acceleration of the C .M . o f sphere.)

For the plank from the Eq. $F_x = mwx$ F - f r = m_1 w_1

In addition, the condition for the absence of slipping of the sphere yields the kinem atical relation between the accelerations :

$w_1 = w_2 + \beta r$



Sim ultaneous solution of the four equations yields :

$$w_1 = \frac{F}{\left(m_1 + \frac{2}{7} m_2\right)} \quad \text{and} \quad w_2 = \frac{2}{7} w_1$$

Q. 262. A uniform solid cylinder of mass m and radius R is set in rotation about its axis with an angular velocity ω_0, then lowered with its lateral surface onto a horizontal plane and released. The coefficient of friction between the cylinder and the plane is equal to k.

Find:
(a) how long the cylinder will move with sliding;
(b) the total work performed by the sliding friction force acting on the cylinder

Ans. (a) Let us depict the forces acting on the cylinder and their point of applications for the cylinder and indicate positive direction of x and φ as shown in the figure.

From the equations for the plane motion of a solid $F_x = m w_{cx}$ and $N_{cz} = I_c \beta_z$:

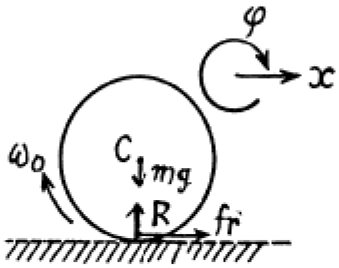
$$k\,mg = m\,w_{cx} \quad \text{or} \quad w_{cx} = kg \qquad (1)$$

$$-kmg\,R = \frac{mR^2}{2}\beta^z \quad \text{or} \quad \beta_z = -2\frac{kg}{R} \qquad (2)$$

Let the cylinder starts pure rolling at t = t_0 after releasing on the horizontal floor at t = 0. From the angular kinematical equation

$$\omega_z = \omega_{oz} + \beta_z t,$$

or $\omega = \omega_0 - 2\frac{kg}{R}t$ (3)



From the equation of the linear kinem atics,

$v_{cx} = v_{ocx} + w_{cx}t$

or $v_c = 0 + kg\,t_0$

But at the moment t = t_O, when pure rolling starts $v_e = \omega R$

so, $kg\,t_0 = \left(\omega_0 - 2\frac{kg}{R}t_0\right)R$

Thus $t_0 = \frac{\omega_0 R}{3\,kg}$

(b) As the cylinder pick, up speed till it starts rolling, the point of contact has a purely translatory movement equal to $\frac{1}{2}w_c\,t_0^2$ in the forward directions but there is also a backward movement of the point of contact of magnitude $(\omega_0\tau_0 - \frac{1}{2}\beta\,t_0^2)\,R.$ Because of slipping the net displacement is backwards. The total work done is then,

$$A_{fr} = kmg\left[\frac{1}{2}w_c\,t_0^2 - (\omega_0 t_0 + \frac{1}{2}\beta t_0^2)\,R\right]$$

$$= kmg\left[\frac{1}{2}kg\,t_0^2 - \frac{1}{2}\left(-\frac{2kg}{R}\right)t_0^2\,R - \omega_0\,t_0\,R\right]$$

$$= kmg\,\frac{\omega_0 R}{3kg}\left[\frac{\omega_0 R}{6} + \frac{\omega_0 R}{3} - \omega_0 R\right] = -\frac{m\omega_0^2 R^2}{6}$$

The sam e result can also be obtained by the work-energy theorem, $A_{fr} = \Delta T$.

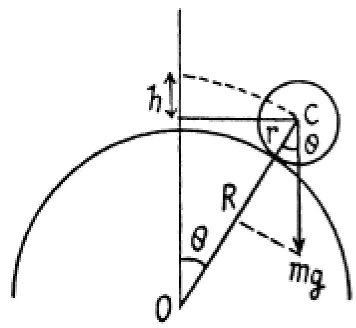
Q. 263. A uniform ball of radius r rolls without slipping down from the top of a sphere of radius R. Find the angular velocity of the ball at the moment it breaks off the sphere. The initial velocity of the ball is negligible.

Ans. Let us write the equation of motion for the centre of the sphere at the moment of breaking-off:

$$mv^2/(R+r) = mg\cos\theta,$$

where v is the velocity of the centre of the sphere at that moment, and θ is the corresponding angle (Fig.). The velocity v can be found from the energy conservation law :

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2,$$



where l is the moment of inertia of the sphere relative to the axis passing through the sphere's

centre. i.e. $I = \frac{2}{5}mr^2$. In addition,

$$v = \omega r\ ;\ h = (R+r)(1-\cos\theta).$$

From these four equations we obtain

$$\omega = \sqrt{10\,g\,(R+r)\,17\,r^2}.$$

Q. 264. A uniform solid cylinder of radius R = 15 cm rolls over a horizontal plane passing into an inclined plane forming an angle α = 30° with the horizontal (Fig. 1.67). Find the maximum value of the velocity v_0 which still permits the cylinder to roll onto the inclined plane section without a jump. The sliding is assumed to be absent.

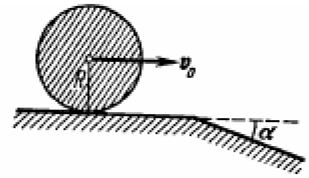


Fig. 1.67.

Ans. Since the cylinder moves without sliding, the centre of the cylinder rotates about the point O, while passing through the common edge of the planes. In other words, the point O becomes the foot of the instantaneous axis of rotation of the cylinder.
It at any instant during this motion the velocity of the C.M. is v_1 when the angle (shown in the figure) is β, we have

$$\frac{m v_1^2}{R} = mg \cos \beta - N,$$

where N is the normal reaction of the edge

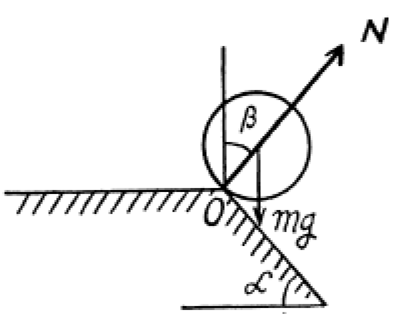
or, $$v_1^2 = gR \cos \beta - \frac{NR}{m} \qquad (1)$$

From the energy conservation law,

$$\frac{1}{2} I_0 \frac{v_1^2}{R^2} - \frac{1}{2} I_0 \frac{v_0^2}{R^2} = mg R (1 - \cos \beta)$$

But $$I_0 = \frac{mR^2}{2} + m R^2 = \frac{3}{2} mR^2,$$

(from the parallel axis theorem)



Thus, $$v_1^2 = v_0^2 + \frac{4}{3} g R (1 - \cos \beta)$$

From (1) and (2)

$$v_0^2 = \frac{g R}{3} (7 \cos \beta - 4) - \frac{NR}{m}$$

The angle β in this equation is clearly smaller than or equal to α so putting β = α we get

$$v_0^2 = \frac{g R}{3} (7 \cos \alpha - 4) - \frac{N_0 R}{M}$$

where N_0 is the corresponding reaction. Note that $N \geq N_0$. No jumping occurs during this turning if $N_0 > 0$. Hence, v_0 must be less than

$$v_{max} = \sqrt{\frac{gR}{3}(7\cos\alpha - 4)}$$

Q. 265. A small body A is fixed to the inside of a thin rigid hoop of radius R and mass equal to that of the body A. The hoop rolls without slipping over a horizontal plane; at the moments when the body A gets into the lower position, the centre of the hoop moves with velocity v_0 (Fig. 1.68). At what values of v_0 will the hoop move without bouncing?

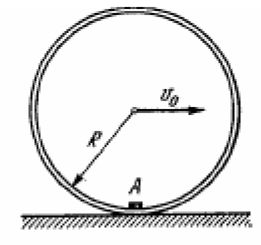


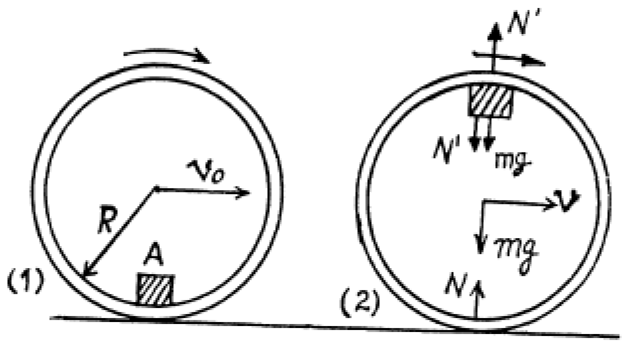
Fig. 1.68.

Ans. Clearly the tendency of bouncing of the hoop will be maximum when the small body A, will be at the highest point of the hoop during its rolling motion. Let the velocity of C.M. of the hoop equal v at this position. The static friction does no work on the hoop, so from conservation of mechanical energy; $E_1 = E_2$

$$0 + \frac{1}{2}mv_0^2 + \frac{1}{2}mR^2\left(\frac{v_0}{R}\right)^2 - mgR = \frac{1}{2}m(2v)^2 + \frac{1}{2}mv^2 + \frac{1}{2}mR^2\left(\frac{v}{R}\right)^2 + mgR$$

or $3v^2 = v_0^2 - 2gR$ **(1)**

$$mg + N' = m\omega^2 R = m\left(\frac{v}{R}\right)^2 R \qquad \textbf{(2)}$$



As the hoop has no acceleration in vertical direction, so for the hoop,

$N + N' = mg$ (3)

From Eqs. (2) and (3),

$N = 2\,mg - \frac{m\,v^2}{R}$ (4)

A s the hoop does not bounce, N $\geq$ 0 (5)

So from Eqs. (1), (4) and (5),

$$\frac{8\,g\,R - v_0^2}{3\,R} \geq 0 \quad \text{or} \quad 8\,g\,R \geq v_0^2$$

Hence $v_0 \leq \sqrt{8\,g\,R}$

Dynamics Of A Solid Body (Part - 3)

Q. 266. Determine the kinetic energy of a tractor crawler belt of mass m if the tractor moves with velocity v (Fig. 1.69).

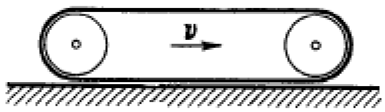


Fig. 1.69.

Solution. 266. Since the lower part of the belt is in contact with the rigid floor, velocity of this part becomes zero. The crawler moves with velocity v, hence the velocity of upper part of the belt becomes 2v by the rolling condition and kinetic energy of upper part $= \frac{1}{2}\left(\frac{m}{2}\right)(2v)^2 = mv^2,$ which is also the sought kinetic energy, assuming that the length of the belt is much larger than the radius of the wheels.

Q. 267. A uniform sphere of mass In and radius r rolls without sliding over a horizontal plane, rotating about a horizontal axle OA (Fig. 1.70). In the process, the centre of the sphere moves with velocity v along a circle of radius R. Find the kinetic energy of the sphere.

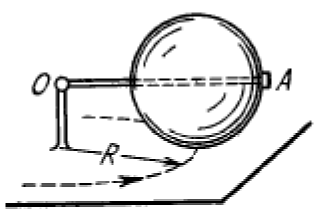


Fig. 1.70.

Solution. 267. The sphere has two types of motion, one is the rotation about its own axis and the other is motion in a circle of radius R. Hence the sought kinetic energy

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 \quad (1)$$

where is the moment of inertia about its own axis, and I2 is the moment of inertia about the vertical axis, passing through O, But, $I_1 = \frac{2}{5}mr^2$ and $I_2 = \frac{2}{5}mr^2 + mR^2$ (using parallel axis theorem,) (2)

In addition to

$\omega_1 = \frac{v}{r}$ and $\omega_2 = \frac{v}{R}$ (3)

Using (2) and (3) in (1), we get $T' = \frac{7}{10}mv^2\left(1 + \frac{2r^2}{7R^2}\right)$

Q. 268. Demonstrate that in the reference frame rotating with a constant angular velocity ω about a stationary axis a body of mass m experiences the resultant

(a) centrifugal force of inertia $F_{cf} = m\omega^2 R_c$, where R_c is the radius vector of the body's centre of inertia relative to the rotation axis;
(b) Coriolis force $F_{cor} = 2m\,[v'_c\omega]$, where is the velocity of the body's centre of inertia in the rotating reference frame

Solution. 268. For a point mass of mass dm, looked at from C rotating frame, the equation is

$$dm\,\vec{w}' = \vec{f} + dm\,\omega^2\,\vec{r}' + 2\,dm\,(\vec{v}' \times \vec{\omega})$$

where $\vec{r}'$ = radius vector in the rotating frame with respect to rotation axis
and $\vec{v}'$ = velocity in the same frame. The total centrifugal force is clearly

$$\vec{F}_{cf} = \sum dm\,\omega^2\,\vec{r}' = m\omega^2\,\vec{R}_c$$

$\vec{R}_c$ is the radius vector of the C.M. of the body with respect to rotation axis, also

$$\vec{F}_{cor} = 2m\,\vec{v}_c' \times \vec{\omega}$$

where we have used the definitions

$$m\,\vec{R}_c = \sum dm\,\vec{r}' \text{ and } m\,\vec{v}_c' = \sum dm\,\vec{v}'$$

Q. 269. A midpoint of a thin uniform rod AB of mass m and length l is rigidly fixed to a rotation axle OO' as shown in Fig. 1.71. The rod is set into rotation with a constant angular velocity ω. Find the resultant moment of the centrifugal forces of inertia relative to the point C in the reference frame fixed to the axle OO' and to the rod.

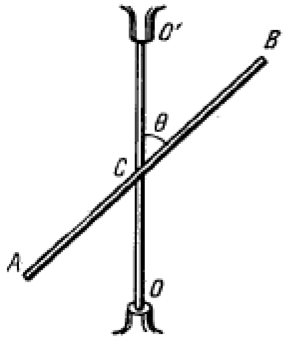
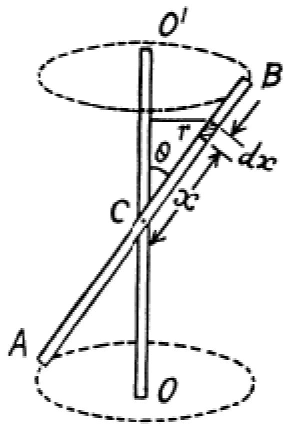


Fig. 1.71.

Solution. 269. Consider a small element of length dx at a distance x from the point C, which is rotating in a circle of radius r = x sin θ



Now, mass of the element $= \left(\frac{m}{l}\right) dx$

So, centrifugal force acting on this element $= \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta$ and moment of this force about C,

$$|dN| = \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta \cdot x\cos\theta$$

$$= \frac{m\,\omega^2}{2l} \sin 2\theta\, x^2\, dx$$

and hence, total moment

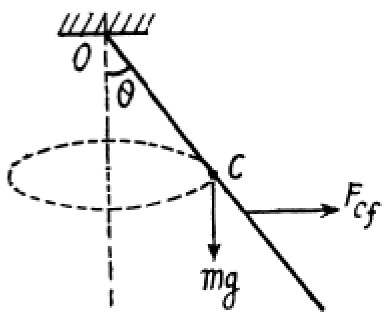
$$N = 2\int_0^{l/2} \frac{m\,\omega^2}{2l} \sin 2\theta\; x^2\, dx = \frac{1}{24} m\,\omega^2 l^2 \sin 2\theta,$$

Q. 270. A conical pendulum, a thin uniform rod of length l and mass m, rotates uniformly about a vertical axis with angular velocity ω (the upper end of the rod is hinged). Find the angle θ between the rod and the vertical.

Solution. 270. Let us consider the system in a frame rotating with the rod. In this frame, the rod is at rest and experiences not only the gravitational force $m\vec{g}$ and the reaction force $\vec{R}$, but also the centrifugal force $\vec{F}_{cf}$.

In the considered frame, from the condition of equilibrium i.e. $N_{0z} = 0$

or, $N_{cf} = mg\frac{l}{2}\sin\theta$ (1)



where Ncf is the moment of centrifugal force about O. To calculate N_{cf}, let us consider an element of length dx, situated at a distance x from the point O. This element is subjected to a horizontal pseudo force $\left(\frac{m}{l}\right) dx\,\omega^2 x \sin\theta.$ The moment of this pseudo force about the axis of rotation through the point O is

$$dN_{cf} = \left(\frac{m}{l}\right) dx\,\omega^2 x \sin\theta\, x\cos\theta$$

$$= \frac{m\omega^2}{l}\sin\theta\cos\theta\, x^2\,dx$$

So $$N_{cf} = \int_0^l \frac{m\omega^2}{l}\sin\theta\cos\theta\, x^2\,dx = \frac{m\omega^2 l^2}{3}\sin\theta\cos\theta \qquad (2)$$

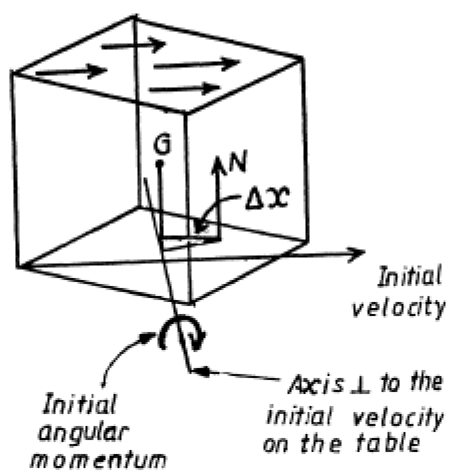
It follows from Eqs. (1) and (2) that,

$$\cos\theta = \left(\frac{3g}{2\omega^2 l}\right) \text{ or } \theta = \cos^{-1}\left(\frac{3g}{2\omega^2 l}\right) \qquad (3)$$

Q. 271. A uniform cube with edge a rests on a horizontal plane whose friction coefficient equals k. The cube is set in motion with an initial velocity, travels some distance over the plane and comes to a stand- still. Explain the disappearance of the angular momentum of the cube relative to the axis lying in the plane at right angles to the cube's motion direction. Find the distance between the resultants of gravitational forces and the reaction forces exerted by the supporting plane.

Solution. 271. When the cube is given an initial velocity on the table in some direction (as shown) it acquires an angular momentum about an axis on the table perpendicular to the initial velocity and (say) just below the C.G.. This angular momentum will disappear when the cube stops and this can only by due to a torque. Frictional forces cannot do this by themselves because they act in the plain containing the axis. But if the force of normal reaction act eccentrically (as shown), their torque can bring about the vanishing of the angular momentum. We can calculate the distance Ax between the point of application of the normal reaction and the C.G. of the cube as follows. Take the moment about C.G. of all the forces. This must vanish because the cube does not turn or tumble on the table. Then if the force of friction is fr

$$fr\ \frac{a}{2} = N\ \Delta x$$



But N = mg and fr = kmg, so

$\Delta x = ka/2$

Q. 272. A smooth uniform rod AB of mass M and length l rotates freely with an angular velocity ω_0, in a horizontal plane about a stationary vertical axis passing through its end A. A small sleeve of mass m starts sliding along the rod from the point A. Find the velocity v' of the sleeve relative to the rod at the moment it reaches its other end B.

Solution. 272. In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence, it follows that

$$\frac{1}{2}\frac{Ml^2}{3}\omega_0^2 = \frac{1}{2}m(\omega^2 l^2 + v'^2) + \frac{1}{2}\frac{Ml^2}{3}\omega^2$$

(ω is the final angular velocity of the rod)

and $\frac{Ml^2}{3}\omega_0 = \frac{Ml^2}{3}\omega + ml^2\omega$

From these equations we obtain

$$\omega = \omega_0 \Big/ \left(1 + \frac{3M}{M}\right) \text{ and}$$

$$v' = \omega_0 l \Big/ \sqrt{1 + 3m/M}$$

Q. 273. A uniform rod of mass m = 5.0 kg and length l = 90 cm rests on a smooth horizontal surface. One of the ends of the rod is struck with the impulse J = 3.0 N•s in a horizontal direction perpendicular to the rod. As a result, the rod obtains the momentum p = 3.0 N·s. Find the force with which one half of the rod will act on the other in the process of motion.

Solution. 273. Due to hitting of the ball, the angular impulse received by the rod about the C.M. is equal to $p\frac{1}{2}$. If ω is the angular velocity acquired by the rod, we have

$$\frac{ml^2}{12}\omega = \frac{pl}{2} \text{ or } \omega = \frac{6p}{ml} \qquad (1)$$

In the frame of C.M., the rod is rotating about an axis passing through its mid point with the angular velocity ω. Hence the force exerted by one half on the other = mass of one half x acceleration of C.M. of that part, in the frame of C.M

$$= \frac{m}{2}\left(\omega^2 \frac{l}{4}\right) = m\frac{\omega^2 l}{8} = \frac{9p^2}{2ml} = 9\text{ N}$$

Q. 274. A thin uniform square plate with side l and mass M can rotate freely about a stationary vertical axis coinciding with one of its sides. A small ball of mass m flying with velocity v at right angles to the plate strikes elastically the centre of it. Find:

(a) the velocity of the ball v' after the impact;
(b) the horizontal component of the resultant force which the axis will exert on the plate after the impact.

Solution. 274. (a) In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence it follows that

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2$$

and $$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{Ml^2}{3}\omega$$

From these equations we obtain

$$v' = \left(\frac{3m-4M}{3m+4M}\right)v. \quad \text{and} \quad \omega = \frac{4v}{l(1+4m/3M)}$$

As $\vec{v}' \uparrow\uparrow \vec{v}$, so in vector form $\vec{v}' = \left(\frac{3m-4M}{3m+4M}\right)\vec{v}$

(b) Obviously the sought force provides the centripetal acceleration to the C.M. of the rod and is

$$F_n = mw_{cn}$$
$$= M\omega^2\frac{l}{2} = \frac{8Mv^2}{l(1+4M/3m)^2}$$

Q. 275. A vertically oriented uniform rod of mass M and length l can rotate about its upper end. A horizontally flying bullet of mass m strikes the lower end of the rod and gets stuck in it; as a result, the rod swings through an angle α. Assuming that m << M, find:

(a) the velocity of the flying bullet;
(b) the momentum increment in the system "bullet-rod" during the impact; what causes the change of that momentum;
(c) at what distance x from the upper end of the rod the bullet must strike for the momentum of the system "bullet-rod" to remain constant during the impact.

Solution. 275. (a) About the axis of rotation of the rod, the angular momentum of the system is conserved. Thus if the velocity of the flying bullet is v.

$$mvl = \left(ml^2 + \frac{Ml^2}{3}\right)\omega$$

$$\omega = \frac{mv}{\left(m + \frac{M}{3}\right)l} \approx \frac{3mv}{Ml} \text{ as } m << M \qquad (1)$$

Now from the conservation of mechanical energy of-the system (rod with bullet) in the uniform field of gravity

$$\frac{1}{2}\left(ml^2 + \frac{Ml^2}{3}\right)\omega^2 = (M+m)g\frac{l}{2}(1-\cos\alpha) \qquad (2)$$

[because C.M. of rod raises by the height $\frac{l}{2}(1-\cos\alpha)$]

Solving (1) and (2), we get

$$v = \left(\frac{M}{m}\right)\sqrt{\frac{2}{3}gl}\,\sin\frac{\alpha}{2} \text{ and } \omega = \sqrt{\frac{6g}{l}}\,\sin\frac{\alpha}{2}$$

(b) Sought $\Delta p = \left[m(\omega l) + M\left(\omega\frac{l}{2}\right)\right] - mv$

where , ωl is the velqcdty of the bullet and $\omega\frac{l}{2}$ equals the velocity of C.M. of the rod after the impact. Putting the value of v and ω we get

$$\Delta p \approx \frac{1}{2}mv = M\sqrt{\frac{gl}{6}}\,\sin\frac{\alpha}{2}$$

This is caused by the reaction at the hinge on the upper end.

$$mvx = \left(\frac{Ml^2}{3} + mx^2\right)\omega' \text{ or } \omega' \approx \frac{3mvx}{Ml^2}$$

Final momentum is

$$p_f = mx\omega' + \int_0^l y\omega'\frac{M}{l}dy \approx \frac{M}{2}\omega' l = \frac{3}{2}mv\frac{x}{l}$$

So, $\Delta p = p_f - p_i \approx mv\left(\frac{3x}{2l} - 1\right)$

This vanishes for $x \approx \frac{2}{3}l$

Q. 276. A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass m. A light thread running down through the hollow axle of the disc is tied to the body. Initially the body was located at the edge of the disc and the whole system rotated with an angular velocity ω_0. Then by means of a force F applied to the lower end of the thread the body was slowly pulled to the rotation axis. Find:

(a) the angular velocity of the system in its final state;
(b) the work performed by the force F.

Solution. 276. (a) As force F on the body is radial so its angular momentum about the axis becomes zero and the angular momentum of the system about the given axis is conserved. Thus

$$\frac{MR^2}{2}\omega_0 + m\omega_0 R^2 = \frac{MR^2}{2}\omega \text{ or } \omega = \omega_0\left(1+\frac{2m}{M}\right)$$

(b) From the equation of the increment of the mechanical energy of the system :

$$\Delta T = A_{ext}$$

$$\frac{1}{2}\frac{MR^2}{2}\omega^2 - \frac{1}{2}\left(\frac{MR^2}{2}+mR^2\right)\omega_0^2 = A_{ext}$$

Putting the value of ω from part (a) and solving we get

$$A_{ext} = \frac{m\omega_0^2R^2}{2}\left(1+\frac{2m}{M}\right)$$

Q. 277. A man of mass ml stands on the edge of a horizontal uniform disc of mass m_2 and radius R which is capable of rotating freely about a stationary vertical axis passing through its centre. At a cer- tain moment the man starts moving along the edge of the disc; he shifts over an angle φ' relative to the disc and then stops. In the process of motion the velocity of the man varies with time as v' (t). Assuming the dimensions of the man to be negligible, find:

(a) the angle through which the disc had turned by the moment the man stopped;
(b) the force moment (relative to the rotation axis) with which the man acted on the disc in the process of motion.

Solution. 277. (a) Let z be the rotation axis of disc and <p be its rotation angle in accordance with right-hand screw rule (Fig.). (φ and φ' are to be measured in the same

sense algebraically.) As M_z of the system (disc + man) is conserved and $M_{z(\text{initial})} = 0$, we have at any instant,

$$0 = \frac{m_2 R^2}{2}\frac{d\varphi}{dt} + m_1\left[\left(\frac{d\varphi'}{dt}\right)R + \left(\frac{d\varphi}{dt}\right)R\right]R$$

or, $$d\varphi = \left[-\frac{m_1}{m_1 + (m_2/2)}\right]d\varphi'$$

On integrating $$\int_0^{\varphi} d\varphi = -\int_0^{\varphi'}\left(\frac{m_1}{m_1 + (m_2/2)}\right)d\varphi'$$

or, $$\varphi = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\varphi' \quad (1)$$

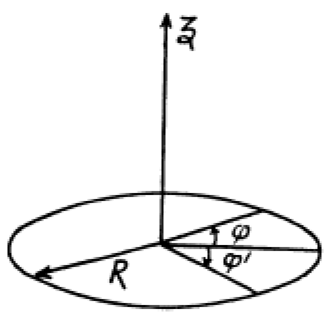
This gives the total angle of rotation of the disc.

(b) From Eq. (1)

$$\frac{d\varphi}{dt} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{d\varphi'}{dt} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{v'(t)}{R}$$

Differentiating with respect to time

$$\frac{d^2\varphi}{dt^2} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$



Thus the sought force moment from the Eq. $N_z = I\beta_z$

$$N_z = \frac{m_2 R^2}{2}\frac{d^2\varphi}{dt^2} = -\frac{m_2 R^2}{2}\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$

Hence $$N_z = -\frac{m_1 m_2 R}{2m_1 + m_2}\frac{dv'(t)}{dt}$$

Q. 278. Two horizontal discs rotate freely about a vertical axis passing through their centres. The moments of inertia of the discs relative to this axis are equal to l_1 and l_2, and the angular velocities to oh and ω_1. When the upper disc fell on the lower one, both discs began rotating, after some time, as a single whole (due to friction). Find:

(a) the steady-state angular rotation velocity of the discs;
(b) the work performed by the friction forces in this process.

Solution. 278. (a) Frome the law of conservation of angular momentum of the system relative to vertical axis z, it follows that:

$$I_1 \omega_{1z} + I_2 \omega_{2z} = (I_1 + I_2)\, \omega_z$$

Hence $\omega_z = (I_1 \omega_{1z} + I_2 \omega_{2z}) / (I_1 + I_2)$

Not that for $\omega_z > 0$, the corresponding vector $\vec{\omega}$ coincides with the poitive direction to the z axis, and vice versa. As both discs rotates about the same vertical axis z, thus in vector form.

$$\vec{\omega} = I_1 \vec{\omega}_1 + I_2 \vec{\omega}_2 / (I_1 + I_2)$$

However, the problem makes sense only if $\vec{\omega}_1 \uparrow\uparrow \vec{\omega}_2$ or $\vec{\omega}_1 \uparrow\downarrow \vec{\omega}_2$

(b) From the equation of increment of mechanical energy of a system: $A_{fr} = \Delta T.$

$$= \frac{1}{2}(I_1 + I_2)\, \omega_z^2 - \frac{1}{2} I_1 \omega_{1z}^2 + \frac{1}{2} I_2 \omega_{2z}^2$$

Using Eq. (1)

$$A_{fr} = -\frac{I_1 I_2}{2(I_1 + I_2)} (\omega_{1z} - \omega_{2z})^2$$

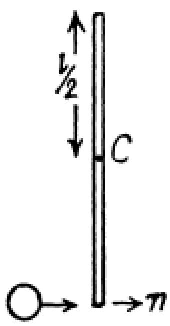
Q. 279. A small disc and a thin uniform rod of length l, whose mass is η times greater than the mass of the disc, lie on a smooth horizontal plane. The disc is set in motion, in horizontal direction and perpendicular to the rod, with velocity v, after which it elastically collides with the end of the rod. Find the velocity of the disc and the angular velocity of the rod after the collision. At what value of η will the velocity of the disc after the collision be equal to zero? reverse its direction?

Solution. 279. For the closed system (disc + rod), the angular momentum is conserved about any axis. Thus from the conservation of angular momentum of the system about the rotation axis of rod passing through its C.M. gives :

$$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{\eta ml^2}{12}\omega \qquad (1)$$

(v' is the final velocity of the disc and co angular velocity of the rod) For the closed system linear momentum is also conserved. Hence

$$v_c = \frac{l\omega}{3} \quad \text{and} \quad v - v' = \eta v_c$$



Applying conservation of kinetic energy, as the collision is elastic

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}\eta m v_c^2 + \frac{1}{2}\frac{\eta ml^2}{12}\omega^2 \qquad (3)$$

or $\qquad v^2 - v'^2 = 4\eta v_c^2$ and hence $v + v' = 4v_c$

Then

$$v' = \frac{4-\eta}{4+\eta}v \quad \text{and} \quad \omega = \frac{12v}{(4+\eta)l}$$

Vectorially, noting that we have taken $\vec{v}'$ parallel to $\vec{v}$

$$\vec{u}' = \left(\frac{4-\eta}{4+\eta}\right)\vec{v}$$

So, $\vec{u}' = 0$ for $\eta = 4$ and $\vec{u}' \downarrow\uparrow \vec{v}$ for $\eta > 4$

Q. 280. A stationary platform P which can rotate freely about a vertical axis (Fig. 1.72) supports a motor M and a balance weight N. The mo- ment of inertia of the platform with the motor and the balance weight relative to this axis is equal to I. A light frame is fixed to the motor's shaft with a uniform sphere A rotating freely with an angular velocity ω_0 about a shaft BB' coinciding with the axis OO'. The moment of inertia of the sphere relative to the rotation axis is equal to I. Find:

(a) the work performed by the motor in turning the shaft BB' through 90'; through 180°;
(b) the moment of external forces which maintains the axis of the arrangement in the vertical position after the motor turns the shaft BB' through 90°.

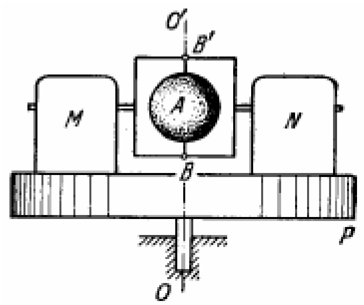


Fig. 1.72.

Solution. 280. See the diagram in the book (Fig. 1.72) (a) When the shaft BB' is turned through 90° the platform must start turning with angular velocity Ω so that the angular momentum remains constant. Here

$$(I + I_0)\Omega = I_0 \omega_0 \quad \text{or,} \quad \Omega = \frac{I_0 \omega_0}{I + I_0}$$

The work performed by the motor is therefore

$$\frac{1}{2}(I + I_0)\Omega^2 = \frac{1}{2}\frac{I_0^2 \omega_0^2}{I + I_0}$$

If the shaft is turned through 180°, angular velocity of the sphere changes sign. Thus from conservation of angular momentum,

$$I\Omega - I_0 \omega_0 = I_0 \omega_0$$

(**Here –** $I_0\omega_0$ is the complete angular momentum of the sphere i. e. we assume that the angular velocity of the sphere is just $-\omega_0$). Then

$$\Omega = 2I_0 \frac{\omega_0}{I}$$

and the work done must be,

$$\frac{1}{2} I \Omega^2 + \frac{1}{2} I_0 \omega_0^2 - \frac{1}{2} I_0 \omega_0^2 = \frac{2I_0^2 \omega_0^2}{I}$$

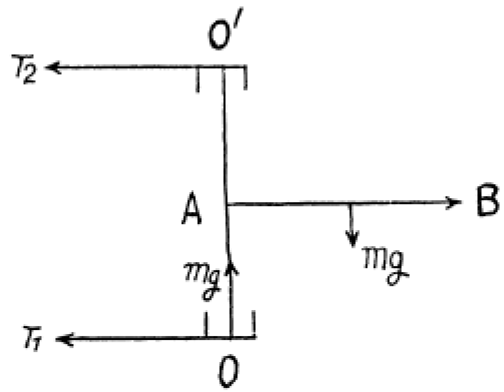
(b) In the case (a), first part, the angular momentum vector of the sphere is precessing with angular velocity Ω. Thus a torque,

$I_0 \omega_0 \Omega = \dfrac{I_0^2 \omega_0^2}{I + I_0}$ is needed.

Dynamics Of A Solid Body (Part - 4)

Q. 281. A horizontally oriented uniform rod AB of mass m = 1.40 kg and length l_0 = 100 cm rotates freely about a stationary vertical axis OO' passing through its end A. The point A is located at the middle of the axis OO' whose length is equal to l = 55 cm. At what angular velocity of the rod the horizontal component of the force acting on the lower end of the axis OO' is equal to zero? What is in this case the horizontal component of the force acting on the upper end of the axis?

Solution. 281. The total centrifugal force can be calculated by,



$$\int_0^{l_0} \frac{m}{l_0}\omega^2 x\,dx = \frac{1}{2} m\, l_0\, \omega^2$$

Then for equilibrium

$$(T_2 - T_1)\frac{l}{2} = mg\frac{l_0}{2}$$

and, $T_2 + T_1 = \frac{1}{2} m\, l_0\, \omega^2$

Thus T_1 vanishes, when

$$\omega^2 = \frac{2g}{l},\ \omega = \sqrt{\frac{2g}{l}} = 6 \text{ rad/s}$$

Then $T_2 = mg\frac{l_0}{l} = 25$ N

Q. 282. The middle of a uniform rod of mass m and length l is rigidly fixed to a vertical axis OO' so that the angle between the rod and the axis is equal to θ (see Fig. 1.71). The ends of the axis OO' are provided with bearings. The system rotates without friction with an angular velocity ω. Find:

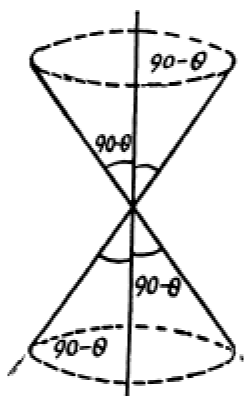
(a) the magnitude and direction of the rod's angular momentum M relative to the point C, as well as its angular momentum relative to the rotation axis;
(b) how much the modulus of the vector M relative to the point C increases during a half-turn;
(c) the moment of external forces N acting on the axle OO' in the process of rotation

Solution. 282. See the diagram in the book (Fig. 1.71).

(a) The angular velocity $\vec{\omega}$ about OO' can be resolved into a component parallel to the rod and a component co sin0 perpendicular to the rod through C. The component parallel to the rod does not contribute so the angular momentum

$$M = I\,\omega \sin\theta = \frac{1}{12} m\, l^2\, \omega \sin\theta$$

Also, $$M_z = M \sin\theta = \frac{1}{12} m\, l^2\, \omega \sin^2\theta$$



This can be obtained directly also,

(b) The modulus of $\vec{M}$ does not change but the modulus of the change of $\vec{M}$ is $|\Delta \vec{M}|$.

$$|\Delta \vec{M}| = 2M \sin(90-\theta) = \frac{1}{12} m l^2 \omega \sin 2\theta$$

(c) Here $M_\perp = M\cos\theta = I\omega\sin\theta\cos\theta$

$$\text{Now } \left|\frac{d\vec{M}}{dt}\right| = I\omega\sin\theta\cos\theta \frac{\omega\, dt}{dt} = \frac{1}{24} ml^2\omega^2\sin^2\theta$$

as $\vec{M}$ precesses with angular velocity ω.

Q. 283. A top of mass m = 0.50 kg, whose axis is tilted by an angle θ = 30° to the vertical, precesses due to gravity. The moment of inertia of the top relative to its symmetry axis is equal to I = 2.0 g•m^2 , the angular velocity of rotation about that axis is equal to ω = 350 rad/s, the distance from the point of rest to the centre of inertia of the top is l = 10 cm. Find:

(a) the angular velocity of the top's precession;
(b) the magnitude and direction of the horizontal component of the reaction force acting on the top at the point of rest

Solution. 283. Here M = I ω is along the symmetry axis. It has tw o components, the part l ω cosθ is constant and the part $M_\perp$ - l ω sinθ presesses, then

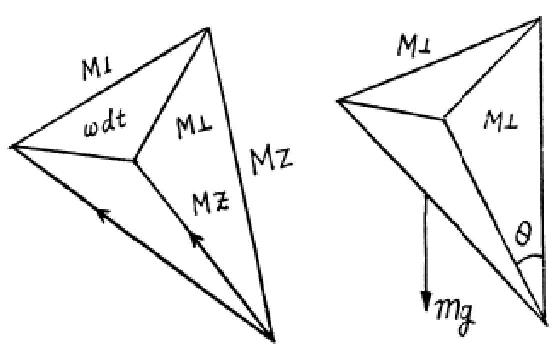
$$\left|\frac{d\vec{M}}{dt}\right| = I\omega\sin\theta\,\omega' = mgl\sin\theta$$

or, $$\omega' = \text{precession frequency} = \frac{mgl}{I\omega} = 0{\cdot}7 \text{ rad/s}$$

(b) This force is the centripetal force due to precession. It acls inward and has the magnitude

$$|\vec{F}| = \left|\sum m_i \omega'^2 \vec{\rho_i}\right| = m\omega'^2 l\sin\theta = 12 \text{ mN}.$$

$\vec{\rho_i}$ is the distance of the i th element from the axis. This is the force that the table will exert on the top. See the diagram in the answer sheet



Q. 284. A gyroscope, a uniform disc of radius R = 5.0 cm at the end of a rod of length l = 10 cm (Fig. 1.73), is mounted on the floor of an elevator car going up with a constant acceleration w = 2.0 m/s^2. The other end of the rod is hinged at the point O. The gyroscope precesses with an angular velocity n = 0.5 rps. Neglecting the friction and the mass of the rod, find the proper angular velocity of the disc.

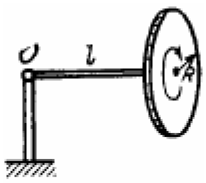


Fig. 1.73.

Solution. 284. See the diagram in the book (Fig. 1.73).

The moment of inertia of the disc about its symmentry axis is $\frac{1}{2} m R^2$. If the angular velocity of the disc is co then the angular momentum is $\frac{1}{2} m R^2 \omega$. The precession frequency being 2 π n,

we have $\left|\frac{d\vec{M}}{dt}\right| = \frac{1}{2} mR^2 \omega \times 2\pi n$

This must equal m (g + w) l, the effective gravitational torques (g being replaced by g + w in the elevator). Thus,

$$\omega = \frac{(g+w)l}{\pi R^2 n} = 300 \text{ rad/s}$$

Q. 285. A top of mass m = 1.0 kg and moment of inertia relative to its own axis I = 4.0 g•m^{2}spins with an angular velocity ω = 310 rad/s. Its point of rest is located on a block which is shifted in a horizontal direction with a constant acceleration w = 1.0 m/s^2. The distance between the point of rest and the centre of inertia of the top equals l = 10 cm. Find the magnitude and direction of the angular velocity of precession ω'.

Solution. 285.

The effective g is $\sqrt{g^2+w^2}$ inclined at angle $\tan^{-1}\frac{w}{g}$ with the vertical. Then with reference to the new " vertical" we proceed as in problem 1.283. Thu

$$\omega' = \frac{ml\sqrt{g^2+w^2}}{I\omega} = 0{\cdot}8 \text{ rad/s.}$$

The vector $\vec{\omega'}$ forms an angle $\theta = \tan^{-1}\frac{w}{g} = 6°$ with the normal vertical.

Q. 286. A uniform sphere of mass m = 5.0 kg and radius R = 6.0 cm rotates with an angular velocity ω = 1250 rad/s about a horizontal axle passing through its centre and fixed on the mounting base by means of bearings. The distance between the bearings equals l = 15 cm. The base is set in rotation about a vertical axis with an angular velocity ω' = 5.0 rad/s. Find the modulus and direction of the gyroscopic forces.

Solution. 286. The moment of inertia of the sphere is $\frac{2}{5}mR^2$ and hence the value of angular momentum is $\frac{2}{5}mR^2\omega.$ Since it precesses at speed ω' the torque required is

$$\frac{2}{5}mR^2\omega\,\omega' = F'\,l$$

So, $F' = \frac{2}{5}mR^2\omega\,\omega'/l = 300\text{ N}$

(The force F ' must be vertical.)

Q. 287. A cylindrical disc of a gyroscope of mass m = 15 kg and radius r = 5.0 cm spins with an angular velocity ω = 330 rad/s. The distance between the bearings in which the axle of the disc is mounted is equal to l = 15 cm. The axle is forced to oscillate about a horizontal axis with a period T = 1.0 s and amplitude φ_m, = 20°. Find the maximum value of the gyroscopic forces exerted by the axle on the bearings.

Solution. 287. The moment of inertia is $\frac{1}{2}mr^2$ and angular momentum is $\frac{1}{2}mr^2\omega.$ The axle oscillates about a horizontal axis making an instantaneous angle.

$$\varphi = \varphi_m \sin\frac{2\pi t}{T}$$

This means that there is a variable precession with a rate of precession $\frac{d\varphi}{dt}$. The maximum value of this is $\frac{2\pi\varphi_m}{T}$. When the angle between the axle and the axis is at its maximumis value, a torque ***I* ω Ω**

$$= \frac{1}{2}mr^2\omega\ \frac{2\pi\varphi_m}{T} = \frac{\pi\, mr^2\,\omega\varphi_m}{T} \text{ acts on it.}$$

The corresponding gyroscopic force will be $\frac{\pi\, mr^2\,\omega\varphi_m}{lT} = 90$ N

Q. 288. A ship moves with velocity v = 36 km per hour along an arc of a circle of radius R = 200 m. Find the moment of the gyroscopic forces exerted on the bearings by the shaft with a flywheel whose moment of inertia relative to the rotation axis equals I = 3.8.10^3 kg•m^2 and whose rotation velocity n = 300 rpm. The rotation axis is oriented along the length of the ship.

Solution. 288. The revolutions per minute of the flywheel being n, the angular momentum of the flywheel

is $I \times 2\pi n$. The rate of precession is $\frac{v}{R}$

Thus $N = 2\pi I N V/R = 5{\cdot}97$ kN. m.

Q. 289. A locomotive is propelled by a turbine whose axle is parallel to the axes of wheels. The turbine's rotation direction coincides with that of wheels. The moment of inertia of the turbine rotor relative to its own axis is equal to I = 240 kg• m^2 . Find the additional force exerted by the gyroscopic forces on the rails when the locomotive moves along a circle of radius R = 250 m with velocity v = 50 km per hour. The gauge is equal to l = 1.5 m. The angular velocity of the turbine equals n = 1500 rpm.

Solution. 289. As in the previous problem a couple 2πInv/R must come in play. This can be done if a force, $\frac{2\pi Inv}{Rl}$ acts on the rails in opposite directions in addition to the centrifugal and other forces. The force on the outer rail is increased and that on the inner rail decreased. The additional force in this case has the magnitude 1.4 kN.m.