DEFINITE INTEGRALS

Q.1)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx$
Sol.1)	$I \ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx$ Here $f(x) \ = \log\left[\frac{2-\sin x}{2+\sin x}\right]$ $f(-x) \ = \log\left[\frac{2-\sin(-x)}{2+\sin(-x)}\right]$ $= \log\left(\frac{2+\sin x}{2-\sin x}\right)$ $= \ -\log\left(\frac{2-\sin x}{2+\sin x}\right)$ ....$\left[\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)\right]$ $f(-x) \ = -f(x)$ $\therefore \ f(x) \rightarrow \ odd\ function$ $\therefore \ I \ = \ 0$ Ans ....
Q.2)	$I \ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2 x\, dx$
Sol.2)	I = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2\, x\, dx$ $f(x) \ = \ sin^2\, x$ $f(-x) \ = sin^2(-x) \ = (-\sin x)^2 \ = sin^2 x = f(x)$ $\therefore \ f(x)$ $\rightarrow$ an even function $\therefore \ I \ = 2\int_0^{\frac{\pi}{2}} sin^2\, x\, dx$ ....$\{\because \ \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx\}$ $I \ = \ \frac{2}{2}\int_0^{\frac{\pi}{2}} 1 - cos\ (2x)\, dx$ $I \ = \ \left[x - \ \frac{sin(2x)}{2}\right]_0^{\frac{\pi}{2}}$ $I \ = \ \left[\frac{\pi}{2} - \frac{sin(\pi)}{2}\right] - [0]$ $I \ = \ \frac{\pi}{2}$ $Ans ....$ $\{\because \ \ sin\ (\pi) = 0\}$
Q.3)	$I \ = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} sin^4 x\, dx$
Sol.3)	$I \ = \ \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} sin^4 x\, dx$ Clearly $f(x)$ is an even function $\therefore \ I \ = 2\int_0^{\frac{\pi}{4}} sin^4 x \ \ dx$ $= \ 2\int_0^{\frac{\pi}{4}} (sin^2 x)^2\, dx$

	$= 2\int_0^{\frac{\pi}{4}} \left(\frac{1-cos(2x)}{2}\right)^2 dx$ $= \frac{2}{4}\int_0^{\frac{\pi}{4}} 1 + cos^2(2x) - 2\,cos(2x)\,dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 + \frac{1+cos(4x)}{2} - 2\,cos\,(2x)\,dx$ $= \frac{1}{4}\int_0^{\frac{\pi}{4}} 3 + cos(4x) - 4\,cos(2x)\,dx$ $= \frac{1}{4}\left[3x + \frac{sin(4x)}{4} - 2\,sin(2x)\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{4}\left[\left(\frac{3\pi}{4} + \frac{sin\,\pi}{4} - 2sin\left(\frac{\pi}{4}\right)\right) - (0 + 0 - 0)\right]$ $= \frac{1}{4}\left[\frac{3\pi}{4} + 0 - 2\right]$ ……… $\{\because sin(\pi) = sin(2\pi) = 0\}$ $= \frac{3\pi}{16} - \frac{1}{2}$ Ans …..
Q.4)	$\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx$
Sol.4)	$I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx$ rationalize $I = \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}}\,dx$ $I = a\int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}}\,dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\,dx$ $Let\ f(x) = \frac{1}{\sqrt{a^2-x^2}}$; $g(x) = \frac{x}{\sqrt{a^2-x^2}}$ $f(-x) = \frac{1}{\sqrt{a^2(-x)^2}}$; $g(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}}$ $f(-x) = \frac{1}{\sqrt{a^2-x^2}}$; $g(-x) = \frac{-x}{\sqrt{a^2-x^2}}$ $f(-x) = f(x)$; $g(-x) = -g(x)$ $f(x) \rightarrow even\ function$ $g(x) \rightarrow odd\ function$ $\therefore I = 2a\int_0^{a} \frac{1}{\sqrt{a^2-x^2}}\,dx - 0$ …… $\left\{\because \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx : f(x) even \\ 0 : f(x) odd \end{cases}\right\}$ $I = 2a\left(sin^{-1}\frac{x}{a}\right)_0^a$ $I = 2a\left(sin^{-1}(1) - sin^{-1}(0)\right)$ $I = 2a\left(\frac{\pi}{2}\right)$ $I = a\,\pi$ ans.
Q.5)	$I = \int_{-\pi}^{\pi} \frac{2x\,(1+\,sin\,x)}{1+cos^2x}\,dx$
Sol.5)	$I = \int_{-\pi}^{\pi} \frac{2x\,(1+\,sin\,x)}{1+cos^2x}\,dx$

	$I = \int_{-\pi}^{\pi} \frac{2x + 2x \sin x)}{1+\cos^2 x} dx$ $I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + 2\int_{-\pi}^{\pi} \frac{x \sin x}{1+ \cos^2 x} dx$ $f(x) = \frac{2x}{1+\cos^2 x}$; $g(x) = \frac{x \sin x}{1+\cos^2 x}$ $f(-x) = \frac{-2x}{1+\cos^2 x}$; $g(-x) = \frac{(-x)\sin(-x)}{1+\cos^2 x}$ $f(-x) = -f(x)$; $g(-x) = \frac{x \sin x}{1+\cos^2 x} = g(x)$ $f(x) \rightarrow odd\ function$; $g(x) \rightarrow odd\ function$ $\therefore\ I = 0 + 4\int_{0}^{\pi} \frac{x \sin x}{1+ \cos^2 x} dx$ ↙ Removal of x (already done) Do yourself $I = \pi^2$ ans.
Q.6)	$I = \int_{0}^{2\pi} \cos^5 x\, dx$
Sol.6)	$I = 2\int_{0}^{\pi} \cos^5 x\, dx$ ……$\left\{\int_{0}^{2a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx\ \ f(2a-x) = f(x)\right\}$ $I = 2\int_{0}^{\pi} \cos^5 x\, dx$ …….(1) $I = 2\int_{0}^{\pi} \cos^5(\pi - x)\, dx$ …….$\left[\int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(a-x)\, dx\right]$ $I = -2\int_{0}^{\pi} \cos^5 x\, dx$ ……(2) ……..$[\because\ \cos(\pi - x) = -\cos x]$ $(1) + (2)$ $2I = 0$ $I = 0$ Ans……
Q.7)	$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x^2 \tan x + \sin x - x^2)\, dx$
Sol.7)	I = $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^3 + x^2 \tan x + \sin x\, dx - \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^2\, dx$ $f(x) = x^3 + x^2 \tan x + \sin x$ $g(x) = x^2$ $f(-x) = -x^3 - x^2 \tan x - \sin x$ $g(-x) = (-x)^2 = x^2$ $f(-x) = -(x^3 + x^2 \tan x + \sin x)$ $g(-x) = g(x)$ $f(-x) = -f(x)$ $g(x) \rightarrow even\ function$ $\therefore\ f(x) \rightarrow odd\ function$ $\therefore I = 0 - 2\int_{0}^{\frac{\pi}{2}} x^2\, dx$ …….{property-VII}

	$= -2\left(\frac{x^3}{3}\right)_0^{\frac{\pi}{2}}$ $= -2\left(\frac{\pi^3}{24}\right)$ $= \frac{-\pi^3}{12}$ ans.
Q.8)	Evaluation using limit as sum $I = \int_2^4 3x - 2\, dx$
Sol.8)	Here $a = 2, b = 4\ and\ nh = 4 - 2 = 2$ $I = \lim_{h\to 0} h[f(2) + f(2+h) + f(2+2h) \ldots\ldots f(2+(n-1)h)]$ $= \lim_{h\to 0} h[(6-2) + (6+3h-2) + (6+6h-2) + \ldots\ldots(6+3(n-1)h-2)]$ $= \lim_{h\to 0} h[4 + (4+3h) + (4+6h) + \ldots\ldots(4+3(n-1)h)]$ $= \lim_{h\to 0} h[(4+4+4\ldots\ldots n\ term) + (3h+6h+\ldots\ldots 3(n-1)h)]$ $= \lim_{h\to 0} h\left[4n + 3h\left(1+2+\ldots\ldots(n-1)\right)\right]$ $= \lim_{h\to 0} h\left[4n + 3h\,.\ \frac{n(n-1)}{2}\right]$ $= \lim_{h\to 0}\left[4nh + 3\ \frac{(nh)(nh-h)}{2}\right]$ $Put\ nh = 2$ $\therefore\ I = \lim_{h\to 0}\left[8 + \frac{3(2)(2-h)}{2}\right]$ $= 8 + \frac{3(2)(2)}{2}$ $I = 14\quad Ans \ldots.$ $Check: I = \int_2^4 3x - 2\, dx = \left[\frac{3x^2}{2} - 2x\right]_2^4 = [(24-8) - (6-4)]$ $= 16 - 2 = 14$
Q.9)	$I = \int_{-1}^{2} 3x^2 + 2x - 5\, dx$
Sol.9)	Here $a = -1,\ b = 2\ and\ nh = 3$ $I = \lim_{h\to 0} h[f(-1) + f(-1+h) + f(-1+2h) + \ldots\ldots f(-1+(n-1)h)]$ $= \lim_{h\to 0} h\big[(3-2-5) + [3(1+h^2-2h) - 2 + 2h - 5] + [3(1+4h^2-4h) - 2 +$ $4h - 5] + \ldots\ldots[3(1+(n-1)^2h^2 - 2(n-1)h) - 2 + 2(n-1)h - 5]\big]$ $= \lim_{h\to 0} h[(-4) + (3n^2 - 4h - h) + (12h^2 - 8h - 4) + \ldots\ldots(3(n-1)^2h^2 -$ $4(n-1)h - 4)]$ $= \lim_{h\to 0} h[(-4-4-4\ldots\ldots n\ terms) + (3h^2 + 12h^2 + \ldots\ldots 3(n-1)^2h^2) + (-4h - 8\ldots\ldots -4(n-1)h)]$ $= \lim_{h\to 0} h\left[-4n + 3h^2(1^2 + 2^2 \ldots\ldots(n-1)^2) - 4h\left(1+2+3\ldots\ldots(n-1)\right)\right]$

	$= \lim_{h\to 0} h\left[-4n + 3h^2 \cdot \frac{n(n-1)(2n-1)}{6} - 4h \cdot \frac{n(n-1)}{2}\right]$ $= \lim_{h\to 0}\left[-4nh + \frac{(nh)(nh-h)(2nh-h)}{2} - 2(nh)(nh-h)\right]$ $Put\ nh = 3$ $\therefore\ I = \lim_{h\to 0}\left[-12 + \frac{(3)(3-h)(6-h)}{2} - 2(3)(3-h)\right]$ $= \left[-12 + \frac{(3)(3)(6)}{2} - 2(3)(3)\right]$ $= -12 + 27 - 18$ $= -30 + 27$ $I = -3 \quad Ans ....$ Check: $I = \int_{-1}^{2} 3x^2 + 2x - 5\,dx$ $= (x^3 + x^2 - 5x)_{-1}^{2} = (8 + 4 - 10) - (-1 + 1 + 5)$ $= 2 - 5 = -3 \quad Ans .....$
Q.10)	$I = \int_2^5 e^{2x+1}\,dx$
Sol.10)	Here $a = 2$, $b = 5$ & $nh = 3$ $I = \lim_{h\to 0} h[f(2) + f(2+h) + f(2+2h) + \ .....(f(2 + (n-1)h))]$ $= \lim_{h\to 0} h[e^{4+1} + e^{4+2h+1} + e^{4+4h+1} + \ ......e^{4+2(n-1)h+1}]$ $= \lim_{h\to 0} h[e^5 + e^{2h+5} + e^{4h+5} + \ .....e^{2(n-1)h-15}]$ $= e^5 \lim_{h\to 0} h[1 + e^{2h} + e^{4h} + \ .....e^{2(n-1)h}]$ $G.P.\ a = 1\ ,\ r = e^{2h}$ $= e^5 \lim_{h\to 0} h\left[1 \cdot \left(\frac{(e^{2h})^n - 1}{e^{2h}-1}\right)\right]$ $= e^5 \lim_{h\to 0}\left[\frac{e^{2nh}-1}{e^{2h}-1}\right]$ $= e^5 \lim_{h\to 0} h\left[\frac{e^{2nh}-1}{\frac{e^{2h}-1}{2h}\times 2h}\right] \quad ......\{adjustment\}$ $= \frac{e^5[e^6-1]}{\lim_{h\to 0}\left(\frac{e^{2h}-1}{2h}\right)\times 2} \quad ....\{\because\ nh = 3\}$ $I = \frac{e^5(e^6-1)}{2} \quad Ans \quad .... \left\{\because\ \lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1\right\}$ $Check:\ I = \int_2^5 e^{2x+1}\,dx$ $= \left(\frac{e^{2x+1}}{2}\right)_2^5 = \frac{1}{2}[e^{11} - e^5] = \frac{e^5(e^6-1)}{2}$ ans,