Patterns

What is the pattern of numbers?

PATTERNS

Patterns arise due to special arrangements of numbers associated with geometrical figures. The numbers are of different kinds like

1. Natural numbers
2. Polygonal numbers
   (a) Triangular numbers
   (b) Square numbers
   (c) Pentagonal numbers
   (d) Hexagonal numbers, etc.

Examples

1. Patterns in natural numbers

(a)

$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$
$$1234 \times 9 + 5 = 11111$$
$$12345 \times 9 + 6 = 111111$$
$$123456 \times 9 + 7 = 1111111$$
$$1234567 \times 9 + 8 = 11111111$$
$$12345678 \times 9 + 9 = 111111111$$

(b)

$$9 \times 9 + 7 = 88$$
$$98 \times 9 + 6 = 888$$
$$987 \times 9 + 5 = 8888$$
$$9876 \times 9 + 4 = 88888$$
$$98765 \times 9 + 3 = 888888$$
$$987654 \times 9 + 2 = 8888888$$
$$9876543 \times 9 + 1 = 88888888$$
$$98765432 \times 9 + 0 = 888888888$$

(c)

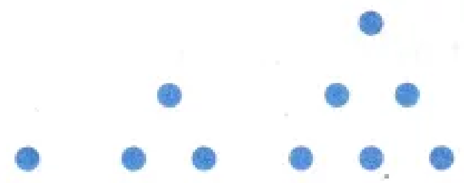
$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$
$$123456 \times 8 + 6 = 987654$$
$$1234567 \times 8 + 7 = 9876543$$
$$12345678 \times 8 + 8 = 98765432$$
$$123456789 \times 8 + 9 = 987654321$$

(d)

$$(11)^2 = 121$$
$$(111)^2 = 12321$$
$$(1111)^2 = 1234321$$
$$(11111)^2 = 123454321$$

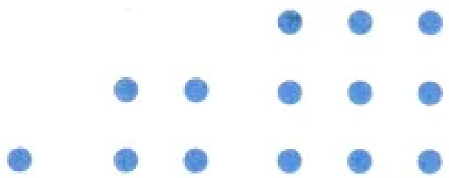
2. Patterns in polygonal numbers

(a) Triangular numbers:



Here the numbers of dots up to any stage are given by the pattern
1, 3, 6, 10, 15, ……
These numbers arise from triangles and so these are called triangular numbers.

(b) Square numbers:

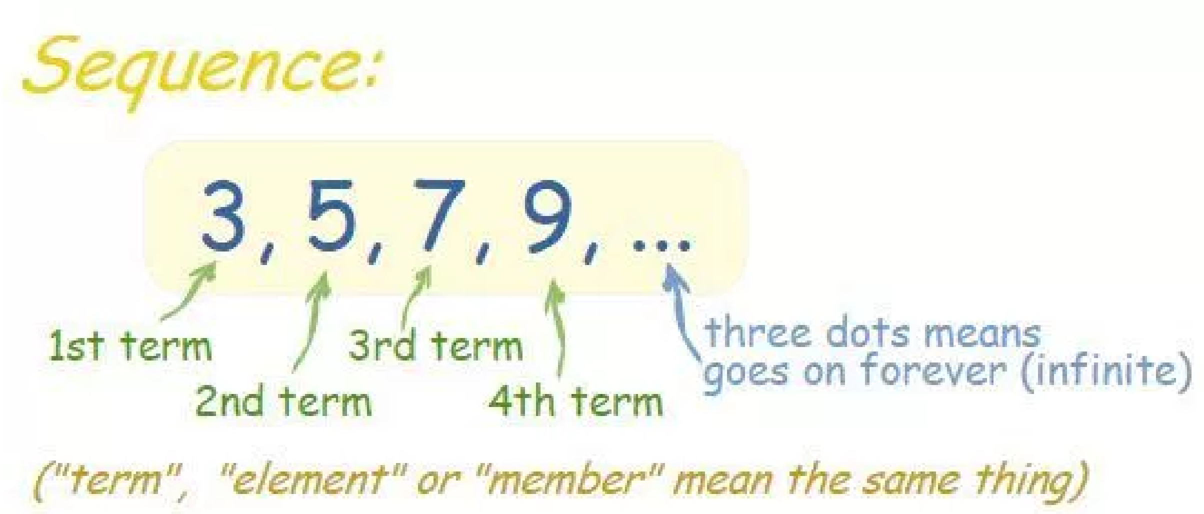


Here the numbers of dots upto any stage are given by the pattern
1, 4, 9, 16, 25, 36, ……
These numbers arise from squares and are called square numbers.

Sequences

In the previous lesson, we learned about pattern of numbers. In this lesson we discuss about Sequences.

A sequence is an ordered list of numbers.



The sum of the terms of a sequence is called a **series**.

- Each number of a sequence is called a term (or element) of the sequence.
- A finite sequence contains a finite number of terms (you can count them). 1, 4, 7, 10, 13
- An infinite sequence contains an infinite number of terms (you cannot count them). 1, 4, 7, 10, 13, . . .
- The terms of a sequence are referred to in the subscripted form shown below, where the natural number subscript refers to the location (position) of the term in the sequence. $\underset{a_1}{1},\ \underset{a_2}{4},\ \underset{a_3}{7},\ \underset{a_4}{10},\ \underset{a_5}{13},\ \underset{a_6}{16},\ \ldots$

(If you study computer programming languages such as C, C++, and Java,
you will find that the first position in their arrays (sequences) start with a subscript of zero.)

The general form of a sequence is represented:

- The domain of a sequence consists of the counting numbers 1, 2, 3, 4, … and the range consists of the terms of the sequence.
- The terms in a sequence may, or may not, have a pattern, or a related formula.
- For some sequences, the terms are simply random.

Let’s examine some sequences that have patterns:

Sequences often possess a definite pattern that is used to arrive at the sequence’s terms.

It is often possible to express such patterns as a formula. In the sequence shown at the left, an explicit formula may be:

$$a_n = 12n$$

where n represents the term's position in the sequence.

$$\overset{a_1}{\underset{12(1)}{12}},\ \overset{a_2}{\underset{12(2)}{24}},\ \overset{a_3}{\underset{12(3)}{36}},\ \ldots$$

Examples:

1. Write the first three terms of the sequence whose n^{th} term is given by the explicit formula:

$$a_n = 2n - 1$$

ANSWER: Remember that n is a natural number (starting with $n = 1$).

$$a_1 = 2(1) - 1 = 1$$
$$a_2 = 2(2) - 1 = 3$$
$$a_3 = 2(3) - 1 = 5$$

Notice that n is replaced with the number of the term you are trying to find.

2. Find the 5^{th} and 10^{th} terms of the sequence whose n^{th} term is given by: $a_n = \dfrac{n}{n+1}$

ANSWER: Remember that n corresponds to the location of the term. Use $n = 5$ and $n = 10$.

$$a_5 = \frac{5}{5+1} = \frac{5}{6}$$
$$a_{10} = \frac{10}{10+1} = \frac{10}{11}$$

3. Write an explicit formula for the n^{th} term of a sequence of negative even integers starting with -2.

ANSWER: Get a visual of the terms. -2, -4, -6, -8, ...
Compare the terms to the numbers associated with their locations and look for a pattern.

Notation	Location	Term
a_1	1	-2
a_2	2	-4
a_3	3	-6
a_4	4	-8

Look for a pattern. In this example, each term can be found by multiplying the location number by -2.
A formula could be:

$$a_n = -2n$$

4. Find the first 4 terms of the sequence $a_n = (-1)^n (n^2 + 3)$

$$a_1 = (-1)^1 (1^2 + 3) = -4$$
$$a_2 = (-1)^2 (2^2 + 3) = +7$$
$$a_3 = (-1)^3 (3^2 + 3) = -12$$
$$a_4 = (-1)^4 (4^2 + 3) = +19$$

Notice how the terms are alternating signs between negative and positive.

Keep this pattern in mind (involving powers of -1) when asked to write formulas for sequences.

$(-1)^n (n^2 + 3)$ yields -4, 7, -12, 19, ...

$(-1)^{n+1} (n^2 + 3)$ yields 4, -7, 12, -19, ...

Arithmetic Sequences and Series

A sequence is an ordered list of numbers.
The sum of the terms of a sequence is called a **series**.

Arithmetic Sequences and Series

Arithmetic Sequence

An arithmetic sequence is of the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad . \quad . \quad .$$

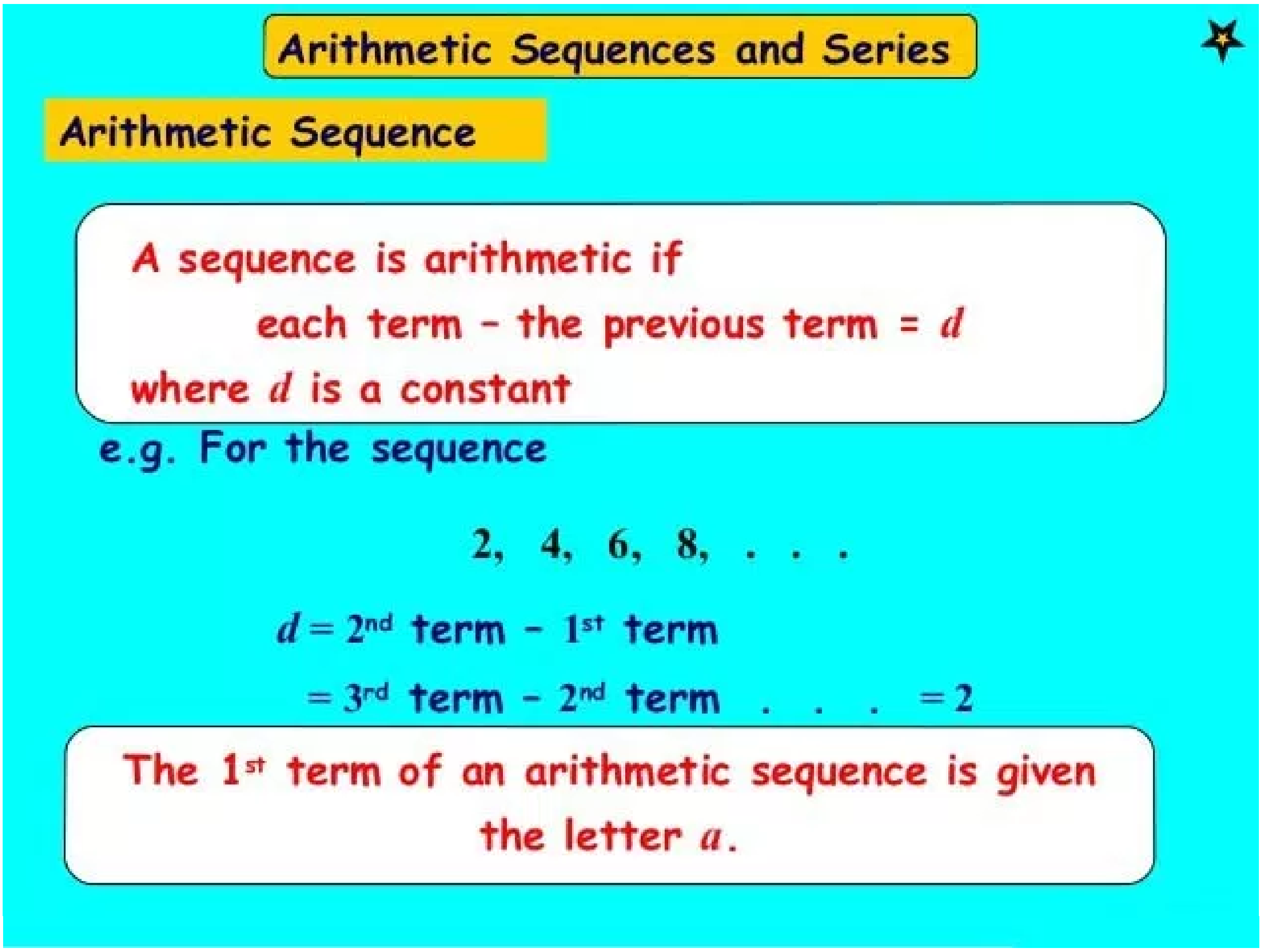
Notice that the 4^{th} term has $3d$ added so, for example, the 20^{th} term will be

$$a + 19d$$

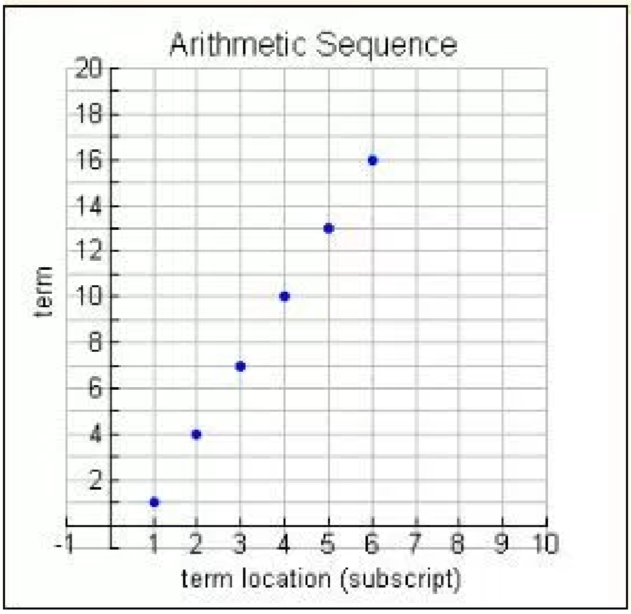
The n^{th} term of an Arithmetic Sequence is

$$u_n = a + (n-1)d$$

An arithmetic sequence is sometimes called an Arithmetic Progression (A.P.)



While some sequences are simply random values, other sequences have a definite pattern that is used to arrive at the sequence's terms.
Two such sequences are the arithmetic and geometric sequences. Let's investigate the arithmetic sequence.



If a sequence of values follows a pattern of adding a fixed amount from one term to the next, it is referred to as an arithmetic sequence. The number added to each term is constant (always the same).

The fixed amount is called the common difference, d, referring to the fact that the difference between two successive terms yields the constant value that was added. To find the common difference, subtract the first term from the second term.
Notice the linear nature of the scatter plot of the terms of an arithmetic sequence. The domain consists of the counting numbers 1, 2, 3, 4, … and the range consists of the terms of the sequence. While the x value increases by a constant value of one, the y value increases by a constant value of 3 (for this graph).

- An arithmetic sequence is of the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad . \; . \; .$$

- The n^{th} term is $u_n = a+(n-1)d$

- The sum of n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}(a+l) \quad \text{or} \quad S_n = \frac{n}{2}(2a+(n-1)d)$$

Examples:

Arithmetic Sequence	Common Difference, d	
1, 4, 7, 10, 13, 16, ...	$d = 3$	add 3 to each term to arrive at the next term, or...the difference $a_2 - a_1$ is 3.
15, 10, 5, 0, -5, -10, ...	$d = -5$	add -5 to each term to arrive at the next term, or...the difference $a_2 - a_1$ is -5.
$1, \frac{1}{2}, 0, -\frac{1}{2}, \ldots$	$d = -\frac{1}{2}$	add -1/2 to each term to arrive at the next term, or....the difference $a_2 - a_1$ is -1/2.

Formulas used with arithmetic sequences and arithmetic series:

To **find any term** of an **arithmetic sequence:**

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term of the sequence, d is the common difference, n is the number of the term to find.

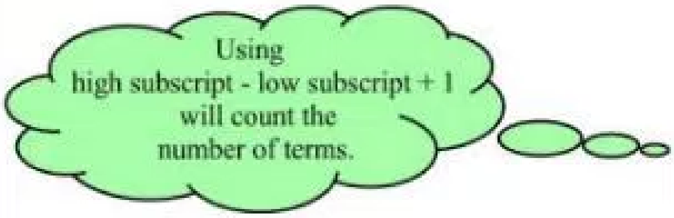
Note: a_1 is often simply referred to as a

To find the **sum of a certain number of terms** of an **arithmetic sequence:**

$$S_n = \frac{n(a_1 + a_n)}{2}$$

where S_n is the sum of n terms (n^{th} **partial sum**), a_1 is the first term, a_n is the n^{th} term.

Examples:

Question	Answer
1. Find the common difference for this arithmetic sequence 5, 9, 13, 17 ...	**1.** The common difference, d, can be found by subtracting the first term from the second term, which in this problem yields **4**. Checking shows that 4 is the difference between all of the entries.
2. Find the common difference for the arithmetic sequence whose formula is $a_n = 6n + 3$	**2.** The formula indicates that 6 is the value being added (with increasing multiples) as the terms increase. A listing of the terms will also show what is happening in the sequence (start with $n = 1$). 9, 15, 21, 27, 33, ... The list shows the common difference to be 6.
3. Find the 10^{th} term of the sequence 3, 5, 7, 9, ...	**3.** $n = 10$; $a_1 = 3$, $d = 2$ $a_n = a_1 + (n-1)d$ $a_{10} = 3 + (10-1)2$ $a_{10} = 21$ The tenth term is 21.
4. Find a_7 for an arithmetic sequence where $a_1 = 3x$ and $d = -x$.	**4.** n = 7; $a_1 = 3x$, $d = -x$ $a_n = a_1 + (n-1)d$ $a_7 = 3x + (7-1)(-x)$ $a_7 = 3x + 6(-x) = -3x$
5. Find t_{15} for an arithmetic sequence where $t_3 = -4 + 5i$ and $t_6 = -13 + 11i$  	**5.** Notice the change of labeling from a to t. The letter used in labeling is of no importance. Get a $\underbrace{\quad}_{t_1}, \underbrace{\quad}_{t_2}, \boxed{\underbrace{-4+5i}_{t_3}, \underbrace{\quad}_{t_4}, \underbrace{\quad}_{t_5}, \underbrace{-13+11i}_{t_6}}$ Using the third term as the "first" term, find the common difference from these known terms. $a_n = a_1 + (n-1)d$ $t_6 = t_3 + (4-1)d$ $-13 + 11i = -4 + 5i + (4-1)d$ $-13 + 11i = -4 + 5i + 3d$ $-9 + 6i = 3d$ $-3 + 2i = d$ Now, from t_3 to t_{15} is 13 terms. $t_{15} = -4 + 5i + (13-1)(-3 + 2i) = -4 + 5i - 36 + 24i$ $= -40 + 29i$
6. Find a formula for the sequence 1, 3, 5, 7, ...	**6.** A formula will relate the subscript number of each term to the actual value of the term. $a_n = 2n - 1$ Substituting $n = 1$, gives 1. Substituting $n = 2$, gives 3, and so on.
7. Find the 25^{th} term of the sequence -7, -4, -1, 2, ...	**7.** $n = 25$; $a_1 = -7$, $d = 3$ $a_n = a_1 + (n-1)d$ $a_{25} = -7 + (25-1)3$ $a_{25} = 65$

8. Find the sum of the first 12 positive even integers.

Notice how BOTH formulas work together.

8. The word "sum" indicates the need for the sum formula.

positive even integers: 2, 4, 6, 8, ...

$n = 12$; $a_1 = 2$, $d = 2$

We are missing a_{12}, for the sum formula, so we use the "any term" formula to find it.

$$a_n = a_1 + (n-1)d$$

$$a_{12} = 2 + (12-1)2$$

$$a_{12} = 24$$

Now, let's find the sum:

$$S_{12} = \frac{12(2+24)}{2} = 156$$

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9. Insert 3 arithmetic means between 7 and 23.

Note: An **arithmetic mean** is the term between any two terms of an arithmetic sequence. It is simply the average (mean) of the given terms.

9. While there are several solution methods, we will use our arithmetic sequence formulas.

Draw a picture to better understand the situation.

7, ____, ____, ____, 23

This set of terms will be an arithmetic sequence.

We know the first term, a_1, the last term, a_n, but not the common difference, d. *This question makes NO mention of "sum", so avoid that formula.*

Find the common difference:

$$a_n = a_1 + (n-1)d$$

$$23 = 7 + (5-1)d$$

$$23 = 7 + 4d$$

$$16 = 4d$$

$$4 = d$$

Now, insert the terms using d.

7, 11, 15, 19, 23

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10. Find the number of terms in the sequence 7, 10, 13, ..., 55.

n must be an integer!

10. $a_1 = 7$, $a_n = 55$, $d = 3$. We need to find n.

This question makes NO mention of "sum", so avoid that formula.

$$a_n = a_1 + (n-1)d$$

$$55 = 7 + (n-1)3$$

$$55 = 7 + 3n - 3$$

$$55 = 4 + 3n$$

$$51 = 3n$$

$$17 = n$$

When solving for n, be sure your answer is a positive integer. There is no such thing as a fractional number of terms in a sequence!

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11. A theater has 60 seats in the first row, 68 seats in the second row, 76 seats in the third row, and so on in the same increasing pattern. If the theater has 20 rows of seats, how many seats are in the theater?

11. The seating pattern is forming an arithmetic sequence.

60, 68, 76, ...

We wish to find "the sum" of all of the seats.

$n = 20$, $a_1 = 60$, $d = 8$ and we need a_{20} for the sum.

$$a_n = a_1 + (n-1)d$$

$$a_{20} = 60 + (20-1)8 = 212$$

Now, use the sum formula:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S_{20} = \frac{20(60+212)}{2} = 2720$$