Exercise 2.6

Chapter 2 Derivatives Exercise 2.6 1E

Consider the following equation:

$9x^2 - y^2 = 1$

(a)

Find y' by implicit differentiation.

Differentiate both sides of the given equation with respect to x.

$$\frac{d}{dx}\left(9x^2 - y^2\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}\left(9x^2\right) - \frac{d}{dx}\left(y^2\right) = 0 \qquad \text{use difference rule and } \frac{d}{dx}(k) = 0.$$

$$9\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(y^2\right) = 0 \qquad \frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}\left(f(x)\right).$$

Let y be a function of x and using the Chain Rule.

$$9\frac{d}{dx}\left(x^2\right) - \frac{d}{dy}\left(y^2\right)\cdot\frac{dy}{dx} = 0$$

$$9(2x) - (2y)\frac{dy}{dx} = 0 \qquad \text{use power rule: } \frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

$$18x - 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = 18x$$

$$\frac{dy}{dx} = \frac{18x}{2y}$$

$$y' = \frac{9x}{y}$$

Therefore, $\boxed{y' = \frac{9x}{y}}$.

(b)

Solve the given equation explicitly for y.

$$9x^2 - y^2 = 1$$
$$9x^2 - 1 = y^2$$
$$y^2 = 9x^2 - 1$$
$$y = \pm\sqrt{9x^2 - 1}$$

So, $\boxed{y = \pm\sqrt{9x^2 - 1}}$.

Find differentiate to get y' in terms of x.

Differentiate $y = \pm\left(9x^2 - 1\right)^{1/2}$ using the chain rule.

$$\begin{aligned}
y' &= \pm\frac{1}{2}\left(9x^2 - 1\right)^{-1/2}\frac{d}{dx}\left(9x^2 - 1\right) \\
&= \pm\frac{1}{2}\left(9x^2 - 1\right)^{-1/2}\left(9(2x)\right) \\
&= \pm\frac{1}{2}\left(9x^2 - 1\right)^{-1/2}\left(18x\right) \\
&= \pm\frac{9x}{\sqrt{9x^2 - 1}}
\end{aligned}$$

Therefore, $\boxed{y' = \pm\frac{9x}{\sqrt{9x^2 - 1}}}$.

(c)

Check:

The solution of part (a) and (b) are consistent by substituting the expression for y into the solution of part (a).

The solution of part (a) is given by

$$\frac{dy}{dx} = \frac{9x}{y}.$$

Substitute $\pm\sqrt{9x^2 - 1}$ for y in the solution of part (a).

$$\begin{aligned}
\frac{dy}{dx} &= \frac{9x}{y} \\
&= \frac{9x}{\pm\sqrt{9x^2 - 1}} \\
&= \pm\frac{9x}{\sqrt{9x^2 - 1}}
\end{aligned}$$

So, $\boxed{\frac{9x}{y} = \pm\frac{9x}{\sqrt{9x^2 - 1}}}$.

Chapter 2 Derivatives Exercise 2.6 2E

The measure of rate of change of a quantity with respect to some other quantity is called as the derivative and the method to determine the derivative is called differentiation.

a.

Consider the equation:

$$2x^2 + x + xy = 1 \quad \ldots\ldots (1)$$

Differentiate both sides of the above equation with respect to *x*:

$$\frac{d}{dx}\left(2x^2 + x + xy\right) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}\left(2x^2\right) + \frac{d}{dx}(x) + \frac{d}{dx}(xy) = 0$$
$$2\frac{d}{dx}x^2 + 1 + x\frac{dy}{dx} + y\frac{d}{dx}x = 0$$
$$2(2x) + 1 + xy' + y = 0$$

Simplify the above equation further and solve for y':

$$4x + 1 + xy' + y = 0$$
$$y' = \frac{-4x - 1 - y}{x} \quad \ldots\ldots (2)$$

Hence, the final expression is $\boxed{y' = \frac{-4x - 1 - y}{x}}$.

b.

Solve the equation (1) for *y*:

$$2x^2 + x + xy = 1$$
$$y = \frac{1 - 2x^2 - x}{x} \quad \ldots\ldots (3)$$

Differentiate the above equation with respect to *x*:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1 - 2x^2 - x}{x}\right)$$
$$y' = \frac{d}{dx}\left[\frac{1}{x} - 2x - 1\right]$$
$$= \frac{d}{dx}\left[x^{-1} - 2x - 1\right]$$
$$= \frac{d}{dx}\left(x^{-1}\right) - \frac{d}{dx}(2x) - \frac{d}{dx}(1)$$

Determine the derivative of the above function and simplify the above equation:

$$y' = (-1)x^{-2} - (2) - 0$$
$$= \frac{-1}{x^2} - 2$$
$$= \frac{-1 - 2x^2}{x^2}$$
$$= \frac{-\left(2x^2 + 1\right)}{x^2}$$

Hence, the final expression is $\boxed{y' = \frac{-\left(2x^2 + 1\right)}{x^2}}$.

c.

Substitute the value of *y* from the equation (3) in equation (2):

$$y = \frac{1 - 2x^2 - x}{x}$$
$$y' = \frac{-4x - 1 - y}{x}$$
$$= \frac{-4x - 1 - \left[\frac{1 - 2x^2 - x}{x}\right]}{x}$$
$$= \frac{-4x^2 - x - 1 + 2x^2 + x}{x^2}$$

Simplify the above equation further:

$$y' = \frac{-2x^2 - 1}{x^2} = \frac{-(2x^2 + 1)}{x^2}$$

Hence, the **solutions are consistent** by substituting the expression for y into the solution of part a.

Chapter 2 Derivatives Exercise 2.6 3E

(b)

Consider the equation:

$\frac{1}{x} + \frac{1}{y} = 1$ Rewrite the equation explicitly for y:

$$\frac{1}{x} + \frac{1}{y} = 1$$

$$\frac{1}{y} = 1 - \frac{1}{x}$$

$$\frac{1}{y} = \frac{x-1}{x}$$

$$\boxed{y = \frac{x}{x-1}}$$

Now, an explicit equation has been obtained, differentiate the equation with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x-1}\right)$$

Use quotient rule:

$$\frac{dy}{dx} = \frac{(x-1)\frac{d}{dx}(x) - (x)\frac{d}{dx}(x-1)}{(x-1)^2}$$
$$= \frac{(x-1)1 - (x)(1-0)}{(x-1)^2}$$
$$= \frac{x-1-x}{(x-1)^2}$$
$$= \frac{-1}{(x-1)^2}$$

Therefore the explicit differentiation gives:

$$\boxed{\frac{dy}{dx} = -\frac{1}{(x-1)^2}}$$

(c)

Put the expression $y = \frac{x}{x-1}$ in $\frac{dy}{dx} = -\frac{y^2}{x^2}$, then:

$$\frac{dy}{dx} = -\frac{\left(\frac{x}{x-1}\right)^2}{x^2}$$
$$= \frac{-x^2}{(x-1)^2 \times x^2}$$
$$= \frac{-1}{(x-1)^2}$$

Therefore, the derivative is $\frac{-1}{(x-1)^2}$ which is same as the solution of part (b).

Hence it is verified that the solutions of both the parts are **consistent**.

Chapter 2 Derivatives Exercise 2.6 4E

(a)

The equation is $\cos x+\sqrt{y}=5$.

The objective is to find y' by implicit differentiation.

Consider the expression,

$$\frac{d}{dx}\left(\cos x+\sqrt{y}\right)=\frac{d}{dx}5$$

$$\frac{d}{dx}\cos x+\frac{d}{dx}\sqrt{y}=\frac{d}{dx}5 \text{ Since } \frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

$$-\sin x+\frac{1}{2}y^{\frac{1}{2}-1}\frac{dy}{dx}=0 \text{ Since } \frac{d}{dx}\cos x=-\sin x, \frac{d}{dx}x^n=nx^{n-1}$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}=\sin x$$

$$\frac{dy}{dx}=2\sqrt{y}\sin x$$

Therefore, the result is $\boxed{y'=2\sqrt{y}\sin x}$.

(b)

The equation is $\cos x+\sqrt{y}=5$.

Find the expression for y and its derivative.

The equation can be written as $\sqrt{y}=5-\cos x$.

Square it on both sides.

$$y=(5-\cos x)^2$$

Differentiate both sides.

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}(5-\cos x)^2\\&=2(5-\cos x)\frac{d}{dx}(5-\cos x)\end{aligned}$$

$$=2(5-\cos x)\left(\frac{d}{dx}5-\frac{d}{dx}\cos x\right) \text{ Since } \frac{d}{dx}(\text{Constant})=0, \frac{d}{dx}\cos x=-\sin x$$

$$=2(5-\cos x)(\sin x)$$

Therefore, the result is $\boxed{y'=2(5-\cos x)(\sin x)}$.

(c)

From part (b), the expression for y is $y=(5-\cos x)^2$.

Substitute this solution from part (a).

$$\begin{aligned}y'&=2\sqrt{y}\sin x\\&=2\sqrt{(5-\sin x)^2}\sin x\\&=2(5-\cos x)\sin x\end{aligned}$$

The solutions from parts (a),(b) are consistent.

Chapter 2 Derivatives Exercise 2.6 5E

Consider the equation,

$x^3 + y^3 = 1$

The objective is to find $\frac{dy}{dx}$ by implicit differentiation.

$$\frac{d}{dx}\left[x^3 + y^3\right] = \frac{d}{dx}[1]$$
$$\frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[y^3\right] = 0$$
$$3x^2 + \frac{d}{dy}\left[y^3\right]\frac{dy}{dx} = 0$$
$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

Solve the last equation for the variable $\frac{dy}{dx}$.

$$3y^2\frac{dy}{dx} = 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2}{3y^2}$$

Therefore, the derivative of the equation as,

$$\boxed{\frac{dy}{dx} = \left(\frac{x}{y}\right)^2}.$$

Chapter 2 Derivatives Exercise 2.6 6E

Consider the function $2\sqrt{x} + \sqrt{y} = 3$.

To find the value of $\frac{dy}{dx}$, use the implicit differentiation to the above equation.

The value of the derivative is,

$$2\sqrt{x} + \sqrt{y} = 3$$
$$2\cdot\frac{1}{2}(x)^{\frac{-1}{2}} + \frac{1}{2}(y)^{\frac{-1}{2}}\frac{dy}{dx} = 0 \qquad \text{(Apply the implicit differentiation)}$$
$$\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \qquad \left(\text{Subtract } \frac{-1}{\sqrt{x}} \text{ on both sides}\right)$$
$$\frac{dy}{dx} = \frac{-2\sqrt{y}}{\sqrt{x}} \qquad \left(\text{Multiply } 2\sqrt{y} \text{ on both sides}\right)$$
$$= -2\cdot\sqrt{\frac{y}{x}}$$

Hence, the value of the derivative is $\boxed{\frac{dy}{dx} = -2\cdot\sqrt{\frac{y}{x}}}$.

Chapter 2 Derivatives Exercise 2.6 7E

Consider the function,

$x^2 + xy - y^2 = 4$

The objective is to find the implicit differentiation.

First, differentiate both sides of the equation with respect to x, treating y as a function of x.
The product rule says that, for a product of two functions,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Use the product rule to take the derivative of the product xy in the equation.

$$\frac{d}{dx}\left(x^2+xy-y^2\right)=\frac{d}{dx}(4)$$
$$2x+y+x\cdot y'-2y\cdot y'=0$$

Then, solve this equation for y'.

$$(x-2y)y'=-(2x+y)$$
$$y'=\boxed{-\frac{2x+y}{x-2y}}$$

Chapter 2 Derivatives Exercise 2.6 8E

Consider the function $2x^3+x^2y-xy^3=2$.

To find the value of $\frac{dy}{dx}$, use the implicit differentiation to the above equation.

The value of the derivative is,

$$2x^3+x^2y-xy^3=2$$
$$6x^2+\left(2xy+x^2\frac{dy}{dx}\right)-\left(y^3+3xy^2\frac{dy}{dx}\right)=0 \quad \text{(Apply the implicit differentiation)}$$
$$x^2\frac{dy}{dx}-3xy^2\frac{dy}{dx}=-6x^2-2xy+y^3$$
$$\left(x^2-3xy^2\right)\frac{dy}{dx}=y^3-6x^2-2xy$$
$$\frac{dy}{dx}=\frac{y^3-6x^2-2xy}{x^2-3xy^2}$$

Hence, the value of the derivative is $\boxed{\frac{dy}{dx}=\frac{y^3-6x^2-2xy}{x^2-3xy^2}}$.

Chapter 2 Derivatives Exercise 2.6 9E

Consider the equation,

$x^4(x+y)=y^2(3x-y)$.

Rewrite the given equation by removing parenthesis as

$x^5+x^4y=3xy^2-y^3$.

Need to find the expression $\frac{dy}{dx}$ by implicit differentiation.

Differentiate both sides of the equation $x^4(x+y)=y^2(3x-y)$ implicitly with respect to *x*.

$$\frac{d}{dx}\left(x^5+x^4y\right)=\frac{d}{dx}\left(3xy^2-y^3\right)$$
$$5x^4+4x^3\cdot y+x^4\cdot\frac{dy}{dx}=3\left(1\cdot y^2+x\cdot 2y\frac{dy}{dx}\right)-3y^2\frac{dy}{dx}$$
$$5x^4+4x^3y+x^4\frac{dy}{dx}=3y^2+6xy\frac{dy}{dx}-3y^2\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$:

$$x^4\frac{dy}{dx}-6xy\frac{dy}{dx}+3y^2\frac{dy}{dx}=3y^2-5x^4-4x^3y$$
$$\left(x^4-6xy+3y^2\right)\frac{dy}{dx}=3y^2-5x^4-4x^3y$$
$$\frac{dy}{dx}=\frac{3y^2-5x^4-4x^3y}{x^4-6xy+3y^2}$$

Therefore, the required derivative of the given equation $x^4(x+y)=y^2(3x-y)$ with respect to *x* is $\boxed{\frac{dy}{dx}=\frac{3y^2-5x^4-4x^3y}{x^4-6xy+3y^2}}$.

Chapter 2 Derivatives Exercise 2.6 10E

$y^5 + x^2y^3 = 1 + x^4y$

Differentiating both sides with respect to x we get

$$(y^5)' + (x^2)'y^3 + x^2(y^3)' = (1)' + (x^4)'y + x^4(y)'$$
$$\Rightarrow 5y^4y' + 2xy^3 + 3x^2y^2y' = 0 + 4x^3y + x^4y'$$

Collecting all terms involving y' we get

$$(5y^4 + 3x^2y^2 - x^4)y' = 4x^3y - 2xy^3$$

Solving for y' we have

$$y' = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}$$

Chapter 2 Derivatives Exercise 2.6 11E

Given curve is $y\cos x = x^2 + y^2$

Differentiating implicitly w.r.t x and y is a function of x, we get

$$\frac{d}{dx}(y\cos x) = \frac{d}{dx}(x^2 + y^2)$$
$$\Rightarrow y\frac{d}{dx}(\cos x) + \cos x\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{d}{dx}y^2$$
$$\Rightarrow -y\sin x + \cos xy' = 2x + 2yy'$$
$$\Rightarrow -y\sin x - 2x = 2yy' - \cos xy'$$
$$\Rightarrow y' = \frac{-y\sin x - 2x}{2y - \cos x}$$
$$= \frac{-(y\sin x + 2x)}{2y - \cos x}$$
$$\therefore\ y' = \frac{y\sin x + 2x}{\cos x - 2y}$$

Chapter 2 Derivatives Exercise 2.6 12E

Given curve is $\cos(xy) = 1 + \sin y$

Differentiating implicitly w.r.t x and y is a function of x, we get

$$\frac{d}{dx}\cos(xy) = \frac{d}{dx}(1 + \sin y)$$
$$-\sin(xy)\frac{d}{dx}(xy) = 0 + \cos y\frac{dy}{dx}$$
$$\Rightarrow\quad -\sin(xy)[xy' + y] = (\cos y)y'$$
$$\Rightarrow\quad -x\sin(xy)y' - y\sin(xy) = (\cos y)y'$$
$$\Rightarrow\quad -x\sin(xy)y' - (\cos y)y' = y\sin(xy)$$
$$\Rightarrow\quad y'[-x\sin(xy) - \cos y] = y\sin(xy)$$
$$\Rightarrow\quad y' = \frac{y\sin(xy)}{-[x\sin(xy) + \cos y]}$$
$$= \frac{-y\sin(xy)}{x\sin(xy) + \cos y}$$
$$\therefore\ y' = \frac{-y\sin(xy)}{x\sin(xy) + \cos y}$$

Chapter 2 Derivatives Exercise 2.6 13E

$$4\cos x \sin y = 1$$

Differentiating with respect to x on both sides and considering y as a function of x, we get

$$\frac{d}{dx}(4\cos x \sin y) = \frac{d}{dx}(1)$$

$$\Rightarrow 4\left[\cos x \frac{d}{dy}(\sin y)\frac{dy}{dx} + \sin y \frac{d}{dx}(\cos x)\right] = 0$$

$$\Rightarrow 4\left[(\cos x \cos y)y' - \sin y \sin x\right] = 0$$

$$\Rightarrow y' = \frac{\sin x \sin y}{\cos x \cos y}$$

$$\Rightarrow \boxed{y' = \tan x \tan y}$$

Chapter 2 Derivatives Exercise 2.6 14E

$$y\sin\left(x^2\right) = x\sin\left(y^2\right)$$

Differentiating both sides with respect to x and considering y as a functions of x, we have

$$\left[y\sin\left(x^2\right)\right]' = \left[x\sin\left(y^2\right)\right]'$$

$$\Rightarrow (y)'\sin\left(x^2\right) + y\left[\sin\left(x^2\right)\right]' = x'\sin\left(y^2\right) + x\left[\sin\left(y^2\right)\right]'$$

$$\Rightarrow y'\sin x^2 + 2xy\cos\left(x^2\right) = \sin y^2 + 2xy\cos\left(y^2\right)y'$$

$$\Rightarrow y'\left[\sin x^2 - 2xy\cos\left(y^2\right)\right] = \sin y^2 - 2xy\cos\left(x^2\right)$$

$$\Rightarrow y' = \frac{\sin y^2 - 2xy\cos\left(x^2\right)}{\sin x^2 - 2xy\cos\left(y^2\right)}$$

Chapter 2 Derivatives Exercise 2.6 15E

Consider the equation,

$$\tan\left(\frac{x}{y}\right) = x + y.$$

The object is to find $\frac{dy}{dx}$ by implicit differentiation.

Differentiate $\tan\left(\frac{x}{y}\right) = x + y$ on both sides with respect to x.

$$\sec^2\left(\frac{x}{y}\right)\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}(x+y)$$

$$\sec^2\left(\frac{x}{y}\right)\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2} = 1 + \frac{dy}{dx}$$

$$\sec^2\left(\frac{x}{y}\right)\frac{y\cdot 1 - x\frac{dy}{dx}}{y^2} = 1 + \frac{dy}{dx}$$

$$\sec^2\left(\frac{x}{y}\right)\left[\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right] = 1 + \frac{dy}{dx}$$

Combine like terms on both sides of the above step, then

$$\frac{1}{y}\sec^2\left(\frac{x}{y}\right)-\frac{x}{y^2}\sec^2\left(\frac{x}{y}\right)\frac{dy}{dx}=1+\frac{dy}{dx}$$

$$\frac{1}{y}\sec^2\left(\frac{x}{y}\right)-1=\frac{dy}{dx}+\frac{x}{y^2}\sec^2\left(\frac{x}{y}\right)\frac{dy}{dx}$$

$$\frac{1}{y}\sec^2\left(\frac{x}{y}\right)-1=\frac{dy}{dx}\left[1+\frac{x}{y^2}\sec^2\left(\frac{x}{y}\right)\right]$$

$$\frac{dy}{dx}=\frac{\frac{1}{y}\sec^2\left(\frac{x}{y}\right)-1}{1+\frac{x}{y^2}\sec^2\left(\frac{x}{y}\right)}$$

$$=\frac{\frac{\sec^2\left(\frac{x}{y}\right)-y}{y}}{\frac{y^2+x\sec^2\left(\frac{x}{y}\right)}{y^2}}$$

$$=\frac{y\sec^2\left(\frac{x}{y}\right)-y^2}{y^2+x\sec^2\left(\frac{x}{y}\right)}$$

Therefore, $\frac{dy}{dx}=\boxed{\frac{y\sec^2\left(\frac{x}{y}\right)-y^2}{y^2+x\sec^2\left(\frac{x}{y}\right)}}$.

Chapter 2 Derivatives Exercise 2.6 16E

The equation is $\sqrt{x+y}=1+x^2y^2$.

The objective is to find $\frac{dy}{dx}$ by implicit differentiation.

Rewrite the equation by squaring it on both sides.

$$x+y=\left(1+x^2y^2\right)^2$$
$$=1+2x^2y^2+x^4y^4$$

Use the following formulas to differentiate.

(i) $\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$

(ii) $\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+f'(x)g(x)$

Differentiate the equation $x+y=1+2x^2y^2+x^4y^4$ on both sides.

$$\frac{d}{dx}(x+y)=\frac{d}{dx}\left(1+2x^2y^2+x^4y^4\right)$$
$$\frac{d}{dx}x+\frac{d}{dx}y=\frac{d}{dx}1+\frac{d}{dx}\left(2x^2y^2\right)+\frac{d}{dx}\left(x^4y^4\right)$$
$$1+\frac{dy}{dx}=0+2\left[x^2\frac{d}{dx}y^2+y^2\frac{d}{dx}x^2\right]+x^4\frac{d}{dx}y^4+y^4\frac{d}{dx}x^4$$
$$1+\frac{dy}{dx}=2\left[2x^2y\frac{dy}{dx}+2xy^2\right]+4x^4y^3\frac{dy}{dx}+4y^4x^3$$
$$1+\frac{dy}{dx}=4x^2y\frac{dy}{dx}+4xy^2+4x^4y^3\frac{dy}{dx}+4y^4x^3$$
$$\frac{dy}{dx}\left[1-4x^2y-4x^4y^3\right]=4xy^2+4y^4x^3-1$$
$$\frac{dy}{dx}=\frac{4xy^2+4y^4x^3-1}{1-4x^2y-4x^4y^3}$$

Therefore, the derivative is $\boxed{\frac{dy}{dx}=\frac{4xy^2+4y^4x^3-1}{1-4x^2y-4x^4y^3}}$.

Chapter 2 Derivatives Exercise 2.6 17E

The equation is $\sqrt{xy}=1+x^2y$.

The objective is to find $\frac{dy}{dx}$

Use the following formulas:

Product rule:

If u,v are any two functions of x then $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

Chain rule:

If $y=f(u)$ and $u=g(x)$ then $\frac{dy}{dx}=\frac{df}{du}\cdot\frac{du}{dx}$

Consider the equation,

$$\sqrt{xy}=1+x^2y$$

Differentiate this with respect to x.

$$\frac{d}{dx}\sqrt{xy}=\frac{d}{dx}\left(1+x^2y\right)$$
$$\frac{1}{2}(xy)^{\frac{1}{2}-1}\frac{d}{dx}(xy)=\frac{d}{dx}1+\frac{d}{dx}x^2y \text{ Use chain rule}$$
$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx}+y(1)\right]=0+x^2\frac{dy}{dx}+y\frac{dx^2}{dx} \text{ Use product rule}$$
$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx}+y(1)\right]=x^2\frac{dy}{dx}+y(2x)$$
$$x\frac{dy}{dx}+y(1)=2\sqrt{xy}\left[x^2\frac{dy}{dx}+2xy\right]$$
$$=2x^2\sqrt{xy}\frac{dy}{dx}+4(xy)^{\frac{3}{2}}$$
$$\frac{dy}{dx}\left(x-2x^2\sqrt{xy}\right)=4(xy)^{\frac{3}{2}}-y$$
$$\frac{dy}{dx}=\frac{4(xy)^{\frac{3}{2}}-y}{x-2x^2\sqrt{xy}}$$

Therefore, the derivative is $\boxed{\frac{dy}{dx}=\frac{4(xy)^{\frac{3}{2}}-y}{x-2x^2\sqrt{xy}}}$.

Chapter 2 Derivatives Exercise 2.6 18E

Given equation is $x\sin y + y\sin x = 1$

Differentiating implicitly w.r.t x and y is a function of x, we get

$$\frac{d}{dx}(x\sin y + y\sin x) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(x\sin y) + \frac{d}{dx}(y\sin x) = 0$$

$$\Rightarrow x\frac{d}{dx}\sin y + \sin y\frac{d}{dx}x + y\frac{d}{dx}\sin x + \sin x\frac{dy}{dx} = 0$$

$$\Rightarrow x(\cos y)y' + \sin y + y\cos x + (\sin x)y' = 0$$

$$\Rightarrow y'[x\cos y + \sin x] = -\sin y - y\cos x$$

$$\Rightarrow y' = \frac{-\sin y - y\cos x}{x\cos y + \sin x}$$

$$\boxed{\therefore y' = \frac{-\sin y - y\cos x}{x\cos y + \sin x}}$$

Chapter 2 Derivatives Exercise 2.6 19E

$y\cos(x) = 1 + \sin(xy)$

$\Leftrightarrow y\left(-\sin\left(x\right)\right) + \left(\cos(x)\right)\cdot\frac{dy}{dx} = 0 + \left(\cos(xy)\right)\left(x\cdot\frac{dy}{dx} + y\cdot 1\right)$, remembering the product rule, and the chain rule to obtain dy/dx.

$\Leftrightarrow \frac{dy}{dx}\cos\left(x\right) - \frac{dy}{dx}x\cos\left(xy\right) = y\sin\left(x\right) + y\cos\left(xy\right)$, isolating terms containing dy/dx on one side of the equation.

$$\Leftrightarrow \frac{dy}{dx}\left(\cos(x) - x\cos(xy)\right) = y\sin\left(x\right) + y\cos\left(xy\right)$$

$\Leftrightarrow \frac{dy}{dx} = \frac{y\sin(x) + y\cos(xy)}{\cos(x) - x\cos(xy)}$, solving for dy/dx.

Chapter 2 Derivatives Exercise 2.6 20E

$$\tan(x-y) = \frac{y}{1+x^2}$$

Differentiating with respect x on both sides implicitly, we have

$$[\tan(x-y)]' = \left(\frac{y}{1+x^2}\right)'$$

$$\Rightarrow \sec^2(x-y)\times(1-y') = \frac{y'(1+x^2) - y(2x)}{(1+x^2)^2}$$

$$\Rightarrow \sec^2(x-y) - y'\sec^2(x-y) = \frac{y'}{1+x^2} - \frac{2xy}{(1+x^2)^2}$$

$$\Rightarrow y'\left[\frac{1}{1+x^2} + \sec^2(x-y)\right] = \sec^2(x-y) + \frac{2xy}{(1+x^2)^2}$$

$$\Rightarrow y'\left[1 + (1+x^2)\sec^2(x-y)\right] = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{1+x^2}$$

$$\Rightarrow y' = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{(1+x^2)\left[1+\sec^2(x-y) + x^2\sec^2(x-y)\right]}$$

$$\Rightarrow y' = \frac{(1+x^2)\sec^2(x-y) + \frac{2xy}{(1+x^2)}}{\left[1+\sec^2(x-y) + x^2\sec^2(x-y)\right]}$$

$$\Rightarrow y' = \frac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{\left[1+(1+x^2)\sec^2(x-y)\right]}$$

Chapter 2 Derivatives Exercise 2.6 21E

Given that $f(x)+x^2[f(x)]^3=10$ and $f(1)=2$

Differentiating both sides with respect to x

$$\frac{d}{dx}\left[f(x)+x^2[f(x)]^3\right]=\frac{d}{dx}(10)$$

$$\frac{d}{dx}f(x)+\frac{d}{dx}\left[x^2(f(x))^3\right]=0$$

Since the derivative of constant is zero.

Now use product rule fist and chain rule next to the second term and by using the notation $\frac{d}{dx}f(x)=f'(x)$ gives

$$f'(x)+x^2\frac{d}{dx}[f(x)]^3+[f(x)]^3\cdot\frac{d}{dx}(x^2)=0$$

$$f'(x)+x^2\cdot 3\cdot[f(x)]^2\cdot f'(x)+[f(x)]^3\cdot 2x=0$$

$$f'(x)\left[1+3x^2[f(x)]^2\right]=-2x[f(x)]^3$$

By taking $f'(x)$ as factor and by simplifying

$$f'(x)=\frac{-2x[f(x)]^3}{1+3x^2[f(x)]^2}$$

Now substituting $x=1$ gives

$$f'(1)=\frac{-2\cdot 1\cdot[f(1)]^3}{1+3\cdot 1^2[f(1)]^2}$$

But $f(1)=2$ so

$$=\frac{-2(2)^3}{1+3\cdot 2^2}$$

$$=\frac{-16}{13}$$

By simplification

$$\therefore f'(1)=\frac{-16}{13}$$

Chapter 2 Derivatives Exercise 2.6 22E

Given $g(x)+x\sin g(x)=x^2$

Differentiating both sides with respect to x, we have

$$g'(x)+\sin g(x)+x\cos g(x).g'(x)=2x$$

$$\Rightarrow g'(x)[1+x\cos g(x)]=2x-\sin g(x)$$

$$\Rightarrow g'(x)=\frac{2x-\sin g(x)}{1+x\cos g(x)}$$

Putting $x=1$, we have

$$g'(1)=\frac{2-\sin g(1)}{1+\cos g(1)}$$

Since $g(1)=0$, so we get

$$g'(1)=\frac{2-\sin 0}{1+\cos 0}=\frac{2}{2}=1$$

Chapter 2 Derivatives Exercise 2.6 23E

Consider the equation,

$x^4y^2 - x^3y + 2xy^3 = 0$.

The object is to find the expression $\frac{dx}{dy}$ by implicit differentiation, by considering *y* as the independent variable and *x* as the dependent variable.

Use implicit differentiation to differentiate both sides of the equation $x^4y^2 - x^3y + 2xy^3 = 0$ implicitly with respect to *y*.

$$\frac{d}{dy}\left(x^4y^2 - x^3y + 2xy^3\right) = \frac{d}{dy}(0)$$

$$x^4\frac{d}{dy}\left(y^2\right) + y^2\frac{d}{dy}\left(x^4\right) - \left(x^3\frac{d}{dy}(y) + y\frac{d}{dy}\left(x^3\right)\right) + 2\left[x\frac{d}{dy}\left(y^3\right) + y^3\frac{d}{dy}(x)\right] = 0$$

$$x^4 \cdot 2y + y^2 4x^3\frac{dx}{dy} - x^3(1) - 3yx^2\frac{dx}{dy} + 2x \cdot 3y^2 + 2y^3\frac{dx}{dy} = 0$$

$$2x^4y + 4x^3y^2\frac{dx}{dy} - 3x^2y\frac{dx}{dy} - x^3 + 2y^3\frac{dx}{dy} + 6xy^2 = 0$$

Solve for $\frac{dx}{dy}$:

$$4x^3y^2\frac{dx}{dy} - 3x^2y\frac{dx}{dy} + 2y^3\frac{dx}{dy} = -2x^4y + x^3 - 6xy^2$$

$$\left(4x^3y^2 - 3x^2y + 2y^3\right)\frac{dx}{dy} = -2x^4y + x^3 - 6xy^2$$

$$\frac{dx}{dy} = \frac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$$

$$\frac{dx}{dy} = \frac{x^3 - 2x^4y - 6xy^2}{y\left(4x^3y + 2y^2 - 3x^2\right)}$$

Therefore, the required derivative of the equation $x^4(x+y) = y^2(3x - y)$ with respect to *x* is

$$\frac{dx}{dy} = \boxed{\frac{x^3 - 2x^4y - 6xy^2}{y\left(4x^3y + 2y^2 - 3x^2\right)}}.$$

Chapter 2 Derivatives Exercise 2.6 24E

$y\sec(x) = x\tan(y)$

$\Leftrightarrow\ y \cdot \sec(x)\tan(x) \cdot \frac{dx}{dy} + \sec(x) \cdot 1 = x \cdot \sec^2(y) + \tan(y) \cdot \frac{dy}{dx}$, differentiating implicitly, remembering the product rule and the chain rule to obtain dx/dy.

$\Leftrightarrow\ y \cdot \sec(x)\tan(x) \cdot \frac{dx}{dy} - \tan(y) \cdot \frac{dx}{dy} = x \cdot \sec^2(y) - \sec(x)$, isolating expressions containing dx/dy on one side of the equation.

$\Leftrightarrow\ \frac{dx}{dy}\left(y \cdot \sec(x)\tan(x) - \tan(y)\right) = x \cdot \sec^2(y) - \sec(x)$, factoring out dx/dy.

$\Leftrightarrow\ \frac{dx}{dy} = \frac{x \cdot \sec^2(y) - \sec(x)}{y \cdot \sec(x)\tan(x) - \tan(y)}$, solving for dx/dy.

Consider the function,

$y\sin 2x = x\cos 2y$.

The objective is to find the equation of the tangent line to the curve at the point $\left(\frac{\pi}{2},\frac{\pi}{4}\right)$.

Differentiating both sides of $y\sin 2x = x\cos 2y$ with respect to x, regarding y as a function of x,

$$\frac{d}{dx}(y\sin 2x) = \frac{d}{dx}(x\cos 2y)$$

$$\sin 2x\frac{d}{dx}(y) + y\frac{d}{dx}(\sin 2x) = \cos 2y\frac{d}{dx}(x) + x\frac{d}{dx}(\cos 2y) \quad \text{(Using product rule)}$$

$$\sin 2x\frac{dy}{dx} + y\left\{\cos 2x\frac{d}{dx}(2x)\right\} = \cos 2y(1) + x(-\sin 2y)\frac{d}{dx}(2y) \quad \text{(Chain rule)}$$

$$\sin 2x\frac{dy}{dx} + \{y\cos 2x(2)\} = \cos 2y - x\sin 2y\left(2\frac{dy}{dx}\right)$$

$$\sin 2x\frac{dy}{dx} + 2y\cos 2x = \cos 2y - 2x\sin 2y\frac{dy}{dx}$$

$$(\sin 2x + 2x\sin 2y)\frac{dy}{dx} = \cos 2y - 2y\cos 2x \quad \left(\text{Take out } \frac{dy}{dx} \text{ terms}\right)$$

$$\frac{dy}{dx} = \frac{\cos 2y - 2y\cos 2x}{\sin 2x + 2x\sin 2y} \quad \text{...... (1)}$$

The slope of the tangent at the point $\left(\frac{\pi}{2},\frac{\pi}{4}\right)$ is,

$$\left.\frac{dy}{dx}\right|_{\text{at}\left(\frac{\pi}{2},\frac{\pi}{4}\right)} = \frac{\cos 2\left(\frac{\pi}{4}\right) - 2\left(\frac{\pi}{4}\right)\cos 2\left(\frac{\pi}{2}\right)}{\sin 2\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\sin 2\left(\frac{\pi}{4}\right)} \quad \text{(Using equation (1))}$$

$$= \frac{0 - \left(\frac{\pi}{2}\right)(-1)}{0 + (\pi)(1)} \quad \left(\cos\frac{\pi}{2} = 0\right)$$

$$= \frac{\frac{\pi}{2}}{\pi}$$

$$= \frac{1}{2}$$

Therefore, the equation of the tangent line to the curve $y\sin 2x = x\cos 2y$ at the point $\left(\frac{\pi}{2},\frac{\pi}{4}\right)$ is,

$$y - \frac{\pi}{4} = \left(\frac{1}{2}\right)\left(x - \frac{\pi}{2}\right) \quad (y - y_1 = m(x - x_1))$$

$$y - \frac{\pi}{4} = \frac{1}{2}x - \frac{\pi}{4}$$

$$\boxed{y = \frac{1}{2}x}.$$

Chapter 2 Derivatives Exercise 2.6 26E

Given curve is $\sin(x+y)=2x-2y, \quad (\pi,\pi)$
Differentiating implicitly w.r.t x and y is a function of x, we get

$$\frac{d}{dx}(\sin(x+y))=\frac{d}{dx}(2x-2y)$$

$$\Rightarrow \cos(x+y)\frac{d}{dx}(x+y)=2-2\frac{dy}{dx}=2-2y'$$

$$\Rightarrow \cos(x+y)[1+y']=2-2y'$$

$$\Rightarrow \cos(x+y)+y'\cos(x+y)=2-2y'$$

$$\Rightarrow 2-\cos(x+y)=y'\cos(x+y)+2y'$$

$$\Rightarrow y'=\frac{2-\cos(x+y)}{2+\cos(x+y)}$$

$\therefore$ Slope of the tangent line is

$$m=\left(\frac{dy}{dx}\right)_{x=\pi}=\frac{2-\cos(\pi+\pi)}{2+\cos(\pi+\pi)}=\frac{2-\cos 2\pi}{2+\cos 2\pi}=\frac{2-1}{2+1}=\frac{1}{3}$$

The equation of the tangent line to the curve $\sin(x+y)=2x-2y$ at the point (π,π) is

$$(y-\pi)=\frac{1}{3}(x-\pi) \quad \Rightarrow \quad y-\pi=\frac{x}{3}-\frac{\pi}{3}$$

$$\Rightarrow \quad y=\frac{x}{3}-\frac{\pi}{3}+\pi=\frac{x}{3}+\frac{2\pi}{3}$$

$$\boxed{\therefore\ y=\frac{x}{3}+\frac{2\pi}{3}}$$

Chapter 2 Derivatives Exercise 2.6 27E

The given curve is $x^2+xy+y^2=3$
Differentiating both sides with respect to x we get,

$$2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}(x+2y)=-(2x+y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(2x+y)}{(x+2y)}$$

The slope of the tangent at (1, 1) is given by

$$m=\left.\frac{dy}{dx}\right|_{x=1,y=1}=\frac{-(2+1)}{1+2}=-1$$

The equation of the tangent at (1, 1) is

$$y-y_1=m(x-x_1)$$

$$\Rightarrow y-1=-1(x-1)$$

$$\Rightarrow y+x=2$$

Chapter 2 Derivatives Exercise 2.6 28E

The given curve is

$$x^2+2xy-y^2+x=2$$

Differentiating both sides with respect to x, we have

$$2x+2y+2xy'-2yy'+1=0$$

$$\Rightarrow y'(2x-2y)=-1-2x-2y$$

$$\Rightarrow y'=\frac{dy}{dx}=\frac{-(1+2x+2y)}{2x-2y}$$

The slope of the tangent at (1, 2) is given by

$$m=\left.\frac{dy}{dx}\right|_{x=1,y=2}=\frac{-(1+2+4)}{2-4}=\frac{7}{2}$$

The equation of the tangent at (1, 2) is given by

$$y-y_1=m(x-x_1)$$

$$y-2=\frac{7}{2}(x-1)$$

$$\boxed{y=\frac{7}{2}x-\frac{3}{2}}$$

Chapter 2 Derivatives Exercise 2.6 29E

Consider the curve

$$x^2+y^2=\left(2x^2+2y^2-x\right)^2 \qquad ......(1)$$

Here y is implicitly defined as a function of x.

In such cases, use implicit differentiation for solving $\frac{dy}{dx}$.

The power rule combined with chain rule is,

$$\frac{d}{dx}[g(x)]^n=n[g(x)]^{n-1}\cdot g'(x) \qquad ......(2)$$

Differentiate the function (1) with respect to x on both sides,

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}\left[\left(2x^2+2y^2-x\right)^2\right]$$

The expression on left hand side can be simplified by the use of sum rule of differentiation given by

$$\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{d}{dx}\left(f(x)\right)+\frac{d}{dx}\left(g(x)\right) \qquad ......(3)$$

$$\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left[\left(2x^2+2y^2-x\right)^2\right] \qquad \text{use sum rule } (2)$$

$$2x+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left[\left(2x^2+2y^2-x\right)^2\right] \qquad ......(4)$$

Use the chain rule to compute $\frac{d}{dx}\left(y^2\right)=2y\frac{dy}{dx}$

Compute $\frac{d}{dx}\left[\left(2x^2+2y^2-x\right)^2\right]$ without expanding the square.

$$\frac{d}{dx}\left[\left(2x^2+2y^2-x\right)^2\right]=2\left(2x^2+2y^2-x\right)\frac{d}{dx}\left(2x^2+2y^2-x\right) \qquad \text{use chain rule } (2)$$

$$=2\left(2x^2+2y^2-x\right)\left[\frac{d}{dx}\left(2x^2\right)+\frac{d}{dx}\left(2y^2\right)-\frac{d}{dx}(x)\right] \qquad \text{use sum rule}(3)$$

$$=2\left(2x^2+2y^2-x\right)\left[2(2x)+2(2y)\frac{dy}{dx}-1\right] \qquad \text{use chain rule}(2)$$

$$=\left(4x^2+4y^2-2x\right)\left[4x+4y\frac{dy}{dx}-1\right]$$

Substitute the above values in (4) to solve the differentiation.

$$2x+2y\frac{dy}{dx}=\left(4x^2+4y^2-2x\right)\left(4x+4y\frac{dy}{dx}-1\right)$$

$$2x+2y\frac{dy}{dx}=\left(4x^2+4y^2-2x\right)\left((4x-1)+4y\frac{dy}{dx}\right)$$

$$2x+2y\frac{dy}{dx}=\left(4x^2+4y^2-2x\right)(4x-1)+\left(4y\left(4x^2+4y^2-2x\right)\frac{dy}{dx}\right)$$

$$2x+2y\frac{dy}{dx}=16x^3+16xy^2-8x^2-4x^2-4y^2+2x+\left(16x^2y+16y^3-8xy\right)\frac{dy}{dx}$$

$$\left(2y-16x^2y-16y^3+8xy\right)\frac{dy}{dx}=16x^3+16xy^2-8x^2-4x^2-4y^2+2x-2x$$

$$\left(2y-16x^2y-16y^3+8xy\right)\frac{dy}{dx}=16x^3+16xy^2-12x^2-4y^2$$

$$\frac{dy}{dx}=\frac{16x^3+16xy^2-12x^2-4y^2}{\left(2y-16x^2y-16y^3+8xy\right)}$$

$$\frac{dy}{dx}=\frac{8x^3+8xy^2-6x^2-2y^2}{y-8x^2y-8y^3+4xy}$$

The tangent is a straight line which touches the curve of function at one point.

Since it is straight line, write the equation of the tangent line.

The equation of the straight line is,

$$y-y_1=m(x-x_1)$$

Here m is the slope of the tangent

Compute the slope of the tangent line at the point $\left(0,\frac{1}{2}\right)$ by using the formula,

$$m=\left(\frac{dy}{dx}\right)_{\left(0,\frac{1}{2}\right)}$$

$$m=\left(\frac{8x^3+8xy^2-6x^2-2y^2}{y-8x^2y-8y^3+4xy}\right)_{\left(0,\frac{1}{2}\right)}$$

$$m=\left(\frac{8\cdot 0^3+8\cdot 0\cdot\left(\frac{1}{2}\right)^2-6\cdot 0^2-2\cdot\left(\frac{1}{2}\right)^2}{\frac{1}{2}-8\cdot 0^2\cdot\frac{1}{2}-8\cdot\left(\frac{1}{2}\right)^3+4\cdot 0\cdot\frac{1}{2}}\right) \quad \text{Substitute } x=0 \text{ and } y=\frac{1}{2}$$

$$m=\left(\frac{-\frac{1}{2}}{\frac{1}{2}-8\cdot\frac{1}{8}}\right)$$

Continue the above simplification:

$$m=\left(\frac{-\frac{1}{2}}{-\frac{1}{2}}\right)$$

$$m=1$$

Therefore, the slope is $m=1$.

The equation of the tangent to the cardioids at $\left(0,\frac{1}{2}\right)$ is,

$$y-y_1=m(x-x_1)$$

$$y-\frac{1}{2}=1(x-0) \quad \text{substitute } x=0, y=\frac{1}{2} \text{ and } m=1$$

$$y=x+\frac{1}{2} \quad \text{Simplify}$$

Therefore, the equation of the tangent to the given curve is $\boxed{y=x+\frac{1}{2}}$.

Chapter 2 Derivatives Exercise 2.6 30E

Given equation of the curve is $x^{2/3}+y^{2/3}=4$

Differentiating with respect to x implicitly, we get

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

Or $$\frac{dy}{dx}=\frac{-x^{1/3}}{y^{-1/3}}$$

Or $$\frac{dy}{dx}=\frac{-y^{1/3}}{x^{1/3}}$$

Slope of the tangent line at $\left(-3\sqrt{3},\ 1\right)$ is

$$\left(\frac{dy}{dx}\right)_{(-3\sqrt{3},\,1)} = \frac{-(1)^{1/3}}{\left(-3\sqrt{3}\right)^{1/3}}$$
$$= \frac{-1}{\left(-3^{3/2}\right)^{1/3}}$$
$$= \frac{1}{\left(3^{1/2}\right)} = \frac{1}{\sqrt{3}}$$

So the equation of the tangent line is

$$(y-1) = \frac{1}{\sqrt{3}}\left(x+3\sqrt{3}\right)$$

Or $\quad y-1 = \frac{x}{\sqrt{3}}+3$

Or $\quad \boxed{y = \frac{x}{\sqrt{3}}+4}$

Chapter 2 Derivatives Exercise 2.6 31E

The equation of the curve is given as

$$2\left(x^2+y^2\right)^2 = 25\left(x^2-y^2\right)$$

Differentiating with respect to x implicitly

$$4\left(x^2+y^2\right)\left\{2x+2y\frac{dy}{dx}\right\} = 25\left(2x-2y\frac{dy}{dx}\right) \qquad \text{[Chain rule]}$$

$$\Rightarrow \quad 8\left(x^2+y^2\right)x+8y\left(x^2+y^2\right)\frac{dy}{dx} = 50x-50y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{8y\left(x^2+y^2\right)+50y\right\} = 50x-8x\left(x^2+y^2\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{50x-8x\left(x^2+y^2\right)}{8y\left(x^2+y^2\right)+50y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{25x-4x\left(x^2+y^2\right)}{4y\left(x^2+y^2\right)+25y}$$

Slope of the tangent line at $(3,\ 1)$ is

$$\left(\frac{dy}{dx}\right)_{(3,1)} = \frac{25\times 3-4\times 3\left(3^2+1^2\right)}{4\times 1\left(3^2+1^2\right)+25\times 1}$$
$$= \frac{75-120}{40+25}$$
$$= \frac{-45}{65}$$
$$= \frac{-9}{13}$$

Equation of tangent line is

$$(y-1) = \frac{-9}{13}(x-3)$$

Or $\quad y-1 = -\frac{9x}{13}+\frac{27}{13}$

Or $\quad y = -\frac{9x}{13}+\frac{27}{13}+1$

Or $\quad \boxed{y = -\frac{9}{13}x+\frac{40}{13}}$

Chapter 2 Derivatives Exercise 2.6 32E

The equation of the curve is given as

$$y^2\left(y^2-4\right)=x^2\left(x^2-5\right)$$

Or $$\left(y^4-4y^2\right)=\left(x^4-5x^2\right)$$

Differentiating with respect to x implicitly

$$\left(4y^3-8y\right)\frac{dy}{dx}=\left(4x^3-10x\right)$$

Or $$\frac{dy}{dx}=\frac{4x^3-10x}{4y^3-8y}$$

Slope of the tangent line at (0, -2) is

$$\left.\frac{dy}{dx}\right|_{(0,-2)}=0$$

Equation of tangent line is

$$(y+2)=0(x-0)$$

Or $$\boxed{y=-2}$$

Chapter 2 Derivatives Exercise 2.6 33E

(a) Given that the equation of the curve as $y^2=5x^4-x^2$

Differentiating, we get

$$2yy'=5(4x^3)-2x$$

$$y'=\frac{20x^3-2x}{2y}$$

$$=\frac{10x^3-x}{y}$$

So the slope of the tangent line at the point $(1,2)$ is

$$y'=\frac{10(1)^3-1}{2}$$

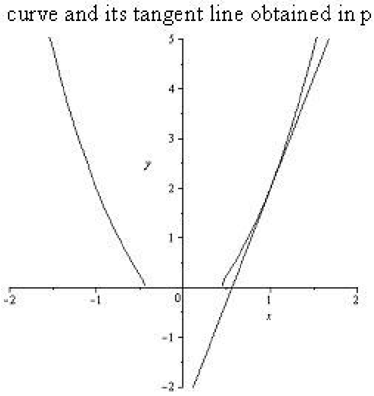
$$y'=\frac{9}{2}$$

So the equation of the tangent line passing through the point $(1,2)$ and slope $y'=\frac{9}{2}$ is

$$y-2=\frac{9}{2}(x-1)$$

$$y=\frac{9}{2}x-\frac{5}{2}$$

(b) Now we graph the curve and its tangent line obtained in part (a) as follows:



Chapter 2 Derivatives Exercise 2.6 34E

(A)

The equation of the curve $y^2=x^3+3x^2$.

We will get the slope of the tangent line at (1, -2).

We have $y^2=x^3+3x^2$

Differentiate both sides with respect to x, we have

$$2yy'=3x^2+6x$$

Then $$\boxed{y'=\frac{3x^2+6x}{2y}}$$

The slope at (1,-2)

$$\Rightarrow y' = \frac{3.1^2 + 6.1}{2(-2)}$$

$$\Rightarrow y' = \frac{3+6}{-4} = \frac{-9}{4}$$

Then the equation of tangent line at (1, -2) is

$$\left(y-(-2)\right) = \frac{-9}{4}(x-1)$$

$$\Rightarrow y+2 = \frac{-9}{4}x + \frac{9}{4}$$

$$\Rightarrow y = \frac{-9}{4}x + \frac{9}{4} - 2$$

$$\Rightarrow \boxed{y = \frac{-9}{4}x + \frac{1}{4}}$$

(B)

When the curve have a horizontal tangent then

$$y' = 0$$

$$\Rightarrow \frac{3x^2+6x}{2y} = 0$$

$$\Rightarrow 3x^2 + 6x = 0$$

$$\Rightarrow 3x + 6 = 0$$

$$\Rightarrow x = -\frac{6}{3} \Rightarrow \boxed{x = -2}$$

Putting this value of x in the equation of the curve, we get

$$y^2 = (-2)^3 + 3(-2)^2$$

$$y^2 = -8 + 12$$

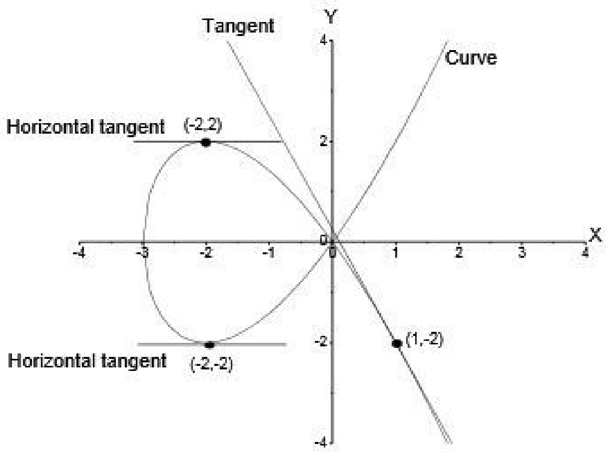
$$y^2 = +4 \Rightarrow y = \sqrt[4]{4}$$

$$\boxed{y = \pm 2}$$

Then at the point (-2, 2) and (-2, -2) the curve have horizontal tangent lines.

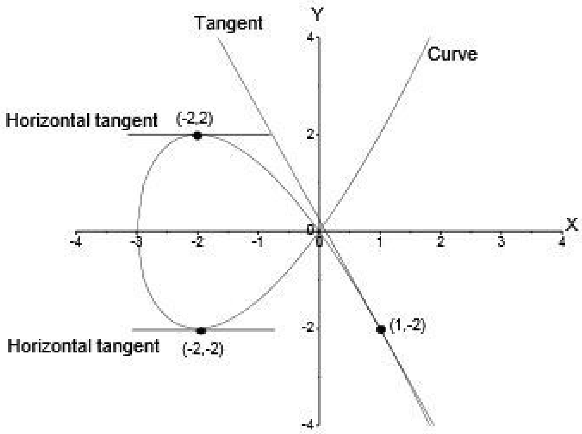
(C)

The graph of the curve $y^2 = x^3 + 3x^2$ and tangent lines are shown in figure below.



(C)

The graph of the curve $y^2 = x^3 + 3x^2$ and tangent lines are shown in figure below.



Here *y* is a function of *x* and using the Chain Rule,

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} \text{ (In Leibniz notation)}$$

$$= (2y)y'$$

Now substitute the above value into equation (1),

$$9(2x) + 2yy' = 0$$

Therefore,

$$\mathbf{18x + 2yy' = 0}.$$

Solve this equation for $\boldsymbol{y'}$.

$$\mathbf{2yy' = -18x}$$

$$\boldsymbol{y' = \frac{-18x}{2y}}$$

$$= -\frac{9x}{y}$$

Therefore, $y' = -\frac{9x}{y}$.

To find $\boldsymbol{y''}$, use implicit differentiation.

Again take $\boldsymbol{\frac{d}{dx}}$ on both sides of $\boldsymbol{y' = -\frac{9x}{y}}$.

$$y'' = \frac{d}{dx}\left(-\frac{9x}{y}\right)$$

$$y'' = -\frac{y\frac{d}{dx}(9x) - 9x\frac{d}{dx}(y)}{y^2} \quad \text{Use the Quotient Rule}$$

$$y'' = -\frac{9y - 9x\frac{dy}{dx}}{y^2}$$

Now substitute $y' = -\dfrac{9x}{y}$ into $y'' = \dfrac{-9y-(-9x)y'}{y^2}$,

$$y'' = -\frac{9y - 9x\left(-\dfrac{9x}{y}\right)}{y^2}$$

$$= -\frac{\dfrac{9(9x^2+y^2)}{y}}{y^2}$$

$$= -\frac{9\times 9}{y^3} \qquad \left[\text{Since } 9x^2+y^2=9\right]$$

$$= -\frac{81}{y^3}$$

Therefore the value of y'' is $\boxed{-\dfrac{81}{y^3}}$.

Chapter 2 Derivatives Exercise 2.6 36E

Given equation is $\sqrt{x}+\sqrt{y}=1$

Differentiating both sides with respect to x gives $\dfrac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\dfrac{d}{dx}(1)$

$$\frac{d}{dx}\sqrt{x}+\frac{d}{dx}\sqrt{y}=0$$

Since by addition rule of differentiation and $\dfrac{d}{dx}(1)=0$ because of 1 is constant

Now using chain rule and remembering that y is a function of x

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx}=-\frac{2\sqrt{y}}{2\sqrt{x}} \qquad \text{Since by simplification}$$

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

First apply this quotient rule and then power rule remembering that y is a function of x

$$y''=-\left[\frac{\sqrt{x}\cdot\dfrac{d}{dx}\sqrt{y}-\sqrt{y}\dfrac{d}{dx}\sqrt{x}}{\left(\sqrt{x}\right)^2}\right]$$

$$=-\left[\frac{\sqrt{x}\cdot\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}-\sqrt{y}\cdot\dfrac{1}{2\sqrt{x}}}{x}\right]$$

Substituting $\frac{dy}{dx}$ as $-\frac{\sqrt{y}}{\sqrt{x}}$

$$y'' = -\left[\frac{\frac{\sqrt{x}}{2\sqrt{y}} \times \frac{-\sqrt{y}}{\sqrt{x}} - \frac{\sqrt{y}}{2\sqrt{x}}}{x}\right]$$

By algebra simplifications

$$= -\left[\frac{\frac{-1}{2} - \frac{\sqrt{y}}{2\sqrt{x}}}{x}\right]$$

$$= -\left[\frac{\frac{\sqrt{x} - \sqrt{y}}{2\sqrt{x}}}{x}\right]$$

$$= \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}}$$

But $\sqrt{x} + \sqrt{y} = 1$ is a given function

$$= \frac{1}{2x\sqrt{x}}$$

$$\therefore y'' = \frac{1}{2x^{\frac{3}{2}}}$$

Chapter 2 Derivatives Exercise 2.6 37E

Consider the equation

$x^3 + y^3 = 1$.

Now find y' by using implicit differentiation.

So take $\frac{d}{dx}$ on both sides of an equation, $x^3 + y^3 = 1$ and remembering that y is a function of x.

Then,

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) = \frac{d}{dx}(1)$$

$$3x^2 + \frac{d}{dx}\left(y^3\right) = 0 \quad \ldots\ldots (1)$$

Here y is a function of x and using the Chain Rule,

$$\frac{d}{dx}\left(y^3\right) = \frac{d}{dy}\left(y^3\right) \cdot \frac{dy}{dx} \text{ (In Leibniz notation)}$$

$$= 3y^2 y'$$

Now substitute the above value into equation (1),

$$3x^2 + 3y^2 y' = 0.$$

Now solve the above equation for y'.

Take the term $3x^2$ to the right side,

$$3y^2 y' = -3x^2.$$

Dividing both sides by $3y^2$,

$$y' = \frac{-3x^2}{3y^2}$$

$$= \frac{-x^2}{y^2}.$$

To find y'', again use implicit differentiation.

So take $\frac{d}{dx}$ on both sides of $y' = \frac{-x^2}{y^2}$.

This is quotient of functions, so use the Quotient Rule, which states

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

Here, the top function is $u(x) = -x^2$ and its derivative is $u'(x) = -2x$.

The bottom function is $v(x) = y^2$

And its derivative is $\frac{d}{dx}[v(x)] = \frac{d}{dx}[y^2]$

Here *y* is a function of *x* and using the Chain Rule,

$$\frac{d}{dx}[y^2] = \frac{d}{dy}(y^2)\frac{dy}{dx} \text{ (In Leibniz notation)}$$

$$= 2yy'$$

Then,

$$v'(x) = 2yy'$$

Now substitute the values of $u(x), u'(x), v(x), v'(x)$ into the Quotient Rule formula,

$$y'' = \frac{y^2(-2x) - (-x^2)2yy'}{y^4}$$

$$= \frac{-2xy^2 + 2x^2yy'}{y^4}$$

Now substitute $y' = -\frac{x^2}{y^2}$ into $y'' = \frac{-2xy^2 + 2x^2yy'}{y^4}$,

$$y'' = \frac{-2xy^2 + 2x^2y\left(-x^2/y^2\right)}{y^4}$$

$$y'' = \frac{-2xy^2 - \frac{2x^4}{y}}{y^4}$$

Multiplying top and bottom by y,

$$y'' = \frac{-2xy^2(y) - \frac{2x^4}{y}(y)}{y^4(y)}$$

$$= \frac{-2xy^3 - 2x^4}{y^5}$$

$$= \frac{-2x(y^3 + x^3)}{y^5}$$

Now, from the original equation, $y^3 + x^3 = 1$ and substitute this value into

$y'' = \frac{-2x(y^3 + x^3)}{y^5}$, to get

$$y'' = \frac{-2x(1)}{y^5}$$

Thus,

$$y'' = \boxed{-\frac{2x}{y^5}}.$$

Given function is $x^4+y^4=a^4$

Differentiating both sides with respect to x

$$\frac{d}{dx}\left(x^4+y^4\right)=\frac{d}{dx}a^4$$

$$\frac{d}{dx}x^4+\frac{d}{dx}y^4=0$$

Since by $\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$ and derivative of constant is zero

$\frac{d}{dx}a^4=0$ where a^4 is a constant.

Now using the formula $\frac{d}{dx}x^n=n\cdot x^{n-1}$ remembering that y is a function of x given

$$4x^3+4y^3\frac{dy}{dx}=0$$

$$4y^3\frac{dy}{dx}=-4x^3$$

$$\frac{dy}{dx}=-\frac{x^3}{y^3}$$

By using quotient rule and differentiating by remembering y is a function of x

$$y''=-\left[\frac{y^3\frac{d}{dx}\left(x^3\right)-x^3\frac{d}{dx}y^3}{\left(y^3\right)^2}\right]$$

$$=-\left[\frac{y^3\cdot 3x^2-x^3\cdot 3y^2\frac{dy}{dx}}{y^6}\right]$$

Substituting $\frac{dy}{dx}=-\frac{x^3}{y^3}$ gives

$$y''=-\left[\frac{3x^2y^3-3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}\right]$$

By simplification using algebra rules

$$y''=-\left[\frac{\frac{3x^2y^4+3x^6}{y}}{y^6}\right]$$

$$=-\left[\frac{3x^2y^4+3x^6}{y^7}\right]$$

$$=-\left[\frac{3x^2\left(y^4+x^4\right)}{y^7}\right]$$

By factoring $3x^2$

$$=-\frac{3x^2\times a^4}{y^7}\qquad (\text{Since } x^4+y^4=a^4)$$

$$y''=-\frac{3x^2a^4}{y^7}$$

$$\therefore y''=-\frac{3a^4x^2}{y^7}$$

Chapter 2 Derivatives Exercise 2.6 39E

The measure of rate of change of a quantity with respect to some other quantity is called as the derivative and the method to determine the derivative is called differentiation.

Consider the equation:

$$xy + y^3 = 1$$

Differentiate the above equation implicitly with respect to *x*:

$$\frac{d}{dx}\left(xy + y^3\right) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}(xy) + \frac{d}{dx}\left(y^3\right) = 0$$
$$x\frac{dy}{dx} + y\frac{dx}{dx} + \frac{d}{dx}\left(y^3\right) = 0$$
$$x\frac{dy}{dx} + y\frac{dx}{dx} + 3y^2\frac{dy}{dx} = 0$$

Simplify the above equation:

$$xy' + y + 3y^2y' = 0$$
$$\left(x + 3y^2\right)y' + y = 0 \quad \text{...... (1)}$$

Solve the above equation for y':

$$y' = -\frac{y}{x + 3y^2}$$

Denote y by y_0 at $x = 0$:

$$\left(y'\right)_{x=0} = \frac{-y_0}{0 + 3y_0^2}$$
$$= \frac{-y_0}{3y_0^2} \quad \text{...... (2)}$$
$$= -\frac{1}{3y_0}$$

Determine the second derivative of *y* by implicit differentiation:

$$\frac{d}{dx}\left(xy' + y + 3y^2y'\right) = \frac{d}{dx}(0)$$
$$\frac{d}{dx}\left(xy'\right) + \frac{dy}{dx} + \frac{d}{dx}\left(3y^2y'\right) = 0$$
$$x\frac{d}{dx}\left(y'\right) + y'\frac{dx}{dx} + \frac{dy}{dx} + 3y^2\frac{d}{dx}\left(y'\right) + y'\frac{d}{dx}\left(3y^2\right) = 0$$
$$xy'' + y' + y' + 3y^2y'' + y'\left(6yy'\right) = 0$$

Simplify the above equation and solve for y'':

$$xy'' + 2y' + 3y^2y'' + 6y\left(y'\right)^2 = 0$$
$$\left(x + 3y^2\right)y'' + 2y' + 6y\left(y'\right)^2 = 0$$
$$y'' = \frac{-2y' - 6y\left(y'\right)^2}{x + 3y^2}$$

Denote y at $x=0$:

$$(y'')_{x=0} = \frac{-2(y')_{x=0} - 6y_0[y']^2_{x=0}}{0+3y_0^2}$$

$$= \frac{-2\left(\frac{-1}{3y_0}\right) - 6y_0\left[\frac{-1}{3y_0}\right]^2}{3y_0^2}$$

$$= \frac{\frac{2}{3y_0} - 6y_0\left(\frac{1}{9y_0^2}\right)}{3y_0^2}$$

$$= \frac{\frac{2}{3y_0} - \frac{2}{3y_0}}{3y_0^2}$$

Simplify the above equation further:

$$(y'')_{x=0} = \frac{0}{3y_0^2}$$
$$= 0$$

Hence, the final expression is $\boxed{(y'')_{x=0} = 0}$.

Chapter 2 Derivatives Exercise 2.6 40E

Consider the function $x^2 + xy + y^3 = 1$.

The value of the variable y at the point $x=1$ is,

$$x^2 + xy + y^3 = 1$$
$$1 + y + y^3 = 1 \quad (\text{Substitute } x=1)$$
$$y(y^2+1) = 0$$
$$y = 0$$

Apply the implicit differentiation first time to the above equation,

$$x^2 + xy + y^3 = 1$$
$$2x + (xy' + y) + (3y^2 y') = 0 \quad ......(1)$$

The value of the derivative at $x=1$ and $y=0$ is,

$$2 + (y' + y) + (3y^2 y') = 0$$
$$2 + (y' + 0) + (3(0)^2 y') = 0 \quad (\text{Substitute } x=1 \text{ and } y=0)$$
$$y' = -2$$

Apply the implicit differentiation second time to the equation – (1),

$$2 + (xy'' + y' + y') + (6y \cdot y' \cdot y' + 3y^2 \cdot y'') = 0$$
$$2 + (xy'' + 2y') + (6y \cdot (y')^2 + 3y^2 \cdot y'') = 0 \quad ......(2)$$

The value of second derivative at $x=1, y=0$ and $y'=-2$ is,

$$2 + (xy'' + 2y') + (6y \cdot (y')^2 + 3y^2 \cdot y'') = 0$$
$$2 + (1 \cdot y'' + 2(-2)) + (6(0) \cdot (-2)^2 + 3(0)^2 \cdot y'') = 0$$
$$2 + y'' - 4 = 0$$
$$y'' = 2$$

Apply the implicit differentiation second time to the equation – (2),

$$2+(xy''+2y')+\left(6y\cdot(y')^2+3y^2\cdot y''\right)=0$$

$$0+(xy'''+y''+2y'')+\left(12y\cdot(y')(y'')+6y'\cdot(y')^2+6y\cdot y'\cdot y''+3y^2\cdot y'''\right)=0$$

$$(xy'''+3y'')+\left(18y\cdot(y')(y'')+6(y')^3+3y^2\cdot y'''\right)=0$$

The value of second derivative at $x=1, y=0, y'=-2$ and $y''=2$ is,

$$(xy'''+3y'')+\left(18y\cdot(y')(y'')+6(y')^3+3y^2\cdot y'''\right)=0$$

$$(1\cdot y'''+3(2))+\left(18(0)\cdot(-2)(2)+6(-2)^3+3(0)^2\cdot y'''\right)=0$$

$$y'''+6+0-48=0$$

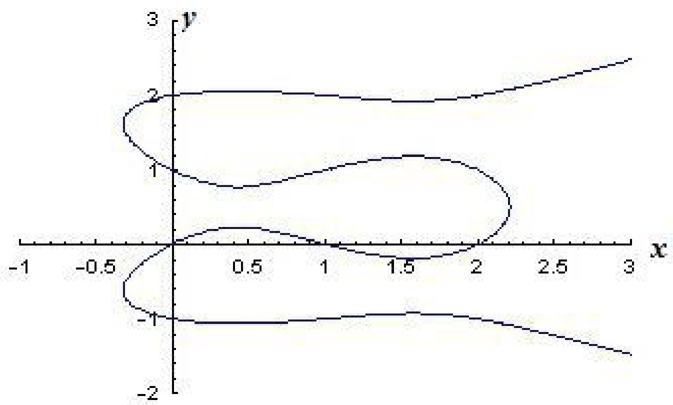
$$y'''=42$$

Hence, the value of the derivative is $\boxed{y''=42}$.

Chapter 2 Derivatives Exercise 2.6 41E

(a)

Graph of the curve with equation $y(y^2-1)(y-2)=x(x-1)(x-2)$ as shown below:



It appears that there are eight points where the tangent line is horizontal, with approximate x-values being $\boxed{x\approx 0.42}$ and $\boxed{x\approx 1.58}$.

(b)

To find the slope of the tangent line at a given point, use implicit differentiation to find y'. To make the differentiation simpler, begin by multiplying out both sides.

$$y^4 - 2y^3 - y^2 + 2y = x^3 - 3x^2 + 2x$$

Now, differentiate both sides with respect to x and solve for y'.

$$\begin{aligned} 4y^3y' - 6y^2y' - 2yy' + 2y' &= 3x^2 - 6x + 2 \\ \left(4y^3 - 6y^2 - 2y + 2\right)y' &= 3x^2 - 6x + 2 \\ y' &= \frac{3x^2 - 6x + 2}{4y^3 - 6y^2 - 2y + 2} \end{aligned}$$

So, the slope of the tangent line at $(0,1)$ is given by

$$\begin{aligned} y' &= \frac{3(0) - 6(0) + 2}{4(1) - 6(1) - 2(1) + 2} \\ &= \frac{2}{6-8} \\ &= \frac{2}{-2} \\ &= -1 \end{aligned}$$

Finally, use point-slope form to find the equation of the tangent line.

The tangent line at $(0,1)$ is given by

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -1(x - 0)$$

$$y - 1 = -x$$

$$\boxed{y = -x + 1}$$

Similarly, the slope of the tangent line at $(0,2)$ is given by

$$\begin{aligned} y' &= \frac{3(0) - 6(0) + 2}{4(8) - 6(4) - 2(2) + 2} \\ &= \frac{2}{34-28} \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

Finally, use point-slope form to find the equation of the tangent line.

The tangent line at $(0,2)$ is given by

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{3}(x - 0)$$

$$y - 2 = \frac{1}{3}x$$

$$\boxed{y = \frac{1}{3}x + 2}$$

(c)

The tangent line is horizontal when $y' = 0$. That is $3x^2 - 6x + 2 = 0$. Then

$$\begin{aligned} x &= \frac{6 \pm \sqrt{6^2 - 4\cdot 3\cdot 2}}{2\cdot 3} \\ &= \frac{6 \pm \sqrt{36-24}}{6} \\ &= \frac{6 \pm \sqrt{12}}{6} \\ &= \frac{6 \pm 2\sqrt{3}}{6} \\ &= \frac{6}{6} \pm \frac{2\sqrt{3}}{6} \end{aligned}$$

By using quadratic formula

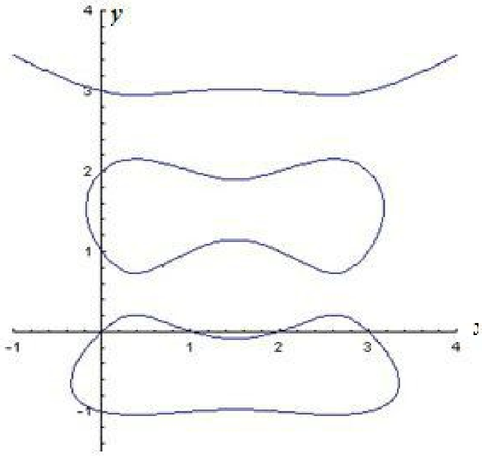
So, the x-coordinates of the points where the tangent line is horizontal are given by

$$\boxed{x = 1 \pm \frac{\sqrt{3}}{3}}.$$

(d)

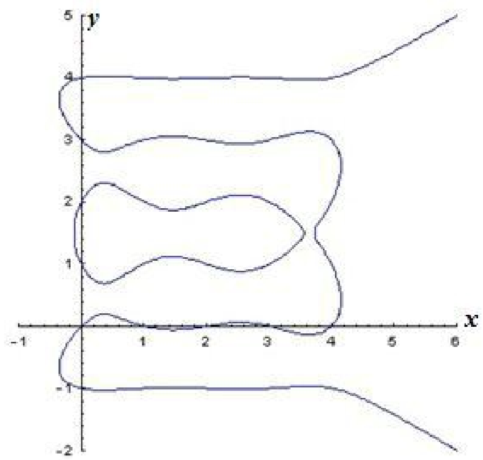
Graph of the modified curve with equation

$y(y^2-1)(y-2)(y-3) = x(x-1)(x-2)(x-3)$ as shown below:



Graph of the modified curve with equation

$y(y^2-1)(y-2)(y-3)(y-4) = x(x-1)(x-2)(x-3)(x-4)$ as shown below:



Chapter 2 Derivatives Exercise 2.6 42E

$$\left(x^2+y^2\right)^2 = ax^2y$$

Differentiating with respect to y on both sides, we get

$$2\left(x^2+y^2\right)\frac{d}{dy}\left(x^2+y^2\right) = \frac{d}{dy}\left(ax^2\right)y + ax^2\frac{dy}{dy}$$

$$\Rightarrow 2\left(x^2+y^2\right)\left(2x\frac{dx}{dy}+2y\right) = 2ax\frac{dx}{dy}.y + ax^2$$

$$\Rightarrow \frac{dx}{dy} = \frac{ax^2-4y\left(x^2+y^2\right)}{4x\left(x^2+y^2\right)-2axy}$$

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The equation of the curve is given as

$$2\left(x^2+y^2\right)^2 = 25\left(x^2-y^2\right)$$

Differentiating with respect to x implicitly

$$4\left(x^2+y^2\right)\left\{2x+2y\frac{dy}{dx}\right\} = 25\left(2x-2y\frac{dy}{dx}\right) \text{ [Chain rule]}$$

$$\Rightarrow \quad 8\left(x^2+y^2\right)x+8y\left(x^2+y^2\right)\frac{dy}{dx} = 50x-50y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left\{8y\left(x^2+y^2\right)+50y\right\} = 50x-8x\left(x^2+y^2\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{50x-8x\left(x^2+y^2\right)}{8y\left(x^2+y^2\right)+50y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{25x-4x\left(x^2+y^2\right)}{4y\left(x^2+y^2\right)+25y}$$

Tangent is horizontal when

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 25x-4x\left(x^2+y^2\right) = 0$$

$$\Rightarrow x^2+y^2 = \frac{25}{4} \qquad ...(1)$$

Putting this value in equation of curve

$$2\left(\frac{25}{4}\right)^2 = 25\left(x^2-y^2\right)$$

$$\Rightarrow \left(x^2-y^2\right) = \frac{25}{8} \quad ....(2)$$

From (1) and (2)

$$2x^2 = \frac{75}{8}$$

OR $x = \pm\frac{5\sqrt{3}}{4}$

And $y^2 = \frac{25}{16}$

OR, $y = \pm\frac{5}{4}$

So the required points are, $\boxed{\left(\pm\frac{5\sqrt{3}}{4},\ \pm\frac{5}{4}\right)}$

Chapter 2 Derivatives Exercise 2.6 44E

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{--- (1)}$$

Differentiate both sides with respect to x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}.y'=0$$

Solving for y',

$$\Rightarrow \frac{2y}{b^2}.y'=-\frac{2x}{a^2}$$

$$\Rightarrow y'=-\frac{2x}{a^2}.\frac{b^2}{2y}$$

$$\Rightarrow \boxed{y'=-\frac{b^2x}{a^2y}}$$

Then the slope of the tangent at (x_o, y_o) is

$$\Rightarrow \boxed{y'=-\frac{b^2x_o}{a^2y_o}}$$

So the equation of the tangent line at (x_o, y_o) is

$$(y-y_o)=-\frac{b^2x_o}{a^2y_o}(x-x_o)$$

$$\Rightarrow \frac{y_o}{b^2}(y-y_o)=-\frac{x_o}{a^2}(x-x_o)$$

$$\Rightarrow \frac{yy_o}{b^2}-\frac{y_o^2}{b^2}=-\frac{xx_o}{a^2}+\frac{x_o^2}{a^2}$$

$$\Rightarrow \frac{yy_o}{b^2}+\frac{x.x_o}{a^2}=\frac{x_o^2}{a^2}+\frac{y_o^2}{b^2}$$

$$\Rightarrow \frac{x_ox}{a^2}+\frac{y_oy}{b^2}=\frac{x_o^2}{a^2}+\frac{y_o^2}{b^2}$$

The point (x_o, y_o) lies on the ellipse so this will satisfy the equation of ellipse.
Thus

$$\Rightarrow \frac{x_o^2}{a^2}+\frac{y_o^2}{b^2}=1$$

So we have

$$\boxed{\frac{x_ox}{a^2}+\frac{y_oy}{b^2}=1}$$

This is the equation of tangent line at (x_o, y_o).

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Equation of the curve is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Differentiating with respect to x implicitly

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

Or $$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Slope of the tangent line at (x_0, y_0) is

$$\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0}$$

Equation x of tangent at (x_o, y_o) is

$$(y - y_0) = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

Or $$\frac{y y_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x x_0}{a^2} - \frac{x_0^2}{a^2}$$

Or $$\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}$$

Or $$\boxed{\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1}$$ $\left[\text{Since } (x_0, y_0) \text{ lies on curve, thus } \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1\right]$

Chapter 2 Derivatives Exercise 2.6 46E

The equation of curve is $\sqrt{y} + \sqrt{x} = \sqrt{C}$.

Then $\sqrt{y} = \sqrt{C} - \sqrt{x}$ --- (1)

Now differentiate both sides with respect to x to get the slope of tangent (y').

$$\frac{1}{2\sqrt{y}}.y' = -\frac{1}{2\sqrt{x}}$$

$$\Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}}$$

From equation (1)

$$\Rightarrow y' = -\frac{\sqrt{C} - \sqrt{x}}{\sqrt{x}}$$

$$\Rightarrow \boxed{y' = \frac{\sqrt{x} - \sqrt{C}}{\sqrt{x}}}$$

Then the equation of tangent line at (x_1, y_1) is

$$(y - y_1) = \frac{\sqrt{x} - \sqrt{C}}{\sqrt{x}}.(x - x_1) \quad \text{--- (2)}$$

Now we get x and y-intercept of the tangent for x-intercept we put $y_1 = 0$ in equation (2).

$$y = \frac{\sqrt{x} - \sqrt{C}}{\sqrt{x}}(x - x_1)$$

$$\Rightarrow (x - x_1) = \frac{\sqrt{x}.y}{\sqrt{x} - \sqrt{C}}$$

$$\Rightarrow x_1 - x = -\frac{\sqrt{x}.y}{\sqrt{x} - \sqrt{C}}$$

$$\Rightarrow x_1 = \frac{-\sqrt{x}.y}{\sqrt{x} - \sqrt{C}} + x$$

$$\Rightarrow x_1 = \frac{\sqrt{x}.y}{-\sqrt{x} + \sqrt{C}} + x$$

$$\Rightarrow x_1 = \frac{\sqrt{x}.\left(\sqrt{C} - \sqrt{x}\right)^2}{\left(\sqrt{C} - \sqrt{x}\right)} + x$$ From equation (1) $\left(\sqrt{y} = \sqrt{C} - \sqrt{x}\right)$

$$\Rightarrow x_1 = \sqrt{x}.\left(\sqrt{C} - \sqrt{x}\right) + x$$

$$\Rightarrow x_1 = \sqrt{Cx} - x + x$$

$$\Rightarrow \boxed{x_1 = \sqrt{Cx}}$$ $\Rightarrow$ x-intercept. --- (3)

Now for y-intercept, we put $x_1 = 0$ in equation (2).

$$y - y_1 = \frac{\sqrt{x} - \sqrt{C}}{\sqrt{x}}.x$$

$$\Rightarrow y - y_1 = \left(\sqrt{x} - \sqrt{C}\right)\sqrt{x}$$

$$\Rightarrow y_1 = y - \left(x - \sqrt{Cx}\right)$$

$$\Rightarrow y_1 = \left(\sqrt{C} - \sqrt{x}\right)^2 - x + \sqrt{Cx} \qquad \text{y-intercept} \qquad (4)$$

Now from equations (3) and (4), the sum of x- and y-intercepts is

$$x_1 + y_1 = 2\sqrt{Cx} - x + \left(\sqrt{C} - \sqrt{x}\right)^2$$

$$= 2\sqrt{Cx} - x + \left(C + x - 2\sqrt{Cx}\right)$$

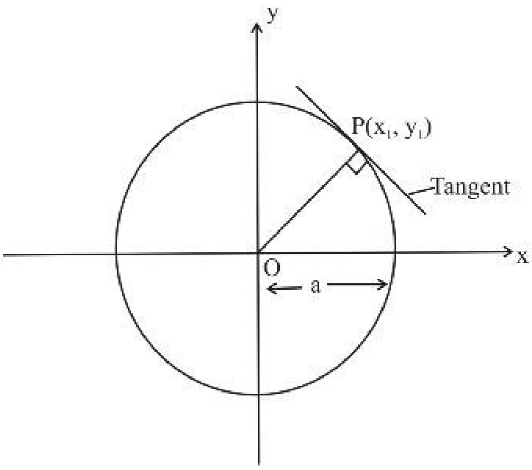
$$= 2\sqrt{Cx} - x + C + x - 2\sqrt{Cx}$$

$$\boxed{x_1 + y_1 = C}$$

Then it is proved that the sum of x- and y-intercepts of any tangent line to the curve, $\sqrt{x} + \sqrt{y} = \sqrt{C}$, is equal to C.

Hence proved

Chapter 2 Derivatives Exercise 2.6 47E



If the radius of a circle centre at origin 0, is a then the equation of the circle is

$$x^2 + y^2 = a^2 \qquad (1)$$

Let the point P has the co-ordinates (x_1, y_1), point P lies on the circle so this will satisfy the equation of the circle thus

$$x_1^2 + y_1^2 = a^2$$

$$\Rightarrow y_1^2 = a^2 - x_1^2$$

$$\Rightarrow y_1 = \pm\sqrt{a^2 - x_1^2}$$

$$\Rightarrow y_1 = \sqrt{a^2 - x_1^2}$$

(Since we took P in first quadrant)

The coordinates of the point P is $\left(x_1, \sqrt{a^2 - x_1^2}\right)$

Then slope of the line OP is

$$= \frac{\sqrt{a^2 - x_1^2} - 0}{x_1 - 0}$$

$$\Rightarrow \boxed{M_1 = \frac{\sqrt{a^2 - x_1^2}}{x_1}} \qquad \text{--- (2)}$$

Where OP is the radius of the circle.

Now differentiate the both sides of the equation of the circle, (1)

$$2x + 2y.y' = 0$$

$$\Rightarrow \boxed{y' = -\frac{x}{y}}$$

Then the slope of the tangent at point P, $\left(x_1, \sqrt{a^2 - x_1^2}\right)$ is

$$\Rightarrow y' = -\frac{x_1}{\sqrt{a^2 - x_1^2}}$$

$$\Rightarrow \boxed{M_2 = -\frac{x_1}{\sqrt{a^2 - x_1^2}}}$$

If two lies with slopes m_1 and m_2, are perpendicular to each other then
$m_1.m_2 = -1$
So here we have to show that
(Slope of the line OP) $\times$(Slope of the tangent at P)= -1

We have (Slope of the line OP) $\times$(Slope of the tangent at P) = M_1, M_2
Now from equation (2) and (3)

$$\Rightarrow M_1 M_2 = \frac{\sqrt{a^2 - x_1^2}}{x_1} \cdot \left(-\frac{x_1}{\sqrt{a^2 - x_1^2}}\right)$$

$$= -\frac{\sqrt{a^2 - x_1^2}}{\sqrt{a^2 - x_1^2}} \cdot \frac{x_1}{x_1}$$

$\boxed{M_1 M_2 = -1}$ Thus tangent at P is perpendicular to radius OP

Chapter 2 Derivatives Exercise 2.6 48E

We have $y^q = x^p$
Differentiate both sides with respect to x

$$\frac{d}{dx}\left(y^q\right) = \frac{d}{dx}\left(x^p\right) \qquad \text{-- (1)}$$

By the power rule of differentiation we have $\frac{d}{dx} x^n = nx^{n-1}$ then equation (1) becomes

$$qy^{q-1}.y' = p.x^{p-1}$$

$$\Rightarrow y' = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}} \qquad \text{--- (2)}$$

We have $y = x^{p/q}$

Then $y^{q-1} = \left(x^{p/q}\right)^{q-1} = x^{p - p/q}$ --- (3)

Then we have $y' = \frac{p}{q} \frac{x^{p-1}}{x^{p-p/q}}$

We have $\boxed{\frac{x^m}{x^n} = x^{m-n}}$ thus

$$y' = \frac{p}{q} x^{p-1-p+p/q}$$

$$\Rightarrow \boxed{y' = \frac{p}{q} x^{p/q-1}}$$

Chapter 2 Derivatives Exercise 2.6 49E

Recall that two curves are **orthogonal** if their lines are perpendicular at each point of intersection.

To find the slope of the tangent line to $x^2 + y^2 = r^2$, use implicit differentiation to find y'

$$\begin{aligned} 2x + 2yy' &= 0 \\ 2yy' &= -2x \\ y' &= \frac{-2x}{2y} \\ m_1 &= -\frac{x}{y} \end{aligned}$$

Similarly, find the slope of the tangent line to $ax + by = 0$.

$$\begin{aligned} a + by' &= 0 \\ by' &= -a \\ y' &= -\frac{a}{b} \\ m_2 &= -\frac{a}{b} \end{aligned}$$

At a point where the curves intersect, solve the linear equation for y, to get

$$\begin{aligned} ax + by &= 0 \\ by &= -ax \\ y &= \frac{-ax}{b} \end{aligned}$$

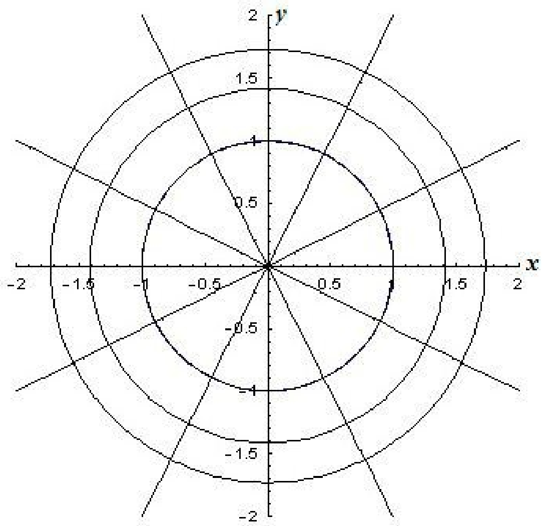
The slope of the tangent line to the linear equation is $-\frac{a}{b}$, and the slope of the other tangent line is

$$\begin{aligned} m_1 &= -\frac{x}{y} \\ &= -\frac{x}{(-ax/b)} \\ &= \frac{bx}{ax} \\ &= \frac{b}{a} \end{aligned}$$

Since $m_1 \cdot m_2 = -1$, the tangent lines are orthogonal.

Therefore, by the recall, the given families of curves are orthogonal trajectories of each other, that is, every curve in one family is orthogonal to every curve in the other family.

Sketch both families of curves on the same axes as shown below:



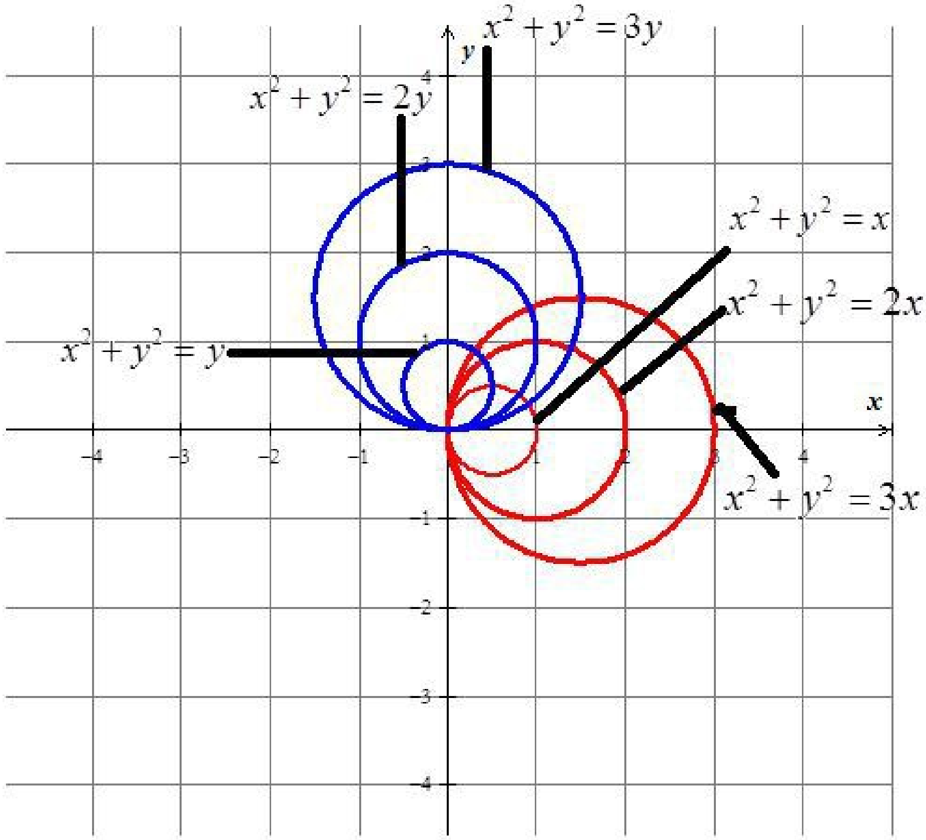
Chapter 2 Derivatives Exercise 2.6 50E

Consider the following family of curves:

$x^2 + y^2 = ax$, a is parameter ...... (1)

$x^2 + y^2 = by$ b is parameter ...... (2)

The graph of a finite sample of these curves is pictured below with $x^2 + y^2 = ax$ in red for values of $a = 1, 2, 3$ and $x^2 + y^2 = by$ in blue for values of $b = 1, 2, 3$



Need to show that these curves are orthogonal to each other.

That is, to say, for any value of the parameter a, and any value of the parameter b, curves are orthogonal to each other.

When the curves in red and blue intersect, their respective tangent lines are perpendicular at the points of intersection.

First to find the points of intersection, let's multiply the equation (2) by – 1

And add it to equation (1)

$$\begin{array}{rl} & x^2 + y^2 = ax \\ + & -x^2 - y^2 = -by \\ \hline & 0 = ax - by \end{array}$$

That is, see that the points (x, y) of intersection lie on the straight line through the origin

$ax - by = 0$

Next let's differentiate each curve implicitly, to find the slope of the tangent lines.

Start with equation (1), take $\frac{d}{dx}$ of both sides as follows:

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[ax]$$
$$\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right]=a\frac{d}{dx}[x]$$

Use the Chain Rule to find the derivative the 2nd term on the left side, and switching notation with $y'=\frac{dy}{dx}$.

$$2x+\frac{d}{dy}\left[y^2\right]\cdot\frac{dy}{dx}=a\cdot 1$$
$$2x+2yy'=a$$
$$2yy'=a-2x$$
$$y'=\frac{a-2x}{2y}$$

So, at any point on curve (1) the slope of the tangent line m_1 is

$$m_1=\frac{a-2x}{2y}.$$

For the equation (2) take $\frac{d}{dx}$ of both sides as follows:

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[by]$$
$$\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right]=b\frac{d}{dx}[y]$$

Use the Chain Rule, to find the derivative the 2nd term on the left side and the one term on the right side. Switch the notation with $y'=\frac{dy}{dx}$.

$$2x+\frac{d}{dy}\left[y^2\right]\cdot\frac{dy}{dx}=b\frac{d}{dy}[y]\cdot\frac{dy}{dx}$$
$$2x+2y\cdot y'=b\cdot 1\cdot y'$$
$$2x+2yy'=by'$$
$$by'=2x+2yy'$$
$$by'-2yy'=2x$$
$$(b-2y)y'=2x$$
$$y'=\frac{2x}{b-2y}$$

So, at any point on curve (2) the slope of the tangent line m_2 is:

$$m_2=\frac{2x}{b-2y}.$$

Fact: Two lines are perpendicular if and only if the product of their slopes is -1.

So, multiply the two slopes found and see if indeed this is the case.

Claim: $m_1\cdot m_2=-1$

$$m_1\cdot m_2=\frac{a-2x}{2y}\cdot\frac{2x}{b-2y}$$
$$=\frac{x(a-2x)}{y(b-2y)} \quad \ldots\ldots (3)$$

But recall that at the points of intersection, then x and y satisfy $ax-by=0$.

Solve this equation for y:

$$y=\frac{a}{b}x.$$

Plug in this into equation (3) and get:

$$\begin{aligned}m_1\cdot m_2&=\frac{x(a-2x)}{\frac{a}{b}x\left(b-2\cdot\frac{a}{b}x\right)}\\&=\frac{x(a-2x)}{ax-\frac{2a^2x^2}{b^2}}\\&=\frac{b^2}{b^2}\cdot\frac{x(a-2x)}{ax-\frac{2a^2x^2}{b^2}}\\&=\frac{ab^2x-2b^2x^2}{ab^2x-2a^2x^2}\end{aligned}$$

Factoring out x and dividing, assuming $x\neq 0$ then:

$$m_1\cdot m_2=\frac{ab^2-2b^2x}{ab^2-2a^2x} \text{ ...... (4)}$$

From equation (1):

$$y^2=ax-x^2.$$

Square both sides of equation (2)

$$\begin{aligned}\left(x^2+y^2\right)^2&=(by)^2\\(x^2)^2+2x^2y^2+(y^2)^2&=b^2y^2\\x^4+2x^2y^2+y^4&=b^2y^2\end{aligned}$$

Substitute this last equation $y^2=ax-x^2$ in for the variable y^2.

$$\begin{aligned}x^4+2x^2\left(ax-x^2\right)+\left(ax-x^2\right)^2&=b^2\left(ax-x^2\right)\\x^4+2ax^3-2x^4+a^2x^2-2axx^2+x^4&=ab^2x-b^2x^2\\a^2x^2&=ab^2x-b^2x^2\\\left(a^2+b^2\right)x^2&=ab^2x\end{aligned}$$

Divide by x, that is assuming $x\neq 0$.

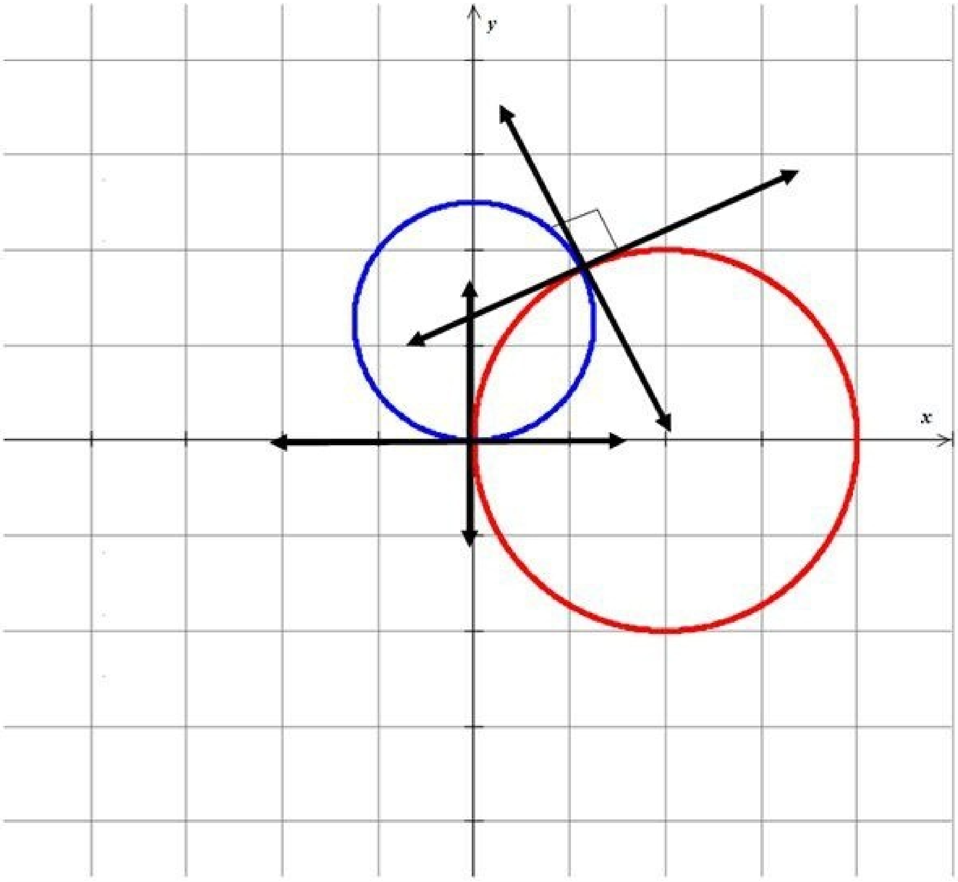
$$\left(a^2+b^2\right)x=ab^2 \text{ ...... (5)}$$

Use equation (5) for ab^2 in (4).

$$\begin{aligned}m_1\cdot m_2&=\frac{ab^2-2b^2x}{ab^2-2a^2x}\\&=\frac{\left(a^2+b^2\right)x-2b^2x}{\left(a^2+b^2\right)x-2a^2x}\\&=\frac{\left(a^2-b^2\right)x}{-\left(a^2-b^2\right)x}\\&=-1,\ \text{assuming } x\neq 0\end{aligned}$$

Therefore, our original claim above is true and hence the tangent lines are indeed perpendicular at the points where the two curves intersect.

Below is a visual aid of the two curves intersecting and the perpendicular tangent lines at the two points of intersection.



The curve in red color indicates the curve $x^2+y^2=ax$ and the curve in blue color indicates the curve of the form $x^2+y^2=by$ and when these two tangents of these curves intersect the tangent seen are perpendicular.

Hence, the family of these given curves is orthogonal trajectories.

Chapter 2 Derivatives Exercise 2.6 51E

Recall that two curves are **orthogonal** if their lines are perpendicular at each point of intersection.

To find the slope of the tangent line of $y=cx^2$, differentiate to find y'.

$$y'=c(2x)$$
$$m_1=2cx$$

To find the slope of the tangent line of $x^2+2y^2=k$, use implicit differentiation to find y'.

$$2x+4yy'=0$$
$$4yy'=-2x$$
$$y'=\frac{-2x}{4y}$$
$$m_2=-\frac{x}{2y}$$

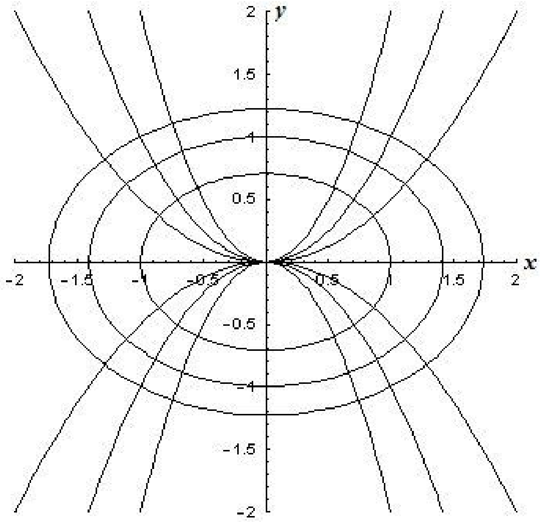
Plugging in the value of y from the first equation, the second slope is

$$m_2=-\frac{x}{2(cx^2)}$$
$$=-\frac{1}{2cx}$$

Since $m_1\cdot m_2=-1$, the tangent lines are orthogonal.

Therefore, by the recall, the given families of curves are orthogonal trajectories of each other, that is, every curve in one family is orthogonal to every curve in the other family.

Sketch both families of curves on the same axes as shown below:



Chapter 2 Derivatives Exercise 2.6 52E

Recall that two curves are **orthogonal** if their lines are perpendicular at each point of intersection.

To find the slope of the tangent line to $y = ax^3$, differentiate to find y'.

$$y' = a\left(3x^2\right)$$
$$m_1 = 3ax^2$$

To find the slope of the tangent line to $x^2 + 3y^2 = b$, use implicit differentiation to find y'.

$$2x + 6yy' = 0$$
$$6yy' = -2x$$
$$y' = -\frac{2x}{6y}$$
$$m_2 = -\frac{x}{3y}$$

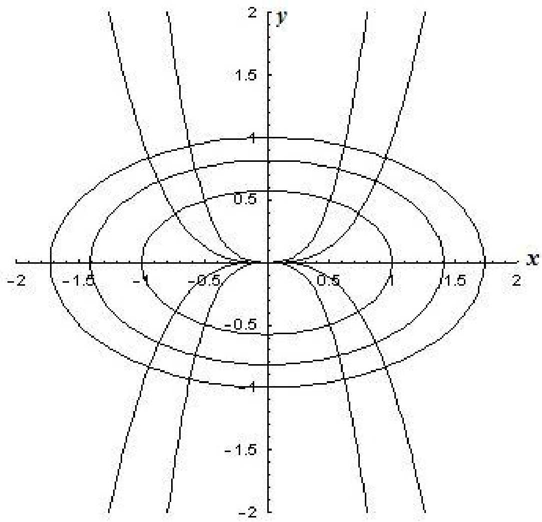
Plugging in the value of y from the first equation, the second slope is

$$m_2 = -\frac{x}{3\left(ax^3\right)}$$
$$= -\frac{1}{3ax^2}$$

Since $m_1 \cdot m_2 = -1$, the tangent lines are orthogonal.

Therefore, by the recall, the given families of curves are orthogonal trajectories of each other, that is, every curve in one family is orthogonal to every curve in the other family.

Sketch both families of curves on the same axes as shown below:



Chapter 2 Derivatives Exercise 2.6 53E

Consider the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

To find the derivative, apply the implicit differentiation to the equation of ellipse.

The derivative of the equation of the ellipse is,

$$\begin{aligned}\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}&=0\\ \frac{2y}{b^2}\cdot\frac{dy}{dx}&=-\frac{2x}{a^2}\\ \frac{dy}{dx}&=-\frac{2x}{a^2}\cdot\frac{b^2}{2y}\\ &=\frac{-xb^2}{ya^2}\end{aligned}$$

Hence, the derivative of the equation of the ellipse is $\boxed{m_1=\frac{-xb^2}{ya^2}}$.

To find the derivative of the equation of hyperbola, apply the implicit differentiation to the equation of the hyperbola $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$.

The derivative of the equation of the hyperbola is,

$$\begin{aligned}\frac{2x}{A^2}-\frac{2y}{B^2}\cdot\frac{dy}{dx}&=0\\ -\frac{2y}{B^2}\cdot\frac{dy}{dx}&=-\frac{2x}{A^2}\\ \frac{dy}{dx}&=\frac{2x}{A^2}\cdot\frac{B^2}{2y}\\ &=\frac{xB^2}{yA^2}\end{aligned}$$

Hence, the derivative of the equation of the ellipse is $\boxed{m_2=\frac{xB^2}{yA^2}}$.

The objective is given that, the equation of the ellipse and hyperbola is orthogonal trajectories.

So,

$$m_1 \cdot m_2 = -1$$
$$\left(\frac{-xb^2}{ya^2}\right)\cdot\left(\frac{xB^2}{yA^2}\right) = -1$$
$$x^2B^2b^2 = y^2A^2a^2$$
$$x^2B^2 = \frac{y^2A^2a^2}{b^2} \quad ......(1)$$

From the equation of the hyperbola,

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$$
$$B^2x^2 - A^2y^2 = A^2B^2$$
$$\frac{y^2A^2a^2}{b^2} - A^2y^2 = A^2B^2$$
$$y^2A^2\left(a^2-b^2\right) = A^2B^2b^2$$
$$y^2 = \frac{B^2b^2}{a^2-b^2} \quad ......(2)$$

Substitute the equation (2) in equation (1),

$$x^2B^2 = \frac{y^2A^2a^2}{b^2}$$
$$x^2B^2 = \frac{\left(\frac{B^2b^2}{a^2-b^2}\right)A^2a^2}{b^2}$$
$$x^2\cancel{B^2} = \frac{\cancel{B^2}\,\cancel{b^2}\cdot A^2a^2}{\left(a^2-b^2\right)\cancel{b^2}}$$
$$x^2 = \frac{A^2a^2}{a^2-b^2} \quad ......(3)$$

Substitute the equation (2) and equation (3) in the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\frac{\frac{A^2\cancel{a^2}}{a^2-b^2}}{\cancel{a^2}} + \frac{\frac{B^2\cancel{b^2}}{a^2-b^2}}{\cancel{b^2}} = 1$$
$$\frac{A^2}{a^2-b^2}+\frac{B^2}{a^2-b^2}=1$$
$$\frac{A^2+B^2}{a^2-b^2}=1$$
$$A^2+B^2 = a^2-b^2$$

Hence, the ellipse and hyperbola have the same foci

Chapter 2 Derivatives Exercise 2.6 54E

Given curves are $y=\frac{1}{x+c}$ and $y=a\left(x+k\right)^{\frac{1}{3}}$

Then $m_1 = y' = \frac{-1}{\left(x+c\right)^2} = -y^2$ and

$$m_2 = y' = \frac{a}{3}\left(x+k\right)^{\frac{-2}{3}}$$
$$= \frac{a}{3}\left[\left(x+k\right)^{\frac{1}{3}}\right]^{-2}$$
$$= \frac{a}{3}\left[\frac{y}{a}\right]^{-2} = \frac{a}{3}\frac{a^2}{y^2} = \frac{a^3}{3y^2}$$

Since curves are orthogonal, so $m_1 m_2 = -1$

$$\left[-y^2\right]\left[\frac{a^3}{3y^2}\right] = -1$$

$$\Rightarrow \quad \frac{a^3}{3} = 1 \quad \Rightarrow \quad a^3 = 3$$

$$\Rightarrow \quad a = \sqrt[3]{3}$$

$$\boxed{\therefore\ a = \sqrt[3]{3}}$$

Chapter 2 Derivatives Exercise 2.6 55E

(a) The vander waals equation for n moles of a gas is $\left(p+\frac{n^2a}{v^2}\right)(v-nb) = nRT$, where

p is the pressure, v is the volume and T is the temperature of the gas.

The constant R is the universal gas constant and a and b are positive constants that are characteristic of a particular gas. T is also constant

Differentiating implicitly w.r.t p and v is a function of v, we get

$$\left(p+\frac{n^2a}{v^2}\right)\frac{d}{dp}(v-nb)+(v-nb)\frac{d}{dp}\left(p+\frac{n^2a}{v^2}\right) = \frac{d}{dp}nRT = 0$$

$$\Rightarrow \quad \left(p+\frac{n^2a}{v^2}\right)\left(\frac{dv}{dp}\right)+(v-nb)+\left[1+n^2a(-2)v^{-3}\frac{dv}{dp}\right] = 0$$

$$\Rightarrow \quad \boxed{\frac{dv}{dp} = \frac{v^3(nb-v)}{pv^3-n^2av+2n^3ab}}$$

(b) Here $n=1$, $v=10L$, $p=2.5$ atm, $a=3.592\ \mathrm{L^2-atm/mole^2}$ and $b=0.04267\ \mathrm{L/mole}$

$$\left(\frac{dv}{dp}\right)_{p=2.5} = \frac{v^3(nb-v)}{pv^3-n^2av+2n^3ab}$$

$$= \frac{(10)^3(0.04267-10)}{2.5(10)^3-(3.592)(10)+2(3.592)(0.04267)}$$

$$= -4.04\ \mathrm{L/atm}$$

$$\boxed{-4.04\ \mathrm{L/atm}}$$

Chapter 2 Derivatives Exercise 2.6 56E

Consider the equation $x^2+xy+y^2+1=0$.

(a)

To find the derivative of the function, apply the implicit differentiation to the above equation.

The derivative of the equation is,

$$x^2+xy+y^2+1=0$$
$$2x+y+xy'+2yy'=0$$
$$2x+y+(x+2y)y'=0$$
$$y' = \frac{-(2x+y)}{(x+2y)}$$

Hence, the derivative of the function is $\boxed{y' = \frac{-(2x+y)}{(x+2y)}}$.

(b)

To sketch the graph of the equation, use the maple software.

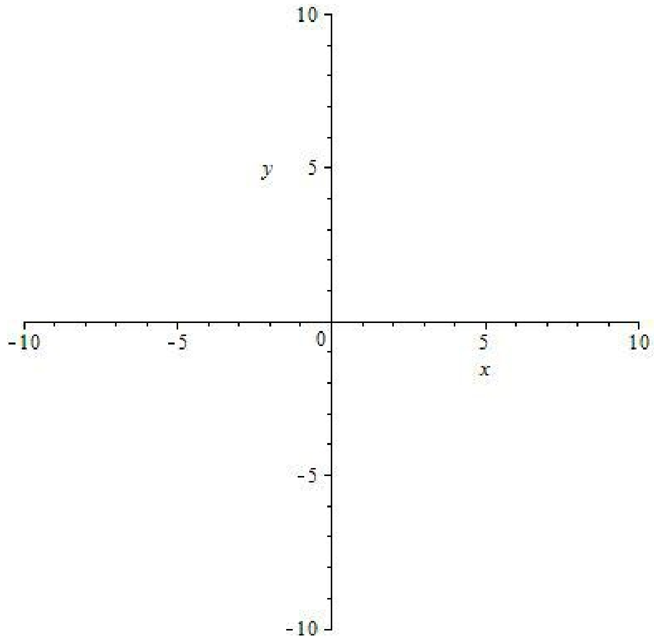
Type the below input into the maple software,

Input:

> $with(plots):$

> $implicitplot(x^2 + x \cdot y + y^2 + 1 = 0, x = -10..10, y = -10..10)$

The output is as follows.



It can be observed that, there is no curve is appearing in the rectangular coordinate system.

If we change the range of the variables x and y then also, no curve is appearing in the graph.

This means that, we need to prove that for every real value of x there does not exist any real value y.

Take $x = k$ where any real number is k and the corresponding value is,

$$x^2 + xy + y^2 + 1 = 0$$
$$k^2 + ky + y^2 + 1 = 0$$
$$y^2 + y(k) + (k^2 + 1) = 0$$

This equation is a quadratic equation in terms of y and it has real roots only when its discriminant is greater than or equal to zero.

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= k^2 - 4(k^2 + 1) \\ &= -3k^2 - 4 \\ &< 0 \text{ for all } k\end{aligned}$$

This shows that the equation $y^2 + y(k) + (k^2 + 1) = 0$ has no real roots, for every value of x there does not exist any real number y.

This is the reason, the maple did not show any curve for the given equation.

(c)

The derivative of the given function at any point is the slope of the tangent at that point.

Here, the given equation does not have any curve in the graph which means that there is no use with the derivative of the given equation.

Chapter 2 Derivatives Exercise 2.6 57E

Given equation is $x^2 - xy + y^2 = 3$ ....(1)

We find x-intercepts of this curve. Here we put $y = 0$ and solve for x.

We get $x^2 = 3$

Or $x = \pm\sqrt{3}$

So at the points $\boxed{(\sqrt{3},\ 0);(-\sqrt{3},\ 0)}$ the ellipse crosses the x-axis.

Differentiating (1) with respect to x implicitly, we get

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

Or $$\frac{dy}{dx}(2y - x) = y - 2x$$

Or $$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

Now the slope of the tangent line at $(\sqrt{3},\ 0)$ is

$$m_1 = \left(\frac{dy}{dx}\right)_{\sqrt{3},0} = \frac{-2\sqrt{3}}{-\sqrt{3}} = 2$$

And the slope of the tangent line at $(-\sqrt{3},\ 0)$ is

$$m_2 = \left(\frac{dy}{dx}\right)_{(-\sqrt{3},\ 0)} = \frac{1\sqrt{3}}{\sqrt{3}} = 2$$

Thus $m_1 = m_2$

Therefore, the tangents are parallel to each other at these points.

Chapter 2 Derivatives Exercise 2.6 58E

(A) Equation of the ellipse $x^2 - xy + y^2 = 3$

Differentiate both sides with respect to x, we get

$$2x - (xy' + y) + 2yy' = 0$$
$$\Rightarrow 2x - xy' + 2yy' - y = 0$$
$$\Rightarrow y'(2y - x) + 2x - y = 0$$
$$\Rightarrow y'(2y - x) = y - 2x$$
$$\Rightarrow \boxed{y' = \frac{y - 2x}{2y - x}}$$

At the point (-1, 1) the slope of tangent line is

$$m = y'_{(-1,1)} = \frac{-1-2}{-2-1} = \frac{-3}{-3} = 1$$

So the slope of the normal line=-1/m = -1

Hence, equation of the normal at (-1, 1) is

$$(y-1) = -1.(x+1)$$
$$\boxed{y = -x}$$

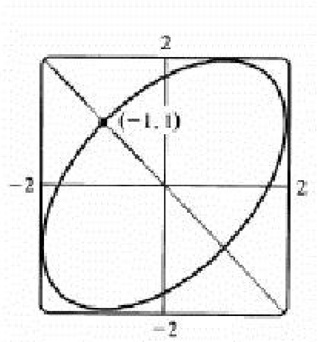
On putting this value of y in equation of ellipse, we get

$$x^2 - (x)(-x) + (-x)^2 = 3$$
$$\Rightarrow x^2 + x^2 + x^2 = 3$$
$$\Rightarrow 3x^2 = 3$$
$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

For $x = 1$, we get $y = -1$.

Hence, another point is (1, -1) at which the normal line intersects the ellipse.

(B)



Chapter 2 Derivatives Exercise 2.6 59E

Equation of the curve is $x^2y^2+xy=2$

Then differentiate both sides with respect to x

$$\frac{d}{dx}\left(x^2y^2\right)+\frac{d}{dx}(xy)=\frac{d}{dx}(2)$$

$$x^2\frac{d}{dx}\left(y^2\right)+y^2\frac{d}{dx}\left(x^2\right)+x\frac{d}{dx}(y)+y\frac{d}{dx}(x)=\frac{d}{dx}(2)$$

$$\Rightarrow x^2.2y.y'+y^2.2x+xy'+y=0$$

$$\Rightarrow 2x^2yy'+2xy^2+xy'+y=0$$

$$\Rightarrow y'(2x^2y+x)+(2xy+1)y=0$$

$$\Rightarrow y'=-\frac{(2xy+1)y}{(2x^2y+x)}$$

$$\Rightarrow y'=-\frac{(2xy+1)y}{(2xy+1)x}$$

$$\Rightarrow \boxed{y'=-\frac{y}{x}}$$

The condition is

The slope of tangent line = -1

$$\Rightarrow y'=-1$$

$$\Rightarrow -\frac{y}{x}=-1$$

$$\Rightarrow \boxed{y=x}$$

Put the values of y in equation of the curve

$$x^2y^2+xy=2$$

$$\Rightarrow x^2.x^2+x.x=2$$

$$\Rightarrow x^4+x^2=2$$

$$\Rightarrow x^4+x^2-2=0$$

$$\Rightarrow x^2\left(x^2+2\right)-1\left(x^2+2\right)=0$$

$$\Rightarrow \left(x^2+2\right)\left(x^2-1\right)=0$$

$\Rightarrow x^2-1=0$ Since other factor can not be 0

$$\Rightarrow x^2=1$$

$$\Rightarrow x=\pm 1$$

Hence we have the points (-1, -1) and (1, 1) at which the slope of the tangent line is -1

$$\Rightarrow x^2\left(x^2+2\right)-1\left(x^2+2\right)=0$$

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Hence we have the points (-1, -1) and (1, 1) at which the slope of the tangent line is -1

Equation of the ellipse is $x^2+4y^2=36$ (1)

Differentiate both sides of the equation with respect to x, we get

$$2x+8yy'=0$$
$$\Rightarrow x+4yy'=0$$
$$\Rightarrow \boxed{y'=-\frac{x}{4y}}$$

Suppose that tangent line touches the ellipse at (a, b) then equations of the tangent is,

$$(y-b)=-\frac{a}{4b}(x-a)$$
$$\Rightarrow 4by-4b^2=-ax+a^2$$
$$\Rightarrow 4by+ax=a^2+4b^2$$

This point (a, b) lies on the ellipse so

$$a^2+4b^2=36 \qquad (2)$$

Thus the equation of the tangent line is

$$\boxed{ax+4by=36}$$

Now the tangent line passes though the point (12,3), so this point will satisfy the equation of tangent line.

$$\Rightarrow 12a+12b=36$$
$$\Rightarrow a+b=3$$
$$\Rightarrow \boxed{a=3-b}$$

On putting this value of a in equations (2), we get

$$(3-b)^2+4b^2=36$$
$$\Rightarrow 9+b^2-6b+4b^2=36$$
$$\Rightarrow 5b^2-6b-27=0$$
$$\Rightarrow 5b^2+9b-15b-27=0$$
$$\Rightarrow b(5b+9)-3(5b+9)=0$$
$$\Rightarrow (5b+9)(b-3)=0$$

i.e. $b=-\frac{9}{5}$ and $b=3$

Thus

$$a=3-b$$
$$\Rightarrow a=3+\frac{9}{5}=\frac{24}{5}$$

And $a=3-3=0$

Hence, the two points on the ellipse are $\left(\frac{24}{5},-\frac{9}{5}\right)$ and (0, 3).

Now, slope of the tangent at $\left(\frac{24}{5},-\frac{9}{5}\right)$ is

$$y'=\frac{-\frac{24}{5}}{4\times-\frac{9}{5}}=\frac{24}{36}=\frac{2}{3}$$

Hence, the equations of tangent at $\left(\frac{24}{5},-\frac{9}{5}\right)$ is

$$\left(y+\frac{9}{5}\right)=\frac{2}{3}\left(x-\frac{24}{5}\right)$$
$$3y+\frac{27}{5}=2x-\frac{48}{5}$$
$$15y+27=10x-48$$
$$10x-15y=75$$
$$\boxed{2x-3y=15} \text{ or } \boxed{y=\frac{2}{3}x-5}$$

Slope of the tangent at (0, 3) is

$$y' = -\frac{0}{12} = 0$$

This means at (0, 3) the tangent is horizontal then the equation of the tangent at (0, 3) is

$$(y-3) = 0(x-0)$$
$$\Rightarrow y-3 = 0$$
$$\Rightarrow \boxed{y=3}$$

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Consider the equation $xy'' + y' + xy = 0$.

(a)

To find $J'(0)$ value, use the Bessel function of zeroed order $J(x)$.

The Bessel function of zeroed order $J(x)$ is $J(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$.

Substitute the value $x = 0$ into the Bessel function,

$$J(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

$$J(0) = 1 - \frac{(0)^2}{2^2} + \frac{(0)^4}{2^2 \cdot 4^2} - \frac{(0)^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$
$$= 1$$

The derivative of the Bessel function is,

$$J(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

$$J'(x) = -\frac{2x}{2^2} + \frac{4x^3}{2^2 \cdot 4^2} - \frac{6x^5}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

Substitute the value $x = 0$ into the derivative of the Bessel function,

$$J'(x) = -\frac{2x}{2^2} + \frac{4x^3}{2^2 \cdot 4^2} - \frac{6x^5}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

$$J'(0) = -\frac{2(0)}{2^2} + \frac{4(0)^3}{2^2 \cdot 4^2} - \frac{6(0)^5}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$
$$= 0$$

Hence, the value of the derivative of the Bessel function at $x = 0$ is 0 i.e. $\boxed{J'(0) = 0}$.

(b)

To find the second derivative, apply the implicit differentiation to the first derivative.

$$J(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\ldots$$
$$J'(x)=-\frac{2x}{2^2}+\frac{4x^3}{2^2\cdot 4^2}-\frac{6x^5}{2^2\cdot 4^2\cdot 6^2}+\ldots$$

Apply the implicit differentiation to the above equation,

$$J'(x)=-\frac{2x}{2^2}+\frac{4x^3}{2^2\cdot 4^2}-\frac{6x^5}{2^2\cdot 4^2\cdot 6^2}+\ldots$$
$$J''(x)=-\frac{2}{2^2}+\frac{12x^2}{2^2\cdot 4^2}-\frac{30x^4}{2^2\cdot 4^2\cdot 6^2}+\ldots$$

Substitute the value $x=0$ into the derivative of the Bessel function,

$$J''(x)=-\frac{2}{2^2}+\frac{12x^2}{2^2\cdot 4^2}-\frac{30x^4}{2^2\cdot 4^2\cdot 6^2}+\ldots$$
$$J''(0)=-\frac{2}{2^2}+\frac{12(0)^2}{2^2\cdot 4^2}-\frac{30(0)^4}{2^2\cdot 4^2\cdot 6^2}+\ldots$$
$$=-\frac{2}{2^2}+0-0+\ldots$$
$$=-\frac{1}{2}$$

Hence, the value of the second derivative of the Bessel function at $x=0$ is $-\frac{1}{2}$ i.e.

$$\boxed{J''(0)=-\frac{1}{2}}.$$

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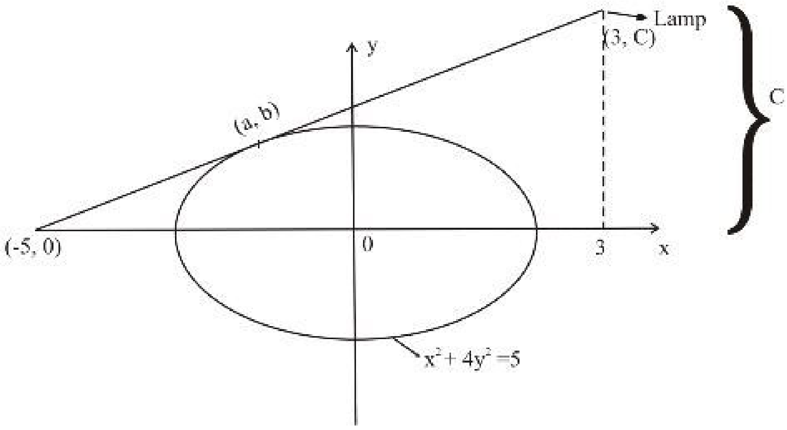

Equations of the ellipse is

$$x^2+4y^2=5 \qquad \text{--- (1)}$$

By differentiating both sides with respect to x, we get

$$2x+8yy'=0$$
$$\Rightarrow x+4yy'=0$$
$$\Rightarrow y'=-\frac{x}{4y} \qquad \text{--- (2)}$$

Let this tangent line touches the ellipse at (a, b).
Then slope of the tangent is

$$y'=-\frac{a}{4b}$$

Then the equation of the tangent at (a, b) is

$$(y-b)=-\frac{a}{4b}(x-a)$$
$$\Rightarrow 4by-4b^2=a^2-ax$$
$$\Rightarrow 4by+ax=a^2+4b^2 \qquad \text{--- (3)}$$

Now this tangent line passes though (-5, 0) so this point will satisfy the equation of tangent.
Thus

$$\Rightarrow 0 - 5a = a^2 + 4b^2$$

$$\Rightarrow -5a = a^2 + 4b^2 \qquad \text{--- (4)}$$

Now the point (a, b) lies on the ellipse of this will satisfy the equation of ellipse

$$a^2 + 4b^2 = 5 \qquad \text{--- (5)}$$

Now the equation (4) becomes

$$-5a = 5$$

$$\Rightarrow a = -1$$

Putting this value of a in the equations (5), we get

$$1 + 4b^2 = 5$$

$$4b^2 = 4$$

$$b^2 = 1 \Rightarrow b = \pm 1$$

We have the point (-1, 1) and (-1, -1), But the lamp is located in positive y direction, so we will consider only the point (-1, 1).

Slope of the tangent at (-1, 1) is $y' = \dfrac{1}{4}$

Then the equation of the tangent is

$$(y-1) = \frac{1}{4}(x+1)$$

Or $\quad 4y - 4 = x + 1$

Or $\quad 4y - x = 5$

Now suppose that the lamp is at a distance C from the x-axis, so that the co-ordinates of the point at which the lamp is located are (3, C).
The tangent line $4y - x = 5$ is passing though this point (3, C). Hence, there co-ordinates will satisfy the equation of the tangent.

Then we have

$$4C - 3 = 5$$

Or $\quad 4C = 5 + 3$

Or $\quad 4C = 8$

Or $\quad C = 2$

Hence, the lamp is located at the distance of $\boxed{\text{2 units}}$ from x-axis.